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# Exact sequences in the Steenrod algebra.

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J. P. Serre [1] has proved that the cohomology algebra $H^*(Z_2; q, Z_2)$ of the Eilenberg-MacLane complex $K(Z_2, q)$ with $Z_2$ coefficients is a polynomial algebra generated by $\mathrm{Sq}^I(u_q)$, where $u_q$ is the generator of $H^q(Z_2; q, Z_2)$ and $I$ runs over the admissible sequences with excess $<q$, $\mathrm{Sq}^I$ being the iterated Steenrod squaring operations. He has proved thereby that $H^{n+q}(Z_2; q, Z_2)$ remains 'stable' for $q>n$, and put $A^n(Z_2, Z_2)=H^{n+q}(Z_2; q, Z_2)$. The graded algebra $\sum_{n=0}^{\infty} A^n(Z_2, Z_2)$ is denoted by $A^*(Z_2, Z_2)$ and is called the *Steenrod algebra* (Cf. Adem [2], [3]). Following Adem [2], we shall denote the generators of $A^*(Z_2, Z_2)$ with $\mathrm{Sq}^I$ instead of $\mathrm{Sq}^I(u_q)$. The multiplication between these generators is determined by Adem's relations (Adem [2], [3]).

$$\mathrm{Sq}^\alpha \mathrm{Sq}^\beta = \sum_{t=0}^{[\alpha/2]} \binom{\beta-t-1}{\alpha-2t} \mathrm{Sq}^{\alpha+\beta-t}\,\mathrm{Sq}^t \quad \mathrm{mod}\ 2, \quad 0 \leqq \alpha < 2\beta . \tag{1}$$

Let $I_0$ be any fixed sequence of integers. We can define a homomorphism $\alpha'_{I_0}$ of $A^*(Z_2, Z_2)$ into itself by $\alpha'_{I_0}\mathrm{Sq}^I=\mathrm{Sq}^{I_0}\mathrm{Sq}^I$, and another homomorphism $\alpha''_{I_0}$ by $\alpha''_{I_0}\mathrm{Sq}^I=\mathrm{Sq}^I\mathrm{Sq}^{I_0}$. If $M$ is a certain fixed submodule of $A^*(Z_2, Z_2)$, then $\mathrm{Sq}^I \to \alpha'_{I_0}\mathrm{Sq}^I$ mod $M$ or $\alpha''_{I_0}\mathrm{Sq}^I$ mod $M$ define respectively cohomology operations. These operations are of interest in view of topological applications. (Cf. Cartan [4], Serre [1]).

In this paper, we consider the operator $\alpha'_n$ defined by $\alpha'_n\mathrm{Sq}^I=\mathrm{Sq}^{2^n}\mathrm{Sq}^I$ $(n=0, 1, \cdots)$. We denote the module generated by the sums of the images of $\alpha'_i$ $(i=0, 1, \cdots, n)$ with $M_n$ for $n \geqq 0$, and put $M_{-2}=M_{-1}=0$. Obviously we have $M_n \supset M_{n-1}$. We shall give explicitly the generators of $M_n$ mod $M_{n-1}$ (Theorem 1) and those of $A^*(Z_2, Z_2)$ mod $M_n$ (Corollary of Theorem 1), and apply this to prove the following result. We can define $\alpha_{n+3}$ and $\beta_{n+3}$ for $n \geqq -2$ so that the following diagram is commutative, where $p_n$ is the natural homomorphism $A^*(Z_2, Z_2) \to A^*(Z_2, Z_2)/M_n$ for $n \geqq 0$, and $p_{-2}=p_{-1}=id$.

$$\begin{array}{ccccc}
A^*(Z_2, Z_2) & \xrightarrow{\alpha'_{n+3}} & A^*(Z_2, Z_2) & \xrightarrow{\alpha'_{n+3}} & A^*(Z_2, Z_2) \\
\Big\downarrow p_n & & \Big\downarrow p_{n+1} & & \Big\downarrow p_{n+2} \\
A^*(Z_2, Z_2)/M_n & \xrightarrow{\beta_{n+3}} & A^*(Z_2, Z_2)/M_{n+1} & \xrightarrow{\beta_{n+3}} & A^*(Z_2, Z_2)/M_{n+2} .
\end{array}$$

Then we shall prove that *the sequence*

$A^*(Z_2, Z_2)/M_n \xrightarrow{\beta_{n+3}} A^*(Z_2, Z_2)/M_{n+1} \xrightarrow{\alpha_{n+3}} A^*(Z_2, Z_2)/M_{n+2}$ *is exact for* $n=-2, -1, 0, 1, \cdots$ (Theorem 2). The exactness of this sequence for $n=-2, -1$ was proved by Professor T. Yamanoshita [5], who suggested to the author to occupy herself with this question. The author wishes to express her sincere thanks to Professor T. Yamanoshita for his kind suggestions and advices and also to Professor S. Iyanaga for his constant encouragement during the preparation of this paper.

In the following, we have often to deal with binomial coefficients mod 2. The following formula of Cartan [3] is fundamental for us. If the dyadic expansions of $n$ and $r$ are respectively $\sum_{i=0}^{n} 2^i a_i$ and $\sum_{j=0}^{m} 2^j b_j$, and $n \geqq m$, then

$$(2) \qquad \binom{n}{r}=\binom{2^n a_n+\cdots+2a_1+a_0}{2^m b_m+\cdots+2b_1+b_0}\equiv\binom{a_n}{0}\binom{a_{n-1}}{0}\cdots\binom{a_{m+1}}{0}\binom{a_m}{b_m}\cdots\binom{a_0}{b_0} \mod 2.$$

In particular, we have

$$(3) \qquad \binom{\beta-t-1}{2^n-2t}\equiv\binom{\beta+2^{n+1}-t-1}{2^n-2t} \mod 2.$$

These binomial coefficients appear in Adem's relation for $Sq^{2^n} Sq^{\beta}$. Hereafter we shall denote $Sq^{i_1} Sq^{i_2} \cdots Sq^{i_r}$ with $(i_1, i_2, \cdots, i_r)$. Often we denote such $(i_1, i_2, \cdots, i_r)$ with $I$. We denote the collection of all admissible sequences of the form $(2^{n+1} k_1, 2^n k_2, \cdots, 2^{n-j+3} k_{j-1}, 2^{n-j+1}(2k_j+1), 2^{n-j} k_{j+1}, \cdots, 2k_n, k_{n+1}, i_{n+2}, \cdots, i_r)$ $(j=1, \cdots, n+1)$ with $N_j^n$ and an arbitrary sequence belonging to $N_j^n$ generally with $I_j^n$. For the above $I_j^n \in N_j^n$ we denote the sequence $(2^{n+1}k_1, 2^n k_2, \cdots, 2k_{n+1}, i_{n+2}, \cdots, i_r)$ with $I^n$. As easily verified, for every admissible sequence $J$ there is uniquely determined a pair of integers $(n, j)$ such that $J=I_j^n \in N_j^n$. Setting $N^n=\bigcup_j N_j^n$ we have $I^n \in N^{n+m}$ for $m>0$. (Here and in what follows $k_1, k_2, \cdots$ denote always non negative integers.)

Now, if we identify $N^n$ with the free module over $Z$ generated by the collection $N^n$, then we have

THEOREM 1. $M_n=N^n \oplus M_{n-1}$, *that is,* $I_j^n$'s $(j=1, \cdots, n+1)$ *are not contained in* $M_{n-1}$ *and generate* $M_n/M_{n-1}$.

PROOF. The case $n=0$ means $I_1^0=(2k_1+1, i_2, \cdots, i_r)=(1, 2k_1, i_2, \cdots, i_r)\equiv 0$ mod $M_0$, and $I^0=(2k_1, i_2, \cdots, i_r)\not\equiv 0$ mod $M_0$. This is easily seen from $(1,1)=0$ and $(1, 2k)=(2k+1)$. Assume, inductively, that the theorem is true for $n-1$, i.e.

$$I_j^{n-1}\equiv 0 \mod M_{n-1} \quad \text{i.e.} \quad I_j^{n-1}\equiv(2^{n-1}, I) \mod M_{n-2} \quad \text{for some } I.$$

$$(4) \qquad I^{n-1}\not\equiv 0 \mod M_{n-1}.$$

According to (1), (2) and the inductive assumption $\sum_{i<n} N^i=M_{n-1}$

$$(2^n, 2^{n+1}k_1)=\sum_{t=0}^{2^{n-1}}\binom{2^{n+1}k_1-t-1}{2^n-2t}(2^n+2^{n+1}k_1-t, t)\equiv\binom{2^{n+1}k_1-1}{2^n}(2^n(2k_1+1))$$

$$=\binom{2^{n+1}(k_1-1)+2^{n+1}-1}{2^n}(2^n(2k_1+1))$$

$$=\binom{2^{n+1}(k_1-1)+2^n+2^{n-1}+\cdots+1}{2^n}(2^n(2k_1+1))$$

$$=\binom{k_1-1}{0}\binom{1}{1}\binom{1}{0}\cdots\binom{1}{0}(2^n(2k_1+1))=(2^n(2k_1+1)) \quad \text{mod } M_{n-1}.$$

Therefore

(5) $\quad I_1^n=(2^n(2k_1+1), 2^{n-1}k_2, \cdots, 2k_n, i_{n+1}, \cdots, i_r)$

$\quad\equiv(2^n, 2^{n+1}k_1, 2^{n-1}k_2, \cdots, 2k_n, i_{n+1}, \cdots, i_r) \qquad \text{mod } M_{n-1},$

(5′) $\quad\equiv 0 \qquad \text{mod } M_n.$

$$(2^n, 2^{n+1}k_0-2^{n-1})=\sum_{t=0}^{2^{n-1}}\binom{2^{n+1}k_0-2^{n-1}-t-1}{2^n-2t}(2^{n+1}k_0+2^{n-1}-t, t)$$

$$\equiv\binom{2^{n+1}k_0-2^n-1}{2^n-2^n}(2^{n+1}k_0, 2^{n-1})=(2^{n+1}k_0, 2^{n-1}) \quad \text{mod } M_{n-1}.$$

This implies

(6) $\quad I_{j+1}^n=(2^{n+1}k_0, I_j^{n-1})=(2^{n+1}k_0, 2^{n-1}, I)\equiv(2^n, 2^{n+1}k_0-2^{n-1}, I) \qquad \text{mod } M_{n-1},$

(6′) $\quad\equiv 0 \qquad \text{mod } M_n$

$(j=1,\cdots,n)$. By (4), (5) and (6), $I_j^n$'s $(j=1,\cdots,n+1)$ are not contained in $M_{n-1}$ but contained in $M_n$. If now $I^n=(2^{n+1}k_1, 2^n k_2, \cdots, 2k_{n+1}, i_{n+2}, \cdots, i_r)\in M_n$, then there would exist, by the inductive hypothesis the relation $(2^{n-1}, 2^{n-1}+2^m)\equiv(2^n, 2^m)$ mod $M_{n-1}$ for $n>m$, an $I$ with $I^n\equiv(2^n, I)$ mod $M_{n-1}$, where $I$ has a form $I^{n-1}=(2^n k_1', 2^{n-1}k_2', \cdots, 2k_n', i_{n+1}, \cdots, i_r)$. And we have

$$(2^n, I^{n-1})=(2^n, 2^n k_1', 2^{n-1}k_2', \cdots, 2k_n', i_{n+1}, \cdots, i_r)$$

$$=\sum_{t=0}^{2^{n-1}}\binom{2^n k_1'-t-1}{2^n-2t}(2^n+2^n k_1'-t, t, 2^{n-1}k_2', \cdots, 2k_n', i_{n+1}, \cdots, i_r).$$

Therefore $(2^n, I^{n-1})$ becomes an $I^n$ mod $M_{n-1}$, only when $t=0$ and $k_1'$ is odd. But in this case the coefficient $\binom{2^n k_1'-1}{2^n}=\binom{2^n(k_1'-1)+2^n-1}{2^n}\equiv 0$ mod 2. Therefore $I^n\notin M_n$, and $I_j^n$'s $(j=1,\cdots,n+1)$ generate $M_n/M_{n-1}$.

COROLLARY. *$I^n$'s generate $A^*(Z_2, Z_2)/M_n$.*

To prepare for the proof of the next theorem, we list here some formulas which are easily proved by (3). The formulas (7) to (14) (which are congruences mod $M_{i-2}$) are used to calculate $\beta_i$-images $(2^i, I)$ $(i=0,1,\cdots)$. Let $I=(a, I')$ be a given sequence. Then $(2^i, I)=(2^i, a, I')$ is contained in $M_{i-2}$, if

$a$ is not a multiple of $2^{i-2}$, by Theorem 1, and the formula $(2^{i-2}, 2^{i-1}+2^{i-2}+2^m)\equiv(2^i, 2^m) \bmod M_{i-2}$ for $m<i-2$, so that we have only to consider the case $a=2^{i-2}b$. By (3), we have $(2^i, 2^{i-2}b)\equiv(2^i, 2^{i-2}b')$ if $b\equiv b' \bmod 2^3$. For $(2^i, 2^{i-2}(2^3k+j))$ $j=0, 1, \cdots, 7$, we have

(7) $(2^i, 2^{i-2}\cdot 2^3k)\equiv(2^{i+1}k+2^i)+(2^{i+1}k+2^{i-1}, 2^{i-1}) \quad \bmod M_{i-2}$

(8) $(2^i, 2^{i-2}(2^3k+1))\equiv(2^{i+1}k+2^i, 2^{i-2}) \quad \bmod M_{i-2}$

(9) $(2^i, 2^{i-2}(2^3k+2))\equiv(2^{i+1}k+2^i, 2^{i-1}) \quad \bmod M_{i-2}$

(10) $(2^i, 2^{i-2}(2^3k+3))\equiv 0 \quad \bmod M_{i-2}$

(11) $(2^i, 2^{i-2}(2^3k+4))\equiv(2^{i+1}k+2^{i-1}\cdot 3, 2^{i-1}) \quad \bmod M_{i-2}$

(12) $(2^i, 2^{i-2}(2^3k+5))\equiv(2^{i+1}(k+1), 2^{i-2}) \quad \bmod M_{i-2}$

(13) $(2^i, 2^{i-2}(2^3k+6))\equiv(2^{i+1}(k+1)+2^{i-1})+(2^{i+1}(k+1), 2^{i-1}) \quad \bmod M_{i-2}$

(14) $(2^i, 2^{i-2}(2^3k+7))\equiv 0 \quad \bmod M_{i-2}$.

The following formulas (15) to (18) are used to calculate $\alpha_i$-images $(2^i, I) \bmod M_{i-1}$ $(i=0, 1, \cdots)$. Similarly as in the above case, we have only to consider $(2^i, 2^{i-1}c)$ with $c \bmod 2^2$, i. e. $(2^i, 2^{i-1}(2^2k+j))$, $j=0, 1, 2, 3$.

(15) $(2^i, 2^{i-1}\cdot 2^2k)\equiv(2^{i+1}k+2^i) \quad \bmod M_{i-1}$

(16) $(2^i, 2^{i-1}(2^2k+1))\equiv(2^{i+1}k+2^i, 2^{i-1}) \quad \bmod M_{i-1}$

(17) $(2^i, 2^{i-1}(2^2k+2))\equiv 0 \quad \bmod M_{i-1}$

(18) $(2^i, 2^{i-1}(2^2k+3))\equiv(2^{i+1}(k+1), 2^{i-1}) \quad \bmod M_{i-1}$.

In calculating $\alpha'_n$ images, we may proceed as follows in utilizing (15) to (18). Let $I=(2^{i-1}c_1, 2^{i-2}c_2, \cdots, i_i, \cdots, i_r)=(2^{i-1}c_1, I')=(2^{i-1}c_1, 2^{i-2}c_2, I'')$ be an admissible sequence. By (15) to (18), the following three cases occur:

1) $(2^i, 2^{i-1}(2^2k_1+1), 2^{i-2}c_2, I'')$

$$\equiv(2^{i-1}(2^2k_1+2), 2^{i-1}, 2^{i-2}c_2, I'') \quad \bmod M_{i-1} \quad (2^2k_1+1\geqq c_2)$$

2) $(2^i, 2^{i-1}(2^2k_1+3), 2^{i-2}c_2, I'')$

$$\equiv(2^{i-1}\cdot 2^2(k_1+1), 2^{i-1}, 2^{i-2}c_2, I'') \quad \bmod M_{i-1} \quad (2^2k_1+3\geqq c_2)$$

3) $(2^i, 2^{i-1}\cdot 2^2k_1, 2^{i-2}c_2, I'')$

$$\equiv(2^{i-1}(2^2k_1+2), 2^{i-2}c_2, I'') \quad \bmod M_{i-1} \quad (2^2k_1\geqq c_2).$$

The right hand side of 3) is obviously admissible. Those of 1) and 2) may not be admissible. Then we transform $(2^{i-1}, 2^{i-2}c_2)$ again by (15) to (18). Let $c_2=2^2k_1+1$ in 1), then $(2^{i-1}(2^2k_1+2), 2^{i-1}, 2^{i-2}c_2, I'')$ $=(2^{i-1}(2^2k_1+2), 2^{i-1}, 2^{i-2}(2^2k_1+1), I'')=(2^{i-1}(2^2k_1+1), 2^ik_1+2^{i-1}, 2^{i-2}, I'')$ by (16), and then $2^{i-1}(2^2k_1+2)-2(2^ik_1+2^{i-1})=0$. Therefore the result satisfies the admissibility condition for the first two terms. The same is also true in case

$2^2k_1+1>c_2$ as is easily seen. Let $c_2=2^2k_1+3$ in 2), then $(2^{i-1}\cdot 2^2(k_1+1), 2^{i-1}, 2^{i-2}c_2, I'') = (2^{i-1}\cdot 2^2(k_1+1), 2^{i-1}, 2^{i-2}(2^2k_1+3), I'') = (2^{i-1}\cdot 2^2(k_1+1), 2^i(k_1+1), 2^{i-2}, I'')$ by (18), and then $2^{i-1}\cdot 2^2(k_1+1)-2\{2^i(k_1+1)\}=0$. Again the result satisfies the admissibility condition for the first two terms, and this is also true in case $2^2k_1+3>c_2$. Thus we may calculate $\alpha'_n$ images straightforwardly beginning by the 'head'.

THEOREM 2. *The sequence*

$$A^*(Z_2, Z_2)/M_n \xrightarrow{\beta_{n+3}} A^*(Z_2, Z_2)/M_{n+1} \xrightarrow{\alpha_{n+3}} A^*(Z_2, Z_2)/M_{n+2}$$

*is exact for* $n=-2, -1, 0, 1, 2, \cdots$.

PROOF. $\mathrm{Im}\,\beta_{n+3}\subset\mathrm{Ker}\,\alpha_{n+3}$ is easily seen by putting $i=n+3$ and $k=0$ in (17). Now we shall show that $\mathrm{Ker}\,\alpha_{n+3}\subset\mathrm{Im}\,\beta_{n+3}$ by induction. If $n=-2$, we obtain by putting $i=1$ in (15) to (18),

$$(2, 2^2k)\equiv(2^2k+2) \quad \mathrm{mod}\ M_0$$
$$(2, 2^2k+1)\equiv(2^2k+2, 1) \quad \mathrm{mod}\ M_0$$
$$(2, 2^2k+2)\equiv 0 \quad \mathrm{mod}\ M_0$$
$$(2, 2^2k+3)\equiv(2^2(k+1), 1) \quad \mathrm{mod}\ M_0\,.$$

Therefore the kernel of $\alpha_1$ is generated by

$$(2^2k+2, i_2, \cdots, i_r)\,,$$
$$(2^2k_1+1, 2k_2+1, i_3, \cdots, i_r)\,,$$
$$(2^2k_1+3, 2k_2+1, i_3, \cdots, i_r) \text{ and}$$
$$(2^2k_1+1, 2k_2, i_3, \cdots, i_r)+(2^2k_1, 2k_2+1, i_3, \cdots, i_r)\,.$$

Put $i=1$ in (7), (9), (11) and (13), then we obtain

$$(2, 2^2k)=(2^2k+2)+(2^2k+1, 1)$$
$$(2, 2^2k+1)=(2^2k+2, 1)$$
$$(2, 2^2k+2)=(2^2k+3, 1)$$
$$(2, 2^2k+3)=(2^2(k+1)+1)+(2^2(k+1), 1)\,.$$

Thus

$$(2, 2^2k_1, 2k_2+1, i_3, \cdots, i_r)=(2^2k_1+2, 2k_2+1, i_3, \cdots, i_r)$$
$$(2, 2^2k_1, 2k_2, i_3, \cdots, i_r)+(2, 2^2(k_1-1)+3, 2k_2+1, i_3, \cdots, i_r)=(2^2k_1+2, 2k_2, i_3, \cdots, i_r)$$
$$(2, 2^2(k_1-1)+3, 2k_2+1, i_3, \cdots, i_r)=(2^2k_1+1, 2k_2+1, i_3, \cdots, i_r)$$
$$(2, 2^2k_1+2, 2k_2, i_3, \cdots, i_r)=(2^2k_1+3, 2k_2+1, i_3, \cdots, i_r)$$
$$(2, 2^2(k_1-1)+3, 2k_2, i_3, \cdots, i_r)=(2^2k_1+1, 2k_2, i_3, \cdots, i_r)+(2^2k_1, 2k_2+1, i_3, \cdots, i_r)\,.$$

This shows that $\mathrm{Ker}\,\alpha_1\subset\mathrm{Im}\,\beta_1$, and therefore $\mathrm{Ker}\,\alpha_1=\mathrm{Im}\,\beta_1$.

Assume, inductively, that the theorem is true for integers $<n+3$. Let $K_t$ denote the kernel of $\alpha_t$ for $t<n+3$. Then by our assumption, the $\alpha'_t$ image of $p_{t-2}^{-1}K_t$ is generated by $I^{t}{}_{j}^{-1}$ $(j=1, \cdots, t)$ mod $M_{t-2}$ and $p_{t-2}^{-1}K_t$ is generated by $(2^t, I^{t-3})$ mod $M_{t-2}$. Under this assumption, we shall prove the following two lemmas. Hereafter we identify $K_t$ with $p_{t-2}^{-1}K_t$.

LEMMA 1. *For simplicity, we denote the numbers of the type* $2^{i-1}(2^2k+j)$

*generally with* $m_i^j$ $(j=0, 1, 2, 3,\ i=1, 2, 3, \cdots)$. ($m_0^0, m_0^2$ *will mean even and odd numbers respectively.*) *Then* $\operatorname{Ker}\alpha_{n+3}$ *is generated by elements of the following type*

(19) $(m_{n+3}^2, i_2, \cdots, i_r)$

(20) $(m_{n+3}^1, K_{n+2})$

(21) $(m_{n+3}^3, K_{n+2})$

(22) $(m_{n+3}^0, I_j^{n+1})$

(23) $(m_{n+3}^1, m_{n+2}^0, i_3, \cdots, i_r)+(m_{n+3}^0, m_{n+2}^2, i_3, \cdots, i_r)$

(24) $(m_{n+3}^{\lambda_0}, m_{n+2}^{\lambda_1}, m_{n+1}^{\lambda_2}, \cdots, m_{n-k+2}^{\lambda_{k+1}}, i_{k+3}, \cdots, i_r)$

$$+(m_{n+3}^{\mu_0}, m_{n+2}^{\mu_1}, m_{n+1}^{\mu_2}, \cdots, m_{n-k+2}^{\mu_{k+1}}, i_{k+3}, \cdots, i_r) \quad k=1, \cdots, n+2\,,$$

*where always* $\lambda_0=1$, $\lambda_{k+1}=0$, $\mu_0=0$, $\mu_{k+1}=2$ *and* $\{\lambda_1, \cdots, \lambda_k\}$ *is any sequence of* $k$ *terms composed of numbers* 1, 3 (*such as* $\{1, 1\}$, $\{1, 3\}$, $\{3, 1\}$, $\{3, 3\}$ *if* $k=2$, *there are* $2^k$ *such sequences*) *and* $\mu_i$ *is* 0 *or* 2 *according as* $\lambda_i$ *is* 3 *or* 1.

PROOF. By (17), we immediately see that elements of the type (19) are in $\operatorname{Ker}\alpha_{n+3}$. As easily seen, we have only to consider as generators of $\operatorname{Ker}\alpha_{n+3}$ the elements of the form $(m_{n+3}^j, I)$ and their sums.

Consider first the elements of the form $(m_{n+3}^j, K_{n+2})$. By (16), our assumption of induction and (5), (6), we have

$$(2^{n+3}, m_{n+3}^1, K_{n+2}) \equiv (2^{n+3}(2k+1), 2^{n+2}, K_{n+2})$$
$$\equiv \sum (2^{n+3}(2k+1), I_j^{n+1}) \equiv \sum I_{j'}^{n+2} \equiv 0 \qquad \bmod M_{n+2}\,.$$

Thus elements of the type (20) are in $\operatorname{Ker}\alpha_{n+3}$. We see in the same way, that also elements of the type (21) are in $\operatorname{Ker}\alpha_{n+3}$. By (15) to (18), these are obviously only elements of the form $(m_{n+3}^j, K_{n+2})$ which are in $\operatorname{Ker}\alpha_{n+3}$.

Now consider the elements of the form $(m_{n+3}^j, I)$ where $I$ is not in $K_{n+2}$. We see immediately by (17) and (15) that elements of the forms (19), (22) are in $\operatorname{Ker}\alpha_{n+3}$, and also that these are only such elements contained in $\operatorname{Ker}\alpha_{n+3}$.

Consider finally elements of the form $(m_{n+3}^j, I)+(m_{n+3}^{j'}, I')$. Generators of $\operatorname{Ker}\alpha_{n+3}$ of this type will be called *compound generators*. By Theorem 1, we must have $j=0$, $j'=1$ or $j=1$, $j'=0$ in the compound generators which are not contained in (20) and (21). To fix the notation, we shall put $j=0$, $j'=1$. Now we have by (15) to (18)

(25) $(2^{n+3}, m_{n+3}^1, m_{n+2}^0, i_3, \cdots, i_r) \equiv (m_{n+3}^2, m_{n+2}^2, i_3, \cdots, i_r) \qquad \bmod M_{n+2}$

(26) $(2^{n+3}, m_{n+3}^1, m_{n+2}^1, i_3, \cdots, i_r) \equiv (m_{n+3}^2, m_{n+2}^2, 2^{n+1}, i_3, \cdots, i_r) \qquad \bmod M_{n+2}$

$(2^{n+3}, m_{n+3}^1, m_{n+2}^2, i_3, \cdots, i_r) \equiv 0 \qquad \bmod M_{n+2}$

(27) $(2^{n+3}, m_{n+3}^1, m_{n+2}^3, i_3, \cdots, i_r) \equiv (m_{n+3}^2, m_{n+2}^0, 2^{n+1}, i_3, \cdots, i_r) \qquad \bmod M_{n+2}\,.$

And similarly

(28) $(2^{n+3}, m^0_{n+3}, m^0_{n+2}, i_3, \cdots, i_r) \equiv (m^2_{n+3}, m^0_{n+2}, i_3, \cdots, i_r) \quad \bmod M_{n+2}$

(29) $(2^{n+3}, m^0_{n+3}, m^1_{n+2}, i_3, \cdots, i_r) \equiv (m^2_{n+3}, m^1_{n+2}, i_3, \cdots, i_r) \quad \bmod M_{n+2}$

(30) $(2^{n+3}, m^0_{n+3}, m^2_{n+2}, i_3, \cdots, i_r) \equiv (m^2_{n+3}, m^2_{n+2}, i_3, \cdots, i_r) \quad \bmod M_{n+2}$

(31) $(2^{n+3}, m^0_{n+3}, m^3_{n+2}, i_3, \cdots, i_r) \equiv (m^2_{n+3}, m^3_{n+2}, i_3, \cdots, i_r) \quad \bmod M_{n+2}.$

Thus we see that the sum of (25) and (30), i. e. (23) is in $\operatorname{Ker} \alpha_{n+3}$. These are obviously only compound generators of $\operatorname{Ker} \alpha_{n+3}$ determined by first two terms. For (26) and (30) we compare further by (15) to (18)

(32) $(2^{n+3}, m^1_{n+3}, m^1_{n+2}, m^0_{n+1}, i_4, \cdots, i_r) \equiv (m^2_{n+3}, m^2_{n+2}, 2^{n+1}, m^0_{n+1}, i_4, \cdots, i_r)$
$\equiv (m^2_{n+3}, m^2_{n+2}, m^2_{n+1}, i_4, \cdots, i_r) \quad \bmod M_{n+2}$

(33) $(2^{n+3}, m^1_{n+3}, m^1_{n+2}, m^1_{n+1}, i_4, \cdots, i_r) \equiv (m^2_{n+3}, m^2_{n+2}, 2^{n+1}, m^1_{n+1}, i_4, \cdots, i_r)$
$\equiv (m^2_{n+3}, m^2_{n+2}, m^2_{n+1}, 2^n, i_4, \cdots, i_r) \quad \bmod M_{n+2}$

(34) $(2^{n+3}, m^1_{n+3}, m^1_{n+2}, m^2_{n+1}, i_4, \cdots, i_r)$
$\equiv (m^2_{n+3}, m^2_{n+2}, 2^{n+1}, m^2_{n+1}, i_4, \cdots, i_r) \equiv 0 \quad \bmod M_{n+2}$

(35) $(2^{n+3}, m^1_{n+3}, m^1_{n+2}, m^3_{n+1}, i_4, \cdots, i_r) \equiv (m^2_{n+3}, m^2_{n+2}, 2^{n+1}, m^3_{n+1}, i_4, \cdots, i_r)$
$\equiv (m^2_{n+3}, m^2_{n+2}, m^0_{n+1}, 2^n, i_4, \cdots, i_r) \quad \bmod M_{n+2}.$

(36) $(2^{n+3}, m^0_{n+3}, m^2_{n+2}, m^0_{n+1}, i_4, \cdots, i_r) \equiv (m^2_{n+3}, m^2_{n+2}, m^0_{n+1}, i_4, \cdots, i_r) \quad \bmod M_{n+2}$

(37) $(2^{n+3}, m^0_{n+3}, m^2_{n+1}, m^1_{n+1}, i_4, \cdots, i_r) \equiv (m^2_{n+3}, m^2_{n+2}, m^1_{n+1}, i_4, \cdots, i_r) \quad \bmod M_{n+2}$

(38) $(2^{n+3}, m^0_{n+3}, m^2_{n+2}, m^2_{n+1}, i_4, \cdots i_r) \equiv (m^2_{n+3}, m^2_{n+2}, m^2_{n+1}, i_4, \cdots, i_r) \quad \bmod M_{n+2}$

(39) $(2^{n+3}, m^0_{n+3}, m^2_{n+2}, m^3_{n+1}, i_4, \cdots, i_r) \equiv (m^2_{n+3}, m^2_{n+2}, m^3_{n+1}, i_4, \cdots, i_r) \quad \bmod M_{n+2}.$

By comparing (32) and (38), we see that

$$(m^1_{n+3}, m^1_{n+2}, m^0_{n+1}, i_4, \cdots, i_r) + (m^0_{n+3}, m^2_{n+2}, m^2_{n+1}, i_4, \cdots, i_r)$$

is in $\operatorname{Ker} \alpha_{n+3}$. In the same way we see that

$$(m^1_{n+3}, m^3_{n+2}, m^0_{n+1}, i_4, \cdots, i_r) + (m^0_{n+3}, m^0_{n+2}, m^2_{n+1}, i_4, \cdots, i_r)$$

is also in $\operatorname{Ker} \alpha_{n+3}$. Thus we obtain as compound generators of $\operatorname{Ker} \alpha_{n+3}$ elements of the form (24) with $k=1$, and these are obviously only compound generators determined by first three terms. Other compound generators are obtained in the same way.

LEMMA 2. $\operatorname{Ker} \alpha_{n+3} \subset \operatorname{Im} \beta_{n+3}$.

PROOF. We can see as follows that the generators of the $\operatorname{Ker} \alpha_{n+3}$ are the elements of $\operatorname{Im} \beta_{n+3}$ by referring to the formulas (7) to (14). For (19), we have

$$(m^2_{n+3}, I) \equiv (2^{n+3}, m^0_{n+3}, I) + (2^{n+3}, 2^{n+4}(k-1) + 2^{n+1} \cdot 6, 2^{n+2}, I)$$
$$+ (2^{n+3}, 2^{n+4}(k-1) + 2^{n+1} \cdot 5, 2^{n+2}, 2^{n+1}, I) \quad \bmod M_{n+1}.$$

For (20), since $(m^1_{n+3}, K_{n+2}) \equiv \sum (m^1_{n+3}, 2^{n+2}, I^{n-1})$ mod $M_n$ by the inductive hypothesis,

$$(m^1_{n+3}, 2^{n+2}, I^{n-1}) \equiv (2^{n+3}, 2^{n+4}(k-1)+2^{n+1}\cdot 6, 2^{n+2}, I^{n-1}) + (2^{n+3}, 2^{n+4}(k-1)+2^{n+1}\cdot 5, 2^{n+2}, 2^{n+1}, I^{n-1}) \quad \text{mod } M_{n+1}.$$

For (21), we have

$$(m^3_{n+3}, K_{n+2}) = \sum (m^3_{n+3}, 2^{n+1}, I^{n-1}) \equiv \sum (2^{n+3}, m^2_{n+3}, I^{n-1}) \quad \text{mod } M_{n+1}.$$

For (22),

$$(m^0_{n+3}, I^{n+1}_j) = (m^0_{n+3}, 2^{n+2}, K_{n+2}) \equiv \sum (m^0_{n+3}, 2^{n+2}, 2^{n+2}, I^{n-1}) \equiv \sum (2^{n+3}, 2^{n+4}(k-1)+2^{n+1}\cdot 5, 2^{n+2}, 2^{n+1}, I^{n-1}) \quad \text{mod } M_{n+1}.$$

For (23), we have

$$(m^1_{n+3}, m^0_{n+2}, I)+(m^0_{n+3}, m^2_{n+2}, I) \equiv (2^{n+3}, 2^{n+4}(k-1)+2^{n+1}\cdot 6, m^0_{n+2}, I)+(2^{n+3}, 2^{n+4}(k-1)+2^{n+1}\cdot 5, m^0_{n+2}, 2^{n+1}, I) \quad \text{mod } M_{n+1}.$$

For (24), we have in case $k=1$,

$$(m^1_{n+3}, m^1_{n+2}, m^0_{n+1}, I)+(m^0_{n+3}, m^2_{n+2}, m^2_{n+1}, I) \equiv (2^{n+3}, m^3_{n+3}, m^1_{n+2}, m^0_{n+1}, I) \quad \text{mod } M_{n+1}.$$

$$(m^1_{n+3}, m^3_{n+2}, m^0_{n+1}, I)+(m^0_{n+3}, m^0_{n+2}, m^2_{n+1}, I) \equiv (2^{n+3}, m^3_{n+3}, m^3_{n+2}, m^0_{n+1}, I)+(2^{n+3}, 2^{n+4}(k-1)+2^{n+1}\cdot 5, m^0_{n+2}, m^0_{n+1}, I) \quad \text{mod } M_{n+1}.$$

Also in case $k>1$ we can proceed in the same way.

By this Lemma 2, we have Ker $\alpha_{n+3} = \text{Im } \beta_{n+3}$. This asserts the exactness of the sequence. Q. E. D.

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