Journal of the Mathematical Society of Japan Vol. 9, No. 3, July, 1957

On similarities and isomorphisms of ideals in a ring

By Yutaka KAWADA

(Received Nov. 24, 1955)
(Revised June 16, 1957)

As for the connections between similarities and isomorphisms of ideals, the investigations known up to this time seem to be restricted only to the case of principal ideal domains. In the present paper we shall study some relations between these two notions of equivalence for ideals for the case of a ring which has a unit element and satisfies the minimum (whence the maximum) condition for left and right ideals. This problem was suggested to me by Prof. K. Morita and I express my hearty thanks to him.

In Section 1, we shall establish our main theorem which asserts that a left [right] similarity between two left [right] ideals implies a left [right] operator-isomorphism of them. In Section 2, we shall deal with the problem: in what ring does every left [right] operator-isomorphism between two left [right] ideals imply a left [right] similarity of them? For the validity of this implication, we shall show that it is not necessary but sufficient that the ring be quasi-Frobeniusean.

Throughout this paper "isomorphism" will mean "operator-isomorphism".

1. Let A be a ring with a unit element 1 satisfying the minimum (whence the maximum) condition for left and right ideals, and let N be its radical. Then we have the following.[1)]

THEOREM 1. *Let L and L' be left ideals of A. If L is left similar to L', that is, the residue class module A/L is A-isomorphic to the residue class module A/L', then L is A-isomorphic to L'. Furthermore, if, by this similarity, the residue class* 1 *(mod L) is mapped onto the residue class a (mod L'), then there exists a regular element a_0 of A such that $a_0 \equiv a$ (mod L') and $L' = La_0$. Convsersely if there exists a regular elements a_0 of A such that $L' = La_0$, then L is left similar to L'. The same is of course true for right ideals.*

1) Recently the most general extension of our Theorem 1 has been obtained in [5].

PROOF. In the first place, we shall remark that the last (converse) part of the theorem is obvious and we have only to prove that there exists a regular element a_0 of A such that $a_0 \equiv a$ (mod L'). This is shown as follows:

Suppose that we possess such a regular element a_0. Then if 1 (mod L) is mapped onto a_0 (mod. L') by this similarity, L must be mapped onto La_0 ($\equiv 0$ mod L'), since $L = L1 \equiv 0$ (mod L). Hence La_0 is contained in L'. On the other hand, since a_0 is regular, the composition length of L is equal to that of La_0, and, since A/L is isomorphic to A/L', the composition length of L is equal to that of L'. Therefore L' has the same composition length as that of La_0, and hence $L' = La_0$ and consequently L is isomorphic to L'.

Now we shall proceed to the proof of the existence of a_0. We distinguish two cases.

Case (1) $N=0$.

Since A is semi-simple, we can assume that $L = Ae_1$ and $L' = Af_1$ where e_1 and f_1 are suitable idempotents. Put $e_2 = 1 - e_1$ and $f_2 = 1 - f_1$.

Now suppose that Ae_1 be left similar to Af_1 and that, by this similarity,

$$1 \ (\text{mod } Ae_1) \longrightarrow a \ (\text{mod } Af_1)$$

and
$$b \ (\text{mod } Ae_1) \longleftarrow 1 \ (\text{mod } Af_1).$$

Then, since $1 = e_1 + e_2 = f_1 + f_2$, the above correspondence is refined as follows:

$$\begin{aligned} 1 \equiv e_2 \quad & (\text{mod } Ae_1) \longrightarrow a \equiv e_2 a \equiv e_2 a f_2 \ (\text{mod } Af_1) \\ b \equiv f_2 b \equiv f_2 b e_2 \ & (\text{mod } Ae_1) \longleftarrow \quad 1 \equiv f_2 \quad (\text{mod } Af_1). \end{aligned}$$

We denote e_2af_2, f_2be_2 by a_2, b_2 respectively. Evidently $a_2b_2 \equiv 1$ (mod Ae_1), whence we have $a_2b_2 = e_2$ by the right multiplication of e_2. Similarly we have $b_2a_2 = f_2$. Hence the mapping: $x \rightarrow xa_2$ ($x \in Ae_2$) gives an isomorphism between Ae_2 and Af_2.

Moreover, from the Krull-Remak-Schmidt theorem it follows that there exists an isomorphism between Ae_1 and Af_1, by which, we suppose that $e_1 \rightarrow a_1, b_1 \leftarrow f_1$ and so $a_1b_1 = e_1, b_1a_1 = f_1$. Put $a_0 = a_1 + a_2$ and $b_0 = b_1 + b_2$. Then it is easily seen that $a_0b_0 = b_0a_0 = 1$, i. e. a_0 is regular in A and that $a_0 \equiv a_2 \equiv a$ (mod Af_1). Thus the existence of a_0 is proved. Therefore, according to the above mentioned remark, our assertion is valid in this csse.

Case (2) $N \neq 0$.

As a preliminary to the proof, we prove the following

LEMMA. *Let φ be an A-isomorphism of A/L onto A/L'. Then $N \cup L/L$ is mapped onto $N \cup L'/L'$ by φ.*

PROOF OF LEMMA. Let, by the given isomorphism between A/L and A/L',

$$\varphi \quad : \quad 1 \pmod{L} \longrightarrow a \pmod{L'}$$
$$\varphi^{-1} : \quad b \pmod{L} \longleftarrow 1 \pmod{L'}.$$

Then, since $N \cup L \equiv N = N1 \pmod{L}$, $N \cup L \pmod{L}$ is mapped onto $Na \pmod{L'}$ which is contained in N ($\equiv N \cup L'$ mod L'). Namely, $N \cup L/L$ is mapped into $N \cup L'/L'$ by φ. Similarly by the inverse isomorphism φ^{-1} $N \cup L'/L'$ is mapped into $N \cup L/L$. Therefore $N \cup L/L$ is mapped onto $N \cup L'/L'$ by the given isomorphism.

Now suppose that there exists a similarity between L and L', by which $1 \pmod{L} \to a \pmod{L'}$. Then, according to the above lemma, $N \cup L/L$ is isomorphic to $N \cup L'/L'$ by the very same correspondence between A/L and A/L'. Hence, as is well known, there exists an isomorphism between $A/L \,/\, N \cup L/L$ and $A/L' \,/\, N \cup L'/L'$, by which $\hat{1} \pmod{N \cup L/L} \to \hat{a} \pmod{N \cup L'/L'}$. Here we denote without confusion the residue class $x \pmod{L}$ as well as $x \pmod{L'}$ by $\hat{x}$.

Furthermore, in virtue of repeated usage of the Isomorphism theorem, we have a chain of A-isomorphisms such that

$$A/N \,/\, N \cup L/N \cong A/N \cup L \cong A/L \,/\, N \cup L/L \cong A/L' \,/\, N \cup L'/L'$$
$$\cong A/N \cup L' \cong A/N \,/\, N \cup L'/N$$

and that

$$\underset{(\text{mod } N \cup L/N)}{\bar{1}} \longrightarrow \underset{(\text{mod } N \cup L)}{1} \longrightarrow \underset{(\text{mod } N \cup L/L)}{\hat{1}} \longrightarrow \underset{(\text{mod } N \cup L'/L')}{\hat{a}} \longrightarrow \underset{(\text{mod } N \cup L')}{a} \longrightarrow \underset{(\text{mod } N \cup L'/N)}{\bar{a}} .$$

Here we understand the residue class $x \pmod{N}$ by $\bar{x}$.

By the mapping: $\bar{1} \pmod{N \cup L/N} \to \bar{a} \pmod{N \cup L'/N}$, $A/N \,/\, N \cup L/N$ is A-(whence A/N-) isomorphic to $A/N \,/\, N \cup L'/N$. Since A/N is semi-simple, the conclusion obtained for the Case (1) is applicable. Thus we have a regular element $\bar{a}'$ of A/N such that $\bar{a}' \equiv \bar{a}$ (mod $N \cup L'/N$) and so $a' \equiv a \pmod{N \cup L'}$. Hence we can express that $a' - a = n + l'$, where $n \in N$ and $l' \in L'$.

Put $a_0 = a + l' = a' - n$. Then we have $a_0 \equiv a \pmod{L'}$ and $a_0 \equiv a'$ $\pmod{N}$ i.e. $\bar{a}_0 = \bar{a}'$. Since $\bar{a}_0$ is regular in A/N, a_0 is of course regular in A. Therefore the existence of a regular element a_0 is proved. In view of the remark at the beginning, this completes our proof.

2. Now we shall propose the problem: in what ring A does every A-isomorphism between two left [right] ideals imply a left similarity of them?

When A is a ring with a unit element satisfying the minimum (whence the maximum) condition for left and right ideals, this condition is, in view of Theorem 1, equivalent to the next one:

(*) Every left [right] ideal A-isomorphic to a given left ideal L [right ideal R] can be expressed as La [aR] by the right [left] multiplication of a regular element a of A. This condition is not always satisfied, which is shown by the following

EXAMPLE. Let $A=e_1P+e_2P+nP$ be an algebra over an arbitrary field P, whose multiplication table is as follows:

	e_1	e_2	n
e_1	e_1	0	0
e_2	0	e_2	n
n	n	0	0

Obviously the radical N is nP. It is easy to see that $Ae_2 \cong Ne_1$ and $A/Ae_2 \not\cong A/Ne_1$.

Recently M. Ikeda [4] has, however, obtained the following[2].

THEOREM 2. *Let A be a quasi-Frobenius ring. Then A satisfies the condition* (*) *for every left ideal L. More precisely, every A-isomorphism between two left* [*right*] *ideals is given by the right* [*left*] *multiplication of a regular element of A.*

We shall give here a new proof of this theorem which is based on Theorem 1.

First, we shall define the character module of a given left ideal: Let L be a left ideal of A. Then all A-homomorphisms from L into A, which we express as *Char L*, form an A-right module if we define $x^{\sigma\cdot a}=(x^{\sigma})a$ for every $x\in L, \sigma\in$ *Char* L and $a\in A$. This A-module, *Char L*, is called the character module of L.

Now suppose that there exists an isomorphism φ between two left ideals L and L', by which $x\rightarrow x^{\varphi}(x\in L)$.

Take, for each element σ' of *Char L*, an element σ of *Char L*

2) Cf. Theorem 2 l. c.

such as $x^{\sigma}=(x^{\varphi})^{\sigma'}$, $(x\in L)$. Then, by the mapping: $\sigma'\rightarrow\sigma$, we get an isomorphism of *Char* L' onto *Char* L.

On the other hand, owing to M. Ikeda [4][3], we know that every A-homomorphism from L into A is given by the right multiplication of an element of A, when A is quasi-Frobeniusean. Hence *Char* L is naturally isomorphic to the residue class module $A/r(L)$ where $r(L)$ is the right annihilator of L in A, and, by this isomorphism, 1 in *Char* L is mapped onto the residue class 1 (mod $r(L)$). Similarly *Char* L' is isomorphic to $A/r(L')$.

Moreover, the isomorphism φ between L and L' must be also given by the right multiplication of an element a of A; $x^{\varphi}=xa$ $(x\in L)$. Therefore we have a chain of isomorphisms such that

$$A/r(L')\cong \mathit{Char}\ L'\cong \mathit{Char}\ L\cong A/r(L)$$

and that

$$1\ (\mathrm{mod}\ r(L'))\rightarrow 1\rightarrow a\rightarrow a\ (\mathrm{mod}\ r(L)).$$

In view of the transitivities of isomorphisms, the mapping: $1\ (\mathrm{mod}\ r(L'))\rightarrow a\ (\mathrm{mod}\ r(L))$ gives an isomorphism between $A/r(L')$ and $A/r(L)$. Hence, according to Theorem 1, we see that there exists a regular element a_0 such that $a_0\equiv a\ (\mathrm{mod}\ r(L))$ i.e. $xa_0=xa$ $(=x^{\varphi})$ for any element x in L. Thus we have our assertion.

Furthermore we have

THEOREM 3. *Let A be an algebra with a finite rank over an algebraically closed field P. If A satisfies the condition* (*) *for any simple left ideal, then A is a quasi-Frobenius algebra*[4].

PROOF. As is well known, we have a direct-sum decomposition of A into directly indecomposable left ideals:

$$A=\sum_{\kappa=1}^{n}\sum_{i=1}^{f(\kappa)}Ae_{\kappa,i}$$

3) Cf. Theorem 1 l.c.

4) Ikeda [4] considers the following condition (α) (instead of (*)) on the ring A with unit satisfying minimum condition for left and right ideals.

(α) Every A-homomorphism between two left ideals of A is given by the right multiplication of an element of A. Theorem 1 l.c. asserts that this condition is necessary and sufficient for A to be quasi-Frobeniusean. Moreover, the following result is proved as Proposition 1 l.c.: Let A be an algebra with a finite rank over a field P. If A has a left unit element and satisfies (α) for simple left ideals, then A is a quasi-Frobenius algebra.

No assumption on P is needed here, whereas in our Theorem 3, the assumption that P is algebraically closed is essential as is shown below; our condition (*) for all simple left ideals is in fact weaker than the condition (α) for the same ideals.

where $1=\sum_{\kappa=1}^{n} E_\kappa$, $E_\kappa=\sum_{i=1}^{f(\kappa)} e_{\kappa,i}$ and $e_{\kappa,i}$ $(\kappa=1,2,\cdots,n;\ i=1,2,\cdots,f(\kappa))$ are mutually orthogonal primitive idempotents such that $Ae_{\kappa,i}\cong Ae_{\kappa,1}=Ae_\kappa$ for $i=1,2,\cdots,f(\kappa)$ and $Ae_{\kappa,i}\not\cong Ae_{\lambda,j}$ if $\kappa\neq\lambda$.

Let $r(N)$ be the right annihilator of the radical N. Then, since A satisfies the condition (*) for any simple left ideals, we see, in the same way as the proof of Lemma 2 of M. Ikeda [4], that $r(N)E_\kappa$ is a simple two-sided ideal for each κ and that there exists a permutation π of $(1,2,\cdots,n)$ such that $r(N)E_\kappa=E_{\pi(\kappa)}r(N)$. Hence, by means of Theorem 1 of T. Nakayama [3], it is enough to prove that $r(N)e_\kappa$ is a simple left ideal for each κ.

Now let L be any simple left subideal of $r(N)e_\kappa$. Then $\sum_{j=1}^{f(\kappa)} Lc_{\kappa,1,j}$ is not only a left ideal but also a two-sided ideal, for we have $A=\sum_{\kappa=1}^{n}\sum_{i,j=1}^{f(\kappa)} c_{\kappa,i,j}P\cup N$ because of our assumption about P, where $c_{\kappa,i,j}$ $(\kappa=1,2,\cdots,n;\ i,j=1,2,\cdots,f(\kappa))$ are matric units such that $c_{\kappa,i,j}c_{\lambda,h,k}=\delta_{\kappa,\lambda}\delta_{j,h}c_{\kappa,i,k}$ for any κ,λ,i,j,h,k and $c_{\kappa,i,i}=e_{\kappa,i}$. Moreover, since $\sum_{j=1}^{f(\kappa)} Lc_{\kappa,1,j}$ is contained in the simple two-sided ideal $r(N)E_\kappa$, we obtain $r(N)E_\kappa=\sum_{j=1}^{f(\kappa)} Lc_{\kappa,1,j}$ and so $r(N)e_\kappa=Lc_{\kappa,1,1}=L$ by the right multiplication of e_κ. Thus $r(N)e_\kappa$ is a simple left ideal for each κ and hence A is quasi-Frobeniusean.

REMARK. In case the ground field P of the algebra A is not algebraically closed, from the assumption (*) for any left and right ideal, we can not conclude that A is quasi-Frobeniusean. For example[5], let $A=e_1P+e_2P+wP+uP+uwP+vP+wvP$ be an algebra over P[6] with the multiplication table:

	e_1	e_2	w	u	uw	v	wv
e_1	e_1	0	0	u	uw	0	0
e_2	0	e_2	w	0	0	v	wv
w	0	w	$-e_2$	0	0	wv	v
u	0	u	uw	0	0	0	0

5) We owe this example to Prof. K. Morita.

6) We assume that P has an extension field of degree 2, i.e. P can be taken as the field of rational numbers.

	e_1	e_2	w	u	uw	v	wv
uw	0	uw	$-u$	0	0	0	0
v	v	0	0	0	0	0	0
wv	wv	0	0	0	0	0	0

Clearly the radical N is $uP+uwP+vP+wvP$ and further we have $Ne_1=e_2N=Av=vA\oplus wvA$ and $Ne_2=e_1N=uA=Au\oplus Auw$. Hence A is not quasi-Frobeniusean. But we can see easily that A satisfies the condition (*) for any left and right ideal.

Chitose High School, Tokyo

References

[1] T. Nakayama: On Frobeniusean algebras I, Ann. of Math., 40 (1939).
[2] T. Nakayama: On Frobeniusean algebras II, Ann. of Math., 42 (1941).
[3] T. Nakayama: Supplementary remarks on Frobeniusean algebras I, Proc. Jap. Acad., 25 (1949).
[4] M. Ikeda: A characterization of quasi-Frobenius rings, Osaka Math. J., 4 (1951).
[5] K. Morita, Y. Kawada and H. Tachikawa: On injective modules (forthcoming in Math. Zeit.)