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Domains spread on a complex space.

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Let X and Y be two connected complex analytic manifolds, and let φ be a holomorphic mapping of X into Y. If φ is a local homeomorphism, we usually call the triple (X, φ, Y), or simply the pair (X, φ), a domain spread on Y by φ. The general theory of spread domains has been established by H. Cartan[1] and others. In the classical theory of functions in several complex variables, only these spread domains have been considered. A general tendency at the present time is, however, to investigate the so-called "complex spaces" and the behavior of functions on them. It is natural to include the algebroidal functional elements in the domain of holomorphic prolongation of holomorphic functions. Thus we are led to introduce the notion of "ramified spread domains".

We shall generalize the notion of spread domains as follows: Let X and Y be two connected normal complex spaces of the same dimension in the sense of H. Cartan[2], and let φ be a holomorphic mapping of X into Y; if φ is non-degenerate at every point of X, that is, if the fiber $\varphi^{-1}(\varphi(x))$ of φ through x is a discrete set in X for every point x of X, then we call (X, φ, Y) or (X, φ) a domain spread on Y by φ. In § 1 we recall briefly the notion of complex spaces and make some remarks for later use. In § 2 the space of (holomorphic-) algebroidal jets of a complex space into another is introduced as a generalization of the space of holomorphic jets. § 3 is devoted to the general theory of spread domains; as in the classical case of "unramified spread domains", the maximal holomorphic prolongation and the intersection of spread domains are defined. Using a space of algebroidal jets we prove the existence theorem of the maximal holomorphic prolongation of a given spread domain with respect to a family of holomorphic mappings. The exposition in § 3 is made after the manner of H. Cartan's seminar.

1) cf. [1].
2) cf. [2], VIII bis.

As for the treatment of domains spread on the complex affine space C^n, it will be reserved for opportunity. We note here only that for domains spread on C^n in our sense the uniqueness theorem, that is, the invariance property of envelope of holomorphy and of domain of holomorphy under analytic isomorphisms, does not hold in general.

§ 1. Complex spaces.

By a *ringed space* we mean a topological space X, together with a subsheaf $\mathcal{O}_X$ of the sheaf of jets[3] of complex-valued continuous functions on X. Every open subset U of a ringed space X is also a ringed space with the induced sheaf $\mathcal{O}_X(U)$. The complex affine space C^n of dimension n, with the usual topology and the sheaf $\mathcal{O}_{C^n}$ of jets of holomorphic functions, is a ringed space. Let E be a subset of an open set G of C^n. If E is closed in G and, around each point $x \in E$, E is defined as the common zeros of holomorphic functions around x, then we say that E is an *analytic subset* of G. On the locally compact space E we can define canonically a subsheaf of the sheaf of jets of continuous functions as follows: For a point $x \in E$ we denote by $\mathcal{I}_x(E)$ the ideal of those elements of the ring $\mathcal{O}_{x,G}$, $\mathcal{O}_G = \mathcal{O}_{C^n}(G)$, whose restrictions on E vanish around x; the collection of $\mathcal{O}_{x,E} = \mathcal{O}_{x,G}/\mathcal{I}_x(E)$ for $x \in E$ forms the desired sheaf $\mathcal{O}_E$ on E. The germ of E at x is irreducible if and only if $\mathcal{O}_{x,E}$ is an integral domain. If $\mathcal{O}_{x,E}$ is an integrally closed integral domain, we say that the germ of E at x is *normal*, or that E is normal at x. If, for each point $x \in E$, the germ of E at x is normal, E is said to be a *normal analytic subset* of $G \subset C^n$. We know that, if E is normal at a point x, E is normal at all points sufficiently near x.[4] Given two ringed space $(X, \mathcal{O}_X)$ and $(X', \mathcal{O}_{X'})$, a mapping φ of X into X' is said a *homomorphism* if φ is continuous and if $f \rightarrow f \circ \varphi$ maps $\mathcal{O}_{\varphi(x),X'}$ into $\mathcal{O}_{x,X}$ for every point $x \in X$. The composition of two homomorphisms is also a homomorphism. The notion of *isomorphism* is defined accordingly.

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3) More precisely *local jets*. Cf. Ch. Ehresmann, Introduction à la théorie des structures infinitésimales et des pseudo-groupes de Lie, Colloque International de Géométrie différentielle de Strasbourg, 1953. We prefer the terminology of "jets" to that of "germs", which is to be reserved for the germ of subset.

4) cf. [2], Exp. X, corollary to theorem 3 bis.

A ringed space $(X, \mathcal{O}_X)$ is called a *complex space* if the following conditions are satisfied:

(CS 1) The space X is a Hausdorff space,

(CS 2) Each point $x \in X$ has an open neighborhood U, which is isomorphic as a ringed space with a normal analytic subset E of an open set in a space C^n (such an isomorphism $\varphi: U \to E$ is called a *chart*).

The sheaf $\mathcal{O}_X$ is then called the sheaf of jets of holomorphic functions; its sections are called holomorphic functions. For complex spaces, homomorphisms and isomorphisms are also said *holomorphic mappings* and *analytic isomorphisms* respectively.

Let $(X, \mathcal{O}_X)$ be a complex space. For a point $x \in X$ the ring $\mathcal{O}_{x,X}$ has a structure of analytic ring over C in the sense of H. Cartan[5]. We note that the ring $\mathcal{O}_{x,X}$ is a Noetherian, integrally closed integral domain. The knowledge of $\mathcal{O}_{x,X}$ determines the structure of X around x. If a point x of X has an open neighborhood isomorphic with an open set in some C^n, we say that the point x is *uniformizable*. A point $x \in X$ is uniformizable if and only if $\mathcal{O}_{x,X}$ is isomorphic with the ring $C\{z_1, \cdots, z_n\}$ of convergent power series in $z_1, \cdots, z_n$ for some n. If all the points of X are uniformizable, X is nothing other than a complex manifold. A connected complex space X is said to be of dimension n if the complex manifold formed by the uniformizable points of X is of dimension n.

An analytic subset Y of an open set U in a complex space X is defined in the same way as in the space C^n. The ringed space Y is not in general a complex space. We can, however, canonically associate with Y a complex space $\tilde{Y}$ as follows (the space $\tilde{Y}$ is called the *parameter space* of Y). Let $\tilde{Y}$ be the set of non-empty irreducible components of Y, with respect to U, at the points of Y. As U is locally isomorphic with a normal analytic subset E of an open set G in some C^n, Y is locally isomorphic with an analytic subset F in G. Generally, let A and A' be two commutative rings with unit elements, φ a homomorphism of A onto A', and N the kernel of φ. Then there is a correspondence between the ideals I of A, which contain N, and the ideals I' of A', such that, if the ideals I and I' correspond to each other, then $\varphi(I) = I'$ and $I = \varphi^{-1}(I')$. Let I and I' be two corresponding proper ideals. Then if one is

5) cf. [2], Exp. VI.

primary, so is the other, and if one is prime, so is the other. Let $\mathfrak{q}$ and $\mathfrak{q}'$ be corresponding primary ideals, and let $\mathfrak{p}$ and $\mathfrak{p}'$ be corresponding prime ideals. Then $\mathfrak{q}$ is $\mathfrak{p}$-primary if and only if $\mathfrak{q}'$ is $\mathfrak{p}'$-primary. Further if the rings A and A' are Noetherian, then the normal decompositions of corresponding ideals I and I' are obtained from each other. From these remarks we see that the irreducible components of F, with respect to E, at the points of F coincide with the irreducible components of F, with respect to G, at the corresponding points. Thus we see from a theorem of K. Oka[6] that $\tilde{Y}$ is a complex space. Let $\mathcal{G}_X$ be the set of non-empty irreducible germs at the points of X. We can introduce the topology of $\mathcal{G}_X$ so that the subsets $\tilde{Y}$ constructed above become elementary open sets. As the space $\mathcal{G}_X$ is shown to be a Hausdorff space, we have

PROPOSITION 1. *The space $\mathcal{G}_X$ is a complex space.*

According to H. Grauert[7] we give an equivalent definition of complex space. A pair $\mathfrak{R}=(R, \Phi)$ is said an *analytic spread* over a domain G in C^n if the following four conditions are satisfied:

(AS 1) R is a locally compact topological space; Φ is a proper[7′] continuous mapping of R onto G. We say that a point $x \in R$ lies over the point $z=\Phi(x)$ of G.

(AS 2) There is an (eventually empty) analytic subset A of G such that $\Phi^{-1}(A)$ is nowhere dense in R, that Φ^{-1} induces a local homeomorphism on $\breve{R}=R-\Phi^{-1}(A)$, and that only a finite number of points of R lie over any point of G.

(AS 3) Each point $x \in R$ has arbitrarily small neighborhoods U such that $U-\Phi^{-1}(A)$ are connected.

Transferring the structure of G to $\breve{R}$, we can speak of holomorphic functions on R, analytic subsets in R, etc. Let x be a point of R, and let $z=\Phi(x)$. Around the point x, Φ is a homeomorphism and so the inverse mapping Φ^{-1} is locally defined. Thus Φ^{-1} defines a jet of isomorphism[8] of G into R, which we denote by $\Phi_{x,z}^{-1}$. If f is a holomorphic function around the point x, we denote by f_x

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6) cf. [2], Exp. X, theorem 2.

7) cf. [3].

7′) We call a mapping Φ proper, if the inverse image of a compact set by Φ is compact.

8) We consider the space R as a ringed space. Also cf. [2].

the jet of f at x. We impose further on $\mathfrak{R}$ the following condition (the condition (C) of H. Grauert):

(AS 4) For any pair of points x, y of $\breve{R}$ $(x \neq y, \Phi(x)=\Phi(y)=z)$ there exists a holomorphic function f on R such that the jets of holomorphic functions $f_x \circ \Phi_{y,z}^{-1}$ and $f_y \circ \Phi_{y,z}^{-1}$ are different.

In the definition of a complex space we can replace (CS 2) by the following condition:

(CS 2′) Each point $x \in X$ has an open neighborhood U, as a ringed space, isomorphic with the space of an analytic spread $\mathfrak{R}=(R, \Phi)$ over a domain G in some C^n.

Namely, (CS 2) implies (CS 2′), for the normal analytic subset E can be identified locally with the parameter space $\tilde{E}$ of a principal analytic subset E of an open set in some C^n. To prove the inverse implication, we may suppose the space X itself to be an analytic spread $\mathfrak{R}=(R, \Phi)$ over a domain G in C^n. From the definition of analytic spread the number of sheets of $\mathfrak{R}$ over $G-A$ is a constant, say r. By the condition (AS 4) we can construct a holomorphic function f on R such that, for any pair of points $x, y \in \breve{R}$ $(x \neq y)$ lying over the same point $z \in G$, the jets of holomorphic functions $f_x \circ \Phi_{x,z}^{-1}$ and $f_y \circ \Phi_{y,z}^{-1}$ are different (f does not depend on the choice of pairs of points x, y). Considering the elementary symmetric functions of $f \circ \Phi^{-1}$, we see that f is a zero of the irreducible pseudo-polynomial $P(\zeta\,;\,z)=\zeta^r+a_1(z)\zeta^{r-1}+\cdots+a_r(z)$, where a_i are holomorphic functions on G. Let W be the principal analytic subset of $C \times G$, defined by $P=0$, and let $\tilde{W}$ be the parameter space of W. By the condition (AS 3) we can establish an isomorphism between the space R and $\tilde{W}$. As the space $\tilde{W}$ is a complex space, R, and hence X, is a complex space.

Proposition 2. *Let X and X' be two complex spaces. Then there exists on $X \times X'$ one and only one structure of complex space such that, if $\varphi: U \to E$ and $\varphi': U' \to E'$ are charts, $\varphi \times \varphi': U \times U' \to E \times E'$ should be a chart in $X \times X'$.*

The space $X \times X'$, with this structure, is called the *product* of the space X and X'.

Proof. We are to show that the product of two normal analytic subsets E and E' of C^n and $C^{n'}$, respectively, is a normal analytic subset of some C^p, or that the product of two analytic spreads is also an analytic spread. The simple verification of (AS 1–4) shows the truth of the latter statement. The other points of the proof are similar to those of Proposition 8 and Cor. in J.-P. Serre [5].

Let X, X', Y and Y' be complex spaces. In order that the mapping $x\rightarrow(\varphi(x), \varphi'(x))$ defined by $\varphi: X\rightarrow Y$ and $\varphi': X\rightarrow Y'$ is a holomorphic mapping of X into $Y\times Y'$, it is necessary and sufficient that φ and φ' are holomorphic. As any constant mapping is holomorphic, any section $x\rightarrow(x, x_0')$, $x_0'\in X'$, is a holomorphic mapping of X into $X\times X'$. The projections $X\times X'\rightarrow X$ and $X\times X'\rightarrow X'$ are obviously holomorphic mappings. In order that $\psi\times\psi: X\times Y\rightarrow X'\times Y'$ are holomorphic (isomorphic), it is necessary and sufficient that ψ and ψ' are holomorphic (isomorphic). Let $\varphi: X\rightarrow Y$ be a holomorphic mapping. Then the graph Γ of φ is an analytic subset of $X\times Y$, for this is the inverse image of the diagonal of $Y\times Y$ by the mapping $\varphi\times 1: X\times Y\rightarrow Y\times Y$. Furthermore the mapping $\psi: X\rightarrow\Gamma$ defined by $\varphi(x)=(x, \varphi(x))$ is an analytic isomorphism.

§ 2. Space of holomorphic jets. Space of algebroidal jets.

Let E and F be two complex spaces. We assign to every non-empty open subset U of E the set $\mathcal{A}_U$ of all holomorphic mappings of U into F. If V is an open subset contained in U, we denote the restriction of elements f of $\mathcal{A}_U$ to V by $\rho_V^U(f)$. For three open subsets U, V and W such that $W\subset V\subset U$ we have $\rho_U^U=$identity and $\rho_W^U=\rho_W^V\circ\rho_V^U$. Let $\mathcal{A}_x$ be the inductive limit of the system $(\mathcal{A}_U, \rho_V^U)$, where U runs over a fundamental system of open nieghborhoods of x. We have then the canonical mapping $\rho_x^U: \mathcal{A}_U\rightarrow\mathcal{A}_x$. On the union $\mathcal{A}(E, F)=\bigcup_{x\in E}\mathcal{A}_x$ we define a topology by means of elementary open subsets: Let U be a non-empty open subset of E, and let $f\in\mathcal{A}_U$. Denoting by $[U, f]$ the set $\{\rho_x^U(f)\,|\,x\in U\}$ we define $[U, f]$ to be an elementary open subset in $\mathcal{A}(E, F)$. An arbitrary open subset is thus a union, or a finite intersection, of the subsets of this type. In fact, we can show that the finite intersection of elementary open subsets is also an elementary open subset. As the theorem of identity holds for holomorphic mappings of complex spaces, the topology of $\mathcal{A}(E, F)$ defined above satisfies the separation axiom of Hausdorff. If we denote by σ the canonical projection of $\mathcal{A}(E, F)$ onto E, σ is a continuous mapping and is a local homeomorphism.

Transferring the structure of complex space of E to $\mathcal{A}(E, F)$, we see that $\mathcal{A}(E, F)$ is a complex space. We call elements of $\mathcal{A}_x$ *holomorphic jets at x*. Define a mapping t of $\mathcal{A}(E, F)$ into F by putting $t(f)=f(\sigma(x))$ for $f\in\mathcal{A}(E, F)$. The mapping t is then holomorphic. We call the space $\mathcal{A}(E, F)$, together with the mappings σ and t, *space of holomorphic jets of E into F*. We can define, as in the case of sheaf, its *sections* over an open subset U, which may be identified with elements of $\mathcal{A}_U$: $\Gamma(U, \mathcal{A}(E, F))=\mathcal{A}_U$.

For later use we generalize the notion of space of holomorphic jets. Let E and F be two complex spaces. Let U be a non-empty open subset of E. An analytic subset A of the analytic space $U\subset E$ is called an (*holomorphic*) *algebroidal mapping of U into F* if the following conditions are satisfied: if we denote the projections $U\times F \to U$ and $U\times F\to F$ by p_U and q respectively, $p_U^{-1}(A')=U$ in $U\times F$ for every irreducible component A' of A and, for any point $x\in U$, $p_U^{-1}(x) \cap A$ is discrete in $p_U^{-1}(x)$, that is, $q(p_U^{-1}(x)\cap A)$ is discrete in F. Let $\tilde{\mathcal{A}}_U$ be the set of all algebroidal mappings of U into F. For any pair of open subsets $V\subset U$ we have the canonical restriction mapping ρ_V^U: $\tilde{\mathcal{A}}_U\to\tilde{\mathcal{A}}_V$.

Let $\tilde{\mathcal{A}}_x'=\lim_{x\in U}\tilde{\mathcal{A}}_U$, and $\tilde{\mathcal{A}}_x$ be the set of non empty irreducible components of the elements of $\tilde{\mathcal{A}}_x'$. As in the case of holomorphic jets we define $\tilde{\mathcal{A}}(E, F)=\bigcup_{x\in E}\tilde{\mathcal{A}}_x$ and the projection σ: $\tilde{\mathcal{A}}(E, F)\to E$. Let A be an algebroidal mapping of U into F, and let $\tilde{A}$ be the complex space associated with the analytic subset A. Define the subsets of the type $\tilde{A}$ to be the elementary open subsets of $\tilde{\mathcal{A}}(E, F)$. We see that $\tilde{\mathcal{A}}(E, F)$ is a complex space (§ 1). Let π: $\tilde{A}\to U\times F$ be the canonical mapping. We define a mapping $\tilde{t}$: $\tilde{\mathcal{A}}(E, F)\to F$ by putting $\tilde{t}(\tilde{\alpha})=q(\pi(\tilde{\alpha}))$ for $\tilde{\alpha}\in\tilde{A}$. The mapping $\tilde{t}$ is then holomorphic. Elements of $\tilde{\mathcal{A}}_x$ are called *algebroidal jets* of E into F at the point x, and $\tilde{\mathcal{A}}(E, F)$, with the mappings $\tilde{\sigma}$ and $\tilde{t}$, is called *the space of algebroidal jets* of E into F.

§ 3. Spread domains. Holomorphic prolongation.

In the following, complex spaces $X, X', Y, E, F,\cdots$ are supposed to be *connected*, unless otherwise mentioned. Let $\varphi\in\Gamma(X, \mathcal{A}(X, X'))$.

The mapping φ defines a mapping ${}^t\varphi : \Gamma(X', \mathcal{A}(X', F)) \to \Gamma(X, \mathcal{A}(X, F))$ by ${}^t\varphi(f') = f' \circ \varphi$ for $f' \in \Gamma(X', \mathcal{A}(X', F))$. If φ is an open mapping, ${}^t\varphi$ is injective. An element $f \in \Gamma(X, \mathcal{A}(X, F))$ is called ***holomorphically prolongable with respect to*** φ if f is in the image of ${}^t\varphi$, that is, if there exists an element $f' \in \Gamma(X', \mathcal{A}(X', F))$ such that $f = {}^t\varphi(f')$. If the mapping φ is open, then f' is determined uniquely. The mapping f' is called the ***holomorphic prolongation of f with respect to*** φ.

Let X and Y be two complex spaces and let φ be a holomorphic mapping of X into Y. The mapping φ is called ***non-degenerate at a point*** $x \in X$, if there exists a neighborhood U of x such that $\varphi^{-1}(\varphi(x)) \cap U$ is discrete in U. A triple (X, φ, Y), or a pair (X, φ), is called a *domain spread on* Y *by* φ, if the mapping $\varphi : X \to Y$ is open, holomorphic and non-degenerate at every point of X. Then the dimensions of X and Y are equal and the fiber $\varphi^{-1}(\varphi(x))$ of φ is discrete in X for any point $x \in X$. The mapping φ and the structure of Y determine the structure of complex space of X. In particular, if Y is a countable union of compact subsets, so is also X.

Given a complex space X, we shall define an order relation in the set of mappings spreading X (on an X' not given in advance): Let $\psi' : X \to X'$ and $\psi'' : X \to X''$ be two spreading mappings; we say that ψ'' *majorizes* ψ' if there exists a mapping θ spreading X' on X'' such that $\psi'' = \theta \circ \psi'$. If ψ' and ψ'' majorize each other, we say that ψ' and ψ'' are equivalent. The equivalent mappings are to be identified. Thus an order relation is defined in the set of mappings spreading X.

Let E be a complex space given once for all. Let there be given a domain (X, φ) spread on E and $f \in \Gamma(X, \mathcal{A}(X, F))$, where F denotes a complex space. We consider only the holomorphic prolongations of f with respect to those $\psi' : X \to X'$ which spread X on X' and are majorized by φ. That is, a ***holomorphic prolongation of f over E*** consists of (i) a mapping ψ' spreading X on X' and a mapping φ' spreading X' on E, in such a way that $\varphi = \varphi' \circ \psi'$ and (ii) $f' \in \Gamma(X', \mathcal{A}(X', F))$ such that $f = f' \circ \psi'$. The mapping ψ' determines φ' and f' if they exist.

Now we prove the following theorem which is a generalization of the classical theorem for "unramified" spread domains.

THEOREM 1. *Let there be given E, together with the domain* (X, φ) *spread on E, and the mapping* $f \in \Gamma(X, \mathcal{A}(X, F))$, *where F is a complex* *space. Then there exists, in the set of those ψ' which spread X and are majorized by φ, and with respect to which f is prolongable holomorphically, an element majorizing all the others.*

We shall define a complex space X^*, a mapping ψ^* spreading X on X^*, a mapping φ^* spreading X^* on E and a mapping $f^* \in \Gamma(X^*, \mathcal{A}(X^*, F))$, in such a way (i) that we have $\varphi = \varphi^* \circ \psi^*$ and $f = f^* \circ \psi^*$ and (ii) that as long as we have a ψ' spreading X on X', a φ' spreading X' on E and $f' \in \Gamma(X', \mathcal{A}(X', F))$ such that $\varphi = \varphi' \circ \psi'$ and $f = f' \circ \psi'$, there exists a θ spreading X' on X^* in such a way that $\psi^* = \theta \circ \psi'$, $\varphi' = \varphi^* \circ \theta$ and $f' = f^* \circ \theta$.

$$\begin{tikzcd}
 & X' \arrow[dl, "f'"'] \arrow[dr, "\varphi'"] \arrow[dd, bend left, "\theta"] & \\
F & X \arrow[l, "f"'] \arrow[u, "\psi'"] \arrow[r, "\varphi"] \arrow[d, "\psi^*"'] & E \\
 & X^* \arrow[ul, "f^*"] \arrow[ur, "\varphi^*"'] &
\end{tikzcd}$$

As the first step we shall define a mapping spreading X on the space of algebroidal jets $\tilde{\mathcal{A}}(E, F)$ of E into F. As a preliminary we note the following:

LEMMA. *Let X and Y be two complex spaces, and let φ be a holomorphic mapping of X into Y. Suppose that φ is non-degenerate at a point $x \in X$. Then there exists an open neighborhood U of x such that the image $\varphi(U)$ is an analytic subset of Y, irreducible at the point $y = \varphi(x)$.*

PROOF. Since φ is non-degenerate at x, there exists an open neighborhood U_0 such that $\varphi(x) \notin \varphi(\bar{U}_0 - U_0)$, where $\bar{U}_0$ denotes the closure of U_0 in X. We may suppose that U_0 has a chart and $\bar{U}_0$ is compact. As $\varphi(\bar{U}_0 - U_0)$ is a closed subset of Y, there exists an open neighborhood V of $y = \varphi(x)$ such that $\varphi(\bar{U}_0 - U_0) \cap \bar{V} = \phi$, where $\bar{V}$ is the closure of V in Y. Let U be the open subset $\varphi^{-1}(V) \cap U_0$. Noting that the spaces X and Y are locally compact, we shall prove that the mapping $\varphi: U \rightarrow V$ is proper, that is, the inverse image φ^{-1} of every compact set in V is a compact set in U. Let K be a compact set in V and let Φ be an ultrafilter on $\varphi^{-1}(K) \cap U$. The filter Φ is a base for an ultrafilter Φ_0 on U_0, which converges to a point $x_0 \in \overline{U}_0$. The point x_0 belongs to U_0, for otherwise $x_0 \in \overline{U}_0 - U_0$ would imply $\varphi(x_0) \in \varphi(\overline{U}_0 - U_0)$ and this is contradictory to $\varphi(\overline{U}_0 - U_0) \cap \overline{V} = \phi$, because the ultrafilter $\varphi(\Phi)$ converges and so $\varphi(x_0) \in V$. Thus the relation $x_0 \in U_0$, together with $\varphi(x_0) \in K$, infers $x_0 \in \varphi^{-1}(K) \cap U$. Hence $\varphi^{-1}(K) \cap U$ is compact and hence the mapping $\varphi: U \rightarrow V$ is proper. Now we know from a theorem of Remmert[9] that $\varphi(U)$ is an analytic subset of V. Further we can take U small enough so that $\varphi(U)$ is irreducible at the point $y = \varphi(x)$.

Proof of the theorem. The graph Δ of the holomorphic mapping $x \rightarrow (\varphi(x), x, f(x))$ is a complex space isomorphic with the space X (§ 1). The restriction τ to Δ of the projection of $E \times X \times F$ onto $E \times F$ is a holomorphic mapping non-degenerate at every point of Δ because the mapping φ is of such a character. Now let x be a point of X. According to the Lemma the image under the mapping τ of the germ of Δ at the point $(\varphi(x), x, f(x))$ defines an algebroidal jet $\tilde{\alpha}$ of E into F at the point $\varphi(x)$. Define a mapping $\rho: X \rightarrow \tilde{\mathcal{A}}(E, F)$ by putting $\rho(x) = \alpha$. We see that the mapping ρ spreads X on $\tilde{\mathcal{A}}(E, F)$ and that $f = \tilde{t} \circ \rho$ and $\varphi = \tilde{\sigma} \circ \rho$, where $\tilde{\sigma}$ is the canonical mapping defined in § 2. Now let X^* be the connected component of $\tilde{\mathcal{A}}(E, F)$ containing the (connected) image of X by ρ. Let ψ^* be the mapping of X into X^* defined by ρ, φ^* the restriction of $\tilde{\sigma}$ to X^*, and f^* be the restriction of $\tilde{t}$ to X^*. The space X^* and the mappings ψ^*, φ^* and f^* satisfy the required conditions. Thus the theorem is proved.

As in the classical case, the domain (X^*, φ^*) spread on E, together with the mapping ψ^* spreading X on X^* and the holomorphic mapping f^* of X^* into F, is called the ***maximal holomorphic prolongation, over E, of the holomorphic mapping f of X into F.***

Next we consider the simultaneous holomorphic prolongation. The notations E and (X, φ) being the same as above, let f_i, $i \in I$, be holomorphic mappings of X into complex spaces F_i. *A holomorphic prolongation of the family* (f_i) consists of (i) a mapping ψ spreading

9) cf. [4].

X on X' and a mapping φ' spreading X' on E, in such a way that $\varphi=\varphi'\circ\psi'$ and (ii) a family of holomorphic mappings $f'\in\Gamma(X', \mathcal{A}(X', F_i))$ such that $f_i=f_i'\circ\psi'$. The mapping ψ' determines φ' and (f_i') if they exist.

Let $\tilde{\mathcal{A}}(E, F_i)$ be the spaces of algebroidal jets of E into F_i, $\tilde{\sigma}_i$ the projections onto E, and $\tilde{t}_i$ the canonical mappings into F_i. By the *direct sum* $\sum_{i\in I}\tilde{\mathcal{A}}(E, F_i)$ of the spaces of algebroidal jets $\tilde{\mathcal{A}}(E, F_i)$, $i\in I$, we mean the set of those elements $(\tilde{\alpha}_i)$ of the space $\prod_{i\in I}\tilde{\mathcal{A}}(E, F_i)$ which satisfy the following conditions: (i) for every $i, j\in I$, $\tilde{\sigma}_i(\tilde{\alpha}_i)=\tilde{\sigma}_j(\tilde{\alpha}_j)=x$ (ii) there exists an open neighborhood U of the point x such that the subspaces around $\tilde{\alpha}_i$ of $\tilde{\mathcal{A}}(E, F_i)$, lying over U with respect to $\tilde{\sigma}_i$, are isomorphic with one another. A structure of complex space is induced on the topological space $\sum_{i\in I}\tilde{\mathcal{A}}(E, F_i)$ by those of $\tilde{\mathcal{A}}(E, F_i)$. Define a mapping $\tilde{\sigma}$ and $\tilde{t}$ by putting $\tilde{\sigma}((\tilde{\alpha}_i))=\tilde{\sigma}_i(\tilde{\alpha}_i)$ and $\tilde{t}((\tilde{\alpha}_i))=(\tilde{t}_i(\tilde{\alpha}_i))$ for $(\tilde{\alpha}_i)\in\sum_{i\in I}\tilde{\mathcal{A}}(E, F_i)$. We see that the mapping $\tilde{\sigma}$ spreads $\sum_{i\in I}\tilde{\mathcal{A}}(E, F_i)$ on E, and the mapping $\tilde{t}$ is holomorphic, considered as a mapping of $\sum_{i\in I}\tilde{\mathcal{A}}(E, F_i)$ into each F_i. Replacing by $\sum_{i\in I}\tilde{\mathcal{A}}(E, F_i)$ the space $\tilde{\mathcal{A}}(E, F)$ in the proof of Theorem 1, we have the following

THEOREM 2. *Let there be given E, together with the domain (X, φ) spread on E, and the family of holomorphic mappings $f_i\in\Gamma(X, \mathcal{A}(X, F_i))$, $i\in I$. Then there exists, in the set of those ψ' which spread X and majorized by φ, and with respect to which the family of mappings (f_i) are prolongable holomorphically, an element majorizing, all the others.*

We can define thus the *maximal simultaneous holomorphic prolongation, over E, of the family of holomorphic mappings (f_i), with respect to the mapping φ spreading X on E.*

Consider a particular case where the mappings f_i spread X on F_i and are majorized by φ with respect to mappings g_i. Thus the mappings g_i spread F_i on E in such a way that $\varphi=g_i\circ f_i$. Theorem 2 affirms then that the mappings f_i have an infimum. This is a mapping f spreading X on a complex space F, which is itself spread on E by a mapping g, in such a way that $g\circ f=\varphi$. The space F spreads on each F_i by a mapping h_i, and the following diagram is commutative:

$$\begin{tikzcd}
& X \arrow[dl, "f_i"'] \arrow[dd, "\varphi"] \arrow[dr, "f"] & \\
F_i \arrow[dr, "g_i"'] & & F \arrow[ll, "h_i"'] \arrow[dl, "g"] \\
& E &
\end{tikzcd}$$

The domain (F, g), spread on E, together with the mappings h_i spreading F on F_i and the mapping f spreading X on F, is called the ***intersection*** of the spread domains (F_i, g_i) with respect to the domain X spread on each F_i by f_i.

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