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On the fundamental conjecture of GLC IV.

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This paper belongs to the series of papers [2], [3], [4]. In [2] the auther has proved the following theorem:

The end-sequence of a normal proof-figure in G^1LC is proved withouot cut.

The logical system G^1LC is a subsystem of GLC defined in [1], where we have enounced the "fundamantal conjecture" that every provable sequence in GLC would be provable without cut. In this paper we shall generalize the above result of [2] in proving a theorem of the same form in GLC, when the meaning of "normal" is also widened than in [2] (even restricted to the case of G^1LC). We shall prove this result in Chap. II after preparations in Chap. I. At the end of the paper, we shall also prove a lemma (as Lemma 2) which we have used in [4] without proof.

Chapter I. The proof-figure of *GLC*

The whole paper is based on GLC as was explained in [1], chapter I. However we shall modify some notions as follows.

§ I. Symbols

As in [1], we use the following symbols:

1.1. Variables

1.1.1. t-variables (t means 'term')

1.1.1.1. t-variables without argument-place, which is called variables of type (0) in [1].

Free ones: $a_0, b_0, c_0, \cdots$

Bound ones: $x_0, y_0, z_0, \cdots$

(In this paper, we have not to distinguish special t-variables and special f-variables, among free t-variables and free f-variables in general.)

1.1.1.2. t-variables of type $(n_1,\cdots,n_i)$ $(i, n_1,\cdots,n_i=1,2,3,\cdots)$, which is called functions of type $(n_1,\cdots,n_i)$ in [1].

Free ones: $a(n_1,\cdots,n_i)$, $b(n_1,\cdots,n_i),\cdots$

Bound ones: $x(n_1,\cdots,n_i)$, $y(n_1,\cdots,n_i),\cdots$

1.1.2. f-variables (f means 'formula')

1.1.2.1. f-variables without argument-place, (which is not used in [1]).

Free ones: $\alpha_0, \beta_0, \gamma_0,\cdots$

Bound ones: $\varphi_0, \psi_0,\cdots$

1.1.2.2. f-variables of type $(n_1,\cdots,n_i)$ $(i, n_1,\cdots,n_i=1,2,3,\cdots)$, which is called variables of type $(n_1,\cdots,n_i)$ in [1].

Free ones: $\alpha(n_1,\cdots,n_i)$, $\beta(n_1,\cdots,n_i),\cdots$

Bound ones: $\varphi(n_1,\cdots,n_i)$, $\psi(n_1,\cdots,n_i),\cdots$

1.2. Logical symbols: $\neg$, $\wedge$, $\forall$.

(We do not use the symbols $\vee$ and $\exists$ in this paper.)

If no confusion is likely to occur, we use $\alpha;\beta;\cdots;\varphi;\psi;\cdots$ for $\alpha_0, \alpha(n_1,\cdots,n_i)$; $\beta_0, \beta(n_1,\cdots,n_i);\cdots$; $\varphi_0, \varphi(n_1,\cdots,n_i)$; $\psi_0, \psi(n_1,\cdots,n_i);\cdots$ respectively as in [1].

§ 2. Several definitions

In this section, the notions and notations are as in [1] § 2, § 3, § 4 and § 5. Now, we define some new concepts.

2.1. t-varieties, f-varieties and words

Terms and functionals will be called t-varieties. Formulas and varieties other than terms will be called f-varieties. We use the notations $T, T_1, T_2,\cdots$ for t-varieties and $F, F_1, F_2,\cdots$ for f-varieties

Let $\mathfrak{a}$ be a free variable (which means a free t-variable or a free f-variable), and L be a t-variety or f-variety. L is said to be of the same type with $\mathfrak{a}$, if $\mathfrak{a}$ is a t-variable and L is a t-variety with same type with $\mathfrak{a}$, or $\mathfrak{a}$ is an f-variable and L is an f-variety with same type with $\mathfrak{a}$.

Let $L(a_1,\cdots,a_n,\alpha_1,\cdots,\alpha_m)$ be a t-variety or an f-variety. Then a figure $L(x_1,\cdots,x_n,\varphi_1,\cdots,\varphi_m)$ is called a *t-word* or an *f-word* respectively, provided that $x_1,\cdots,x_n,\varphi_1,\cdots,\varphi_m$ are not contained in $L(a_1,\cdots,a_n,\alpha_1,\cdots\alpha_m)$. A t-word or an f-word is called a *word*, too. A word is called an *essential word*, if it is neither a t-variety nor an f-variety.

2.2. Let $*$ be a logical symbol or an f-variable in a formula or an f-variety E. $*$ is called *improper* in E, if and only if $*$ is contained in an argument-place of an f-variable or a t-variable in E. $*$ is called *proper* in E in all other cases. Moreover, $*$ is called *degenerate* in E, if and only if $*$ is contained in an argument-place of a t-variable in E; *non-degenerate* in E in all other cases.

2.3. The indication $L(\mathfrak{a})$ is called *void*, if and only if the indicated place of $\mathfrak{a}$ in $L(\mathfrak{a})$ is void.

2.4. Indication of t- or f-varieties

Let $\mathfrak{a}$ be free variable, and L be t- or f-varieties of same type with $\mathfrak{a}$. If M is a t- or f-variety and is equal to $N(\mathfrak{a})\binom{L}{\mathfrak{a}}$, then we call the totality of M, $N(\mathfrak{a})$, L and $\mathfrak{a}$, which is denoted by $\{N(\mathfrak{a}); L; \mathfrak{a}\}$, 'an indication of L for M'. If no confusion is likely to occur, we say that this indication is of the form $N(L)$.

An indication $\{N(\mathfrak{a}); L; \mathfrak{a}\}$ is called *void* or *non-void*, according as the indicated place of $\mathfrak{a}$ in $N(\mathfrak{a})$ is void or non-void.

§ 3. Proof-figure

The concept of proof-figure is explained as in [1], § 6. We list here the inference-schemata. Only $\wedge$-right schema is modified.

3.1. Inference-schemata

I) Inference-schemata on structure of sequences

'Weakening'

left: $\dfrac{\Gamma \rightarrow \Delta}{D, \Gamma \rightarrow \Delta}$ right: $\dfrac{\Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta, D}$

'Contraction'

left: $\dfrac{D, D, \Gamma \rightarrow \Delta}{D, \Gamma \rightarrow \Delta}$ right: $\dfrac{\Gamma \rightarrow \Delta, D, D}{\Gamma \rightarrow \Delta, D}$

'Exchange'

left: $\dfrac{\Gamma, C, D, \Pi \rightarrow \Delta}{\Gamma, D, C, \Pi \rightarrow \Delta}$ right: $\dfrac{\Gamma \rightarrow \Delta, C, D, \Lambda}{\Gamma \rightarrow \Delta, D, C, \Lambda}$

'Version'

$$\frac{\Gamma \rightarrow \Delta}{\tilde{\Gamma} \rightarrow \tilde{\Delta}}$$

In these inference-figures, C, D in the upper sequence are called the *subformulas* of the inference-figure, and C, D in the lower sequence are called the *chief-formulas* of the inference.

II) 'Cut'

$$\frac{\Gamma \rightarrow \Delta, D \qquad D, \Pi \rightarrow \Lambda}{\Gamma, \Pi \rightarrow \Delta, \Lambda}$$

III) Inference-schemata on logical symbols

$\neg$

left: $\dfrac{\Gamma \rightarrow \Delta, A}{\neg A, \Gamma \rightarrow \Delta}$ right: $\dfrac{A, \Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta, \neg A}$

$\wedge$

left (1): $\dfrac{A, \Gamma \rightarrow \Delta}{A \wedge B, \Gamma \rightarrow \Delta}$ right: $\dfrac{\Gamma \rightarrow \Delta, A \qquad \Pi \rightarrow \Lambda, B}{\Gamma, \Pi \rightarrow \Delta, \Lambda, A \wedge B}$

left (2): $\dfrac{B, \Gamma \rightarrow \Delta}{A \wedge B, \Gamma \rightarrow \Delta}$

$\forall$ on t-variable

left: $\dfrac{F(T), \Gamma \rightarrow \Delta}{\forall x F(x), \Gamma \rightarrow \Delta}$

(T is an arbitrary t-variety of the same type with x.)

right: $\dfrac{\Gamma \rightarrow \Delta, F(a)}{\Gamma \rightarrow \Delta, \forall x F(x)}$

(There is no a in the lower sequence.) a is the ***eigen-t-variable*** of this inference.

$\forall$ on f-variable

left: $\dfrac{F(G), \Gamma \rightarrow \Delta}{\forall \varphi F(\varphi), \Gamma \rightarrow \Delta}$

(G is an arbitrary f-variety of the same type with φ.)

right: $\dfrac{\Gamma \rightarrow \Delta, F(\alpha)}{\Gamma \rightarrow \Delta, \forall \varphi F(\varphi)}$

(There is no α in the lower sequence.) α is the ***eigen-f-variable*** of this inference.

3.2 Let $\dfrac{F(L), \Gamma \rightarrow \Delta}{\forall \mathfrak{x} F(\mathfrak{x}), \Gamma \rightarrow \Delta}$ be an inference $\forall$ left. Then, the indication $\{F(\mathfrak{a}) ; L ; \mathfrak{a}\}$ for the subformula of this inference is called the indication of this inference.

3.3. formula in a proof-figure

As in [4], we take acconut of the place occupied by a formula (or a sequence or an inference) A in a proof-figure $\mathfrak{P}$, when we speak of A *in* $\mathfrak{P}$.

Let A be a formula in a proof-figure $\mathfrak{P}$. If A is in the right side or in the left side of a sequence in $\mathfrak{P}$, then A is called in the right side or in the left side in $\mathfrak{P}$ respectively.

3.4. Successor

We define the *successor* of a formula A in the upper sequence of the inferences I), II) and III) as the formula in the lower sequence of the same inferences defined as follows. (cf. [2])

3.4.1. If A is a cut-formula, then there is no successor of A.

3.4.2. If A is a subformula of the inference other than cut and exchange, then the successor of A is the chief-formula of the inference.

3.4.3. If A is a subformula of exchange, then the successor of A is a chief-formula with the same form as A in this exchange.

3.4.4. If A is a k-th formula of Γ, Π, Δ or Λ in the upper sequence, then the successor of A is the k-th formula of Γ (or $\tilde{\Gamma}$), Π, Δ (or $\tilde{\Delta}$), Λ in the lower sequence respectively.

3.5. We use the definitions in [2], 2.1, 2.2, 2.3, 2.4, 2.7, 2.8, 2.10, 6.1, 6.2 and in [3], 2.1, 2.2.

Let $\mathfrak{T}$ be the fibre through a formula A in a proof-figure. Then the part of $\mathfrak{T}$ beginning with the beginning formula of $\mathfrak{T}$ and ending with A, is called a *fibre to A*.

§ 4. Original formula

4.1. Extension of indication

Let A be a formula in a proof-figure $\mathfrak{P}$, $F(H)$ an indication for A, and B the predecessor of A. Then we define the indication I of H for B over $F(H)$ as follows.

4.1.1. If B is equivalent to A, then I is same as $F(H)$.

4.1.2. Let A be the chief-formula of an inference $\neg$ and $F(\alpha)$ have a proper logical symbol, that is, $F(\alpha)$ be of the form $\neg G(\alpha)$. We define the indication I as $G(H)$.

4.1.3. Let A be the chief-fromula of an inference $\wedge$ and $F(\alpha)$ have a proper logical symbol, that is, $F(\alpha)$ be of the form $G_1(\alpha)\wedge G_2(\alpha)$. Then we define the indication I as $G_1(H)$ or $G_2(H)$, according as B is the first or the second predecessor of A.

4.1.4. Let A be the chief-formula of an inference $\forall$ and $F(\alpha)$ have a proper logical symbol, that is, $F(\alpha)$ be of the form $\forall \mathfrak{x} G(\alpha, \mathfrak{x})$, and B the subformula of the form $G(H, L)$ of this inference where L is a free variable or a variety of the same type with $\mathfrak{x}$. Then we define the indication I as $\{G(\alpha, L); H; \alpha\}$.

4.1.5. Let A be a chief-formula of a logical inference and $F(\alpha)$ have no proper logical symbol. Then we define the indication I as the void indication, that is, as $\{B; H; \alpha\}$.

Let T be a fibre to A, and I an indication for A. Let A' be the predecessor of A in T, A'' the predecessor of A' in $T, \cdots$. Then we have the indications I' for A' over I, I'' for A'' over $I', \cdots$, These indications I', $I'', \cdots$ are called *over-indications* of I in T.

4.2. Original formula

Let A be a formula in a proof-figure $\mathfrak{P}$ and $I=\{F(\alpha); H; \alpha\}$ be a non-void indication for A. Let H be of the form $\{\varphi_1, \cdots, \varphi_n\}$ $G(\varphi_1, \cdots, \varphi_n)$. If $\mathfrak{T}$ is a fibre to A, then for every formula of T the over-indication of I is defined. Then there arise the following three cases:

4.2.1. There exists a formula D in $\mathfrak{T}$, for which the over-indication of I is $\{\alpha[L_1(\alpha), \cdots, L_n(\alpha)]; H; \alpha\}$. In this case the undermost formula B with this property is called the *original formula* in $\mathfrak{T}$ for the indication I. Clearly, if $\mathfrak{T}$ has an original formula for the indication I, then it is uniquely determined.

4.2.2. There exists no formula with the property stated in 4.1 and a non-void indication of H is defined for the beginning formula or the weakening formula of $\mathfrak{T}$.

4.2.3. There exists no formula with the property stated in 4.1 and the indication of H for the beginning formula or the weakening formula of $\mathfrak{T}$ is void. In this case we say that the indication I *vanishes* in $\mathfrak{T}$. Then there exists the overmost formula C in $\mathfrak{T}$, for which the non-void indication is defined. Then clearly C is a subformula of an inference $\wedge$.

'B is an original formula of the indication I for A' means that there exists a fibre $\mathfrak{T}$, which contains B and A and the original formula in $\mathfrak{T}$ for I is B. An original formula of the indication of $\forall$ left on f-variable is called an original formula of this inference.

§ 5. Logical symbol in an f-word

Let $\sharp$ be a proper logical symbol in an f-word A. Then we define recursively as follows;

5.1. If $\sharp$ is an outermost logical symbol of A, then $\sharp$ is ***positive*** in A.

5.2. Let A be of the form $\neg B$ and $\sharp$ a logical symbol of B. Then $\sharp$ is ***positive*** or ***negative*** in A, according as $\sharp$ is negative or positive in B.

5.3. Let A be of the form $B \wedge C$ and $\sharp$ a logical symbol in B or C. If $\sharp$ is positive in B or C, then $\sharp$ is ***positive*** in A. If $\sharp$ is negative in B or C, then $\sharp$ is ***negative*** in A.

5.4. Let A be of the form $\forall x G(x)$ or $\forall \varphi F(\varphi)$ and $\sharp$ a logical symbol of $G(x)$ or $F(\varphi)$. Then $\sharp$ is ***positive*** or ***negative*** in A, according as $\sharp$ is positve or negative in $G(x)$ or $F(\varphi)$ respectively.

Let $\sharp$ be a proper logical symbol in an arbitrary f-variety $\{\varphi_1, \cdots, \varphi_n\} F(\varphi_1, \cdots, \varphi_n)$. Then we say that $\sharp$ is positive or negative in $\{\varphi_1, \cdots, \varphi_n\} F(\varphi_1, \cdots, \varphi_n)$ according as $\sharp$ is positive or negative in $F(\varphi_1, \cdots, \varphi_n)$.

Let $\sharp$ and $\natural$ be two proper logical symbols in an f-variety or an f-word A. If $\sharp$ and $\natural$ are both positive in A or $\sharp$ and $\natural$ are both negative in A, then we say that $\sharp$ is positive to $\natural$ or $\natural$ is positive to $\sharp$. Otherwise we say that $\sharp$ is negative to $\natural$ or $\natural$ is negative to $\sharp$.

Chapter II. The normal proof-figure

§ 1. The normal proof-figure

A proof-figure $\mathfrak{P}$ satisfying the following conditions 1.1 and 1.2 are called *normal.*

1.1. Let A be a beginning formula with proper logical symbols in $\mathfrak{P}$ and suppose that a fibre $\mathfrak{T}$ begins with A and ends with a cut-formula in a cut $\mathfrak{J}$. Moreover, let $\mathfrak{T}'$ be an arbitrary fibre beginning with a beginning formula and ending with another cut-formula of $\mathfrak{J}$.

Then the beginning formula of $\mathfrak{T}'$ contains no proper logical symbol.
1.2. Let $\mathfrak{J}$ be an arbitrary implicit inference $\forall$ left on f-variable in $\mathfrak{P}$. Let $\mathfrak{J}$ be of the following form

$$\frac{F(H), \Gamma \rightarrow \Delta}{\forall \varphi F(\varphi), \Gamma \rightarrow \Delta}$$

Moreover, let $\mathfrak{T}$ be a fibre through the chief-formula of $\mathfrak{J}$ beginning with a beginning formula A. Then every proper $\forall$ on f-variable in $\forall \varphi F(\varphi)$ is positive to $\forall \varphi F(\varphi)$ and A contains no proper logical symbol.

The aim of this chapter is to prove the following theorem:

THEOREM 1. The end-sequence of a normal proof-figure is provable without cut.

This is clearly a generelization of the result of [2]. As all the circumstances are as in [2], we confine ourselves to give necessary remarks on the modification of the proof.

§ 2. Rank of a formula

We define the *rank* of a formula A as follows.

2.1. If A contains no proper logical symbol, then the rank of A is zero.

2.2. If A is of the form $\neg B$, $\forall x C(x)$ or $\forall \varphi F(\varphi)$, then the rank of A is $r+1$, where r is the rank of B, $C(a)$ or $F(\alpha)$ respectively.

2.3. If A is of the form $B \wedge C$, then the rank of A is $r+1$, where r is the maximal number of the ranks of B and C.

§ 3. Degree of a formula in a normal proof-figure

We define the *degree* of a formula D in a normal proof-figure as follows.

3.1. The degree of a beginning formula or a weakening formula is one.

3.2. If D is not the chief-formula of an inference on logical symbol or a contraction, then the degree of D is equal to the degree of the predecessor of D.

3.3. If D is the chief-formula of a contraction, then the degree of D is the maximal number of the degrees of the predecessors of D.

3.4. If D is the chief-formula of an inference on the logical symbol other than $\forall$ left on f-variable, then the degree of D is $d+1$, where

d is the maximal number of the degrees of the predecessors of D.

3.5. Let D be the chief formula of an inference $\mathfrak{J}$ $\forall$ left on f-variable and of the form $\forall\varphi F(\varphi)$. We define the degree of D as the number $\max(a+b, c+1)$ where a is the rank of $\forall\varphi F(\varphi)$ and b is the maximal number of the degrees of the original formulas of $\mathfrak{J}$ (If there is no original formulas of $\mathfrak{J}$, then put $b=1$), and c is the degree of the predecessor of D.

We define the degree of a cut as the maximal number of the degrees of the cut-formulas of this cut.

§ 4. Potential

A normal proof-figure is called a proof-fgire with ***potential***, if to each sequence of this proof-figure is assigned the natural number called its potential satisfying the following conditions.

4.1. If a sequence $\mathfrak{S}_1$ is above a sequence $\mathfrak{S}_2$, then the potential of $\mathfrak{S}_1$ is not less than the potential of $\mathfrak{S}_2$.

4.2. If a sequence $\mathfrak{S}_2$ is an upper sequence of an inference other than cut and a sequence $\mathfrak{S}_2$ is the lower sequence of this inference, then the potential of $\mathfrak{S}_1$ is equal to the potential of $\mathfrak{S}_2$.

4.3. If $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are two upper sequences of a cut, then the potential of $\mathfrak{S}_1$ is equal to the potential of $\mathfrak{S}_2$.

4.4. If a sequence $\mathfrak{S}$ is an upper sequence of a cut, then the potential of $\mathfrak{S}$ is not less than the degree of this cut.

4.5. If a beginning sequence $D\rightarrow D$ contains proper logical symbols, and a fibre $\mathfrak{T}$ beginning with one of these D's ends with a cut-formula of a cut $\mathfrak{J}$, then the potential of the upper sequence of $\mathfrak{J}$ is not less than $\max(a, b+c)+1$, where a is the degree of $\mathfrak{J}$ and b is the maximal number of the degrees of any formulas related to one of two D's and c is the logical length of $\mathfrak{T}$.

We see easily that every normal proof-figure may be considered as a proof-figure with potential by introducing a potential. Therefore, to prove the theorem 1, we have only to prove that the end-sequence of a proof-figure with potential is provable without cut.

§ 5. The proof of theorem 1.

In this number, we shall prove the theorem 1. The proof is the same as 3.4–6.6 in [2] except using the following lemma instead of 6.6.1 in [2].

Lemma 1. Let A be a formula in a proof-figure $\mathfrak{P}$, and I an indication for A. Let $\mathfrak{T}$ be a fibre to A; $B_{\mathfrak{T}}$ will denote the original formula for I in $\mathfrak{T}$ if such formula exists; otherwise the beginning formula or the weakening formula of $\mathfrak{T}$. We suppose that, for every fibre $\mathfrak{T}$ to A, the part from $B_{\mathfrak{T}}$ to A is not affected by inference $\forall$ left on f-variable. Put furthermore
a the degree of A,
b the maximal number of the degrees of the original formulas for I (If there is no original formula for I, then put $b=1$),
c the maximal number of the logical lengths from $B_{\mathfrak{T}}$ to A,
d the rank of A.
Then we have

$$a \leqq b+d \qquad \text{and} \qquad c \leqq d.$$

This lemma is easily proved by induction on d.

§ 6.

Now, we prove the lemma of [4] in a generelized form.

Let A be a formula or an f-variety and $\sharp$ a proper logical symbol $\forall$ on f-variable in A. $\sharp$ is called '*semi-simple* in A', if and only if the following condition is fulfilled:

If $\sharp$ ties a proper $\forall$ on f-variable denoted by $\natural$, then $\natural$ is positive to $\sharp$.

A formula or an f-variety A is called 'semi-simple' if and only if every proper $\forall$ on f-variable in A is semi-simple in A.

According to the definition of normal proof-figure in § 1 in this chapter, we have clearly the following proposition.

6.1. Let $\mathfrak{P}$ be a proof-figure and suppose that every implicit beginning formula in $\mathfrak{P}$ contains no proper logical symbol. If every implicit formula in $\mathfrak{P}$ is semi-simple, then $\mathfrak{P}$ is normal.

Moreover, we prove easily the following propositions.

6.2. If $\neg A$ is semi-simple, then A is semi-simple.

6.3. If $A \wedge B$ is semi-simple, then A and B are semi-simple.

6.4. If $\forall x A(x)$ is semi-simple, then $A(a)$ is semi-simple.

6.5. If $\forall \varphi F(\varphi)$ is semi-simple, then $F(\alpha)$ is semi-simple.

Then by 6.1–6.5 and 6.8 in [1], we have the following lemma.

Lemma 2. The end-sequence of a proof-figure, whose every implicit formula is semi-simple, is provable without cut.

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