Journal of the Mathematical Society of Japan Vol. 9, No. 1, January, 1957

# Minimal complexs of fibre spaces.

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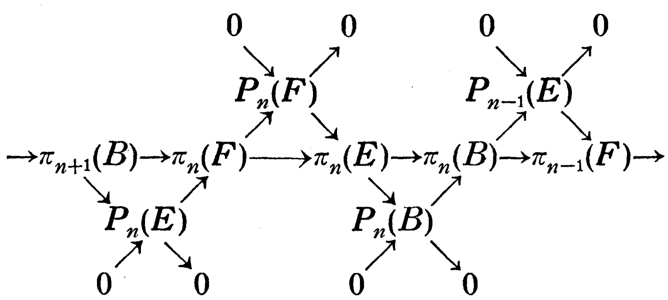
(Received Nov. 10, 1956)

In this note we shall give the algebraic description of several fibre spaces introduced by H. Cartan and J. P. Serre. Our general idea may be characterized as follows. Let $(E, p, B, F)$ be a fibre space in the sense of Serre, and let its homotopy exact sequence be as follows



If we denote the minimal complex of $E, B$ and $F$ with $K(E)$, $K(B)$ and $K(F)$. Then we have an expression of the form

$$
\begin{array}{llllllll}
 & K(F) & \cdots\times K(P_n(E),n) & \overset{\bar{e}^{n+1}}{\times} K(P_n(F),n) & \overset{\bar{k}^{n+1}}{\times} K(P_{n-1}(E),n-1) & \times\cdots \\
(*)K(E)\simeq & \times c \equiv & \times u^{n+1} \quad f^{n+1} & \times e^{n+1}\ c^{n+1} & \times u^n & \times\cdots \\
 & K(B) & \cdots\times K(P_n(E),n+1) & \underset{\underline{k}^{n+2}}{\times} K(P_n(B),n) & \underset{\underline{e}^{n+1}}{\times} K(P_{n-1}(E),n) & \times\cdots
\end{array}
$$

where $K(\pi, n)$ denotes the Eilenberg-McLane complex and the meanings of notations $u^{n+1}, c^{n+1}, k^{n+1}$, etc. are to be explained later (cf. § 3). $K(F)\overset{c}{\times}K(B)$ may be considered as a fibre bundle with the fibre

$$K(B)\equiv \quad \cdots\times K(P_n(E),n+1)\overset{\underline{k}^{n+2}}{\times}K(P_n(B),n)\overset{\underline{e}^{n+1}}{\times}K(P_{n-1}(E),n) \quad \times\cdots$$

over the base

$$K(F)\equiv \quad \cdots\times K(P_n(E),n) \quad \overset{\bar{e}^{n+1}}{\times}K(P_n(F),n)\overset{\bar{k}^{n+1}}{\times}K(P_{n-1}(E),n-1)\times\cdots$$

but it may be also decomposed vertically at an arbitrary place, i. e. it may be considered e. g. as a fibre bundle with fibre

$$\begin{array}{c} K(P_n(E), n) \\ \times u^{n+1} \\ K(P_n(E), n+1) \end{array}$$

over the base

$$\begin{array}{lllll} K(P_n(F), n) & \overset{\bar{k}^{n+1}}{\times} & K(P_{n-1}(E), n-1) & \times & \cdots\cdots \\ \quad \times k^{n+1} & & c^{n+1} & \times u^n & \\ K(P_n(B), n) & \underset{\underline{e}^{n+1}}{\times} & K(P_{n-1}(E), n) & \times & \cdots\cdots \end{array}$$

These fibre bundle structures give rise to corresponding spectral sequences (cf. Proposition 2.9), by means of which we can reproduce among other things the result of Cartan-Serre [2]. The expansion (*) is constructed from three kinds of " products ", on which we have Propositions 2.2, 2.3 and 2.4 respectively which in turn are direct consequences of Proposition 2.1. This last Proposition is the same as Théorème 1 in Exposé 21 in Séminaire de H. Cartan 1955. We had access to the mimeographed note of this Séninaire after we had first written up the bulk of this paper. We notice that there are not a few overlapping parts between our paper and this seminar note, but we allow ourselves to publish this paper in original form for the sake of completenese. Throughout this paper, we assume for simplicity that the fundamental groups of all the spaces to be considered operate trivially on the homotopy groups. The auther wishes to express his thanks to Prof. Iyanaga and also to Mr. N. Yoneda, Mr. T. Yamanosita and Mr. Akio Hattori for their valuable suggestions during the preparation of this paper.

## 1. Preliminaries.

In this paper we use the following terminologies due to S. Eilenberg and S. MacLane. A *complete semi-simplicial complex* $K$—abbreviated as c. s. s. c. in the sequel—is an aggregate of *q-th chain groups* $C_q(K)$ $(q \geqq 0)$, *i-th face operators* $F_i$ and *i-th degeneracy operators* $D_i$ with the following properties

i) $C_q(K)$ is a free abelian group with generators called *q-dimensional cells*

ii) We have $F_i: C_q(K) \to C_{q-1}(K)$ and $D_i: C_q(K) \to C_{q+1}(K)$. $F_i(0 \leqq i \leqq q)$ maps each $q$-cell into a $(q-1)$-cell (which is a *face* of the original $q$-cell) $D_i(0 \leqq i \leqq q)$ maps each $q$-cell into a $(q+1)$-cell (of which the original $q$-cell is a face). The following relations are satisfied;

$$F_iF_j = F_{j-1}F_i \qquad i<j$$

$$D_iD_j = D_{j+1}D_i \qquad i \leqq j$$

$$F_iD_j = D_{j-1}F_i \qquad i<j$$

$$F_iD_i = F_iD_{i-1} = id.$$

$$F_iD_j = D_jF_{i-1} \qquad i-1>j$$

An $F$–$D$ map $f$ of a c. s. s. c. $K$ into another c. s. s. c. $L$ is an aggregate of homomorphisms $f_q$; $C_q(K) \to C_q(L)$ satisfying $F_i f_q = f_q F_i$ and $D_i f_q = f_q D_i$ $(0 \leqq i \leqq q)$ and transforming $q$-cells into $q$-cells.

EXAMPLE 1. We denote by $\Delta(p)$ the c. s. s. c. whose $q$-cells are non-decreasing subsequences $\alpha = \{a_0, a_1, \cdots, a_q\}$ of $\Delta_p = \{0, 1, \cdots, p\}$ $(p \geqq 0)$ and $F_i, D_i$ are given as follows $F_i(a_0, \cdots, a_q) = (a_0, \cdots, \hat{a}_i, \cdots, a_q)$ and $D_i(a_0, \cdots, a_q) = (a_0, \cdots, a_i, a_i, \cdots, a_q)$. The mapping $i \to a_i (0 \leqq i \leqq q)$ induces in an obvious way an $F$–$D$ map from $\Delta(q)$ into $\Delta(p)$ which will be denoted by $\hat{\alpha}$.

EXAMPLE 2. If $\sigma$ is any $q$-cell in a c. s. s. c. $K$, then we define an $F$–$D$ map $\hat{\sigma}$ of $\Delta(q)$ into $K$ as the one satisfying $\hat{\sigma}(\alpha) = (\hat{\sigma} \circ \hat{\alpha})(0, 1, \cdots, r)$ for any $r$-cell $\alpha$ of $\Delta(q)$. $\hat{\sigma}$ is called the *characteristic map* associated with $\sigma$.

The boundary operator $\partial_q: C_q(K) \to C_{q-1}(K)$ are defined by $\partial_q = \sum_{i=0}^{q} (-1)^i F_i$. Notions such as $q$-cycles, $Z_q$, $q$-cochains, $C^q$, and $q$-cocycles, $Z^q$, etc. are defined accordingly.

Let $\sigma$ be an arbitrary $q$-cell in a c. s. s. c. $K$, then we define the *non-degeneracy* of $\sigma$ as the minimum of dimensions of cells $\rho$ for which we have a relation $\sigma = D_{\nu_1} \cdots D_{\nu_{q-p}} \rho$, $q > \nu_1 > \cdots > \nu_{q-p} \geqq 0$. We call $\sigma$ *degenerate* if the non-degeneracy of $\sigma$ is smaller than the dimension of $\sigma$. All the degenerate cells of $K$ form a subcomplex called the *degenerate subcomplex* of $K$ and denoted by $D(K)$. The relative complex $K/D(K)$ is called the *normalized complex* of $K$ and is denoted by $K_N$.

Now, let $\pi$ denote an abelian group and $n$ be an integer $\geqq 0$.

The Eilenberg-MacLane complex $K(\pi, n)$ is a c. s. s. c. defined by

i) $C_q(K(\pi, n))$ (which will be abbreviated as $K_q(\pi, n)$ in the sequel) is a free agelian groug whose generators are all cocycles $\sigma \in Z^n(\Delta(q)_N; \pi)$

ii) $F_i\sigma = \sigma\varepsilon_i$ where $\varepsilon_i: (0,\cdots, q-1) \rightarrow (0,\cdots, \hat{i},\cdots, q)$

$D_i\sigma = \sigma\eta_i$ where $\eta_i: (0,\cdots, q+1) \rightarrow (0,\cdots, i, i,\cdots, q)$

For two $q$-cells $\sigma_1, \sigma_2$ in $K(\pi, n)$ i. e. for $\sigma_1, \sigma_2 \in Z^n(\Delta(q)_N; \pi)$, $\sigma_1 + \sigma_2$ is defined as an element of $Z^n(\Delta(q)_N; \pi)$. Furthermore we define a $q$-cell $1_q$ by $0 \in Z^n(\Delta(q)_N; \pi)$. Then $K_q(\pi, n)$ has a ring structure with unit $1_q$ and with properties $F_i(\sigma_1 \cdot \sigma_2) = F_i\sigma_1 \cdot F_i\sigma_2$ and $D_i(\sigma_1 \cdot \sigma_2) = D_i\sigma_1 \cdot D_i\sigma_2$. This is called the *R-complex structure* of $K(\pi, n)$ (see [6]). (In this paper the product notation instead of the additive notation sometimes will be used also in $C^n(\Delta(q)_N; \pi)$ when no confusion is likely to occur.)

The *suspension operator* $S: K_{q-1}(\pi, n) \rightarrow K_q(\pi, n+1)$ is defined by the formula

$$(S\sigma)(i_0,\cdots, i_{n+1}) = \begin{cases} \sigma(i_1-1,\cdots, i_{n+1}-1) & 0 = i_0 < i_1 < \cdots < i_{n+1} \leqq q \\ 0 & 0 < i_0 < i_1 < \cdots < i_{n+1} \leqq q \end{cases}$$

where $\sigma \in K_{q-1}(\pi, n)$ and $S\sigma \in K_q(\pi, n+1)$ are identified with corresponding elements in $Z^n(\Delta(q-1)_N; \pi)$ and $Z^{n+1}(\Delta(q)_N; \pi)$ respectively. Now we can easily prove that the map

$$\psi: K_{q-1}(\pi, n) \otimes K_{q-1}(\pi, n+1) \rightarrow K_q(\pi, n+1)$$

defined by

$$\psi(\sigma_{q-1} \otimes \overset{*}{\sigma}) = S\sigma_{q-1} \cdot D_0\overset{*}{\sigma} \qquad \text{for } \overset{*}{\sigma} \in K_{q-1}(\pi, n+1)$$

gives an isomorphism between $K_{q-1}(\pi, n) \otimes K_{q-1}(\pi, n+1)$ and $K_q(\pi, n+1)$. If we denote $\psi(\sigma_{q-1} \otimes \overset{*}{\sigma})$ by $\langle \sigma_{q-1}, \overset{*}{\sigma} \rangle$, then by repetition of $\psi^{-1}$ we obtain the isomorphism

$$K_q(\pi, n+1) \cong K_{q-1}(\pi, n) \otimes K_{q-2}(\pi, n) \otimes \cdots \otimes K_0(\pi, n).$$

This isomorphism of $K_q(\pi, n+1)$ onto $K_{q-1}(\pi, n) \otimes \cdots \otimes K_0(\pi, n)$ will be denoted by $w$. We shall write $w^{-1}(\sigma_{q-1} \otimes \cdots \otimes \sigma_0)$ by $\langle \sigma_{q-1},\cdots, \sigma_0 \rangle$. Expansion of every element of in the form of $\langle \sigma_{q-1},\cdots, \sigma_0 \rangle$ will be called the *W-decomposition* of the element.

Let $G$ be an arbitrary abelian group. The $r$-th cochain group $C^r(K(\pi, n); G)$ and cocycle group $Z^r(K(\pi, n); G)$ of the Eilenberg-MacLane complex $K(\pi, n)$ with the coefficient group $G$ are denoted by $C^r(\pi, n; G)$, $Z^r(\pi, n; G)$ respectively.

Let $L$ be a c. s. s. c. and $k^{n+1}$ be an element of $Z^{n+1}(L_N; \pi)$. Then the complex $K(L, k^{n+1}, \pi)$ is defined as follows. Let $\sigma$ be any $q$-cell of

$L$ and $\hat{\sigma}$ be the characteristic map of $\Delta(q)$ into $L$ corresponding to $\sigma$. Now we set for simplicity $k^{n+1}(\sigma)=\hat{\sigma}^{\#}k^{n+1}\in Z^{n+1}(\Delta(q);\pi)$. Then $K(L, k^{n+1}, \pi)$ is the c. s. s. c. whose $q$-cells are by all the pairs $(\upsilon, \sigma)$ of $\upsilon\in C^n(\Delta(q)_N;\pi)$ and $\sigma\in L_q$ satisfying $\delta\upsilon-k^{n+1}(\sigma)=0$. Now we define $F_i\upsilon=\upsilon\varepsilon_i$, $F_i(\upsilon,\sigma)=(F_i\upsilon, F_i\sigma)$ and $D_i\upsilon=\upsilon\eta_i$, $D_i(\upsilon,\sigma)=(D_i\upsilon, D_i\sigma)$.

Then we can define recursively $K(\pi_1, d_1, {}^1k^{d_2+1}, \pi_2, d_2,\cdots, \pi_{n-1}, d_{n-1}, {}^{n-1}k^{d_n+1}, \pi_n, d_n)$ by $K(K(\pi_1, d_1, {}^1k^{d_2+1}, \pi_2, d_2,\cdots, \pi_{n-1}, d_{n-1}), {}^{n-1}k^{d_n+1}, \pi_n)$ for given abelian groups $\pi_1, \pi_2,\cdots, \pi_n$ and ${}^{i-1}k^{d_i+1}\in Z^{d_i+1}(K(\pi_1, d_1, {}^1k^{d_2+1}, \pi_2, d_2, \cdots, \pi_{i-1}, d_{i-1})_N;\pi_i)$ $i=2, 3,\cdots, n$. When a infinite sequence of abelian groups $\pi_1, \pi_2,\cdots$, and a corresponding sequence of ${}^1k^{d_2+1}, {}^2k^{d_3+1},\cdots$ are given, then the direct limit of $K(\pi_1, d_1, {}^1k^{d_2+1}, \pi_2, d_2,\cdots, \pi_n, d_n)$ is denoted by $K(\pi_1, d_1, {}^1k^{d_2+1}, \pi_2, d_2,\cdots)$. If $d_i=i$ we denote $K(\pi_1, 1, {}^1k^3, \pi_2, 2,\cdots)$ by $K(\pi_1, k^3, \pi_2,\cdots)$ and similar abbreviation will be used sometimes in the sequel e. g. $K(\pi_0, k^2, \pi_1,\cdots, \pi_n)$, $K(\pi_m, d^{n+1}, \pi_n)$ etc. As well known [9] these complexes can be determined uniquely within isomorphism by $\pi_i, d_i$ and the cohomology classes of ${}^{i-1}k^{d_i+1}(i-2, 3,\cdots)$.

As shown by Eilenberg [5] or Postnikov [10] a minimal complex of the space with the $i$-th homotopy group $\pi_i$ and $(i-1)$-th Postnikov's invariant $k^{i+1}$ is isomorphic to $K(\pi_1, k^3, \pi_2,\cdots)$.

In the sequel we consider always sequences $\pi_1, \pi_2,\cdots$ and $k^3, k^4,\cdots$ as given once for all and denote $K(\pi_1, k^3, \pi_2,\cdots)$ by $K$ and $K(\pi_1, k^3, \pi_2, \cdots, \pi_n)$ by $K(^n_0)$. These notations will be used also the complexes of type $K(\pi_0, k^2, \pi_1, k^3, \pi_2,\cdots)$. If $\pi_i=0, i\neq m, n(m<n)$ it can be verified that $K(^{n-1}_{\ 0})\cong K(\pi_m, m)$ and $K\cong K(\pi_m, k^{n+1}, \pi_n)$, where $k^{n+1}\in Z^{n+1}(K(^{n-1}_{\ 0})_N; \pi_n)$ is identified with the corresponding element in $Z^{n+1}(\pi_m, m;\pi_n)$ in terms of above isomorphism.

## 2. Representation of minimal complexes by means of $K(\pi, n)$

Let $L$ be a c. s. s. c. and $k^{n+1}$ is a cocycle in $Z^{n+1}(L_N;\pi)$. Then we construct a complex $K=K(L, k^{n+1}, \pi)$ as before and its $q$-cells are all pairs $(\upsilon, \sigma)$ of

$$\upsilon\in C^n(\Delta(q)_N;\pi)$$

and $\sigma\in L_q$ satisfying the coboundary relation

$$\delta\upsilon-k^{n+1}(\sigma)=0 \tag{1}$$

Clearly the *q-simplex* $\Delta(q)$ is acyclic

$$s_{\#}(i_0,\cdots,i_r)=(0,i_0,\cdots,i_r) \qquad 0\leqq i\leqq q$$

defines a chain deformation $s_{\#}: C_r(\Delta(q))\rightarrow C_{r+1}(\Delta(q))$ satisfying

$$\partial s_{\#}+s_{\#}\partial = id-\eta\varepsilon$$

where $id$=identity, $\varepsilon$=augmentation and $\eta$=inverse of $\varepsilon$ (cf. [1] Exposé 2) and, the dual of $s_{\#}=s^{\#}: C^{r+1}(\Delta(q)_N)\rightarrow C^r(\Delta(q)_N)$ satisfies

$$s^{\#}\delta+\delta s^{\#}=id-\varepsilon^{\#}\eta^{\#}$$

Since

$$\delta s^{\#}k^{n+1}(\sigma)=\delta s^{\#}\partial^{\#}k^{n+1}=\partial^{\#}k^{n+1}-s^{\#}\delta\partial^{\#}k^{n+1}=k^{n+1}(\sigma)$$

the *standard q-cell* $(s^{\#}k^{n+1}(\sigma),\sigma)\in K$, furthermore it is apparent that

$$\delta(\upsilon-s^{\#}k^{n+1}(\sigma))=0 \tag{2}$$

or equivalently

$$\tau=\upsilon-s^{\#}k^{n+1}(\sigma)\in Z^n(\Delta(q)_N;\pi) \tag{2'}$$

If we identify each $\tau\in Z^n(\Delta(q)_N;\pi)$ with the corresponding generator of $K_q(\pi,n)$ we have the following isomorphism[1)]

$$\iota: K_q\cong K_q(\pi,n)\otimes L_q \tag{3}$$

defined by $\iota(\upsilon,\sigma)=\tau\otimes\sigma$. This suggests us that $K$ has a fibre bundle structure with the fibre complex $K(\pi,n)$ over the base complex $L$. In fact we have

PROPOSITION 2.1.

$$K(L,k^{n+1},\pi)\cong K(\pi,n)\overset{k^{n+1}}{\times}L$$

*where* $K(\pi,n)\overset{k^{n+1}}{\times}L$ *is a c. s. s. c. with the q-dimensional generators* $\tau\times\sigma(=\tau\otimes\sigma)$, $\sigma$ *and* $\tau$ *being q-cells in* $L$ *and* $K(\pi,n)$ *respectively, and F-D operators are defined as follows*

$$F_0(\tau\times\sigma)=k^n_{q-1}(\sigma)\cdot F_0\tau\times F_0\sigma$$

*with* $k^{n+1}(\sigma)=\langle k^n_{q-1}(\sigma),k^n_{q-2}(\sigma),\cdots,k^n_0(\sigma)\rangle$

$$F_i(\tau\times\sigma)=F_i\tau\times F_i\sigma \qquad 0<i\leqq q$$

$$D_i(\tau\times\sigma)=D_i\tau\times D_i\sigma$$

PROOF. If we identify $\tau,\sigma$ etc. with the corresponding cells in $K_q(\pi,n)$, $L_q$ etc., then the formula (2′) is replaced by

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1) $K_q(\pi,n)$ and $L_q$ denote $C_q(K(\pi,n))$ and $C_q(L)$ respectively.

$$\tau = \upsilon \cdot s^{\#} k^{n+1}(\sigma)^{-1} \tag{2''}$$

making use of the $R$-complex notation (cf. § 1). Assume now that $(\upsilon, \sigma)$ corresponds to $\tau \times \sigma$ by the isomorphism $\iota$ in (3) then it is clear that $F_i\upsilon$ and $D_i\upsilon$ correspond to $F_i\upsilon \cdot s^{\#}k^{n+1}(F_i\sigma)^{-1} \times F_i\sigma$ and $D_i\upsilon \cdot s^{\#}k^{n+1}(D_i\sigma)^{-1} \times D_i\sigma$ respectively $(0 \leqq i \leqq q)$. This permits us to define $F$–$D$ operators in the free group $\sum_{q \geqq 0} K_q(\pi, n) \otimes L_q$ by formulas

$$\begin{aligned} F_i(\tau \times \sigma) &= F_i\upsilon \cdot s^{\#}k^{n+1}(F_i\sigma)^{-1} \times F_i\sigma \\ D_i(\tau \times \sigma) &= D_i\upsilon \cdot s^{\#}k^{n+1}(D_i\sigma)^{-1} \times D_i\sigma \end{aligned} \tag{4}$$

where $\iota(\upsilon, \sigma) = \tau \times \sigma$, and we denote by $K(\pi, n) \overset{k^{n+1}}{\times} L$ the complex thus obtained. Then the isomorphism $\iota$ in (3) becomes a $F$–$D$ isomorphism

$$\iota : K \cong K(\pi, n) \overset{k^{n+1}}{\times} L$$

Next, operating $F_i$ on both side of (2), we have

$$F_i\tau = F_i\upsilon \cdot F_i s^{\#} k^{n+1}(\sigma)^{-1} \tag{5}$$

On the other hand we know that $\rho = S\rho_{q-1} \cdot D_0 F_0 \rho$ for any $\rho = \langle \rho_{q-1}, \overset{*}{\rho} \rangle \in K_q(\pi, n+1)$, so that we have the following formulas

$$F_0 s^{\#} \rho = F_0 s^{\#} S \rho_{q-1} \cdot F_0 s^{\#} D_0 F_0 \rho = \rho_{q-1} \cdot s^{\#} F_0 \rho \tag{$6_0$}$$

since $F_0 s^{\#} S = id$ and $F_0 s^{\#} D_0 = s^{\#}$. From (5) and $(6_0)$ we obtain

$$F_0\tau = F_0\upsilon \cdot k^n_{q-1}(\sigma)^{-1} \cdot s^{\#} F_0 k^{n+1}(\sigma)^{-1} = k^n_{q-1}(\sigma)^{-1} \cdot F_0\upsilon \cdot s^{\#} k^{n+1}(F_0\sigma)^{-1}$$

or equivalently

$$k^n_{q-1}(\sigma) F_0\tau \times F_0\sigma = F_0\upsilon \cdot s^{\#} k^{n+1}(F_0\sigma)^{-1} \times F_0\sigma = F_0(\tau \times \sigma)$$

In case $F_i(\tau \times \sigma)$ $(0 < i \leqq q)$ replacing $(6_0)$ by

$$F_i s^{\#} \rho = s^{\#} F_i \rho \tag{$6_i$}$$

we have

$$F_i\tau = F_i\upsilon \cdot s^{\#} F_i k^{n+1}(\sigma) = F_i\upsilon \cdot s^{\#} k^{n+1}(F_i\sigma)$$

and consequently

$$F_i\tau \times F_i\sigma = F_i(\tau \times \sigma) .$$

Similarly we have

$$D_i\tau \times D_i\sigma = D_i(\tau \times \sigma) \qquad 0 \leqq i \leqq q$$

From this proposition follows immediately

PROPOSITION 2.2. *We have*

$$K(\pi_m, k^{n+1}, \pi_n) \cong K(\pi_n, n) \overset{k^{n+1}}{\times} K(\pi_m, m)$$

Let us denote by $u^{n+1}$ the fundamental cocycle in $K(\pi, n+1)$ so that $u^{n+1}(<\sigma_{q-1}, \sigma_{q-2}, \cdots, \sigma_0>)$ can be identified with $<\sigma_{q-1}, \sigma_{q-2}, \cdots, \sigma_0>$ (The notation $u^{n+1}$ for the fundamental cocycle of $K(\pi, n+1)$ will be used throughout this paper). Then we have

PROPOSITION 2.3. *The c. s. s. c.*

$$K(\pi, u^{n+1}, \pi) \cong K(\pi, n) \overset{u^{n+1}}{\times} K(\pi, n+1)$$

*with F-D operators*

$$\begin{aligned} F_0(\sigma_q \times <\sigma_{q-1}, \sigma_{q-2}, \cdots, \sigma_0>) &= \sigma_{q-1} \cdot F_0\sigma_q \times F_0 <\sigma_{q-1}, \sigma_{q-2}, \cdots, \sigma_0> \\ &= \sigma_{q-1} \cdot F_0\sigma_q \times <\sigma_{q-2}, \cdots, \sigma_0> \\ F_i(\sigma_q \times <\sigma_{q-1}, \sigma_{q-2}, \cdots, \sigma_0>) &= F_i\sigma_q \times F_i <\sigma_{q-1}, \sigma_{q-2}, \cdots, \sigma_0> \qquad 0 < i \leqq q \\ &= F_i\sigma_q \times <F_{i-1}\sigma_{q-1}, \cdots, F_1\sigma_{q-i+1}, \\ &\qquad F_0\sigma_{q-i} \cdot \sigma_{q-i-1}, \cdots, \sigma_0> \\ D_i(\sigma_q \times <\sigma_{q-1}, \sigma_{q-2}, \cdots, \sigma_0>) &= D_i\sigma_q \times D_i <\sigma_{q-1}, \sigma_{q-2}, \cdots, \sigma_0> \qquad 0 \leqq i \leqq q \\ &= D_i\sigma_q \times <D_{i-1}\sigma_{q-1}, \cdots, D_0\sigma_{q-i}, \sigma_{q-i-1}, \cdots, \sigma_0> \end{aligned}$$

*is acyclic and its chain deformation D is given as follows*

$$D(\sigma_q \times <\sigma_{q-1}, \sigma_{q-2}, \cdots, \sigma_0>) = 1_{q+1} \times <\sigma_q, \sigma_{q-1}, \cdots, \sigma_0>.$$

PROPOSITION 2.4. *Let*

$$0 \longrightarrow \pi' \overset{\alpha}{\longrightarrow} \pi \overset{\beta}{\longrightarrow} \pi'' \longrightarrow 0$$

*be an exact sequence of abelian groups and* $k^{n+1} \in Z^{n+1}(\pi'', n; \pi')$ *the* $(n-1)$*-fold suspension of the Eilenberg invariant with the opposite sign* (cf. [3], I) *of this sequence. Then we have*

$$K(\pi, n) \cong K(\pi', n) \overset{k^{n+1}}{\times} K(\pi'', n).$$

PROOF. Since the complex of the right-hand side is isomorphic to $K(K(\pi'', n), k^{n+1}, \pi')$ by the Proposition 2.1., we have only to establish an isomorphism between this complex and $K(\pi, n)$. Let $\rho$ be any $q$-cell of $K(\pi, n)$, $\rho$ can be identified with an element of $Z^n(\Delta(q)_N; \pi)$. On the other hand, any $q$-cell of $K(K(\pi'', n), k^{n+1}, \pi')$ can be identified with a pair $(\upsilon, \sigma)$ of an element $\upsilon$ of $C^n(\Delta(q)_N; \pi')$ and $\sigma \in Z^{n+1}(\Delta(q)_N; \pi'')$ satisfying $\delta\upsilon - k^{n+1}(\sigma) = 0$ or equivalently $(\upsilon, \sigma)$ satisfying $\tau = \upsilon - s^\# k^{n+1}(\sigma)$ for some $\sigma \in Z^{n+1}(\Delta(q)_N; \pi'')$ and $\tau \in Z^n(\Delta(q)_N; \pi')$.

Now let $f$ be a map from $\pi''$ into $\pi$ such that $\beta \circ f = id$ and define $f^n$ the element of $C^n(\pi'', n; \pi)$ by $f^n(\sigma) \cdot (a_0, \cdots, a_n) = (\hat{\sigma}^\# f^n)(a_0, \cdots, a_n) = (f\hat{\sigma})(a_0, \cdots, a_n)$ for $0 \leqq a_0 < \cdots < a_n \leqq q$, Furthermore we define $\alpha^n \in Z^n(\pi, n; \pi'')$ and $\beta^n \in Z^n(\pi', n; \pi)$ by $\alpha^n = \mathrm{Hom}(id, \alpha)u^n$, $\beta^n = \mathrm{Hom}(id, \beta)u^n$ or equivalently by $\alpha^n(\tau) = \mathrm{Hom}(id, \alpha)\tau$, $\beta^n(\rho) = \mathrm{Hom}(id, \beta)\rho$ for $\tau \in Z^n(\Delta(q)_N; \pi')$, $\rho \in Z^n(\Delta(q)_N; \pi)$. Notice that $\mathrm{Hom}(id, \alpha)$ is a monomorphism from $C^n(\Delta(q)_N; \pi')$ into $C^n(\Delta(q)_N; \pi)$.

Now put $\sigma = \beta^n(\rho)$, then $\rho - f^n(\sigma)$ is clearly the $\mathrm{Hom}(id, \alpha)$-image of an element $\upsilon$ of $C^n(\Delta(q); \pi')$. The correspondence $\rho \to (\upsilon, \sigma)$ establishes obviously the desired isomorphism.

REMARK. For $n=1$, this proposition can be generalized with slightly different form to the case of extension of non-abelian groups and we can reproduce the theory of Eilenberg-MacLane [3] and Hochshcild-Serre [8].

PROPOSITION 2.5. *Let*

$$0 \longrightarrow \pi' \xrightarrow{\alpha} \pi \xrightarrow{\beta} \pi'' \longrightarrow 0$$

*be an exact sequence of abelian groups and*

$$\alpha^{n+1} = \mathrm{Hom}(id, \alpha)u^{n+1} \in Z^{n+1}(\pi', n+1; \pi) \text{ i. e.}$$

$$\alpha^{n+1}(\sigma) = \hat{\sigma}^\# \alpha^{n+1} = \mathrm{Hom}(id, \alpha)\sigma \in Z^{n+1}(\Delta(q)_N; \pi)$$

$$\text{for } \sigma \in Z^{n+1}(\Delta(q)_N; \pi')$$

*then we have*

$$K(\pi'', n) \simeq K(\pi, n) \overset{\alpha^{n+1}}{\times} K(\pi', n+1)$$

PROOF. See § 3.

PROPOSITION 2.6. *Let*

$$0 \longrightarrow \pi' \xrightarrow{\alpha} \pi \xrightarrow{\beta} \pi'' \longrightarrow 0$$

*be an exact sequence of abelian groups and*

$$\beta^n = \mathrm{Hom}(id, \beta)u^u \in Z^n(\pi, n; \pi'') \text{ i. e.}$$

$$\beta^n(\sigma) = \hat{\sigma}^\# \beta^n = \mathrm{Hom}(id, \beta)\sigma \in Z^n(\Delta(q)_N; \pi'')$$

$$\text{for } \sigma \in Z^n(\Delta(q)_N; \pi)$$

*we have*

$$K(\pi', n) \simeq K(\pi'', n-1) \overset{\beta n}{\times} K(\pi, n)$$

PROOF. See § 3.

COROLLARY 2.7. $K(0, n) \simeq K(\pi, n) \overset{un+1}{\times} K(\pi, n+1)$

REMARK $K(\pi, n) \overset{un+1}{\times} K(\pi, n+1)$ was shown to be acyclic in the Proposition 2.2. Corollary 2. clarifies the meaning of this acyclicity.

PROPOSITION 2.8. *Let $L$ be any c.s.s.c. and $k^{n+1} \in Z^{n+1}(L_N, \pi)$. Define a map*

$$T(k^{n+1}) : L \to K(\pi, n+1)$$

*by $T(k^{n+1})\sigma = k^{n+1}(\sigma)$ for $\sigma \in L$. If $\sigma$ is a q-cell, $k^{n+1}(\sigma)$ is in $Z^{n+1}(\Delta(q)_N; \pi)$ by the definition and can be identified with a q-cell of $K(\pi, n+1)$. Then*

$$id \times T(k^{n+1}) : K(\pi, n) \overset{kn+1}{\times} L \to K(\pi, n) \overset{un+1}{\times} K(\pi, n+1)$$

*is a F-D map. In other words $k(\pi, n) \overset{kn+1}{\times} L \cong K(L, k^{n+1}, \pi)$ can be considered as the fibre bundle induced from $K(\pi, n) \overset{un+1}{\times} K(\pi, n+1)$ by the map $T(k^{n+1})$.*

Proof is done by the straightforward computation.

As shown in Proposition 2.1., $K(\pi, n) \overset{kn+1}{\times} L$ has the fibre bundle structure with the fibre $K(\pi, n)$ over the base $L$. Now we introduce a filtration in $M = K(\pi, n) \overset{kn+1}{\times} L$ by subcomplexes $F_p(M)$ $(0 \leqq p \leqq \infty)$ of $M$ as follows a $q$-cell $\tau \times \sigma$ of $M$ belongs to $F_p(M)$ if and only if the non-degeneracy of $\sigma \leqq p$. Then, denoting for brevity $C(F_p(M))$ by $A_p$ and $C(M)$ by $A$, $\{A_p\}$ defines a filtration on the differential module $A$ and

$$_rC_p = A_p \cap d^{-1}(A_{p-r}) \qquad _\infty C_p = A_p \cap d^{-1}(0)$$

$$_rB_p = A_p \cap d(A_{p+r}) \qquad _\infty B_p = A_p \cap d(A)$$

defines a spectral sequence $_rE_p = {_rC_p}/({_{r-1}C_p} \cup {_{r-1}B_p})$.

PROPOSITION 2.9. *Under these conventions we have*

$$_2E_p \cong H_p(L, H(\pi, n))$$

PROOF. Let $\rho$ be any non-degenerate $p$-cell in $L$. Then we define a c. s. s. c. $M_\rho$ as an isomorphic copy of $K(\pi, n) \overset{k^{n+1}(\rho)}{\times} \Delta(p)$. We shall denote with the $i_\rho$ the isomorphic map of $M_\rho$ onto $K(\pi, n) \overset{k^{n+1}(\rho)}{\times} \Delta(p)$, $(id \times \hat{\rho}) \circ i_\rho$ is then an $F$-$D$ map of $M_\rho$ into $M$. We put $\varphi_\rho = (id \times \hat{\rho}) \circ i_\rho :$ $M_\rho \to M$. We can introduce a filtration $\{A_p(M_\rho)\}$, $\{A_p(\underset{\rho}{\times} M_\rho)\}$ on $C(M_\rho)$ and $C(\underset{\rho}{\times} M_\rho)$ respectively just as the above filtration $\{A_p\}$ on $C(M)$. Then we can see easily that

$$\varphi = \times \varphi_\rho : \underset{\rho}{\times} M_\rho \to M$$

induces an isomorphism

$$\varphi_* : H(A_p(\underset{\rho}{\times} M_\rho)/A_{p-1}(\underset{\rho}{\times} M_\rho)) \cong H(A_p/A_{p-1})$$

On the other hand $k^{n+1}(\sigma) \sim 0$ in $\Delta(p)$ and we have $M_p \cong K(\pi, n) \overset{k^{n+1}(\rho)}{\times} \Delta(p)$ $\cong K(\pi, n) \times \Delta(p)$. Since $H(A_p(\underset{\rho}{\times} M_\rho)/A_{p-1}(\underset{\rho}{\times} M_\rho)) \cong \sum_\rho H(A_p(M_\rho)/A_{p-1}(M_\rho))$ and $H(A_p(M_\rho)/A_{p-1}(M_\rho)) \cong H(\pi, n)$ by the theorem of Künneth [7], we obtain an isomorphism

$$\varphi_p : {}_1E_p \cong H(\pi, n) \otimes C_p(L_N)$$

Direct computation shows that the following diagram is commutative

$$\begin{array}{ccc} {}_1E_p & \xrightarrow{\varphi_p} & H(\pi, n) \otimes C_p(L_N) \\ \downarrow d_1 & & \downarrow id \otimes \partial \\ {}_1E_{p-1} & \xrightarrow{\varphi_{p-1}} & H(\pi, n) \otimes C_{p-1}(L_N) \end{array}$$

Hence we have the required result

$${}_2E_p \cong H_p(L_N, H(\pi, n)) \cong H_p(L, H(\pi, n))\,.$$

REMARK 1. It can be easily verified that $K(\pi, n) \overset{k^{n+1}}{\times} L$ is a $DGA$-module over the $DGA$-algebra $K(\pi, n)$ in the sense of Cartan [1].

2. Making use of the Propositions 2.3. and 2.9., we can construct the equivalence map between the "$W$-construction" and the "bar-construction" (cf. [6], [1]).

## 3. Minimal complexes of loop space[2)]

We shall define now the c. s. s. c. $\Omega K$ for our complex $K=K(\pi_1, k^3, \pi^2, \cdots)$.[3)] $\Omega K$ is the direct limit of $(\Omega K)\binom{n}{0}$ which are defined successively together with the suspension operators $S$: $(\Omega K)\binom{n}{0}_q \to K\binom{n+1}{0}_{q+1}$ successively as follows,

0) $(\Omega K)\binom{0}{0}=K(\pi_1, 0)$ and $S$ coincides with the suspension operator previously defined on $K(\pi_1, 0)$

$n-1$) $(\Omega K)\binom{n-2}{0}$ and $S: (\Omega K)\binom{n-2}{0}_q \to K\binom{n-1}{0}_{q+1}$ being determined, we define the $(n-1)$-th group of $(\Omega K)\binom{n-1}{0}$ to be $\pi_n$, $n$-th invariant $(\omega k)^n$ by the formulas

$$S(\omega k)^n\sigma = k^{n+1}(S\sigma)$$

for $\sigma \in (\Omega K)\binom{n-2}{0}$, and $(\Omega K)\binom{n-1}{0}$ to be $K((\Omega K)\binom{n-2}{0}, (\omega k)^n, \pi_n) \cong K(\pi_n, n-1) \overset{(\omega k)^n}{\times} (\Omega K)\binom{n-2}{0}$. Furthermore $S: (\Omega K)\binom{n-1}{0}_q \to K\binom{n}{0}_{q+1}$ is defined by

$$S(\tau\times\sigma) = D_0(\omega k)^n(\sigma)^{-1}\cdot S\tau \times S\sigma$$

Thus we have

$$\Omega K \equiv \cdots \overset{(\omega k)^{n+1}}{\times} K(\pi_n, n-1) \overset{(\omega k)^n}{\times} K(\pi_{n-1}, n-2) \overset{(\omega k)^{n-1}}{\times} \cdots \overset{(\omega k)^2}{\times} K(\pi_1, 0)$$

for

$$K \equiv \cdots \overset{k^{n+2}}{\times} K(\pi_n, n) \overset{k^{n+1}}{\times} K(\pi_{n-1}, n-1) \overset{k^n}{\times} \cdots \overset{k^3}{\times} K(\pi_1, 1)$$

REMARK. It is proved without difficulty that $\Omega K$ thus defined is a minimal complex of $\Omega\mathcal{K}$ i. e. the loop space over $\mathcal{K}$, if $\pi_1=0$ and $K$ is realized by a topological space $\mathcal{K}$. More generally if $K$ is realized by $\mathcal{K}$ and $\mathcal{K}$ has the universal covering space $\tilde{\mathcal{K}}$, $\Omega\tilde{\mathcal{K}}$ has the minimal complex

$$\Omega\tilde{K} \equiv \cdots \overset{l^{n+1}}{\times} K(\pi_n, n-1) \overset{l^n}{\times} K(\pi_{n-1}, n-2) \overset{l^{n-1}}{\times} \cdots \overset{l^3}{\times} K(\pi_2, 1)$$

where $l^n=(\omega k)^n\psi$ with the natural injection

$$\psi: \Omega\tilde{K} \to \Omega K.$$

Notice that $\tilde{\mathcal{K}}$ has the minimal complex

2) cf. Remak 1. §3. Proposition 3.12.

3) From the Proposition 2.1, the following identity is almost trivial $K(\pi_1, k^3, \pi_2, \cdots) \cong \cdots \overset{k^{n+2}}{\times} K(\pi_n, n) \overset{k^{n+1}}{\times} K(\pi_{n-1}, n-1) \overset{k^n}{\times} \cdots \overset{k^3}{\times} K(\pi_1, 1)$.

$$\tilde{K} \equiv \cdots \overset{h^{n+2}}{\times} K(\pi_n, n) \overset{h^{n+1}}{\times} K(\pi_{n-1}, n-1) \overset{h^n}{\times} \cdots \overset{h^4}{\times} K(\pi_2, 2)$$

where $h^{n+1} = k^{n+1}\varphi$ with the natural injection

$$\varphi : \tilde{K} \to K$$

and above $l^n$ coincides with $(\omega h)^n$.

Let $K_{ij}$ be given semi-simplicial complexes, then the matrix $(K_{ij})$ will denote a graded free module whose $q$-dimensional component is generated by all matrices $(\sigma_{ij})$, $\sigma_{ij}$ being $q$-cells in $K_{ij}$. We often make use of c. s. s. c. whose chain groups are $(K_{ij})$. The $F$–$D$ operators of these c. s. s. c. will be indicated each time appropriately. The expression in Propositions 2.1, 2.2, 2.3, 2.4 are examples of such complexes for 1-rowed matrix.

Now we consider $2 \times 2$ matrix

$$\begin{pmatrix} K(\pi_n, n-1) & K(\pi_m, m-1) \\ K(\pi_n, n) & K(\pi_m, m) \end{pmatrix}$$

with the generic element

$$\begin{pmatrix} \bar{\tau} & \bar{\sigma} \\ \underline{\tau} & \underline{\sigma} \end{pmatrix}$$

and introduce the following $F$–$D$ operators, where $k^n_q(\sigma) = < k^{n-1}_{q,q-1}(\sigma), \cdots, k^{n-1}_{q,0}(\sigma) >$, where $\sigma = < \bar{\sigma}, \underline{\sigma} >$

$$F_0 \begin{pmatrix} \bar{\tau} & \bar{\sigma} \\ \underline{\tau} & \underline{\sigma} \end{pmatrix} = \begin{pmatrix} k^{n-1}_{q,q-1}(\sigma)^{-1} \cdot \underline{\tau}_{q-1} \cdot F_0\bar{\tau} & \underline{\sigma}_{q-1} \cdot F_0\bar{\sigma} \\ k^n_{q-1}(\underline{\sigma}) F_0 \underline{\tau} & F_0 \underline{\sigma} \end{pmatrix}$$

$$F_i \begin{pmatrix} \bar{\tau} & \bar{\sigma} \\ \underline{\tau} & \underline{\sigma} \end{pmatrix} = \begin{pmatrix} F_i\bar{\tau} & F_i\bar{\sigma} \\ F_i\underline{\tau} & F_i\underline{\sigma} \end{pmatrix} \qquad 0 < i \leqq q$$

$$D_i \begin{pmatrix} \bar{\tau} & \bar{\sigma} \\ \underline{\tau} & \underline{\sigma} \end{pmatrix} = \begin{pmatrix} D_i\bar{\tau} & D_i\bar{\sigma} \\ D_i\underline{\tau} & D_i\underline{\sigma} \end{pmatrix} \qquad 0 \leqq i \leqq q$$

The c. s. s. c. thus obtained will be denoted as

$$\begin{matrix} \Omega K & K(\pi_n, n-1) & \overset{(\omega k)^n}{\times} & K(\pi_m, m-1) \\ \times u \equiv & \times u^n & & \times u^m \\ K & K(\pi, n) & \underset{k^{n+1}}{\times} & K(\pi_m, m) \end{matrix}$$

The same complex will be denoted by $\Omega K \overset{u}{\times} K$. This complex has clearly the fibre bundle structure not only with the fibre $\Omega K$ over the base $K$ but also with the fibre $K(\pi_n, n-1)\overset{u^n}{\times}K(\pi_n, n)$ over the base $K(\pi_m, m-1)\overset{u^m}{\times}K(\pi_m, m)$. Hence we can introduce a filtration on this complex just we have done on $M=C(\Omega K\overset{u}{\times}K)$, in the Proposition 2.9. and define a spectral sequence ${}_rE_p, {}_rE'_p$. And we have just as there

$$ {}_2E_p \cong H_p(K, H(\Omega K)) $$

$$ {}_2E'_p \cong H_p(K(\pi_m, m-1)\overset{u^m}{\times}K(\pi_m, m), H(K(\pi_n, n-1)\overset{u^n}{\times}K(\pi_n, n))) $$

Since $K(\pi_m, m-1)\overset{u^m}{\times}K(\pi_m, m)$ and $K(\pi_n, n-1)\overset{u^n}{\times}K(\pi_n, n)$ are acyclic we see instantly that $\Omega K\overset{u^m}{\times}K$ is acyclic. Thus we have

PROPOSITION 3.1. *The c. s. s. c.* $\Omega K\overset{u}{\times}K$ *is acyclic.*

REMARK 1. The chain deformation $D$ for $\Omega K\overset{u}{\times}K$ is given by

$$ D\begin{pmatrix} \bar{\tau} & \bar{\sigma} \\ \underline{\tau} & \underline{\sigma} \end{pmatrix} = \begin{pmatrix} 1_{q+1} & , & 1_{q+1} \\ D_0k_q^{n+1}<\bar{\sigma}, \underline{\sigma}>^{-1}\cdot<\bar{\tau}, \underline{\tau}>, & & <\bar{\sigma}, \underline{\sigma}> \end{pmatrix} $$

2. To study the relationship between the $W$-construction and $\Omega K\overset{u}{\times}K$ in Proposition 3.1, it is convenient to introduce the following notation

$$ <\bar{\tau}\times\bar{\sigma}, \underline{\tau}\times\underline{\sigma}> = D_0k_q^{n+1}<\bar{\sigma}, \underline{\sigma}>^{-1}\cdot<\bar{\tau}, \underline{\tau}>\times<\bar{\sigma}, \underline{\sigma}> $$

3. The invariant $k^{n+1}$ determines the operation of $\pi_m$ on $\Omega K\overset{u}{\times}K$ for $m=1$ and the multiplicative structure of $\Omega K$ for $m>1$.

4. The following Propositions and Corollaries 3.2.—3.10 are all proved by the same technique as the proof of Proposition 3.1 by introducing appropriate fibre bundle structures.

DEFINITION 3.2. Let us denote $K(\pi_0, k^2, \pi_1, k^3, \pi_2, \cdots)$ by $K$. Then we define the c. s. s. c. $K\binom{n}{m+1}$ as follows

$$ K\binom{n}{m+1} \equiv K(\pi_n, n) \overset{h^{n+1}}{\times} \cdots \overset{h^{m+3}}{\times} K(\pi_{m+1}, m+1) \qquad n \geqq m+1 \geqq 0 $$

where $h^i(\sigma)=k^i(\psi\sigma)$ and $\psi$ is the natural injection

$$ \psi : K\binom{n}{m+1} \rightarrow K\binom{n}{0} $$

Furthermore, in the case $\pi_0=0$, we define the c. s. s. c. $K'\binom{n}{m+1}$ as follows

$$K'\binom{n}{m+1} \equiv \begin{array}{c} (\Omega K)\binom{m-1}{0} \\ \times u \\ K(\pi_n, n) \end{array}$$

$$\equiv \begin{array}{ccccc} & & K(\pi_m, m-1) \overset{(\omega k)^m}{\times} \cdots & \overset{(\omega k)^1}{\times} K(\pi_1, 0) \\ & & \times u^m & u^1 \\ K(\pi_n, n) \underset{k^{n+1}}{\times} \cdots & \underset{k^{m+3}}{\times} K(\pi_{m+1}, m+1) & \underset{k^{m+2}}{\times} K(\pi_m, m) \underset{k^{m+1}}{\times} \cdots & \underset{k^2}{\times} K(\pi_1, 1) \end{array}$$

PROPOSITION 3.3. *The natural injection*

$$\varphi : K\binom{n}{m+1} \rightarrow K'\binom{n}{m+1}$$

*induces a chain equivalence and*

$$K'\binom{n}{m+1}/(\Omega K)\binom{m-1}{0} = K\binom{n}{0}$$

*that is to say* $K'\binom{n}{m+1}$ *has a fibre bundle structure with the fibre* $(\Omega K)\binom{m-1}{0}$ *over the base* $K\binom{n}{0}$. *Consequently* $K\binom{n}{m+1}$ *has the same homological structure as the fibre bundle with the fibre* $(\Omega K)\binom{m-1}{0}$ *over the base* $K\binom{n}{0}$.

Since $\Omega(K\binom{n}{l+1}) = (\Omega K)\binom{n-1}{l}$ and $K\binom{n}{l+1}\binom{n}{m+1} = K\binom{n}{m+1}$ $(0 \leqq l \leqq m)$, we have

COROLLARY 3.4.

$$K\binom{n}{m+1} \simeq K\binom{n}{l+1}\binom{n}{m+1}, \qquad K\binom{n}{l+1}'\binom{n}{m+1}/(\Omega K)\binom{m-1}{l} = K\binom{n}{l+1} \qquad 0 \leqq l \leqq m \leqq n$$

COROLLARY 3.5.

$$K\binom{\infty}{m+1} \simeq K'\binom{\infty}{m+1}, \qquad K'\binom{\infty}{m+1}/K(\pi_m, m-1) = K\binom{\infty}{m} \qquad 0 \leqq m$$

DEFINITION 3.6. *We set*

$$(\Omega K)^{n-1}\binom{m-1}{0} \equiv \begin{array}{c} (\Omega K)\binom{n-1}{0} \\ \times u \\ K\binom{n}{m+1} \end{array}$$

$$\equiv \begin{array}{ccc} K(\pi_n, n-1) \overset{(\omega k)^n}{\times} \cdots & \overset{(\omega)^{m+2}}{\times} K(\pi_{m+1}, m) & \overset{(\omega k)^{m+1}}{\times} K(\pi_m, m-1) \overset{(\omega k)^m}{\times} \cdots \overset{(\omega k)^2}{\times} K(\pi_1, 0) \\ \times u^n & \times u^{m+1} & \\ K(\pi_n, n) \underset{k^{n+1}}{\times} \cdots & \underset{k^{m+3}}{\times} K(\pi_{m+1}, m+1) & \end{array}$$

COROLLARY 3.7. *We have*

$$(\Omega K)\binom{m-1}{l} \simeq (\Omega(K\tbinom{\infty}{l+1}))^{n-1}\binom{m-1}{0}$$

$$(\Omega(K\tbinom{\infty}{l+1}))^{n-1}\binom{m-1}{0}/(\Omega K)\binom{n-1}{l} = K\binom{n}{m+1} \qquad 0 \leqq l \leqq m \leqq n$$

COROLLARY 3.8.

$$(\Omega K)\binom{m-1}{0} \simeq \Omega K^m\binom{m-1}{0} \quad \Omega K^m\binom{m-1}{0}/(\Omega K)\binom{m}{0} = K(\pi_{m+1}, m+1) \qquad 0 \leqq m$$

Define $K^m\binom{m-1}{0}$ for general $K$ just as $(\Omega K)^m\binom{m-1}{0}$ for $\Omega K$, then we have

COROLLARY 3.9.

$$K\binom{m-1}{0} \simeq K^m\binom{m-1}{0} \qquad K^m\binom{m-1}{0}/K\binom{m}{0} = K(\pi_m, m+1)$$

COROLLARY 3.10. *Let*

$$0 \longrightarrow \pi' \longrightarrow \pi \longrightarrow \pi'' \longrightarrow 0$$

*be an exact sequence of abelian groups, then we have*

$$K(\pi'', n) \simeq K^{\pi'}(\pi'', n) \equiv \begin{array}{c} K(\pi', n) \overset{kn+1}{\times} K(\pi'', n) \\ \times u^{n+1} \\ K(\pi', n+1) \end{array}$$

$$K^{\pi'}(\pi'', n)/K(\pi', n) \overset{kn+1}{\times} K(\pi'', n) = K(\pi', n+1)$$

*and*

$$K(\pi', n) \simeq K_{\pi''}(\pi', n) \equiv \begin{array}{r} K(\pi'', n-1) \\ \times u \\ K(\pi', n) \underset{kn+1}{\times} K(\pi'', n) \end{array}$$

$$K_{\pi''}(\pi', n)/K(\pi'', n-1) = K(\pi', n) \overset{kn+1}{\times} K(\pi'', n)$$

*Since*

$$K(\pi, n) \cong K(\pi', n) \overset{kn+1}{\times} K(\pi'', n)$$

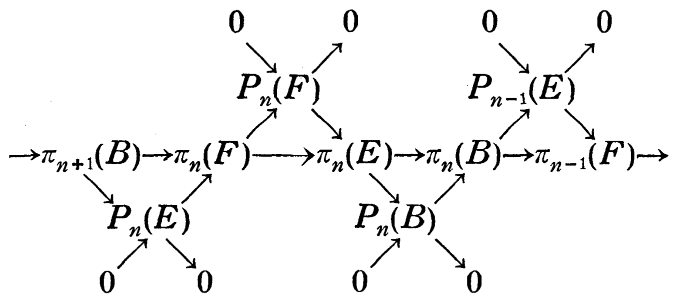
*we can conclude that*

i) $K(\pi'', n)$ *has the same homological structure as a fibre bundle with the fibre* $K(\pi, n)$ *over the base* $K(\pi', n+1)$,

ii) $K(\pi', n)$ *has the same homological structure as a fibre bundle with the fibre* $K(\pi'', n-1)$ *over the base* $K(\pi, n)$.

REMARK 1. All above the Propositions can be considered as special cases of a general proposition given below.

Let $E$ be a fibre space in the sense of Serre [12] with the fibre $F$ over the base $B$ and let the exact sequence of homotopy groups of this fibre space be given as follows:




For any arcwise connected topological space $X$ let $K(X)$ denote the minimal complex of $X$. Then $K(B)$ and $K(F)$ can be expressed in the form

$$K(B)=\cdots\times K(P_n(E),n+1)\overset{\underline{k}^{n+2}}{\times}K(P_n(B),n)\overset{\underline{e}^{n+1}}{\times}K(P_{n-1}(E),n)\quad\times\cdots$$

$$K(F)=\cdots\times K(P_n(E),n)\quad\overset{\bar{e}^{n+1}}{\times}K(P_n(F),n)\overset{\bar{k}^{n+1}}{\times}K(P_{n-1}(E),n-1)\times\cdots$$

Now we define the c. s. s. c. denoted by

$$\begin{array}{ccc} K(\pi,n) & \overset{\bar{k}}{\times} & K(\pi',n') \\ |k\ \times & c & \times k| \\ K(\pi'',n'') & \underset{\underline{k}}{\times} & K(\pi^*,n^*) \end{array}$$

where

$$\bar{k}\in Z^{n+1}(\pi',n';n) \qquad |k\in Z^{n+1}(\pi'',n'';n)$$
$$\underline{k}\in Z^{n''+1}(\pi^*,n^*;n'') \qquad k|\in Z^{n'+1}(\pi^*,n^*;n')$$
$$c\in Z^{n+1}\left(\begin{array}{c}K(\pi',n')\\ \times k|\\ K(n^*,n^*)\end{array};\pi\right)$$

satisfying $c_{q-1}(\bar{\sigma},1_q)=\bar{k}_{q-1}(\bar{\sigma})$.

The $F$-$D$ operator of this complex are given as follows

$$F_0\begin{pmatrix}\bar{\tau} & \bar{\sigma}\\ \underline{\tau} & \underline{\sigma}\end{pmatrix}=\begin{pmatrix}c_{q-1}(\bar{\sigma},\underline{\sigma})\cdot|k_{q-1}(\underline{\tau})\cdot F_0\bar{\tau} & k|_{q-1}(\underline{\sigma})\cdot F_0\bar{\sigma}\\ \underline{k}_{q-1}(\underline{\sigma})F_0\underline{\tau} & F_0\underline{\sigma}\end{pmatrix}$$

$$F_i\begin{pmatrix}\bar{\tau} & \bar{\sigma}\\ \underline{\tau} & \underline{\sigma}\end{pmatrix}=\begin{pmatrix}F_i\bar{\tau} & F_i\bar{\sigma}\\ F_i\underline{\tau} & F_i\underline{\sigma}\end{pmatrix}\qquad 0<i\leqq q$$

$$D_i\begin{pmatrix}\bar{\tau} & \bar{\sigma}\\ \underline{\tau} & \underline{\sigma}\end{pmatrix}=\begin{pmatrix}D_i\bar{\tau} & D_i\bar{\tau}\\ D_i\underline{\tau} & D_i\underline{\tau}\end{pmatrix}\qquad 0\leqq i\leqq q$$

Then we have

PROPOSITION 3.12. *Let E be a fibre space with the fibre F over the base B. The situation being as above, we have*

$$
\begin{array}{lllll}
 & K(F) & \cdots\times K(P_n(E),n) & \overset{\bar{e}^{n+1}}{\times K(P_n(F),n)} & \overset{\bar{k}^{n+1}}{\times K(P_{n-1}(E),n-1)}\times\cdots \\
K(E)\simeq & \times c \;\equiv & \times u^{n+1} & f^{n+1} \quad \times e^{n+1}\, c^{n+1} & \times u^{n} \\
 & K(B) & \cdots\times K(P_n(E),n+1) & \underset{\underline{k}^{n+2}}{\times K(P_n(B),n)} & \underset{\underline{e}^{n+1}}{\times K(P_{n-1}(E),n)} \quad \times\cdots
\end{array}
$$

Proof is performed by a straightforward method. We shall omitt here the detail which is somewhat long and cumbersome.

Moreover we can prove

PROPOSITION 3.13. *If the bundle E is acyclic, then the fibre F has the same singular homotopy type as the loop space $\Omega B$ over the base B.*

REMARK 2. In virtue of the above proposition we can introduce a filtration of the $K(F)\overset{c}{\times}K(B)$ by the non-degeneracy of the $K(B)$-component. The we obtain the spectral sequence $\{{}_rE_{p,q}\}$ such that ${}_2E_{p,q}$ is isomorphic to $H_p(B,H_q(F))$ and ${}_\infty E_{p,q}$ gives the filtration of $H_{p+q}(E)$. This spectral sequence is naturally embedded isomorphically into the spectral sequence introduced by Serre [12].

REMARK 3. In this above proposition $c^{n+1}$ may be considered as generalizations of characteristic classes of fibre bundles.

REMARK 4. By the same method as in the Proposition 3.3. and Corollary 3.11., we can reproduce the "Cartan-Leray's spectral sequence" concerning the covering spaces of arcwise-connected spaces.

## Bibliography


[1] H. Cartan, Séminaire E. N. S., 1954–1955.

[2] H. Cartan et J. P. Serre, *Espace fibrès et gronpes d'homotopie* I., C. R. Acad. Sci. Paris, 234 (1952) pp. 288–290, II, ibid., pp. 393–395.

[3] S. Eilenberg and S. MacLane, *Cohomology theory in abstract gronps* I, Ann. of Math., 48 (1947), pp. 51–78; II, 48 (1947) pp. 326–341.

[4] ———, *Homology of shaces with operators* I, Trans. Amer. Math. Soc., 61 (1947), pp. 378–417; II, ibid., 65 (1949) pp. 49–99.

[5] ———, *Rclations between homology and homotopy gronps of spaces* I, Ann. of Math. 46 (1945) pp. 480–509; II, ibid., 51 (1950) 514–533.

[6] ———, *On the gronps $H(\pi,n)$* I, Ann. of Math., 58 (1953) pp. 55–106; II, ibid., 60 (1954) pp. 49–139.

[7] S. Eilenberg and J. Zilber, *On product of complexes,* Amer. J. of Math. 75 (1953) pp. 200–204.

[8] G. Hochschild and J. P. Serre. *Cohomology of gronp extensions,* Trans. Amer. Math. Soc., 74 (1953) pp. 110-134.

[9] K. Mizuno, *On the minimal complexes.* J. Inst. Polyt. Osaka City Univ., 5 (1954) pp. 41-51.

[10] M. Postnikov, *Determination of the homology gronps of a space by means of the homotopy invariants* (in Russian) Doklady Akad. Nauk S. S. S. R. 19 (76) (1951) pp. 359-362.

[11] ———, *On the homotopy type of polyhedra* (in Russian) Doklady Akad. Nauk S. S. S. R. 76 (1951) pp. 789-791.

[12] J. P. Serre, *Homologie singulière des espaces fibrès. applications.* Ann. of Math., 54 (1951) pp. 425-505