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On the P-extension of topology.

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Let $(X, \mathfrak{T})$ be a topological space with the underlying point set X and the topology $\mathfrak{T}$. For a subset S of X, $(S, \mathfrak{T})$ will mean the subspace of $(X, \mathfrak{T})$ with the topology induced by $\mathfrak{T}$. A property of topological spaces such as compact, connected, etc., will be generally denoted by $\boldsymbol{P}$ (especially the property of being compact will be denoted by $\boldsymbol{K}$). If we have to consider spaces $X, Y, \cdots$, we shall denote with $\sum_{P}^{X}, \sum_{P}^{Y}, \cdots$, the families of all the subspaces of $X, Y, \cdots$ respectively, having the property $\boldsymbol{P}$. We denote with $\mathfrak{T}_{P}$ the topology of X defined as follows: A subset A of X is closed for $\mathfrak{T}_{P}$ if and only if $A \cap S$ is closed in $(S, \mathfrak{T})$ for every $S \in \sum_{P}^{X}$. Such topology was considered by D. E. Cohen [2] who called it *weak topology* for $\sum_{P}^{X}$. In case $\boldsymbol{P}=\boldsymbol{K}$ the topology $\mathfrak{T}_{P}$ was called *k-extention* of $\mathfrak{T}$ by J.L. Kelley. In fact $\mathfrak{T}_{P}$ is always an extension of $\mathfrak{T}$, i.e. a topology weaker than $\mathfrak{T}$. We shall call $\mathfrak{T}_{P}$ the ***P**-extention* of $\mathfrak{T}$. It was proved in [3] that $(X \times Y, \mathfrak{T}_{K} \times \mathfrak{T}_{K}^{\prime})=(X \times Y, (\mathfrak{T}_{K} \times \mathfrak{T}_{K}^{\prime})_{K})$ if $(Y, \mathfrak{T}_{K}^{\prime})$ is locally compact and compact sets in $(X, \mathfrak{T}_{K}), (Y, \mathfrak{T}_{K})$ are regular. In [3] a necessary and sufficient condition for $\mathfrak{T}=\mathfrak{T}_{K}$ is obtained. We shall obtain analogous results to these ones for the $\boldsymbol{P}$-extension in the following lines.

PROPOSITION 1. *$\mathfrak{T}_{P}$ is the weakest topology among all topologies of X which induce the same topology as $\mathfrak{T}$ on each $S, S \in \sum_{P}^{X}$.*

PROOF. We shall prove that $(S, \mathfrak{T})$ is homeomorphic to $(S, \mathfrak{T}_{P})$ by the identity map for $S \in \sum_{P}^{X}$. Clearly the identity map is continuous. Now let K be a closed subset of $(S, \mathfrak{T}_{P})$, then $K=S \cap K^{\prime}$ where $K^{\prime}$ is closed in $(X, \mathfrak{T}_{P})$. Since $S \in \sum_{P}^{X}, S \cap K^{\prime}$ is closed in $(S, \mathfrak{T})$. Thus the identity map is a homeomorphism. Let $\mathfrak{T}^{\prime}$ be any topology on X such that $(S, \mathfrak{T}^{\prime})=(S, \mathfrak{T})$ for every $S \in \sum_{P}^{X}$. If K is closed for $\mathfrak{T}^{\prime}, K \cap S$ is closed in $(S, \mathfrak{T}^{\prime})$, hence $K \cap S$ is closed in $(S, \mathfrak{T})$.

REMARK. It is clear from the above proof that a subspace of $(X, \mathfrak{T})$ having the property $\boldsymbol{P}$ is also a subspace of $(X, \mathfrak{T}_{P})$ having the property $\boldsymbol{P}$, therefore $(X, \mathfrak{T}_{P})=(X, (\mathfrak{T}_{P})_{P})$.

We shall call $(X, \mathfrak{T})$ a *semi local **P**-space*, if for each point x of X there exists $S \in \sum_{P}^{X}$ which contains a nbd of x. Then we obtain

PROPOSITION 2. *If $(X, \mathfrak{T})$ is a semi local **P**-space, then $\mathfrak{T}_{P}=\mathfrak{T}$.*

PROOF. It is sufficient to show that every closed subset K of $(X, \mathfrak{T}_{P})$ is also closed in $(X, \mathfrak{T})$. For $x \in X$, there exist a nbd $U(x)$ of x and an element of $\sum_P^X$, such that $U(x) \subset S$. If $x \notin K$, since $S \cap K$ is clearly closed in $(S, \mathfrak{T})$, there exists a nbd $V(x)$ of x in $(X, \mathfrak{T})$ such that $V(x) \cap S \cap K = \phi$. Now put $W(x) = V(x) \cap U(x)$, then $W(x)$ is a nbd of x in $(X, \mathfrak{T})$ and $W(x) \cap K = V(x) \cap U(x) \cap K = \phi$. Thus K is closed in $(X, \mathfrak{T})$. This completes the proof.

In general the converse of the Proposition 2 is not true but we can find a necessary and sufficient condition for $(X, \mathfrak{T}) = (X, \mathfrak{T}_P)$ in case $\boldsymbol{P}$ has the following property (A): A space consisting of only one point has the property $\boldsymbol{P}$, and every continuous image of a space with the property $\boldsymbol{P}$ has also the property $\boldsymbol{P}$. For the purpose we shall use the following notion. Let P and $(X, \mathfrak{T})$ be topological spaces and f a continuous map of P onto $(X, \mathfrak{T})$. Then we shall call $(X, \mathfrak{T})$ the *identification of P by f* if and only if $\mathfrak{T}$ is compatible with the identification topology by f. Then we obtain

Proposition 3. *Let $\boldsymbol{P}$ have the property* (A). *We have $(X, \mathfrak{T}) = (X, \mathfrak{T}_P)$ if and only if $(X, \mathfrak{T})$ is an identification of a semi local $\boldsymbol{P}$-space by a suitably defined map f of P onto $(X, \mathfrak{T})$.*

Proof. Sufficiency: Let f be the map of P onto $(X, \mathfrak{T})$ and let H be an open subset in $(X, \mathfrak{T}_P)$. Now if $f^{-1}(H) \ni p$ and $f(p) = x \in X$, then there exist a nbd $V(p)$ of p in P and $S \in \sum_P^P$ such that $V(p) \subset S$. Since $f(S) \in \sum_P^X, f(S) \cap H$ is open in $(f(S), \mathfrak{T})$ and $f(S) \cap H \ni x$, moreover there exist a nbd $U(x)$ of x in $(X, \mathfrak{T})$ such that $U(x) \cap f(S) \subset f(S) \cap H$ and a nbd $U(p)$ of p in P such that $f(U(p) \cap S) \subset U(x) \cap f(S) \subset f(S) \cap H$. On the other hand we have $f(U(p) \cap V(p) \subset f(U(p) \cap S) \subset f(S) \cap H \subset H$. Thus H is open in $(X, \mathfrak{T})$.

Necessity: We shall construct P and f for a given $(X, \mathfrak{T})$ as follows: P consists of all pairs (S, x) where $S \in \sum_P^X, x \in S$. For fixed S, the map $(S, x) \to x$ is to be a homeomorphism onto S (with the given topology), and the set $\{(S, x), x \in S\}$ for fixed S is to be both open and closed in P. This clearly defines a topology. Let $f: P \to X$ be the map $(S, x) \to x$. Then P is clearly a semi local $\boldsymbol{P}$-space and it is easy to show that f is continuous and onto. If $(X, \mathfrak{T}) = (X, \mathfrak{T}_P)$, $(X, \mathfrak{T})$ is clearly the identification of P by f. This completes the proof.

Now we shall prove an analogous theorem to D. E. Cohen's one [2]. The property $\boldsymbol{P}$ is called *productive*, if the product space of two topological space with the property $\boldsymbol{P}$ has also the property $\boldsymbol{P}$. Then

Proposition 4. *If $\boldsymbol{P}$ is productive and satisfies (A) and furthermore $(Y, \mathfrak{T}')$ is a Hausdorff, locally compact and semi local $\boldsymbol{P}$-space, then $(X \times Y, \mathfrak{T}_P \times \mathfrak{T}'_P) = (X \times Y, (\mathfrak{T} \times \mathfrak{T}')_P)$.*

Proof. We shall consider another topology $\mathfrak{T}^*$ on the point set $X \times Y$ which is defined as follows: A subset A of $X \times Y$ is closed for $\mathfrak{T}^*$ if and only if $A \cap (S \times S')$ is closed in $(S \times S', \mathfrak{T} \times \mathfrak{T}')$ for every $S \in \sum_P^X, S' \in \sum_P^Y$ and prove $\mathfrak{T}_P \times \mathfrak{T}'_P = \mathfrak{T}^* = (\mathfrak{T} \times \mathfrak{T}')_P$. First we shall show $\mathfrak{T}^* = (\mathfrak{T} \times \mathfrak{T}')_P$. Let H be an open subset of $(X \times Y, (\mathfrak{T} \times \mathfrak{T}')_P)$ and $S \in \sum_P^X, S' \in \sum_P^Y$. Since $S \times S' \in \sum_P^{X \times Y}$ by the productivity of $\boldsymbol{P}, H \cap (S \times S')$ is open in $(S \times S', \mathfrak{T} \times \mathfrak{T}')$, therefore H is open in $(X \times Y, \mathfrak{T}^*)$. Conversely let H be an open subset of $(X \times Y, \mathfrak{T}^*)$ and $T \in \sum_P^{X \times Y}$. If we denote the image of T by the natural projection onto each factor space X, Y with $P_X(T), P_Y(T)$ respectively, then $H \cap (P_X(T) \times P_Y(T))$ is open in $(P_X(T) \times P_Y(T), \mathfrak{T} \times \mathfrak{T}')$ by the definition of $\mathfrak{T}^*$ and the property (A) of $\boldsymbol{P}$. Now let (x_0, y_0) be any point of $H \cap T$, then there exist a nbd $U(x_0)$ of x_0 in $(X, \mathfrak{T})$ and a nbd $U(y_0)$ of y_0 in $(Y, \mathfrak{T}')$ such that $H \cap (P_X(T) \times P_Y(T)) \supset (P_X(T) \cap U(x_0) \times P_Y(T) \cap U(y_0))$, therefore $H \cap T = H \cap T \cap (P_X(T) \times P_Y(T)) \supset T \cap (P_X(T) \cap U(x_0) \times P_Y(T) \cap U(y_0)) \ni (x_0, y_0)$. Since $T \cap (P_X(T) \cap U(x_0) \times P_Y(T) \cap U(y_0))$ is a nbd of (x_0, y_0) in $(T, \mathfrak{T} \times \mathfrak{T}'), H$ is open in $(X \times Y, (\mathfrak{T} \times \mathfrak{T}')_P)$.

Secondly we shall show $\mathfrak{T}_P \times \mathfrak{T}'_P = \mathfrak{T}^*$. Let H be an open subset of $(X \times Y, \mathfrak{T}_P \times \mathfrak{T}'_P)$ and $S \in \sum_P^X, S' \in \sum_P^Y$. If (x_0, y_0) is a point of $H \cap (S \times S')$, there exist a nbd $U(x_0)$ of x_0 in $(X, \mathfrak{T}_P)$ and a nbd $U(y_0)$ of y_0 in $(Y, \mathfrak{T}'_P)$ such that $H \supset (U(x_0) \times U(y_0))$. On the other hand $(x_0, y_0) \in (U(x_0) \cap S \times U(y_0) \cap S') \subset H \cap (S \times S')$ and $(U(x_0) \cap S \times U(y_0) \cap S')$ is open in $(S \times S', \mathfrak{T} \times \mathfrak{T}')$ by the definition of $\mathfrak{T}_P, \mathfrak{T}'_P$, hence H is open in $(X \times Y, \mathfrak{T}^*)$. Conversely let H be an open subset of $(X \times Y, \mathfrak{T}^*)$ and (x_0, y_0) be any point of H. If we put $(x_0 \times Y) \cap H = x_0 \times \tilde{H}, \tilde{H}$ is open in $(Y, \mathfrak{T}'_P)$, because for every $S' \in \sum_P^Y$, we have $x_0 \times (\tilde{H} \cap S') = (x_0 \times \tilde{H}) \cap (x_0 \times S') = (x_0 \times Y) \cap H \cap (x_0 \times S') = H \cap (x_0 \times S')$, so that $x_0 \times \tilde{H} \cap x_0 \times S'$ is open in $((x_0 \times S'), \mathfrak{T} \times \mathfrak{T}')$. By the assumption on $(Y, \mathfrak{T}')$ there exist a nbd $W(y_0)$ of y_0 in $(Y, \mathfrak{T}')$ and $S' \in \sum_P^Y$ such that $\tilde{H} \supset \overline{W(y_0)}$ and $\overline{W(y_0)}$ is compact and $\overline{W(y_0)} \subset S'$. Now we put $H^* = (x \mid x \times \overline{W(y_0)} \subset H, x \in X)$, then for $S \in \sum_P^X$, $(H^* \cap S \times \overline{W(y_0)}) \subset H \cap (S \times S')$ and $H \cap (S \times S')$ is open in $(S \times S', \mathfrak{T} \times \mathfrak{T}')$. Since $\overline{W(y_0)}$ is compact, H^* is open in $(X, \mathfrak{T}_P)$ and $H \supset H^* \times W(y_0) \ni (x_0, y_0)$, thus H is open in $(X \times Y, \mathfrak{T}_P \times \mathfrak{T}'_P)$. This completes the proof.

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References

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