Journal of the Mathematical Society of Japan Vol. 10, No. 1, January, 1958

Remark on the fundamental conjecture of *GLC*.

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(Received July 4, 1957)

Since 1953, the author has worked on the fundamental conjecture of *GLC* [3]. But it seems that some facts concerning the implication of this conjecture, which appeared clear to the author, remain ununderstood by readers. Following the advice of his friends, the author wishes to clearify these points in the following lines.

For the convenience of the reader we begin with giving an explanation about *GLC* and the fundamental conjecture. The *GLC* (Generalized Logic Calculus) was introduced in [2], as a generalization of Gentzen's *LK* [1]. The latter is a particularly workable formalization of Hilbert's "Engerer Funktionenkalkül". The *GLC* is obtained from the *LK* in adjoining to it bound and free variables of predicates and functions of higher orders. For these new variables the inference schemata for $\forall$, $\exists$ are set up in the same form as in *LK*. Gentzen [1] proved for *LK* the fundamental theorem: Every provable sequence in *LK* is provable without cut. Our fundamental conjecture of *GLC* means that the corresponding "cut-elimination theorem" is also valid in *GLC*.

PROPOSITION 1. *If the fundamental conjecture of GLC holds, then every system of axioms in LK, which is consistent in LK, is also consistent in GLC.*

PROOF. Let Γ_0 be a consistent system of axioms in *LK*. Suppose Γ_0 to be inconsistent in *GLC*. Then, in virtue of the fundamental conjecture, there exists a proof-figure P without cut in *GLC*, with end-sequence $\Gamma_0 \rightarrow$. To prove the proposition, we have only to show that P is a proof-figure of *LK*. Now suppose that P is not a proof-figure of *LK*. If every formula in P is a formula of *LK*, then every inference in P must be an inference of *LK* and, moreover, P must be a proof-figure of *LK*. Therefore there exists a lowermost sequence S in P containing a formula not belonging to *LK*. Since the end-sequence $\Gamma_0 \rightarrow$ is a sequence of *LK*, S is anyway not the end-sequence of P. S must be therefore an upper sequence of a certain inference I. Since the upper sequence S of I does not belong to *LK* whereas the lower sequence of I does, I must be a cut, which is a contradiction.

PROPOSITION 2. *If the fundamental conjecture of GLC holds, then the analysis is consistent.*

Remark. We shall explain what we mean by "analysis" and on what depends our proof, before giving the formal proof. Our proof depends on the theorem 9.9 of [2], which means that if a system of axioms containing the equality axiom is consistent in *GLC* without bound functions, then the system remains consistent after adjoining the concept of sets of elements in the domain, defined by the given system of axioms. This theorem allows thus to elevate the "order" (Stuf) of sets. We mean by "analysis" the domain of natural numbers, and sets of natural numbers, sets of sets of natural numbers,$\cdots$.

Proof of Proposition 2. If the fundamental conjecture of *GLC* holds, then, by Proposition 1, the following system of axioms is consistent in *GLC*:

$$\forall x(x=x)$$
$$\forall x\forall y(x=y \vdash y=x)$$
$$\forall x\forall y\forall z(x=y\wedge y=z \vdash x=z)$$
$$\forall x\neg(x'=1)$$
$$\forall x\forall y(x=y \vdash x'=y')$$

Then by 7.22 of [2], $\Gamma_a, \tilde{\Gamma}_e$, $\forall\varphi\forall x(\varphi[1]\wedge\forall y(\varphi[y]\vdash\varphi[y'])\vdash\varphi[x])$ are consistent in *GLC*, where $\tilde{\Gamma}_e$ is a system of equality axioms (See [2] p. 65 for $\tilde{\Gamma}_e$), especially Γ_a, Γ'_e, $\forall\varphi\forall x(\varphi[1]\wedge\forall y(\varphi[y]\vdash\varphi[y'])\vdash\varphi[x])$ are consistent in *GLC* (See [2] p. 66 for Γ'_e). Hence by applying 9.9 of [2] on this system of axioms, we see that the theory Γ_1, containing axioms on the set of natural numbers, is consistent. Moreover, applying again 9.9 on Γ_1, we see that the theory Γ_2, containing axioms on the set of the sets of natural numbers, etc. Therefore we have consistency-proof of $\Gamma_1, \Gamma_2, \cdots$, whence follows the consistency of analysis.

References

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