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On functions starlike in one direction.

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§ 1. Introduction.

Let a function

$$f(z)=\sum_{n=1}^{\infty} a_n z^n \qquad (a_1=1) \tag{1.1}$$

be regular in the unit circle. It is well known that (a) if $f(z)$ is starlike with respect to the origin in the unit circle, all the partial sums of the form

$$\sum_{n=1}^{m} a_n z^n \qquad (1\leqq m\leqq\infty) \tag{1.2}$$

are starlike with respect to the origin for $|z|<1/4$ and convex for $|z|<1/8$, (b) if $f(z)$ is convex in the unit circle, all the partial sums of the form (1.2) are starlike with respect to the origin for $|z|<1/2$ and convex for $|z|<1/4$, and (c) these bounds are all sharp [1], [2].

In this paper we shall consider, in § 4, the partial sums of the form

$$\sum_{n=0}^{m} a_{2n+1} z^{2n+1} \qquad (0\leqq m\leqq\infty), \tag{1.3}$$

which consists of the odd terms, and we shall show the following: (1) If $f(z)$ is starlike with respect to the origin in the unit circle, then the partial sum $\sum_{n=0}^{\infty} a_{2n+1}z^{2n+1}$ is starlike with respect to the origin for $|z|<(3-2\sqrt{2})^{1/2}$ and convex for $|z|<(11-2\sqrt{30})^{1/2}$, and all the partial sums of the form (1.3) are starlike with respect to the origin for $|z|<1/3$ and convex for $|z|<1/3\sqrt{3}$. (2) If $f(z)$ is convex in the unit circle, then the partial sum $\sum_{n=0}^{\infty} a_{2n+1}z^{2n+1}$ is starlike with respect to the origin for $|z|<1$ and convex for $|z|<(3-2\sqrt{2})^{1/2}$, and all the partial sums of the form (1.3) are starlike with respect to the origin for $|z|<1/\sqrt{3}$ and convex for $|z|<1/3$. (3) These bounds are all sharp.

In §§ 2 and 3, we shall study k-fold symmetric functions starlike in the direction of one ray for general k and $k=2$ respectively, and in § 4 we shall prove the above statement by using the results of §§ 2 and 3.

§ 2. k-fold symmetric functions starlike in the direction of one ray.

A function $f(z)$ is said to be starlike in k symmetric directions, if either (1) $f(z)$ is regular in $|z|\leqq 1$ and does not vanish on $|z|=1$, and the image curve of $|z|=1$ by $f(z)$ cuts each ray $\arg w=\alpha+2n\pi/k$ $(n=0,1,\cdots,k-1)$ in exactly one point, or (2) $f(z)$ is regular in $|z|<1$ and for every $\rho(<1)$ sufficiently close to one, the function $f(\rho z)$ satisfies the conditions of (1).

In particular, when $k=1$ or 2, $f(z)$ is said to be starlike in the direction of one ray or one straight line, and these functions were introduced by S. Ogawa and the author [4] and M. S. Robertson [5] respectively.

Now, let a k-fold symmetric function $f_k(z)=\sum_{n=0}^{\infty} a_{kn+1}z^{kn+1}$ $(a_1=1)$ be starlike in the direction of one ray, then $f_k(z)$ is necessarily starlike in k symmetric directions. And if we put

$$f(z^k)=f_k(z)^k, \tag{2.1}$$

then $f(z)$ is evidently starlike in the direction of one ray.

Recently the author [3], [11] investigated certain classes of functions containing the above ones. The following properties of $f_k(z)$ are special ones of the results obtained there for a more general function.

Lemma 1. *Let a k-fold symmetric function $f_k(z)=\sum_{n=0}^{\infty} a_{kn+1}z^{kn+1}$ $(a_1=1)$ be regular in the unit circle and starlike in the direction of one ray, then we have for $|z|=r<1$*

$$\left|z\frac{f_k'(z)}{f_k(z)}-\frac{1+r^{2k}}{1-r^{2k}}\right|\leqq\frac{6r^k}{1-r^{2k}}, \tag{2.2}$$

$$r\frac{(1-r^k)^{2/k}}{(1+r^k)^{4/k}}\leqq|f_k(z)|\leqq r\frac{(1+r^k)^{2/k}}{(1-r^k)^{4/k}}, \tag{2.3}$$

$$\frac{(1-r^k)^{-1+2/k}}{(1+r^k)^{1+4/k}}(1-6r^k+r^{2k})\leqq|f_k'(z)|\leqq\frac{(1+r^k)^{-1+2/k}}{(1-r^k)^{1+4/k}}(1+6r^k+r^{2k}), \tag{2.4}$$

$$\frac{1-6r^k+r^{2k}}{1-r^{2k}}\leqq\left|z\frac{f_k'(z)}{f_k(z)}\right|\leqq\frac{1+6r^k+r^{2k}}{1-r^{2k}}, \tag{2.5}$$

$$\frac{1-6r^k+r^{2k}}{1-r^{2k}}\leqq\Re z\frac{f_k'(z)}{f_k(z)}\leqq\frac{1+6r^k+r^{2k}}{1-r^{2k}}. \tag{2.6}$$

The left-hand equalities of (2.4) *and* (2.5) *are attained for $|z|\leqq(3-2\sqrt{2})^{1/k}$ by the function*

$$F_k(z)=z(1+z^k)^{2/k}/(1-z^k)^{4/k}, \tag{2.7}$$

and the other equalities are all attained for $|z|<1$ by the same function.

Furthermore $f_k(z)$ *is starlike with respect to the origin for*

$$|z|<(3-2\sqrt{2})^{1/k}, \tag{2.8}$$

and this bound is sharp, namely the largest possible.

The main purpose of this section is to find the radius of convexity of $f_k(z)$. For this purpose, we further prepare the following lemma.

LEMMA 2. *Let* $f(z)=\sum_{n=1}^{\infty} a_n z^n$ $(a_1=1)$ *be regular in the unit circle and starlike in the direction of one ray, then we have for* $|z|=r<3-2\sqrt{2}$

$$\left|1+z\frac{f''(z)}{f'(z)}-z\frac{f'(z)}{f(z)}\right|\leqq\frac{2(3r-2r^2+3r^3)}{(1-r^2)(1-6r+r^2)}, \tag{2.9}$$

$$p\Re\left\{1+z\frac{f''(z)}{f'(z)}-\left(1-\frac{1}{p}\right)z\frac{f'(z)}{f(z)}\right\}$$
$$\geqq\frac{1-6(p+2)r+2(2p+19)r^2-6(p+2)r^3+r^4}{(1-r^2)(1-6r+r^2)}, \tag{2.10}$$

where p *is a positive number and both equalities are attained by the function*

$$F(z)=z(1+z)^2/(1-z)^4. \tag{2.11}$$

PROOF. Without loss of generality, we may assume $f(z)$ to be regular in $|z|\leqq 1$. For an arbitrary z such that $0<|z|=r<1$, we put

$$g(\zeta)=rf\left(\frac{\zeta+z}{1+\bar{z}\zeta}\right)\Big/f(z)=r(1+b_1\zeta+b_2\zeta^2+\cdots),\qquad r\neq 0, \tag{2.12}$$

where

$$b_1=\frac{f'(z)}{f(z)}(1-r^2),\quad b_2=\frac{f''(z)}{f(z)}\frac{(1-r^2)^2}{2}-\frac{f'(z)}{f(z)}\bar{z}(1-r^2). \tag{2.13}$$

Then $g(\zeta)$ also is starlike in the direction of one ray. We may further assume, without loss of generality, that this ray is the positive real axis. We denote by $g(e^{i\alpha})$ the point in which the image curve of $|\zeta|=1$ by $g(\zeta)$ cuts the positive real axis, and construct the function

$$h(\zeta)=-e^{-i\alpha}\frac{(e^{i\alpha}-\zeta)^2}{(\zeta+z)(1+\bar{z}\zeta)}g(\zeta).$$

Then $h(\zeta)$ is regular in $|\zeta|\leqq 1$ and the image curve of $|\zeta|=1$ by $h(\zeta)$ does not cut the positive real axis, since

$$-e^{-i\alpha}\frac{(e^{i\alpha}-\zeta)^2}{(\zeta+z)(1+\bar{z}\zeta)}=4\sin^2\frac{\alpha-\theta}{2}\Big/|1+\bar{z}e^{i\theta}|^2\qquad\text{for }\zeta=e^{i\theta}.$$

On the other hand, from the assumption we see by the argument principle that $f(z)$ has only one simple zero at the origin. Therefore $h(\zeta)$ has no zeros in $|\zeta|<1$. Furthermore $h(\zeta)$ has only one double zero on $|\zeta|=1$ at the point $e^{i\alpha}$.

Consequently, if we denote by $\sqrt{h(\zeta)}$ its suitable branch, $\sqrt{h(\zeta)}$ is regular in $|\zeta|\leqq 1$ and there $\Im\sqrt{h(\zeta)}\geqq 0$. We now consider its reciprocal

$$\frac{1}{\sqrt{h(\zeta)}}=c-c\left\{\frac{1}{2}\left(b_1-\frac{1+r^2}{z}\right)-e^{-i\alpha}\right\}\zeta-c\left\{\frac{1}{2}b_2-\frac{3}{8}b_1{}^2+\frac{b_1(1+r^2)}{4z}\right.$$
$$\left.+\frac{1}{2}e^{-i\alpha}\left(b_1-\frac{1+r^2}{z}\right)+\frac{1}{8}\left(\frac{1+r^2}{z}\right)^2-\frac{\bar{z}}{2z}-e^{-2i\alpha}\right\}\zeta^2+\cdots,$$

where $c=(-r^{-1}e^{-i\alpha}z)^{1/2}$.

Since $1/\sqrt{h(\zeta)}$ is regular and $\Im(1/\sqrt{h(\zeta)})<0$ in $|\zeta|<1$, by Carathéodory's theorem we have

$$\left|\frac{1}{2}\left(b_1-\frac{1+r^2}{z}\right)-e^{-i\alpha}\right|\leqq 2, \tag{2.14}$$

$$\left|\frac{1}{2}b_2-\frac{3}{8}b_1{}^2+\frac{b_1(1+r^2)}{4z}+\frac{1}{2}e^{-i\alpha}\left(b_1-\frac{1+r^2}{z}\right)\right.$$
$$\left.+\frac{1}{8}\left(\frac{1+r^2}{z}\right)^2-\frac{\bar{z}}{2z}-e^{-2i\alpha}\right|\leqq 2. \tag{2.15}$$

Next, we consider the function

$$H(\zeta)=-e^{-i\alpha}(e^{i\alpha}-\zeta)^2\frac{(\zeta+z)(1+\bar{z}\zeta)}{\zeta^2}g(\zeta).$$

The image curve of $|\zeta|=1$ by $H(\zeta)$ also does not cut the positive real axis, since

$$-e^{-i\alpha}(e^{i\alpha}-\zeta)^2\frac{(\zeta+z)(1+\bar{z}\zeta)}{\zeta^2}=4\sin^2\frac{\alpha-\theta}{2}|1+\bar{z}e^{i\theta}|^2 \quad \text{for } \zeta=e^{i\theta}.$$

Furthermore $H(\zeta)$ has two double zeros and one double pole at the points $\zeta=-z, e^{i\alpha}$, and $\zeta=0$ respectively, and has no other zeros and poles in $|\zeta|\leqq 1$. Consequently, if we denote by $\sqrt{H(\zeta)}$ its suitable branch, $\sqrt{H(\zeta)}$ is meromorphic in $|\zeta|\leqq 1$ and $\Im\sqrt{H(\zeta)}\geqq 0$ on $|\zeta|=1$.

On the other hand, $H(\zeta)$ has, in $0<|\zeta|\leqq 1$, the following expansion.

$$\sqrt{H(\zeta)}=\frac{c_1}{\zeta}+c_1\left\{\frac{1}{2}\left(b_1+\frac{1+r^2}{z}\right)-e^{-i\alpha}\right\}+c_1\left\{\frac{1}{2}b_2-\frac{1}{8}b_1{}^2+\frac{b_1(1+r^2)}{4z}\right.$$
$$\left.-\frac{1}{8}\left(\frac{1+r^2}{z}\right)^2-\frac{1}{2}e^{-i\alpha}\left(b_1+\frac{1+r^2}{z}\right)+\frac{\bar{z}}{2z}\right\}\zeta+\cdots,$$

where $c_1=(-re^{i\alpha}z)^{1/2}$.

Hence by Robertson's lemma [6] we have

$$\left|\frac{1}{2}b_2-\frac{1}{8}b_1{}^2+\frac{b_1(1+r^2)}{4z}-\frac{1}{8}\left(\frac{1+r^2}{z}\right)^2\right.$$
$$\left.-\frac{1}{2}e^{-i\alpha}\left(b_1+\frac{1+r^2}{z}\right)+\frac{\bar{z}}{2z}\right|-1\leqq 2\left|\frac{1}{2}\left(b_1+\frac{1+r^2}{z}\right)-e^{-i\alpha}\right|. \tag{2.16}$$

By combining (2.16) with (2.15), we find

$$\left|b_2-\frac{1}{2}b_1^2+\frac{b_1(1+r^2)}{2z}-e^{-i\alpha}\frac{1+r^2}{z}-e^{-2i\alpha}\right| \tag{2.17}$$
$$\leqq\left|b_1+\frac{1+r^2}{z}-2e^{-i\alpha}\right|+3.$$

In view of (2.14), we may write

$$(1+r^2)/z=b_1-2e^{-i\alpha}-4\varepsilon,\ |\varepsilon|\leqq 1. \tag{2.18}$$

Substituting this equality in (2.17), we have

$$|b_2-2\varepsilon b_1-2e^{-i\alpha}b_1|\leqq 2|b_1|+16,$$

so that

$$\left|\frac{b_2}{b_1}-\left(\frac{1}{2}b_1-e^{-i\alpha}-\frac{1+r^2}{2z}\right)-2e^{-i\alpha}\right|\leqq 2+\frac{16}{|b_1|},\quad b_1\neq 0.$$

Hence

$$\left|\frac{b_2}{b_1}-\frac{b_1}{2}+\frac{1+r^2}{2z}\right|\leqq 3+\frac{16}{|b_1|},\quad b_1\neq 0.$$

Substituting for b_1 and b_2 from (2.13) in this inequality, we obtain

$$\left|1+z\frac{f''(z)}{f'(z)}-z\frac{f'(z)}{f(z)}\right|\leqq\frac{6r}{1-r^2}+\frac{32r^2}{(1-r^2)^2}\left|\frac{f(z)}{zf'(z)}\right|,\quad f'(z)\neq 0.$$

From this inequality and (2.5) with k replaced by 1, we obtain (2.9) for $|z|=r<3-2\sqrt{2}$.

Next, from (2.9) we have for $|z|=r<3-2\sqrt{2}$

$$p\Re\left(1+z\frac{f''(z)}{f'(z)}-z\frac{f'(z)}{f(z)}\right)\geqq\frac{-2p(3r-2r^2+3r^3)}{(1-r^2)(1-6r+r^2)}.$$

On the other hand, from (2.6) we have for $k=1$

$$\Re z\frac{f'(z)}{f(z)}\geqq\frac{1-6r+r^2}{1-r^2},\quad |z|=r<1.$$

Combining these two inequalities, we obtain (2.10) for $|z|=r<3-2\sqrt{2}$.

Finally, the statement concerning equalities can be easily verified for the function (2.11). Thus we complete the proof.

THEOREM 1. *Let a k-fold symmetric function $f_k(z)=\sum_{n=0}^{\infty}a_{kn+1}z^{kn+1}$ $(a_1=1)$ be regular in the unit circle and starlike in the direction of one ray, then we have for $|z|=r<(3-2\sqrt{2})^{1/k}$*

$$\left|1+z\frac{f_k''(z)}{f_k'(z)}-z\frac{f_k'(z)}{f_k(z)}\right|\leqq\frac{2k(3r^k-2r^{2k}+3r^{3k})}{(1-r^{2k})(1-6r^k+r^{2k})}, \tag{2.19}$$

$$\Re\left(1+z\frac{f_k''(z)}{f_k'(z)}\right)\geqq\frac{1-6(k+2)r^k+2(2k+19)r^{2k}-6(k+2)r^{3k}+r^{4k}}{(1-r^{2k})(1-6r^k+r^{2k})}, \tag{2.20}$$

equalities being attained by the function (2.7). *Accordingly $f_k(z)$ is convex for*

$$|z|<(K-\sqrt{K^2-1})^{1/k},\quad K=(3k+6+\sqrt{9k^2+32k})/2, \tag{2.21}$$

and this bound is sharp.

Proof. From the relation (2.1), we have

$$z\frac{f_k'(z)}{f_k(z)}=z^k\frac{f'(z^k)}{f(z^k)},$$

$$1+z\frac{f_k''(z)}{f_k'(z)}-z\frac{f_k'(z)}{f_k(z)}=k\left(1+z^k\frac{f''(z^k)}{f'(z^k)}-z^k\frac{f'(z^k)}{f(z^k)}\right).$$

Hence by Lemma 2, we obtain (2.19) and (2.20) with equalities for the function (2.7).

§ 3. Odd functions starlike in the direction of one ray.

When $k=2$, the function $f_k(z)$ studied in the preceding section becomes a function of the heading. Obviously this function $f_2(z)$ is necessarily starlike in the direction of diametral line.

Remark. If $f(z)$ is regular in $|z|\leqq 1$, there is at least one straight line such that for a suitable ξ, $|\xi|=1$, the three points $f(\xi), f(-\xi)$, and the origin lie on it. This straight line is called a diametral line of $f(z)$. When $f(z)$ is regular in $|z|\leqq 1$, the meaning that $f(z)$ is starlike in the direction of diametral line is clear. Even when $f(z)$ is regular in $|z|<1$, if for every $\rho(<1)$ sufficiently close to one, the image curve of $|z|=1$ by $f(\rho z)$ cuts a diametral line of $f(\rho z)$ in exactly two points, then $f(z)$ is said to be starlike in the direction of diametral line.

The author [3] showed that if $f(z)=\sum_{n=1}^{\infty} a_n z^n$ $(a_1=1)$ is starlike in the direction of one ray or one straight line, then all the partial sums $\sum_{n=1}^{m} a_n z^n$ $(1\leqq m\leqq\infty)$ are starlike for $|z|<1/12$ or $|z|<1/8$, and convex for $|z|<1/24$ or $|z|<1/16$ respectively, and these bounds are sharp. In this section, we shall study the same kind of problem for the above function $f_2(z)$. We now need the following lemma.

Lemma 3. *Let an odd function $f_2(z)=\sum_{n=0}^{\infty} a_{2n+1}z^{2n+1}$ $(a_1=1)$ be regular in the unit circle and starlike in the direction of one ray, then*

$$f_2(z)\ll F_2(z), \tag{3.1}$$

where

$$F_2(z)=z(1+z^2)/(1-z^2)^2=\sum_{n=0}^{\infty}(2n+1)z^{2n+1}, \tag{3.2}$$

and (3.1) *is sharp.*

PROOF. Since $f_2(z)$ is starlike in the direction of diametral line as stated before, by Ozaki's theorem [7] we have

$$|a_{2n+1}|\leqq 2n+1, \quad n\geqq 0,$$

whence (3.1) holds. And the sharpness is evident from the fact that $F_2(z)$ satisfies the hypotheses of this lemma.

THEOREM 2. *Let an odd function* $f_2(z)=\sum_{n=0}^{\infty} a_{2n+1}z^{2n+1}$ $(a_1=1)$ *be regular in the unit circle and starlike in the direction of one ray, then all the partial sums* $\sum_{n=0}^{m} a_{2n+1}z^{2n+1}$ $(0\leqq m\leqq\infty)$ *are starlike with respect to the origin for*

$$|z|<1/3, \tag{3.3}$$

and convex for

$$|z|<1/3\sqrt{3}. \tag{3.4}$$

These bounds are sharp.

PROOF. (1) Proof for the radius of starlikeness. We denote by $f_{2,m}(z)$ the m-th partial sum of $f_2(z)$ and put

$$f_2(z)=f_{2,m}(z)+R_m(z), \qquad f_{2,m}(z)=\sum_{n=0}^{m-1} a_{2n+1}z^{2n+1}.$$

By virtue of Lemmata 1 and 3, we have, for $m\geqq 3$ and $|z|\leqq r_0=1/3$, the following inequalities.

$$\Re\{zf_2'(z)/f_2(z)\}\geqq(1-6r_0^2+r_0^4)/(1-r_0^4)=0.35,$$

$$|f_2(z)/z|\geqq(1-r_0^2)/(1+r_0^2)^2=0.72,$$

$$|zf_2'(z)/f_2(z)|\leqq(1+6r_0^2+r_0^4)/(1-r_0^4)=1.7,$$

$$|R_m(z)/z|\leqq(1+r_0^2)/(1-r_0^2)^2-(1+3r_0^2+5r_0^4)=0.011\cdots,$$

$$|R_m'(z)|\leqq(1+6r_0^2+r_0^4)/(1-r_0^2)^3-(1+9r_0^2+25r_0^4)=0.081\cdots.$$

Hence we have, for $m\geqq 3$ and $|z|\leqq 1/3$,

$$\Re z\frac{f_{2,m}'(z)}{f_{2,m}(z)}\geqq\Re z\frac{f_2'(z)}{f_2(z)}-\frac{|zf_2'(z)/f_2(z)|\cdot|R_m(z)/z|+|R_m'(z)|}{|f(z)/z|\sim|R_m(z)/z|}>0,$$

which shows that all the partial sums $f_{2,m}(z)$, $m\geqq 3$, are starlike with respect to the origin for $|z|\leqq 1/3$.

Next, we consider the second partial sum $f_{2,2}(z)=z+a_3z^3$. Since $|a_3|\leqq 3$, we have

$$\Re z\frac{f_{2,2}'(z)}{f_{2,2}(z)}\geqq 1-\frac{2|a_3z^2|}{1-|a_3z^2|}>0 \qquad \text{for } |z|<1/3,$$

whence $f_{2,2}(z)$ is also starlike with respect to the origin for $|z|<1/3$. Furthermore it is easily verified that the bound 1/3 can not be replaced by any larger number for the second partial sum of the function (3.2).

(2) Proof for the radius of convexity. From (2.19) and (2.2), we have for $|z|=r<(3-2\sqrt{2})^{1/2}$

$$\left|z\frac{f_2''(z)}{f_2'(z)}\right|\leqq\frac{2r^2(9-21r^2+3r^4+r^6)}{(1-r^4)(1-6r^2+r^4)}, \tag{3.5}$$

and the right-hand side is monotone-increasing as r increases in $0\leqq r\leqq 1/3\sqrt{3}$.

By virtue of Lemmata 1, 3, Theorem 1, and (3.5), we have, for $m\geqq 3$ and $|z|\leqq r_1=1/3\sqrt{3}$, the following inequalities.

$$\Re\left(1+z\frac{f_2''(z)}{f_2'(z)}\right)\geqq\frac{1-24r_1^2+46r_1^4-24r_1^6+r_1^8}{(1-r_1^4)(1-6r_1^2+r_1^4)}=0.22\cdots,$$

$$\left|z\frac{f_2''(z)}{f_2'(z)}\right|\leqq\frac{2r_1^2(9-21r_1^2+3r_1^4+r_1^6)}{(1-r_1^4)(1-6r_1^2+r_1^4)}=0.78\cdots,$$

$$|f_2'(z)|\geqq(1-6r_1^2+r_1^4)/(1+r_1^2)^3=0.69\cdots,$$

$$|R_m'(z)|\leqq\frac{1+6r_1^2+r_1^4}{(1-r_1^2)^3}-(1+9r_1^2+25r_1^4)=0.0026\cdots,$$

$$|zR_m''(z)|\leqq\frac{2r_1^2(9+14r_1^2+r_1^4)}{(1-r_1^2)^4}-(18r_1^2+100r_1^4)=0.016\cdots.$$

Hence we have, for $m\geqq 3$ and $|z|\leqq 1/3\sqrt{3}$,

$$\Re\left(1+z\frac{f_{2,m}''(z)}{f_{2,m}'(z)}\right)\geqq\Re\left(1+z\frac{f_2''(z)}{f_2'(z)}\right)-\frac{|zf_2''(z)/f_2'(z)|\cdot|R_m'(z)|+|zR_m''(z)|}{|f_2'(z)|-|R_m'(z)|}>0,$$

which shows that all the partial sums $f_{2,m}(z)$, $m\geqq 3$, are convex for $|z|\leqq 1/3\sqrt{3}$.

Next, just as in (1), it is easily verified that the second partial sum is convex for $|z|<1/3\sqrt{3}$ and this bound is the largest possible. We thus complete the proof.

§ 4. Functions starlike in the direction of diametral line.

In this section, we shall study the partial sums of the form $\sum_{n=0}^{m} a_{2n+1}z^{2n+1}$ $(0\leqq m\leqq\infty)$ for functions of the heading and some ones related to them.

THEOREM 3. *Let $f(z)=\sum_{n=0}^{\infty} a_n z^n$ $(a_1=1)$ be regular in the unit circle and starlike in the direction of diametral line. Then the partial sum $\sum_{n=0}^{\infty} a_{2n+1}z^{2n+1}$ is also starlike in the direction of diametral line, and it is starlike with respect to the origin for*

$$|z|<(3-2\sqrt{2})^{1/2}=0.414\cdots, \tag{4.1}$$

and convex for

$$|z|<(11-2\sqrt{30})^{1/2}=0.213\cdots. \tag{4.2}$$

Furthermore all the partial sums of the form $\sum_{n=0}^{m} a_{2n+1}z^{2n+1}$ $(0\leqq m\leqq\infty)$ *are starlike with respect to the origin for*

$$|z|<1/3, \tag{4.3}$$

and convex for

$$|z|<1/3\sqrt{3}=0.192\cdots. \tag{4.4}$$

These four bounds are all sharp.

PROOF. Without loss of generality, we may assume that $f(z)$ is regular in $|z|\leqq 1$ and starlike in the direction of a diametral line $\overline{f(e^{i\alpha}), f(-e^{i\alpha})}$. Moreover we may assume that this diametral line is the real axis and $\Im f(e^{i\theta})>0$ for $\alpha<\theta<\alpha+\pi$, $\Im f(e^{i\theta})<0$ for $\alpha+\pi<\theta<\alpha+2\pi$. Then we see that

$$\Im\{f(e^{i\theta})-f(-e^{i\theta})\}>0 \quad \text{for } \alpha<\theta<\alpha+\pi,$$

$$\Im\{f(e^{i\theta})-f(-e^{i\theta})\}<0 \quad \text{for } \alpha+\pi<\theta<\alpha+2\pi.$$

Hence the function $f(z)-f(-z)=2\sum_{n=0}^{\infty} a_{2n+1}z^{2n+1}$ is starlike in the direction of the real axis, which is also a diametral line of $f(z)-f(-z)$. Thus the partial sum $\sum_{n=0}^{\infty} a_{2n+1}z^{2n+1}$ is starlike in the direction of diametral line. Accordingly by Lemma 1, Theorems 1 and 2, we obtain all the statements of this theorem with sharpness for the function $z/(1-z)^2$.

From Lemma 1, we have at once the following corollary.

COROLLARY 1. *Let* $f(z)=\sum_{n=0}^{\infty} a_n z^n$ $(a_1=1)$ *be regular in the unit circle and starlike in the direction of diametral line, then we have for* $|z|=r<1$

$$\left|z\frac{f'(z)+f'(-z)}{f(z)-f(-z)}-\frac{1+r^4}{1-r^4}\right|\leqq\frac{6r^2}{1-r^4}, \tag{4.5}$$

$$\frac{2r(1-r^2)}{(1+r^2)^2}\leqq|f(z)-f(-z)|\leqq\frac{2r(1+r^2)}{(1-r^2)^2}, \tag{4.6}$$

$$\frac{2(1-6r^2+r^4)}{(1+r^2)^3}\leqq|f'(z)+f'(-z)|\leqq\frac{2(1+6r^2+r^4)}{(1-r^2)^3}. \tag{4.7}$$

The left-hand equality of (4.7) *is attained for* $|z|\leqq(3-2\sqrt{2})^{1/2}$ *by the function* $z/(1-z)^2$, *and the other equalities are all attained for* $|z|<1$ *by the same function.*

COROLLARY 2. *Let* $f(z)=\sum_{n=0}^{\infty} a_n z^n$ $(a_1=1)$ *be regular in the unit circle, and let* $f(z)$ *be* (1) *starlike with respect to the origin, or* (2) *typically-real, or* (3) *convex* *in one direction. Then we have the same conclusions as in Theorem* 3 *and Corollary* 1.

Proof. In case $f(z)$ satisfies the condition (1) or (2), $f(z)$ is evidently starlike in the direction of diametral line, and therefore the statement of this corollary is true for these two functions.

Next, we suppose that $f(z)$ satisfies the condition (3). We may assume $f(z)$ to be regular in $|z|\leqq 1$. It is easily seen that there exists a straight line parallel to the direction of convexity which cuts the image curve of $|z|=1$ by $f(z)$ in two points $f(e^{i\alpha}), f(e^{i\beta})$ such that $e^{i\alpha}=-e^{i\beta}$. We take a point w_0 on this straight line, and put $g(z)=f(z)-w_0$. Then $g(z)$ is starlike in the direction of its diametral line $\overline{g(e^{i\alpha}), g(e^{i\beta})}$. By applying Theorem 3 and Corollary 1 to $g(z)$, we see that this corollary is also true for the third function $f(z)$, since $g(z)-g(-z)=f(z)-f(-z)$.

Corollary 3. *Let* $f(z)=\sum_{n=0}^{\infty} a_n z^n$ $(a_1=1)$ *be regular in the unit circle and convex in the direction perpendicular to diametral line. Then the partial sum* $\sum_{n=0}^{\infty} a_{2n+1}z^{2n+1}$ *is convex for* $|z|<(3-2\sqrt{2})^{1/2}$, *and all the partial sums of the form* $\sum_{n=0}^{m} a_{2n+1}z^{2n+1}$ $(0\leqq m\leqq\infty)$ *are convex for* $|z|<1/3$, *and these bounds are sharp. Furthermore we have for* $|z|=r<1$

$$\left| z\frac{f''(z)-f''(-z)}{f'(z)+f'(-z)}-\frac{2r^4}{1-r^4}\right|\leqq\frac{6r^2}{1-r^4}, \tag{4.8}$$

$$\frac{2(1-r^2)}{(1+r^2)^2}\leqq|f'(z)+f'(-z)|\leqq\frac{2(1+r^2)}{(1-r^2)^2}, \tag{4.9}$$

and all equalities are attained by the function $z/(1-z)$.

Proof. In this case, the function $zf'(z)$ is starlike in the direction of diametral line, and therefore from Theorem 3 and Corollary 1, we have the conclusions of this corollary.

Corollary 4. *Let* $f(z)=\sum_{n=0}^{\infty} a_n z^n$ $(a_1=1)$ *be regular in the unit circle, and let* $f(z)$ *be* (1) *convex, or* (2) *typically-real and convex in the direction of the imaginary axis. Then we have the same conclusions as in Corollary* 3.

Proof. This is an immediate consequence of Corollary 3, since each function is convex in the direction perpendicular to diametral line.

For convex functions, we have further the following theorem.

Theorem 4. *Let* $f(z)=\sum_{n=0}^{\infty} a_n z^n$ $(a_1=1)$ *be regular and convex in the unit circle, then the partial sum* $\sum_{n=0}^{\infty} a_{2n+1}z^{2n+1}$ *is starlike with respect to the origin for* $|z|<1$, *and we have, in addition to the conclusions of Corollary* 3, *the following*:

(1) *All the partial sums of the form* $\sum_{n=0}^{m} a_{2n+1}z^{2n+1}$ $(0\leqq m\leqq\infty)$ *are starlike with respect to the origin for*

$$|z|<1/\sqrt{3}, \tag{4.10}$$

and this bound is sharp.

(2) *There hold, for* $|z|=r<1$, *the relations*

$$\left|\frac{f(z)-f(-z)}{z}-\frac{2}{1-r^4}\right|\leqq\frac{2r^2}{1-r^4}, \tag{4.11}$$

$$\left|z\frac{f'(z)+f'(-z)}{f(z)-f(-z)}-\frac{1+r^4}{1-r^4}\right|\leqq\frac{2r^2}{1-r^4}, \tag{4.12}$$

$$\frac{2r}{1+r^2}\leqq|f(z)-f(-z)|\leqq\frac{2r}{1-r^2}, \tag{4.13}$$

and all equalities are attained by the function $z/(1-z)$.

PROOF. We may assume, without loss of generality, that $f(z)$ is regular in $|z|\leqq1$. By means of a geometrical consideration, we see that $\arg\{f(z)-f(-z)\}$ increases monotonously as z traverses on the unit circle in the positive direction. On the other hand, $f(z)-f(-z)$ has only one simple zero in the unit circle at the origin. Hence $f(z)-f(-z)$ is starlike with respect to the origin for $|z|<1$, and therefore from some known properties of odd functions starlike with respect to the origin in the unit circle, we obtain (1) and (2), etc. [8], [9], [10].

The statements concerning sharpness and equalities are easily verified for the function $z/(1-z)$.

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