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Analytic vector functions of several complex variables.

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In this paper, we shall consider a system of k functions, which we shall call a vector function following Bochner-Martin*), of k complex variables. We shall show that various theorems of the theory of functions of a complex variable can be generalized to the case of vector functions. In our previous paper [2] in collaboration with Prof. S. Ozaki, we have established the expansion theorem and the estimation of derivatives for vector functions in polycylindrical domains. Now we shall study such functions in more general domains.

In § 1, we shall prove the expansion theorem and the residue theorem, and give a representation of derivatives and coefficients.

In § 2, we shall consider bounded vector functions, and generalize Gutzmer's inequality, Schur's estimation of coefficients and Landau-Dieudonné's theorem concerning the univalence radius of a hypersphere, etc. The estimation of coefficients was given by E. Peschl and F. Erwe [3] in the case of systems of functions of a complex variable. About the univalence radius some results were obtained by S. Takahashi [4].

In § 3, we shall generalize the argument principle in the case of a complex variable and obtain a formula giving the number of zero points of vector functions. The set of zero points of a single function of several complex variables forms a manifold, but the zero points of vector functions are in general isolated, so that we can speak of the number of them.

§ 1. General considerations

1. *Distance and norm.* We introduce the real coordinates $x_1, y_1, \cdots, x_k, y_k$ in the $2k$-dimensional Euclidean space and put $z_j = x_j + iy_j$,

*) See Bochner-Martin [1], Chap. VIII. § 5.

$j=1,\cdots,k$ and designate the coordinate of any point in the space as

$$z=\begin{pmatrix} z_1 \\ \vdots \\ z_k \end{pmatrix}.$$

Particularly, we denote by (0) a vector or matrix whose elements are all zero.

The distance between two points z, z' is defined by

$$|z-z'|=\sqrt{(z-z')^*(z-z')}\,.$$

Here and in the following, vectors and matrices marked with the symbol * denote the transposed conjugate vectors or matrices. The norm of any matrix $A=(a_{ij})$, $(i=1,\cdots,k\,; j=1,\cdots,n)$ is defined in the following two ways:

$$||A||=\underset{|t|>0}{\text{l. u. b.}}\,(|At|/|t|)=\underset{|u|=1}{\text{l. u. b.}}\,\sqrt{u^*A^*Au}\,,$$

$$[A]=\sqrt{\text{Tr.}\,(A^*A)}=\sqrt{\sum_{i,j}|a_{ij}|^2}\,,$$

where t and u are both k-tuple vectors. As is well-known, the former is the square root of the maximal characteristic value of A^*A, and the latter is that of the sum of the characteristic values of A^*A and so $||A||\leqq[A]$. In particular, we have for the unit matrix E:

$$||E||=1\,,\quad [E]=\sqrt{n}\,,$$

but, for any vector z, we have

$$||z||=[z]=|z|\,.$$

2. *Analyticity.* We assume that a complex function $f(z,\bar{z})=f(z_1, \bar{z}_1,\cdots,z_k,\bar{z}_k)$ is continuous and has the first partial derivatives in a connected domain of the z-space, and we write symbolically,

$$\frac{\partial}{\partial z_j}=\frac{1}{2}\left(\frac{\partial}{\partial x_j}-i\frac{\partial}{\partial y_j}\right),\quad \frac{\partial}{\partial \bar{z}_j}=\frac{1}{2}\left(\frac{\partial}{\partial x_j}+i\frac{\partial}{\partial y_j}\right),\ j=1,\cdots,k\,.$$

If $f(z,\bar{z})$ is regular with respect to every variable z_j, we have

$$\frac{\partial f}{\partial \bar{z}_j}=0\,, \quad j=1,\cdots,k\,.$$

If $w_1(z),\cdots, w_k(z)$ is a system of k regular functions, we call the vector function

$$w(z)=\begin{pmatrix} w_1(z) \\ \vdots \\ w_k(z) \end{pmatrix}$$

regular with respect to z.

Now we define the powers of a vector z as

$$z^n=\begin{pmatrix} z_1^n \\ \vdots \\ \sqrt{\dfrac{n!}{n_1!\cdots n_k!}}\, z_1^{n_1}\cdots z_k^{n_k} \\ \vdots \\ z_k^n \end{pmatrix} \tag{1.1}$$

where $(n_1,\cdots, n_k)$ runs over all the non-negative integers such that $n_1+\cdots+n_k=n$ and ${}_kH_n\left(=\binom{k+n-1}{n}\right)$ monomials of degree n in $z_1,\cdots,z_k$ are arranged in a certain determined way (e. g., in the lexicographical order) to form a ${}_kH_n$-tuple vector.

Moreover, we define the n-th differentiation of a vector function $w(z)$ with respect to z as

$$\begin{aligned} \frac{d^n w(z)}{dz^n} &= \frac{d^n}{dz^n}\times w(z) \\ &= \left(\frac{\partial^n}{\partial z_1^n},\cdots, \sqrt{\frac{n!}{n_1!\cdots n_k!}}\,\frac{\partial^n}{\partial z_1^{n_1}\cdots\partial z_k^{n_k}},\cdots,\frac{\partial^n}{\partial z_k^n}\right)\times w(z)\,, \end{aligned} \tag{1.2}$$

where $\dfrac{\partial^n}{\partial z_1^n},\cdots,\dfrac{\partial^n}{\partial z_k^n}$ are arranged in the order corresponding to $z_1^n,\cdots,z_k^n$ in (1.1) and the sign $\times$ designates the Kronecker product. Thus (1.2) is a matrix of k columns and ${}_kH_n$ rows. Then we have Taylor expansion by the method used by H. Cartan [6]:

THEOREM 1. *If $w(z)$ is a one-valued and regular vector function in a connected domain D of the z-space and a is any fixed point in D, then $w(z)$ is expanded in the form of the following diagonal power series:*

$$w(z)=w(a)+\frac{dw(a)}{dz}(z-a)+\cdots+\frac{1}{n!}\frac{d^n w(a)}{dz^n}(z-a)^n+\cdots. \tag{1.3}$$

This series is absolutely and uniformly convergent in $|z-a|<r$ in the sense of the diagonal series where r is the distance of a from the boundary of D.

PROOF. Let r' $(r'<r)$ be any positive number, and we put $R=(r+r')/(2r')$ $(R>1)$. Then $w(a+(z-a)t)$ is regular with respect to a complex variable t in $|t|\leqq R$ for $|z-a|<r'$. And so by the residue theorem for the function of a complex variable, we have

$$w(z)=\frac{1}{2\pi i}\int_{|t|=R} w(a+(z-a)t)\frac{dt}{t-1}, \tag{1.4}$$

where the integration is done for each component of $w(z)$.

As $1/(t-1)$ is equal to an absolutely and uniformly convergent power series $\sum_{n=0}^{\infty} t^{-n-1}$, we can interchange the integration with the summation in (1.4). Then we have

$$w(z)=\sum_{n=0}^{\infty}\frac{1}{n!}\left[\frac{d^n}{dt^n}w(a+(z-a)t)\right]_{t=0}$$

$$=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\sum_{l=1}^{k}(z_l-a_l)\frac{\partial}{\partial z_l}\right)^n w(a)$$

$$=\sum_{n=0}^{\infty}\frac{1}{n!}\begin{pmatrix} \vdots \\ \displaystyle\sum_{n_1+\cdots+n_k=n}\sqrt{\frac{n!}{n_1!\cdots n_k!}}\,\frac{\partial^n w_j(a)}{\partial z_1^{n_1}\cdots\partial z_k^{n_k}}\sqrt{\frac{n!}{n_1!\cdots n_k!}}\,(z_1-a_1)^{n_1}\cdots(z_k-a_k)^{n_k} \\ \vdots \end{pmatrix}.$$

Thus the proof is completed.

COROLLARY. *The first derivative of $w(z)$ in Theorem 1 is expanded as follows:*

$$\frac{dw(z)}{dz}=\sum_{n=1}^{\infty}\frac{1}{n!}\frac{d^n w(a)}{dz^n}\cdot\frac{d}{dz}(z-a)^n. \tag{1.5}$$

PROOF. Because of the uniform continuity of the function

$$f(z)=\frac{1}{2\pi i}\int_{|t|=R} w(a+(z-a)t)\,\frac{dt}{t-1},$$

we easily obtain this expansion by differentiating both sides of (1.4) with respect to z.

We call a vector function analytic in D when it is expanded in the series as in Theorem 1 at every point in D.

For various purposes it is sometimes more convenient to use other definitions of the powers of z and the differentiations with respect to z as follows:

$$z^n=\left(\begin{array}{c} z_1^n \\ \vdots \\ z_1^{n_1}\cdots\cdots z_k^{n_k} \\ \vdots \\ z_k^n \end{array}\right) \tag{1.1$'$}$$

and

$$\frac{d^n w(z)}{dz^n}=\left(\frac{\partial^n}{\partial z_1^n},\cdots,\frac{n!}{n_1!\cdots n_k!}\,\frac{\partial^n}{\partial z_1^{n_1}\cdots\partial z_k^{n_k}},\cdots,\frac{\partial^n}{\partial z_k^n}\right)\times w(z), \tag{1.2$'$}$$

or

$$z^n=\left(\begin{array}{c} z_1^n \\ \vdots \\ \dfrac{n!}{n_1!\cdots n_k!}\, z_1^{n_1}\cdots\cdots z_k^{n_k} \\ \vdots \\ z_k^n \end{array}\right) \tag{1.1$''$}$$

and

$$\frac{d^n w(z)}{dz^n} = \left(\frac{\partial^n}{\partial z_1^n}, \cdots, \frac{\partial^n}{\partial z_1^{n_1} \cdots \partial z_k^{n_k}}, \cdots, \frac{\partial^n}{\partial z_k^n} \right) \times w(z), \tag{1.2}''$$

where $n_1 + \cdots + n_k = n$.

Even if we use these definitions, we have the expansion theorems of the same form as in Theorem 1 and its corollary. According to the definitions (1.1)″ and (1.2)″, we have

$$\frac{dw(z)}{dz}(z-a) = \frac{dw(a)}{dz}(z-a) + \cdots + \frac{1}{(n-1)!}\frac{d^n w(a)}{dz^n}(z-a)^n + \cdots,$$

and also, in the case of $k=2$, we have

$$\frac{dw(z)}{dz} = \frac{dw(a)}{dz} + \cdots + \frac{1}{(n-1)!}\frac{d^n w(a)}{dz^n}\begin{pmatrix}(z-a)^{n-1} & 0 \\ 0 & (z-a)^{n-1}\end{pmatrix} + \cdots,$$

where the last matrix $\begin{pmatrix}(z-a)^{n-1} & 0 \\ 0 & (z-a)^{n-1}\end{pmatrix}$ has $(n+1)$ rows and 2 columns, each column consisting of $(z-a)^{n-1}$, which has n rows, and a single zero. Unless otherwise stated, we shall use the definitions (1.1) and (1.2).

3. *Green's formula.* We denote by D a connected domain bounded by smooth hypersurfaces $C_1, \cdots, C_m$ in the z-space. We suppose that C_i is representable by an equation of the form $f_i(z, \bar{z}) = 0$ with a real-valued function f_i. For simplicity we shall denote by C the collection of $C_1, \cdots, C_m$ with suitable orientations and by $f=0$ those equations $f_1 = 0, \cdots, f_m = 0$. Then we have from Green's theorem

$$\int_C B(z, \bar{z}) \frac{\partial f}{\partial \bar{z}_j} \left[\frac{df}{dz} \left(\frac{df}{dz} \right)^* \right]^{-\frac{1}{2}} dS = 2 \int_D \frac{\partial B(z, \bar{z})}{\partial \bar{z}_j} dV, \quad j = 1, \cdots, k, \tag{1.6}$$

where $B(z, \bar{z})$ is any single or vector function which is continuous and has the first partial derivatives in D, dS is the surface element on C and dV is the volume element in D. P. R. Garabedian [8] made use of this formula in proving the existence of the generalized Green's function. Taking the conjugates of both sides of (1.6), we obtain, because of the arbitrariness of $B(z, \bar{z})$, the following formula:

$$\int_C B(z, \bar{z}) \frac{\partial f}{\partial z_j} \left[\frac{df}{dz} \left(\frac{df}{dz} \right)^* \right]^{-\frac{1}{2}} dS = 2 \int_D \frac{\partial B(z, \bar{z})}{\partial z_j} dV, \quad j = 1, \cdots, k. \tag{1.7}$$

By these formulas we obtain the following fundamental lemma which is an extension of the following formula for one complex variable:

$$\int_{|z|=r} z^n dz = \begin{cases} 0, & for \quad n \neq -1 \\ 2\pi i, & for \quad n = -1 . \end{cases}$$

LEMMA 1. *For a spherical hypersurface K: $|z|=R$, and non-negative integers $n_1, m_1, \cdots, n_k, m_k$, we have*

(1.8) $$\int_K z_1^{n_1}\bar{z}_1^{m_1}\cdots z_k^{n_k}\bar{z}_k^{m_k} dS = \begin{cases} 0, & for\ (n_1-m_1)^2+\cdots+(n_k-m_k)^2>0 , \\ \dfrac{n_1!\cdots n_k!}{n!}\cdot\dfrac{R^{2n}\omega}{{}_kH_n}, & \\ & for\ (n_1-m_1)^2+\cdots+(n_k-m_k)^2=0 , \end{cases}$$

where $n_1+\cdots+n_k=n$ and ω is the area of K, that is, $2\pi^k R^{2k-1}/(k-1)!$.

PROOF. As the equation of the boundary of C in (1.6) and (1.7) is $|z|^2=R^2$, $\dfrac{\partial f}{\partial \bar{z}_j}\left[\dfrac{df}{dz}\left(\dfrac{df}{dz}\right)^*\right]^{-\frac{1}{2}}$ is equal to $\dfrac{z_j}{R}$. If we denote by D the hypersphere $|z|<R$, we have from (1.6) and (1.7), respectively,

(1.6)′ $$\int_K B\frac{z_j}{R}\,dS = 2\int_D \frac{\partial B}{\partial \bar{z}_j}\,dV ,$$

and

(1.7)′ $$\int_K B\frac{\bar{z}_j}{R}\,dS = 2\int_D \frac{\partial B}{\partial z_j}\,dV , \quad j=1,\cdots,k .$$

Now we denote by P the left side of (1.8).

i) In case $n_j \neq m_j$ for some j, by (1.6)′ we have

(1.9) $$\begin{aligned} P &= R\int_K z_1^{n_1}\bar{z}_1^{m_1}\cdots z_j^{n_j-1}\bar{z}_j^{m_j}\cdots z_k^{n_k}\bar{z}_k^{m_k}\cdot\frac{z_j}{R}\,dS \\ &= 2m_jR\int_D z_1^{n_1}\bar{z}_1^{m_1}\cdots z_j^{n_j-1}\bar{z}_j^{m_j-1}\cdots z_k^{n_k}\bar{z}_k^{m_k}dV . \end{aligned}$$

Also by (1.7)′, we have

$$P=R\int_K z_1^{n_1}\bar z_1^{m_1}\cdots z_j^{n_j}\bar z_j^{m_j-1}\cdots z_k^{n_k}\bar z_k^{m_k}\cdot\frac{\bar z_j}{R}\,dS$$

(1.10)

$$=2n_jR\int_D z_1^{n_1}\bar z_1^{m_1}\cdots z_j^{n_j-1}\bar z_j^{m_j-1}\cdots z_k^{n_k}\bar z_k^{m_k}dV.$$

Accordingly, by (1.9) and (1.10),

$$2(m_j-n_j)R\int_D z_1^{n_1}\bar z_1^{m_1}\cdots z_j^{n_j-1}\bar z_j^{m_j-1}\cdots z_k^{n_k}\bar z_k^{m_k}dV=0.$$

For $n_j\neq m_j$, the above integral vanishes and so does P.

ii) In case $n_j=m_j\ (j=1,\cdots,k)$, we can first show by the same method as used in i) the identity:

$$\int_K |z_1|^{2n_1}\cdots|z_k|^{2n_k}dS=\frac{n_k}{n_1+1}\int_K |z_1|^{2(n_1+1)}\cdots|z_k|^{2(n_k-1)}dS$$

and thus we get

$$\int_K |z_1|^{2(n_1+n_k)}\cdots|z_{k-1}|^{2n_{k-1}}dS=\frac{n_1!\,n_{k-1}!}{(n_1+n_{k-1})!}\int_K |z_1|^{2(n_1+n_k)}\cdots|z_{k-1}|^{2n_{k-1}}dS,$$

where the number of variables in the right side has diminished by one.

Repeating this process, we have

$$\text{(1.11)}\quad \int_K |z_1|^{2n_1}\cdots|z_k|^{2n_k}dS=\frac{n_1!\cdots n_k!}{(n_1+\cdots+n_k)!}\int_K |z_1|^{2(n_1+\cdots+n_k)}dS.$$

From this follows

$$\text{(1.12)}\quad \int_K \sum_{n_1+\cdots+n_k=n}\frac{n!}{n_1!\cdots n_k!}|z_1|^{2n_1}\cdots|z_k|^{2n_k}dS={}_kH_n\int_K |z_1|^{2n}dS.$$

As the left side of (1.12) is equal to $R^{2n}\omega$, we obtain

$$\int_K |z_1|^{2n}dS=R^{2n}\omega/{}_kH_n,$$

and so, substituting this value into the right side of (1.11), we have the proof of the latter part of Lemma 1.

THEOREM 2. *The expansion of* $w(z)$ *in Theorem* 1 *is unique and* $\frac{d^n w(a)}{dz^n}$ *is expressed as follows:*

$$\frac{d^n w(a)}{dz^n} = \frac{n!\,{}_kH_n}{R^{2n}\omega}\int_{|z-a|=R} w(z)\{(z-a)^n\}^* dS. \tag{1.13}$$

PROOF. Let $w(z)$ be analytic in D and be expanded in a uniformly convergent diagonal power series for $|z-a|<r$ $(r<R)$:

$$w(z)=A_0+A_1(z-a)+\cdots+A_n(z-a)^n+\cdots. \tag{1.14}$$

From Lemma 1 we easily obtain

$$\int_{|z-a|=r}(z-a)^n\{(z-a)^m\}^* dS=(0), \quad \text{for } n\neq m,$$

and

$$\int_{|z-a|=r}(z-a)^n\{(z-a)^n\}^* dS=\frac{r^{2n}\omega}{{}_kH_n}E_{{}_kH_n}.$$

Multiplying $\{(z-a)^n\}^*$ on both sides of (1.14) and using these results, we have

$$A_n=\frac{{}_kH_n}{r^{2n}\omega}\int_{|z-a|=r} w(z)\{(z-a)^n\}^* dS.$$

This shows that the coefficients are unique, and letting r tend to R, it is clear that the representation of the derivatives is given as in the theorem.

§ 2. Bounded vector functions

For a vector function $w(z)$ analytic in a connected domain D, we call $w(z)$ bounded in D, if there exists a positive constant M, for which $|w(z)|\leqq M$ in D. For these bounded analytic functions, we can generalize Gutzmer's inequality and Schwarz' lemma as follows.

THEOREM 3. (*Generalized Gutzmer's inequality*) *Let*

$$w(z)=A_0+A_1z+\cdots+A_nz^n+\cdots \tag{2.1}$$

be analytic and bounded, and suppose $|w(z)|\leqq M$ *in* $|z|<R$, *then the two inequalities hold*:

$$[A_0]^2+\frac{R^2}{{}_kH_1}[A_1]^2+\cdots+\frac{R^{2n}}{{}_kH_n}[A_n]^2+\cdots\leqq M^2\,, \tag{2.2}$$

and

$$||A_0||^2+\frac{R^2}{{}_kH_1}||A_1||^2+\cdots+\frac{R^{2n}}{{}_kH_n}||A_n||^2+\cdots\leqq M^2\,. \tag{2.3}$$

PROOF. From the uniform convergence of (2.1), for r $(r<R)$,

$$N\equiv\int_{|z|=r}w(z)^*w(z)dS=\sum_{m,n=0}\int_{|z|=r}(z^n)^*A_n^*A_mz^mdS\,.$$

It follows from Theorem 2 that all the terms of the right side vanish for $m\neq n$ and so we have

$$N=\sum_{n=0}^{\infty}\int_{|z|=r}(z^n)^*A_n^*A_nz^ndS\,.$$

Denoting by a_j^n $(j=1,\cdots,{}_kH_n)$ the row vector of A_n and using Theorem 2 again, we obtain

$$N=\sum_{n=0}^{\infty}\int_{|z|=r}(z^n)^*\begin{pmatrix}(a_1^n)^*a_1^n & & 0\\ & \ddots & \\ 0 & & (a_{{}_kH_n}^n)^*a_{{}_kH_n}^n\end{pmatrix}z^ndS$$

$$=\sum_{n=0}^{\infty}\frac{r^{2n}\omega}{{}_kH_n}\left(\sum_{j=1}^{{}_kH_n}(a_j^n)^*a_j^n\right)=\sum_{n=0}^{\infty}\frac{r^{2n}\omega}{{}_kH_n}\,\mathrm{Tr.}\,(A_n^*A_n)=\sum_{n=0}^{\infty}\frac{r^{2n}\omega}{{}_kH_n}[A_n]^2\,.$$

On the other hand, it is clear that $N\leqq M^2\omega$, and so we obtain (2.2) by $r\to R$. Remarking that $||A_n||\leqq[A_n]$, the validity of (2.3) follows from (2.2).

From this theorem we have the following results, taking $R=1$.

COROLLARY 1. *Let the function* $w(z)$ *be analytic in* $|z|<1$ *and suppose* $|w(z)|\leqq M$, *then the inequalities hold*:

$$[A_0]\leqq M\,,\quad [A_n]\leqq\sqrt{{}_kH_n}\,M\,, \tag{2.4}$$

$$||A_0||\leqq M\,,\quad ||A_n||\leqq\sqrt{{}_kH_n}\,M\,,\qquad n=1,2,\cdots \tag{2.5}$$

Moreover, the equality sign holds if and only if

$$w(z)=A_0\,, \qquad w(z)=A_n z^n\,,$$

respectively.

Remark. If $w(z)=z+$(*higher powers*) and $|w(z)|<1$ in $|z|<1$, then $w(z)=z$, as $[A_1]=\sqrt{k}$. This is a special case of H. Cartan's uniqueness theorem [7]. But as far as the hypersphere is concerned, we obtain a sharper result as follows.

Corollary 2. *If* $w(z)=A_1z+$(*higher powers*) *and* $|w(z)|<1$ *in* $|z|<1$ *and* $[A_1]=\sqrt{k}$ *or* $\|A_1\|=\sqrt{k}$, *then* $w(z)$ *is a linear transformation.*

We shall now estimate the norm of the coefficient matrices of a bounded vector function.

Theorem 4. *Let* $w(z)$ *be an analytic vector function in* $|z|<1$ *and* $w(0)=(0)$ *such that* $|w(z)|<1$ *for* $|z|<1$. *Then the following inequality holds*:

$$\left\|\frac{dw(0)}{dz}\right\|\leqq 1\,. \tag{2.6}$$

The equality sign holds, for instance, for $w(z)=Uz$, *where* U *is a unitary matrix.*

Proof. From Schwarz' lemma for vector functions of several complex variables [9], we have

$$|w(z)|^2\leqq|z|^2\,.$$

Substituting $w(z)=\dfrac{dw(0)}{dz}z+$(*higher powers*) into the left side of this equality, we get

$$\left|z^*\left(\frac{dw(0)}{dz}\right)^*\frac{dw(0)}{dz}z+O(|z|^3)\right|\leqq|z|^2\,.$$

If we put $|z|=r$ and $z=ru$ for $z\neq 0$, and divide both sides by r^2, and let r tend to zero, we have

$$\left|u^*\left(\frac{dw(0)}{dz}\right)^*\frac{dw(0)}{dz}u\right|\leqq 1\,.$$

Here, u is an arbitrary k-tuple complex vector whose length is 1, and so the validity of (2.6) is assured by the definition of the norm of $\frac{dw(0)}{dz}$.

It is to be noted that the equality sign may hold also for a function which is not of the form Uz; for example, in the case of $k=2$, the equality sign holds for

$$w(z)=\frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}z, \quad \text{or} \quad w(z)=\begin{pmatrix} z_1+\frac{z_2^2}{4} \\ \frac{z_2^2}{2}\end{pmatrix}.$$

COROLLARY. *A necessary and sufficient condition that a vector function $w(z)$ satisfying the condition of Theorem 4 (not necessary $w(0)=0$) is of the form Uz, where U is a unitary matrix, is that the two identities hold:*

$$\left\| \frac{dw(0)}{dz} \right\|=1 \quad \textit{and} \quad \left[\frac{dw(0)}{dz}\right]=\sqrt{k}.$$

PROOF. Clearly these conditions are necessary, and so we have only to show that these are sufficient.

Let the characteristic values of $\left(\frac{dw(0)}{dz}\right)^* \frac{dw(0)}{dz}$ be $\lambda_1,\cdots,\lambda_k$ $(\lambda_1 \geqq \cdots \geqq \lambda_k \geqq 0)$. Then from the condition $\left\| \frac{dw(0)}{dz} \right\|=1$, we have

$$\lambda_1=1.$$

Moreover, from the condition $\left[\frac{dw(0)}{dz}\right]=\sqrt{k}$ we get

$$\lambda_1+\cdots+\lambda_k=k.$$

Accordingly,

$$\lambda_1=\cdots=\lambda_k=1.$$

This shows that $\frac{dw(0)}{dz}$ is a unitary matrix and from Corollary 2 of Theorem 3, $w(z)$ is a linear transformation. This completes the proof.

THEOREM 5. *Let a vector function $w(z)$ be analytic in $|z|<1$ and suppose $|w(z)|<1$, then we have the inequality:*

$$\left\|\Gamma(w(z))\frac{dw(z)}{dz}\right\| \leqq \left\|\Gamma(w(z))\frac{dw(z)}{dz}\Gamma(z)^{-1}\right\| \leqq \frac{1-|w(z)|^2}{1-|z|^2}, \tag{2.7}$$

where $\Gamma(w(z))=\sqrt{1-|w(z)|^2}E+\{(1-\sqrt{1-|w(z)|^2})/|w(z)|^2\}w(z)w(z)^*$.

The equality sign holds for $\Gamma(a)(z-a)(1-a^*z)^{-1}$ *where a is an arbitrary point in* $|z|<1$.

PROOF. It is easy to see that the transformation

$$f(z)=\Gamma(b)(z-b)(1-b^*z)^{-1}, \qquad (|b|<1)$$

is a one-to-one and analytic mapping which maps the hypersphere $|z|<1$ onto the hypersphere $|f|<1$.

Now, if we put, for any fixed point z in $|z|<1$

$$S(u)=\Gamma(w(z))\{w(u)-w(z)\}\{1-w(z)^*w(u)\}^{-1},$$

we have $|S(u)|<1$, because of the assumption: $|w(u)|<1$ for $|u|<1$. Accordingly, from Theorem 4, we have

$$\left\|\frac{dS(z(0))}{du}\right\| \leqq 1 \tag{2.8}$$

and by simple calculation,

$$\frac{dS(z(0))}{du}=\frac{1-|z|^2}{1-|w(z)|^2}\Gamma(w(z))\frac{dw(z)}{dz}\Gamma(z)^{-1}, \tag{2.9}$$

where we notice that $\det\{\Gamma(z)\}=(\sqrt{1-|z|^2})^{k-1}$ and so, there exists the inverse of $\Gamma(z)$.

Using the property of the norm: $\|AB\|\leqq\|A\|\cdot\|B\|$ for any matrices A and B, and $\|\Gamma(z)\|=1$, we have

$$\begin{aligned}\left\|\Gamma(w(z))\frac{dw(z)}{dz}\right\| &= \left\|\Gamma(w(z))\frac{dw(z)}{dz}\Gamma(z)^{-1}\Gamma(z)\right\| \\ &\leqq \left\|\Gamma(w(z))\frac{dw(z)}{dz}\Gamma(z)^{-1}\right\|.\end{aligned} \tag{2.10}$$

Thus we obtain the inequality (2.7) from (2.8), (2.9) and (2.10).

COROLLARY 1. *For the function $w(z)$ in the theorem, we have*

$$\left\| \frac{dw(z)}{dz} \right\| \leqq \frac{\sqrt{1-|w(z)|^2}}{1-|z|^2} \tag{2.11}$$

and

$$\left\| \frac{dw(z)}{dz} \right\| \leqq \frac{1}{1-|z|^2}. \tag{2.12}$$

PROOF. Using the relation:

$$\left\| \Gamma(w(z)) \frac{dw(z)}{dz} \right\| \geqq \left\| \frac{dw(z)}{dz} \right\| \Big/ \|\Gamma(w(z))^{-1}\|,$$

and

$$\|\Gamma(w(z))^{-1}\| \geqq 1/\sqrt{1-|w(z)|^2},$$

we have (2.11) from Theorem 5, and (2.12) follows from (2.11).

COROLLARY 2. *Let $w(z)=A_0+A_1z+\cdots$ be analytic in $|z|<1$ and suppose $|w(z)|<1$. Then the two inequalities hold:*

$$|A_0| \leqq 1, \quad \|\Gamma(A_0)A_1\| \leqq 1-|A_0|^2.$$

PROOF. This corollary follows easily, if we put $z=(0)$ in (2.7).

REMARK. In the case of functions of one complex variable, we have the well-known condition of Schur [10] for the bounded family of analytic functions, that is;

Let $w(z)=c_0+c_1z+\cdots$ be bounded: $|w(z)|<1$ in $|z|<1$, then

$$|c_0| \leqq 1, \quad |c_1| \leqq 1-|c_0|^2, \cdots\cdots.$$

The inequalities in Corollary 2 correspond to the first two inequalities of Schur.

THEOREM 6. *If $w(z)=z+A_2z^2+\cdots$ is analytic and $|w(z)|<M$ in $|z|<1$, then $w(z)$ is univalent in $|z|<1/2(k+1)M$, and the image of the latter hypersphere contains a univalent hypersphere*

$$|w|<1/4(k+1)M.$$

PROOF. To prove the first part, it suffices to show

$$\left\| \frac{dw(z)}{dz} - E \right\| < 1 \quad \text{for } |z|<1/2(k+1)M, \tag{2.13}$$

as this is a sufficient condition for the univalency of analytic functions obtained by S. Takahashi [5]. According to the corollary of Theorem 1, we have

$$\frac{dw(z)}{dz}-E=\sum_{n=2}^{\infty} A_n \frac{d}{dz} z^n .$$

From the inequality (2.5),

$$\|A_n\|\leqq\sqrt{{}_kH_n}M$$

and also by simple calculation,

$$\left|\frac{d}{dz} z^n\right|=nr^{n-1}$$

where $r=|z|$, and so we have

$$\left\|\frac{dw(z)}{dz}-E\right\|\leqq\sum_{n=2}^{\infty}\|A_n\|\cdot\left\|\frac{d}{dz} z^n\right\|\leqq\sum_{n=2}^{\infty} n\sqrt{{}_kH_n}Mr^{n-1} .$$

Using Schwarz' inequality, we get

$$\left\|\frac{dw(z)}{dz}-E\right\|^2\leqq M^2\left(\sum_{n=2}^{\infty} nr^{n-1}\right)\left(\sum_{n=2}^{\infty} n_kH_nr^{n-1}\right)$$

$$=M^2\left\{\frac{1}{(1-r)^2}-1\right\}\left\{\frac{k}{(1-r)^{k+1}}-k\right\} .$$

Hence (2.13) will be obtained in putting $r=1/2(k+1)M$, and in noticing $M\geqq 1$ which follows from (2.2), if we can prove the following inequality

$$(2.14)\quad 2k(k+1)\left\{1+\frac{k(k-1)}{3\,!}\frac{1}{(2k+2)^2}+\cdots\right\}<(2k+2)^2\left(1-\frac{1}{2k+2}\right)^{k+3} .$$

Now the left side of this inequality (2.14) is less than

$$P=2k(k+1)\left\{1+\frac{1}{3\,!\,2^2}+\frac{1}{5\,!\,2^4}+\cdots\right\}$$

$$=4k(k+1)\sinh\frac{1}{2}=2.084\cdots k(k+1) ,$$

and the right side is larger than

$$Q=(2k+2)^2\left\{1-\frac{k+3}{2k+2}+\frac{(k+3)(k+2)}{2!\,(2k+2)^2}-\frac{(k+3)(k+2)(k+1)}{3!\,(2k+2)^3}\right\}$$

$$=\frac{29}{12}k^2+\frac{25}{12}k+\frac{1}{2}.$$

Obviously we see $P<Q$, and so (2.14) holds.

We are now going to prove the latter part of this theorem. For $|z|=r$, we have

$$|w(z)|\geqq r-\sum_{n=2}^{\infty}\|A_n\|r^n\geqq r-M\sum_{n=2}^{\infty}\sqrt{{}_kH_n}\,r^n.$$

Applying Schwarz' inequality again, we get

$$\left(\sum_{n=2}^{\infty}\sqrt{{}_kH_n}\,r^n\right)^2\leqq\frac{r^2}{1-r}\left\{\frac{1}{(1-r)^k}-(1+kr)\right\}$$

$$\leqq\frac{r^2}{(1-r)^{k+1}}\left\{\frac{(k+1)k}{2!}+\frac{3(k+1)k(k-1)}{4!}r^2+\cdots\right\}.$$

And so, the theorem will be proved if we show that this last expression is not larger than $1/4(k+1)M$ for $r=1/2(k+1)M$. To show this, we shall prove the following inequality in the same way as in the proof of the first part of this theorem:

$$4k(k+1)\left\{\frac{1}{2!}+\frac{3}{4!\,2^2}+\frac{5}{6!\,2^4}+\cdots\right\}$$

(2.15)

$$<(2k+2)^2\left\{1-\frac{k+1}{2(k+1)}+\frac{(k+1)k}{2!\,(2k+2)^2}-\frac{(k+1)k(k-1)}{3!\,(2k+2)^3}\right\}.$$

The left side of (2.15) is equal to

$$4k(k+1)\left\{4+2\sinh\frac{1}{2}-4\cosh\frac{1}{2}\right\}=2.12\cdots k(k+1)$$

and the right side is $\frac{29k^2}{12}+\frac{55k}{12}+2$. Accordingly, the inequality (2.15) holds.

COROLLARY. *Let* $w(z)=A_1z+A_2z^2+\cdots\cdots$ *be analytic and suppose* $|w(z)|<M$ *in* $|z|<R$ *and* $\det A_1\neq 0$. *Then* $w(z)$ *is univalent in* $|z|<R^2/\{2(k+1)\|A_1^{-1}\|M\}$ *and the image of this hypersphere contains a univalent hypersphere* $|w|<R^2/\{4(k+1)\|A_1^{-1}\|^2M\}$.

PROOF. From the assumption: $\det A_1\neq 0$, there exists the inverse A_1^{-1} of A_1. Putting $z=Rz'$ $(|z'|<1)$, we get

$$f(z')=A_1^{-1}w(Rz')/R=z'+(\textit{higher powers})$$

and

$$|f(z')|\leqq M\|A_1^{-1}\|/R.$$

If we apply Theorem 6 for $f(z')$ analytic in $|z'|<1$, this corollary is easily obtained ([11]).

REMARK. The condition $\det A_1\neq 0$ is necessary. For, it is a necessary condition for the univalency of $w(z)$ in the neighborhood of $z=(0)$ that the Jacobian does not vanish at $z=(0)$, and the value of the Jacobian at the point is equal to $\left|\det \dfrac{dw(0)}{dz}\right|^2=|\det A_1|^2$. Nevertheless, the univalence radii in the above theorem and its corollary are not best possible. Also we have not yet succeeded in generalizing to our case Landau-Dieudonné's theorem on the univalence radius of starlike functions of one complex variable.

§ 3. The number of zero points

We shall introduce the following formula in order to obtain the identity which designates the number of zero points of vector functions of several complex variables.

Putting $B(z,\bar{z})=\dfrac{\partial}{\partial z_j}P(z,\bar{z})$ $(j=1,\cdots,k)$ in Green's formula (1.6) of Section 1, where $P(z,\bar{z})$ is continuous and has the first and the second partial derivatives, and summing up, we have

$$\int_C \frac{d}{dz}P(z,\bar{z})\left(\frac{df}{dz}\right)^*\left[\frac{df}{dz}\left(\frac{df}{dz}\right)^*\right]^{-\frac{1}{2}}dS=\frac{1}{2}\int_D \Delta P(z,\bar{z})dV, \tag{3.1}$$

where ΔP is the Laplacian of $P(z,\bar{z})$.

As noticed in Section 1, we use the definition of z^n and $\dfrac{d^n w(z)}{dz^n}$ in (1.1)′ and (1.2)′.

If all the elements of a vector function $w(z)$ vanish at a point a that is, $w(a)^* w(a)=0$, we call a a zero point of $w(z)$. We shall obtain some conditions that $w(z)$ has isolated zero points. For this purpose we first prove the following elementary lemma.

LEMMA 2. *For positive integers k $(k \geqq 2)$ and n $(n \geqq 1)$, we have the inequality:*

$$k \cdot {}_kH_{(k-1)(n-1)} \geqq {}_kH_{kn-k+1},$$

where the H's are the number of homogeneous products taken from k letters.

PROOF. In the case of $k=2$, we can easily assure that the equality holds in the above inequality. For $k \geqq 3$, it suffices to show the inequality:

$$k(kn-n)(kn-n-1)\cdots(kn-n-k+2)$$
$$\geqq kn(kn-1)\cdots(kn-k+2),$$

that is,

$$(k-1)\{(k-1)(n-1)+k-2\}\cdots\{(k-1)(n-1)+1\}$$
$$\geqq \{k(n-1)+k-1\}\{k(n-1)+k-2\}\cdots\{k(n-1)+2\}.$$

The left side of the above inequality is equal to

$$(k-1)!\left\{1+\frac{k-1}{k-2}(n-1)\right\}\left\{1+\frac{k-1}{k-3}(n-1)\right\}\cdots\left\{1+(k-1)(n-1)\right\},$$

and the right side is equal to

$$(k-1)!\left\{1+\frac{k}{k-1}(n-1)\right\}\left\{1+\frac{k}{k-2}(n-1)\right\}\cdots\left\{1+\frac{k}{2}(n-1)\right\}.$$

Using the facts:

$$\frac{k-1}{j}>\frac{k}{j+1}, \quad \text{for } j=1,\cdots,k-2,$$

we see that every factor of the left side is not less than the corresponding factor of the right side, which concludes the proof.

THEOREM 7. *Let a vector function* $w(z)$ *be analytic in a neighborhood* $K: |z-a|<r$ *and*

$$w(a)=(0),\ \frac{dw(a)}{dz}=(0),\cdots,\ \frac{d^{n-1}w(a)}{dz^{n-1}}=(0)\,, \tag{3.2}$$

and the rank of the following matrix be ${}_kH_{kn-k+1}$:

$$\frac{d^{(kn-k+1)}}{dz^{(kn-k+1)}}\left\{\frac{d^n w(a)}{dz^n}(z-a)^n \times (z-a)^{(k-1)(n-1)}\right\}, \tag{3.3}$$

where the sign $\times$ *means the Kronecker product. Then* $w(z)$ *does not vanish in the suitable neighborhood of* a *except at* a.

PROOF. Without loss of generality we may put $a=(0)$. It follows from (3.3) that $\dfrac{d^n w(a)}{dz^n}$ has at least a non-vanishing element in every row. Using the Weierstrass' preparation theorem ([1], pp. 183–190), and a suitable non-singular linear transformation: $z=Lz'$, we have

$$w(z(z'))=\begin{pmatrix} \Omega_1(z') & & 0 \\ & \ddots & \\ 0 & & \Omega_k(z') \end{pmatrix} \tag{3.4}$$
$$\cdot\begin{pmatrix} 1, H_1^1(z_2^1,\cdots,z_k'),\cdots\cdots, H_1^n(z_2',\cdots,z_k') \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 1, H_k^1(z_2',\cdots,z_k'),\cdots\cdots, H_k^n(z_2',\cdots,z_k') \end{pmatrix}\begin{pmatrix} z_1'^n \\ z_1'^{n-1} \\ \vdots \\ 1 \end{pmatrix},$$

where the functions H's are analytic in a neighborhood of $z_2'=\cdots=z_k'=0$ and

$$H_j^l(0,\cdots,0)=0,\ j=1,\cdots,k,\ l=1,\cdots,n$$

and $\Omega_j(z')$ is analytic and non-vanishing in that neighborhood.

Expanding $w(z(z'))$ with respect to z',

$$w(z(z'))=(a_{ij}')z'^n+(\textit{higher powers}),$$

and comparing the coefficients on both sides, we have

$$w(z(z')) = \begin{pmatrix} 1+(\textit{higher powers}), & & 0 \\ & \ddots & \\ 0 & & 1+(\textit{higher powers}) \end{pmatrix}$$

$$\cdot \begin{pmatrix} a'_{11},\ a'_{12}+(\textit{higher powers}),\cdots,\ a'_{1,{}_kH_n}+(\textit{higher powers}) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a'_{k1},\ a'_{k2}+(\textit{higher powers}),\cdots,\ a'_{k,{}_kH_n}+(\textit{higher powers}) \end{pmatrix} z'^n .$$

If we replace z' by $L^{-1}z$, we see that $w(z)$ is equal to the right side in which the primes are taken away, that is,

$$\text{(3.5)} \qquad \begin{pmatrix} a_{11}+(\textit{higher powers}),\cdots,\ a_{1,{}_kH_n}+(\textit{higher powers}) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{k1}+(\textit{higher powers}),\cdots,\ a_{k,{}_kH_n}+(\textit{higher powers}) \end{pmatrix} z^n = (0) .$$

It suffices to show that $w(z)$ vanishes if and only if the vector of the second factor vanishes.

Now, in this system of equations with respect to the unknowns $z_1,\cdots,z_k$, we multiply both sides by the homogeneous products of order $(k-1)(n-1)$ of these k unknowns. Then we obtain a system of $k\cdot {}_kH_{(k-1)(n-1)}$ equations. From Lemma 2, we see that this number is not less than ${}_kH_{kn-k+1}$ which is the number of the unknowns in the new system of equations, where every homogeneous product $z_1^{kn-k+1}, z_1^{kn-k}z_2,\cdots, z_k^{kn-k+1}$ is considered as an unknown.

From our assumption (3.3) follows that the rank of

$$\text{(3.6)} \qquad \frac{d^{(kn-k+1)}}{dz^{(kn-k+1)}}((a_{ij})z^n \times z^{(k-1)(n-1)}) = A(a_{ij})$$

is ${}_kH_{kn-k+1}$. Therefore the matrix

$$\text{(3.7)} \qquad A(a_{ij}+(\textit{higher powers}))$$

obtained from (3.6) by substituting a_{ij} by corresponding elements in (3.5), has the same rank ${}_kH_{kn-k+1}$ in a sufficiently small neighborhood of $z=(0)$.

On the other hand, if (3.5) had a solution other than $z=(0)$, then we could apply Sylvester's elimination [12] to the above new system of equations, and would find that the rank of (3.7) is less than ${}_kH_{kn-k+1}$, which is a contradiction. This concludes the proof.

We call the point a in Theorem 7 the zero point of the n^k-th order.

REMARK. In the case of $k=2$, the condition (3.3) is expressed in a considerably simpler form. That is,

$$\det\begin{pmatrix} \dfrac{d^n w(a)}{dz^n} & 0\cdots\cdots\cdots\cdots 0 \\ & 0\cdots\cdots\cdots\cdots 0 \\ \begin{matrix}0\\0\end{matrix} & \dfrac{d^n w(a)}{dz^n} \qquad 0 \\ \vdots & \ddots \\ \begin{matrix}0\\0\end{matrix} & 0 \qquad \dfrac{d^n w(a)}{dz^n} \end{pmatrix} \neq 0,$$

where this matrix is of the form $(2n)\times(2n)$ and $\dfrac{d^n w(a)}{dz^n}$, which is of the form $2\times(n+1)$, is situated from the first to $(n+1)$-th column in the first two rows, from second to $(n+2)$-th column in the next two rows, etc. For instance, in the case of $n=2$, putting

$$w(z)=\begin{pmatrix} A & 2B & C \\ D & 2E & F \end{pmatrix}\begin{pmatrix} z_1^2 \\ z_1 z_2 \\ z_2^2 \end{pmatrix}+(\textit{higher powers}),$$

we have

$$\det\begin{pmatrix} A & 2B & C & 0 \\ D & 2E & F & 0 \\ 0 & A & 2B & C \\ 0 & D & 2E & F \end{pmatrix} \neq 0.$$

LEMMA 3. *Let $w(z)$ be analytic in N: $|z-a|<r$, where a is a fixed point and* $\det \dfrac{dw}{dz}=D(z)\neq 0$ *in N. Suppose, moreover, $w(a)=0$.*

i) *In case* $D(a)\neq 0$, *we have*

$$\left(\frac{dw}{dz}\right)^{-1} w(z)=(z-a)+(\textit{higher powers}),$$

ii) *In case* $D(a)=0$, *assume that all the elements of* $(\varDelta_{ij}(z))'w(z)$ *are divisible by* $D(z)$ *in* N, *where* $(\varDelta_{ij}(z))'$ *is the transposed matrix of the matrix which consists of the minor determinants of* $\dfrac{dw}{dz}$, *and also assume the conditions* (3.2) *and* (3.3) *of Theorem* 7. *Then we have*

$$\left(\frac{dw}{dz}\right)^{-1} w(z)=\frac{1}{n}(z-a)+(\textit{higher powers})$$

where the left side is to be understood in the following sense. From the assumption $\left(\dfrac{dw}{dz}\right)^{-1}$ *exists in* N *except for the set* $S=\{z: D(z)=0\}$ *and for* $z_0\in S$, $\lim\limits_{z\to z_0}\left(\dfrac{dw}{dz}\right)^{-1} w(z)$ *exists when* z *tends to* z_0 *from outside* S. $\left[\left(\dfrac{dw}{dz}\right)^{-1} w(z)\right]_{z=z_0}$ *will mean this limit.*

PROOF. i) In case $D(a)\neq 0$, $D(z)$ does not vanish in a suitable neighborhood of a and so there exists the inverse of $\dfrac{dw}{dz}$ and $\left(\dfrac{dw}{dz}\right)^{-1} w(z)$ is analytic there. If we put

$$\left(\frac{dw}{dz}\right)^{-1} w(z)=f(z), \quad \text{that is,} \quad w(z)=\frac{dw}{dz}f(z),$$

$f(z)$ vanishes if and only if $w(z)$ vanishes.

Now, noticing

$$w(z)=A_1(z-a)+(\textit{higher powers}),$$

$$\frac{dw(z)}{dz}=A_1+(\textit{higher powers}),$$

and

$$f(z)=B_0+B_1(z-a)+(\textit{higher powers}),$$

where $\det A_1 \neq 0$, we have

$$A_1(z-a)+\cdots=(A_1+\cdots)\{B_0+B_1(z-a)+\cdots\}.$$

From this we obtain $B_0=(0)$ and $B_1=E$. Thus the first part of this lemma is proved.

ii) Without loss of generality we may assume for convenience that $a=(0)$. As $(\Delta_{ij}(z))'w(z)$ is divisible by $D(z)$, $(1/D(z))(\Delta_{ij}(z))'w(z)$ is analytic in N. If we denote this function by $f(z)$, we get

$$(\Delta_{ij}(z))'w(z)=D(z)f(z).$$

Multiplying $\frac{dw}{dz}$ on both sides, we have

$$D(z)w(z)=D(z)\frac{dw}{dz}f(z),$$

and so, because of the assumption: $D(z)\not\equiv 0$, we have

$$w(z)=\frac{dw(z)}{dz}f(z). \tag{3.8}$$

It follows from this equality that $w(z)$ vanishes if $f(z)$ vanishes.

On the other hand, $w(z)$ vanishes only at (0) in N from the conditions of Theorem 7, and the order of $w(z)$ and $\frac{dw(z)}{dz}$ with respect to z are n and $n-1$, respectively. Accordingly, the order of $f(z)$ is one. Namely $f(z)$ vanishes only at (0) in N.

Now, if we put

$$w(z)=A_nz^n+(\textit{higher powers}),$$

$$\frac{dw(z)}{dz}=A_n\frac{dz^n}{dz}+(\textit{higher powers}),$$

and

$$f(z)=B_1z+(\textit{higher powers}),$$

and substituting these into (3.8) and comparing the terms of the least order on both sides, we have

$$A_n\left(z^n - \frac{dz^n}{dz} \cdot B_1 z\right) = (0)\,.$$

From the condition (3.3) of Theorem 7, we see that this equality holds if and only if the second factor of the left side vanishes. That is,

$$z^n = \frac{dz^n}{dz} \cdot B_1 z\,.$$

Putting $B_1 = (b_{ij})$ and remarking that

$$\frac{dz^n}{dz} = \begin{pmatrix} nz_1^{n-1}, 0, \cdots\cdots\cdots\cdots, 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ n_1 z_1^{n_1-1} \cdots z_k^{n_k}, \cdots\cdots, n_k z_1^{n_1} \cdots z_k^{n_k-1} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ 0, \cdots\cdots\cdots\cdots, nz_k^{n-1} \end{pmatrix},$$

we have

$$z_1 \cdots z_k = \sum_{j=1}^{k} n_j b_{jj} z_1 \cdots z_k + \sum_{j,l(j \neq l)} n_j b_{jl} z_1 \cdots z_{j-1} z_{j+1} \cdots z_k\,.$$

From this we obtain

$$nb_{jj} = 1, \text{ and } b_{jl} = 0,\ j, l = 1, \cdots, k;\ j \neq l\,.$$

This shows that $B_1 = (1/n)E$ and thus the latter part of this lemma is proved.

All the assumptions of (ii) of Lemma 3 are satisfied, for instance, by the function $w(z) = A_n z^n$, where A_n satisfies (3.2) and (3.3). Among these assumptions it is desirable to weaken the condition that $(\Delta_{ij}(z))' w(z)$ is divisible by $D(z)$, but we have not yet succeeded in finding a suitable condition in its place.

THEOREM 8. *Let D be a connected domain bounded by smooth hypersurfaces C in the z-space of k complex dimensions, where C is represented by real equations $f(z, \bar{z}) = 0$, and let a vector function $w(z)$ be single-valued and analytic in D and have the second continuous partial derivatives with respect to $z, \bar{z}$, in the closure of D. And assume, moreover, that* det $\dfrac{dw(z)}{dz} \equiv D(z)$ *does not vanish identically in D, and* *that in a neighborhood of a point a where* $D(a)=0$, *the assumptions* (ii) *of Lemma* 3 *are satisfied.*

Then the number $n(0)$ *of zero points of* $w(z)$ *in* D *is expressed as follows:*

$$n(0)=\frac{1}{S_E}\int_C\left\{\left(\frac{dw}{dz}\right)^{-1}w\right\}^*\frac{d}{dz}\left\{\left(\frac{dw}{dz}\right)^{-1}w\right\}\alpha\left|\left(\frac{dw}{dz}\right)^{-1}w\right|^{-2k}dS$$

(3.9)

$$+\frac{1}{2(k-1)S_E}\int_D\Delta\left(\left|\left(\frac{dw}{dz}\right)^{-1}w\right|^{-2k+2}\right)dV,$$

where S_E *is the whole area* $2\pi^k/(k-1)!$ *of the unit spherical hypersurface in the* $2k$*-dimensional Euclidean space,* α *is the complex direction cosine vector* $\left(\frac{df}{dz}\right)^*\left[\frac{df}{dz}\left(\frac{df}{dz}\right)^*\right]^{-\frac{1}{2}}$ *on* C,*) Δ *means the Laplacian, and* dS *or* dV *is the surface element on* C *or the volume element in* D, *respectively. Here the zero point of* $w(z)$, *for which* (3.2) *and* (3.3) *hold, is to be counted with the multiplicity* n^{2k-2}.

PROOF. It follows from Theorem 7 and Lemma 3 that the zero points of $w(z)$ are all isolated and coincide with those of $\left(\frac{dw}{dz}\right)^{-1}w(z)$.

We denote by $a^1,\cdots,a^p$ the zero points of $w(z)$ in D, whose orders are $n_1,\cdots,n_p$ respectively. We cut off the hyperspheres $|z-a^l|<\varepsilon$ $(l=1,\cdots,p)$ from D and make a new domain D_ε. Applying Green's formula (3.1) for D_ε and $P=\left|\left(\frac{dw}{dz}\right)^{-1}w\right|^{-2k+2}$, we have

$$\int_C\frac{dP}{dz}\alpha dS-\sum_l\int_{|z-a^l|=\varepsilon}\frac{dP}{dz}\cdot\frac{z-a^l}{\varepsilon}dS=\frac{1}{2}\int_{D_\varepsilon}\Delta(P)dV, \tag{3.10}$$

where

$$\frac{dP}{dz}=-(k-1)\left\{\left(\frac{dw}{dz}\right)^{-1}w\right\}^*\frac{d}{dz}\left\{\left(\frac{dw}{dz}\right)^{-1}w\right\}\left|\left(\frac{dw}{dz}\right)^{-1}w\right|^{-2k}.$$

In the second term of the left side we have

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*) α is equal to $(z-a)/R$ when D is the hypersphere $|z-a|<R$.

$$Q = -\frac{dP}{dz} \cdot \frac{z-a^l}{\varepsilon} dS$$

$$= (k-1)\left\{\left(\frac{dw}{dz}\right)^{-1} w\right\}^* \frac{d}{dz}\left\{\left(\frac{dw}{dz}\right)^{-1} w\right\} \frac{z-a^l}{\varepsilon} \varepsilon^{2k-1} \left|\left(\frac{dw}{dz}\right)^{-1} w\right|^{-2k} dS_E\,,$$

where dS_E is the surface element of the unit spherical hypersurface in the $2k$-dimensional Euclidean space.

Using the fact:

$$\left(\frac{dw}{dz}\right)^{-1} w = \frac{1}{n_l}(z-a^l) + (\textit{higher powers})$$

on $|z-a^l| = \varepsilon$, we obtain

$$Q = \left\{\frac{1}{n_l}(z-a^l) + (\textit{higher powers})\right\}\left\{\frac{1}{n_l} + (\textit{higher powers})\right\}$$

$$\cdot (z-a^l)\varepsilon^{2k-2} n_l^{2k} \varepsilon^{-2k} dS_E\,,$$

and letting $\varepsilon \to 0$, we have $Q \to n_l^{2k-2} dS_E$.

Accordingly, we see from (3.10) that $\sum_l n_l^{2k-2} = n(0)$ is equal to the right side of (3.9). This completes the proof.

REMARK. The above formula corresponds to the following fact known in the case of $k=1$:

$$n(0) - n_1(0) = \frac{1}{2\pi i}\int_C \frac{w'}{w} dz - \frac{1}{2\pi i}\int_C \frac{w''}{w'} dz\,,$$

where $n_1(0)$ is the sum of the indices of branch points and $n(0) - n_1(0)$ is the number of distinct zero points. Here $\left|\left(\frac{dw}{dz}\right)^{-1} w\right|^{-2k+2}$ in (3.9) is replaced by $\log |w/w'|$.

As for the multiplicity of the zero points satisfying (3.2) and (3.3), it would seem more natural to count it as n^k, whereas we had to count it as n^{2k-2} in the above theorem. The both ways agree for $k=2$, but disagree for $k \geq 3$.

In order to obtain a formula which gives the number of a-points of $w(z)$ for any fixed a, it is sufficient to consider $w(z) - a$ in place of $w(z)$ in (3.9). It is to be noticed that the formula (3.9) does not give the number of zero points of $w(z)$, if the orders of the components of $w(z)$ are not equal, because the condition of (ii) of Lemma 3 does not hold. For instance, this occurs with the function:

$$w(z)=\begin{pmatrix} z_1+z_2 \\ z_1 z_2 \end{pmatrix}.$$

Another difficulty with the above formula is that the term of the volume integral remains. It depends on the fact that, for an analytic vector function $f(z)$, $|f(z)|^{-2k+2}$ is not in general harmonic with respect to $z, \bar{z}$, whereas $|z|^{-2k+2}$ is harmonic. In this regard we have the following theorem.

THEOREM 9. *The Laplacian of* $\left|\left(\frac{dw}{dz}\right)^{-1} w\right|^{-2k+2}$ *is always real-valued and vanishes identically if and only if* $\left(\frac{dw}{dz}\right)^{-1} w$ *is equal to* $a+cUz$, *where* a *is a* k*-tuple constant vector,* c *is a constant complex number and* U *is a unitary matrix of order* k.

PROOF. If we put $\left(\frac{dw(z)}{dz}\right)^{-1} w(z)=f(z)$ and $f(z)^* f(z)=u$, we have from the notation (1.2),

$$\frac{du}{dz}=f^* \frac{df}{dz}, \quad \left(\frac{d}{dz}\right)^* u=\left(\frac{df}{dz}\right)^* f,$$

and

$$\frac{d}{dz}\left(\frac{d}{dz}\right)^* u=\mathrm{Tr.}\left\{\frac{df}{dz}\left(\frac{df}{dz}\right)^*\right\},$$

and so

$$\frac{1}{4}\Delta(u^{-k+1})=(k-1)\left\{k\frac{du}{dz}\left(\frac{d}{dz}\right)^* u-u\frac{d}{dz}\left(\frac{d}{dz}\right)^* u\right\}u^{-k-1}$$

$$=(k-1)f^*\left[k\frac{df}{dz}\left(\frac{df}{dz}\right)^*-\mathrm{Tr.}\left\{\frac{df}{dz}\left(\frac{df}{dz}\right)^*\right\}\cdot E\right]f(f^*f)^{-k-1}.$$

The middle matrix $k\frac{df}{dz}\left(\frac{df}{dz}\right)^*-\mathrm{Tr.}\left\{\frac{df}{dz}\left(\frac{df}{dz}\right)^*\right\}\cdot E\equiv A$ in the last side being Hermitian, $\Delta(u^{-k+1})$ is real-valued and the Laplacian vanishes if and only if all the elements of A are zero.

Putting

$$k\frac{df}{dz}\left(\frac{df}{dz}\right)^*=\mathrm{Tr.}\left\{\frac{df}{dz}\left(\frac{df}{dz}\right)^*\right\}\cdot E,$$

and taking the determinants on both sides, we have

$$k^k\det\left\{\frac{df}{dz}\left(\frac{df}{dz}\right)^*\right\}=\left[\mathrm{Tr.}\left\{\frac{df}{dz}\left(\frac{df}{dz}\right)^*\right\}\right]^k.$$

In case $\det\frac{df}{dz}=0$, we see that $\frac{df}{dz}=0$, that is,

$$\left(\frac{dw}{dz}\right)^{-1}w=a.$$

In case $\det\frac{df}{dz}\neq 0$, $\left\{\det\frac{df}{dz}\right\}^{\frac{1}{k}}$ is analytic and we have

$$\mathrm{Tr.}\left\{\frac{df}{dz}\left(\frac{df}{dz}\right)^*\right\}=k\left|\det\frac{df}{dz}\right|^{\frac{2}{k}}.$$

Accordingly, $\frac{df}{dz}\left\{\det\frac{df}{dz}\right\}^{-\frac{1}{k}}$ is a unitary matrix and so it is constant*), then we can put

$$\frac{df}{dz}=Ug(z),$$

where U is a unitary matrix (u_{ij}) and $g(z)$ is an analytic single function of z. From this we get

$$\frac{\partial f_j}{\partial z_l}=u_{jl}g(z),\quad j,l=1,\cdots,k.$$

Differentiating partially with respect to z_m, we have

$$\frac{\partial^2 f_j}{\partial z_m\partial z_l}=u_{jl}\frac{\partial g}{\partial z_m},\quad \text{for } l\neq m,$$

*) For example, see Bochner-Martin [1], pp. 154-156.

and in the same way we have

$$\frac{\partial^2 f_j}{\partial z_l \partial z_m} = u_{jm} \frac{\partial g}{\partial z_l}.$$

Accordingly,

$$u_{jl} \frac{\partial g}{\partial z_m} = u_{jm} \frac{\partial g}{\partial z_l}.$$

If we multiply $\bar{u}_{jl}$ on both sides and sum up with respect to j from 1 to k, we have

$$\frac{\partial g}{\partial z_m} = 0, \quad m = 1, \cdots, k.$$

This shows that $g(z)$ is constant and $\frac{df}{dz} = cU$. Thus the proof is completed.

The function $w(z)$ which satisfies the conditions of Theorem 9 is not always linear, since the function $A_n z^n$ ($n \geqq 2$) also satisfies it, but anyway from the above theorem it seems natural to consider the functions whose Laplacians are positive or negative definite. We obtain the following theorem corresponding to Jensen's formula in the theory of meromorphic functions of a complex variable. In the case of the systems of functions of a complex variable, the study in this direction was done by H. and J. Weyl [13] and L. V. Ahlfors [14].

THEOREM 10. *Let $w(z)$ be analytic in a hypersphere $|z| < R$ $(0 < R < +\infty)$ in the z-space of k complex dimensions where $w(z)$ does not vanish at* (0) *and satisfy the conditions in Theorem* 8 *and let $n(0, r)$ denote the number of zero points of $w(z)$ in $|z| < r$. If $\left|\left(\frac{dw}{dz}\right)^{-1} w\right|^{-2k+2}$ is subharmonic or superharmonic in $|z| < R$, then the following inequalities hold, respectively*:

$$(3.11) \qquad \left|\left(\frac{dw(0)}{dz}\right)^{-1} w(0)\right|^{-2k+2} - \frac{1}{S_E}\int_{|z|=R} \left|\left(\frac{dw}{dz}\right)^{-1} w\right|^{-2k+2} dS_E$$
$$\leqq 2(k-1)\int_0^R \frac{n(0, r)}{r^{2k-1}}\, dr,$$

or,

$$\left|\left(\frac{dw(0)}{dz}\right)^{-1} w(0)\right|^{-2k+2} - \frac{1}{S_E}\int_{|z|=R}\left|\left(\frac{dw}{dz}\right)^{-1} w\right|^{-2k+2} dS_E \geqq 2(k-1)\int_0^R \frac{n(0,r)}{r^{2k-1}}\,dr, \tag{3.12}$$

where S_E is the whole area of the unit spherical hypersurface in the $2k$-dimensional Euclidean space and dS_E is its surface element.

PROOF. If we put $\left|\left(\frac{dw}{dz}\right)^{-1} w\right|^{-2k+2}=P$, we have from (3.9) in Theorem 8,

$$n(0,r)=-\frac{1}{(k-1)S_E}\int_{|z|=r}\frac{dP}{dz}\cdot\frac{z}{r}\,dS+\frac{1}{2(k-1)S_E}\int_{|z|=r}\Delta P dV.$$

Taking the conjugate of both sides and using the fact that the Laplacian is real-valued, we have

$$n(0,r)=-\frac{1}{(k-1)S_E}\int_{|z|=r}\frac{z^*}{r}\left(\frac{dP}{dz}\right)^* dS+\frac{1}{2(k-1)S_E}\int_{|z|=r}\Delta P dV.$$

Summing up these equalities and noticing the relation:

$$\frac{dP}{dz}\cdot\frac{z}{r}+\frac{z^*}{r}\left(\frac{dP}{dz}\right)^*=\frac{\partial P}{\partial r},$$

we get in the case of $\Delta P\geqq 0$,

$$n(0,r)\geqq-\frac{1}{2(k-1)S_E}\int_{|z|=r}\frac{\partial P}{\partial r}\,r^{2k-1}dS_E.$$

If we divide this inequality by r^{2k-1} and integrate with respect to r from r_0 to R and let $r_0\to 0$, we obtain the inequality (3.11). We can prove (3.12) in the same way as above.

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