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On cohomology operations of the second kind.

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Introduction.

Let A, B be abelian groups and $n, p \geqq 1$ be two integers. A cohomology operation $\theta_1(A, B, n, p)$ of the first kind is a function θ_1, defined for every c.s.s. pair (K, L), of the cohomology group $H^n(K, L; A)$ into $H^p(K, L; B)$, which satisfies the naturality condition. Given such a cohomology operation $\theta_1(A, B, n, p)$, an abelian group C and an integer $q \geqq 1$, *a cohomology operation of the second kind relative to* $\{\theta_1(A, B, n, p), C, q\}$[1] is a function

$$\theta_2 : H^n(K, L; A) \supseteqq \mathrm{Ker}(\theta_1) \to H^q(K, L; C)/G_{\theta_2}(K, L),$$

defined for every c.s.s. pair (K, L), of $\mathrm{Ker}(\theta_1)$ into a factor group of $H^q(K, L; C)$ by a subgroup $G_{\theta_2}(K, L)$, where $G_{\theta_2}(K, L)$ are determined by θ_2 in such a way that

$$G_{\theta_2}(K, L) \supseteqq f^* G_{\theta_2}(K', L')$$

for every simplicial map $f: (K, L) \to (K', L')$. Furthermore, we require that θ_2 satisfies the naturality condition, i.e. the following diagram is commutative:

$$\begin{array}{ccc} H^n(K', L'; A) \supseteqq \mathrm{Ker}(\theta_1) & \xrightarrow{\quad f^* \quad} & \mathrm{Ker}(\theta_1) \subseteqq H^n(K, L; A) \\ \downarrow \theta_2 & & \downarrow \theta_2 \\ H^q(K', L'; C)/G_{\theta_2}(K', L') & \xrightarrow{f^*} & H^q(K, L; C)/G_{\theta_2}(K, L). \end{array}$$

The cohomology operations introduced by J. Adem [2], N. Shimada [8] and T. Yamanoshita [9] are of the second kind.

It is well known that there exists a 1—1 correspondence between the cohomology operations relative to $\{A, B, n, p\}$ and the elements of the Eilenberg-MacLane cohomology group $H^p(A, n; B)$ ($\mathrm{n}^\circ 14$, [3]), i.e. in the terminology of J. F. Adams [1], the example-spaces of the first kind[2] examplify the cohomology operations of the first kind. Our purpose of this note is to show that the example-spaces of the first and the second kind examplify the cohomology operations of the second kind defined in the above.

1) Cf. § 3. 6, [6].

2) An example-space of the n-th kind is a space with precisely n non-vanishing homotopy groups and is simple in all dimensions.

1. Preliminalies. A c.s.s. complex X is a direct sum $\sum_{q\geqq 0} X_q$ of free abelian groups together with face and degeneracy opeators $\partial_i: X_q \to X_{q-1}$, $s_i: X_q \to X_{q+1}$ $(0\leqq i\leqq q)$ which are homomorphisms and satisfy the following conditions: (i) For each q, the base of the group X_q is given (the elements of this base are called q-simplices and are denoted by σ_q, ρ_q etc.). (ii) The operators ∂_i and s_i map each simplex into a simplex and satisfy the FD-commutation rules (§ 2, [4]). A simplicial map $f: X\to Y$ of a c.s.s. complex X into another Y is a homomorphism which transforms a q-simplex into a q-simplex for each q and commutes with ∂_i and s_i. Throughout this paper, simplicial maps will be refered to simply as maps. Two maps f and $g: X\to Y$ are called homotopic if there is a map $h: X\times I\to Y$ such that $hk_0=f$, $hk_1=g$, where k_0 and $k_1: X\to X\times I$ are maps of X into the base and the top of $X\times I$ respectively. We shall denote by Δ_n the c.s.s. complex whose p-simplices are $(p+1)$-tuples of integers $(i_0, i_1,\cdots, i_p)$ with $0\leqq i_0\leqq i_1\leqq\cdots\leqq i_p\leqq n$. The operators ∂_i and s_i of Δ_n are defined by the usual manner. The non-degenerate n-simplex will be denoted by the same letter Δ_n.

Let Π be an abelian group and $n\geqq 0$ be an integer. The c.s.s. complex $M(\Pi, n)$ is defined as the complex whose q-simplices are the normalized cochains of $C_N^n(\Delta_q, \Pi)$. The Eilenberg-MacLane complex $K(\Pi, n)$ is a subcomplex of $M(\Pi, n)$. Let X be a c.s.s. complex. For a normalized cocycle $k\in Z_N^{n+1}(X, \Pi)$, the c.s.s. complex $K(X, \Pi, n; k)$ is a subcomplex of the cartesian product $X\times M(\Pi, n)$ defined as follows: For a q-simplex σ of X, there is a unique map $\hat{\sigma}: \Delta_q\to X$ with $\hat{\sigma}(\Delta_q)=\sigma$ and $\hat{\sigma}$ induces $\hat{\sigma}^{\natural}: C^*(X, \Pi)\to C^*(\Delta_q, \Pi)$. Then the q-simplices of $K(X, \Pi, n; k)$ are the q-simplices $(\sigma, \rho)\in X\times M(\Pi, n)$ satisfying $\delta\rho+\hat{\sigma}^{\natural}k=0$ in $C^{n+1}(\Delta_q, \Pi)$. Define a simplicial map $\lambda: K(X, \Pi, n; k)\times K(\Pi, n)\to K(X, \Pi, n; k)$ by $\lambda((\sigma, \rho)\times\rho')=(\sigma, \rho+\rho')$. For maps $f: K\to K(X, \Pi, n; k)$ and $g: K\to K(\Pi, n)$, a map $\lambda(f\times g): K\to K(X, \Pi, n; k)$ is defined by $\lambda(f\times g)(\sigma)=\lambda(f(\sigma)\times g(\sigma))$.

Let A, B be abelian groups and $n\geqq 1, p\geqq 1$ be integers. We put $X=K(K(A, n), B, p-1; k)$, $k\in Z^p(A, n; B)$, and $X'=K(B, p-1)$. Denote by the same letter 1_0 the 0-simplex of X' or $K(A, n)$ defined by $1_0(\Delta_0)=0$. Also, denote by 1_0 the 0-simplex $(1_0, 1_0)$ of X and by D the subcomplex of X, X' or $K(A, n)$ generated by all $1_q=s_{q-1}\cdots s_0 1_0$ with $q\geqq 0$. Let (K, L) be a c.s.s. pair and $f: (K, L)\to(X, D)$ and $g: (K, L)\to(X', D)$ be maps. Define a chain map

$$R(f, g): (K, L)\to X$$

as the composite of three chain maps:

$$(K, L)\xrightarrow{e}(K, L)\times(K, L)\xrightarrow{R(f)\times R(g)}X\times X'\longrightarrow X,$$

where e is the diagonal map, $R(f)\times R(g)$ is the cartesian product of $R(f)$

and $R(g)$ which are defined by

$$R(f)(\rho_q)=f(\rho_q)-1_q\,, \qquad R(g)(\rho_q')=g(\rho_q')-1_q\,,$$

and λ is the map defined in the above. Let C be an abelian group, $q\geqq 1$ be an integer and $\mathfrak{y}\in H^q(X,C)$ be a cohomology class. We shall define an element $\mathfrak{y}(f,g)\in H^q(K,L;C)$ by

$$\mathfrak{y}(f,g)=R(f,g)^*\mathfrak{y}\,.$$

Corresponding to this notation, we shall denote by $\mathfrak{y}(f)$ for the element $R(f)^*\mathfrak{y}$. Since the element $\mathfrak{y}(f,g)$ depends only on the homotopy classes of maps f and g, and the homotopy class of g is determined by the element $\xi=g^*\boldsymbol{b}_{p-1}\in H^{p-1}(K,L;B)$, where $\boldsymbol{b}_{p-1}\in H^{p-1}(B,p-1;B)$ is the basic cohomology class, then we shall denote by $\mathfrak{y}(f,\xi)$ for $\mathfrak{y}(f,g)$. The proofs of the following lemmas are analogous to that of Theorems 7.1 and 10.2 of [5] respectively.

Lemma 1. *(Naturality) Let (K,L) and (K',L') be c. s. s. pairs and $U:(K',L')\to(K,L)$ be a map. Then*

$$U^*(\mathfrak{y}(f,\xi))=\mathfrak{y}(fU,U^*\xi)\,,\quad U^*(\mathfrak{y}(f))=\mathfrak{y}(fU)\,.$$

Lemma 2. *(Additivity) $\mathfrak{y}(\lambda(f\times g))=\mathfrak{y}(f,\xi)+\mathfrak{y}(f)+i^*\mathfrak{y}\vdash\xi$, where $\xi=g^*\boldsymbol{b}_{p-1}$, $i^*: H^q(X,C)\to H^q(X',C)$ is induced by the inclusion map $i:X'\to X$ and $\vdash$ is the operation of Eilenberg-MacLane* [5].

Let $\eta:X\to K(A,n)$ be the projection and $c_{p-1}\in C^{p-1}(X,B)$ be the basic cochain which is defined by $c_{p-1}((\sigma,\rho))=\rho(\varDelta_{p-1})$. A map $f:(K,L)\to(X,D)$ is determined by the map ηf and the cochain $c_f=c_{p-1}f\in C^{p-1}(K,L;B)$ which satisfy the condition:

(1) $$k(\eta f(\sigma_p))+\delta c_f(\sigma_p)=0 \qquad \text{(cf. Lemma 1.1, [7]).}$$

It follows from (1) that, for any two maps f and $f':(K,L)\to(X,D)$ such that $\eta f=\eta f'$, the cochain $z=c_{f'}-c_f$ is a cocycle, and if $g:(K,L)\to(X',D)$ is a map such that $g(\rho_{p-1})(\varDelta_{p-1})=z(\rho_{p-1})$, then $f'=\lambda(f\times g)$. Conversely, for a cocycle $z\in Z^{p-1}(K,L;B)$, if $g:(K,L)\to(X',D)$ is a map such that $g(\rho_{p-1})(\varDelta_{p-1})=z(\rho_{p-1})$, then $c_{\lambda(f\times g)}=c_f+z$. Now, consider the set $\{\mathfrak{y}(\lambda(f\times g),\xi)\}\subseteqq H^q(K,L;C)$ consisting of all elements $\mathfrak{y}(\lambda(f\times g),\xi)$ with a map $g:(K,L)\to(X',D)$. It is easy to see that, if $\boldsymbol{b}_n\in H^n(A,n;A)$ is the basic cohomology class, this set depends only on the cohomology class $\zeta=(\eta f)^*\boldsymbol{b}_n$ and ξ. Then we shall denote by $\mathfrak{y}(\zeta,\xi)$ for this set. The definition of the set $\mathfrak{y}(\zeta)$ is similar.

2. Classification of cohomology operations of the second kind.

Let A,B be abelian groups and $n\geqq 1, p\geqq 1$ be integers. A cohomology operation of the first kind $\theta_1(A,B,n,p)$ is determined by an element $\mathfrak{K}_{\theta_1}\in H^p(A,n;B)$, i. e. for each element $\zeta\in H^n(K,L;A)$, there is a map $f:(K,L)\to$

$(K(A, n), D)$ such that $\zeta=f^*\boldsymbol{b}_n$ and $\theta_1\zeta=\mathfrak{K}_{\theta_1}\vdash\zeta$. We choose a cocycle k_{θ_1} representing $\mathfrak{K}_{\theta_1}$ and construct the complex $X=K(K(A, n), B, p-1; k_{\theta_1})$.

Let C be an abelian group and $\mathfrak{y}\in H^q(X, C)$, $q\geqq 1$, be a cohomology class. For any c. s. s. pair (K, L), we shall define a subgroup $G_{\mathfrak{y}}(K, L)\subseteqq H^q(K, L; C)$ as the subgroup generated by all the sets $\mathfrak{y}(\zeta, \xi)+i^*\mathfrak{y}\vdash\xi$ with elements $\xi\in H^{p-1}(K, L; B)$ and $\zeta\in H^n(K, L; A)$ such that $\theta_1\zeta=0$. From the naturality of the operations $\mathfrak{y}(*, *)$ and $\vdash$, we have

$$f^*G_{\mathfrak{y}}(K', L')\subseteqq G_{\mathfrak{y}}(K, L)$$

for every map $f: (K, L)\to(K', L')$.

Two cohomology classes $\mathfrak{y}$ and $\mathfrak{z}\in H^q(X, C)$ will be called to *be equivalent* if

$$G_{\mathfrak{y}}(X)=G_{\mathfrak{z}}(X) \quad\text{and}\quad \mathfrak{y}-\mathfrak{z}\in G_{\mathfrak{y}}(X).$$

It is clear that this relation is an equivalence relation. We shall denote by $[\mathfrak{y}]$ the equivalent class containing $\mathfrak{y}$ and call it *a characteristic class.*

LEMMA 3. *The group $G_{\mathfrak{y}}(K, L)$ is generated by all elements $\lambda(f\times g)^*\mathfrak{y}-f^*\mathfrak{y}$ with maps $f: (K, L)\to(X, D)$ and $g: (K, L)\to(K(B, p-1), D)$.*

PROOF. Since $\lambda(f\times g)^*\mathfrak{y}=\mathfrak{y}(\lambda(f\times g))$ and $f^*\mathfrak{y}=\mathfrak{y}(f)$, it follows from the additivity formula that $\mathfrak{y}(\lambda(f\times g))-\mathfrak{y}(f)=\mathfrak{y}(f, g)+i^*\mathfrak{y}\vdash g$.

LEMMA 4. *Let $\mathfrak{y}, \mathfrak{z}\in H^q(X, C)$ be two cohomology classes. If $\mathfrak{y}-\mathfrak{z}\in G_{\mathfrak{y}}(X)$, then $G_{\mathfrak{y}}(K, L)=G_{\mathfrak{z}}(K, L)$. Especially, $\mathfrak{y}$ and $\mathfrak{z}$ are equivalent.*

PROOF. This readily follows from Lemma 3, the naturality of $G_{\mathfrak{y}}(K, L)$ and the relation:

$$(\lambda(f\times g)^*-f^*)(\mathfrak{y}+\alpha)=(\lambda(f\times g)^*-f^*)\mathfrak{y}+\lambda(f\times g)^*\alpha-f^*\alpha,$$

for $\alpha\in G_{\mathfrak{y}}(X)$ and maps $f: (K, L)\to(X, D)$ and $g: (K, L)\to(K(B, p-1), D)$.

LEMMA 5. *An element $\mathfrak{y}\in H^q(X, C)$ defines a cohomology operation of the second king relative to $\{\theta_1(A, B, n, p), C, q\}$. If $\mathfrak{y}$ and $\mathfrak{z}\in H^q(X, C)$ are equivalent, they define a same cohomology operation of the second kind.*

PROOF. Define a transformation

$$\theta_2: H^n(K, L; A)\supseteqq \mathrm{Ker}(\theta_1)\to H^q(K, L; C)/G_{\mathfrak{y}}(K, L)$$

by

$$\theta_2(\zeta)=\text{the element of } H^q(K, L; C)/G_{\mathfrak{y}}(K, L) \text{ containing } \mathfrak{y}(\zeta).$$

The naturality of θ_2 is clear. The last proposition follows from Lemma 4.

The cohomology operation θ_2 defined in the proof in the above, which is fully determined by the characteristic class $[\mathfrak{y}]$, is called to be defined by $\mathfrak{y}$ or $[\mathfrak{y}]$.

Let

$$\psi_2: H^n(K, L; A)\supseteqq \mathrm{Ker}(\theta_1)\to H^q(K, L; C)/G_{\psi_2}(K, L)$$

be a cohomology operation of the second kind relative to $\{\theta_1(A, B, n, p), C, q\}$.

We say that ψ_2 is *minimal* if the following condition is satisfied:

(*M*) *If there is a cohomology operation of the second kind* ϕ_2 *relative to* $\{\theta_1(A, B, n, p), C, q\}$ *such that*

$$G_{\phi_2}(K, L) \subseteqq G_{\psi_2}(K, L) \quad \textit{and} \quad \psi_2 = \tau \circ \phi_2 , \tag{2}$$

where $\tau: H^q(K, L; C)/G_{\phi_2}(K, L) \rightarrow H^q(K, L; C)/G_{\psi_2}(K, L)$ *is the factorization homomorphism, then we always have*

$$G_{\phi_2}(K, L) = G_{\psi_2}(K, L) .$$

THEOREM 1. *Let* θ_2 *be a cohomology operation of the second kind relative to* $\{\theta_1(A, B, n, p), C, q\}$. *Then there is a minimal cohomology operation* ϕ_2 *relative to* $\{\theta_1(A, B, n, p), C, q\}$ *such that*

$$G_{\phi_2}(K, L) \subset G_{\theta_2}(K, L) \quad \textit{and} \quad \theta_2 = \tau \circ \phi_2 ,$$

where $\tau: H^q(K, L; C)/G_{\phi_2}(K, L) \rightarrow H^q(K, L; C)/G_{\theta_2}(K, L)$ *is the factorization homomorphism.*

PROOF. Let $\boldsymbol{c} \in H^n(X, C)$ be the cohomology class of the cocycle c which is defined by $c((\sigma, \rho)) = \sigma(\Delta_n)$. Since $\boldsymbol{c} = \eta^* \boldsymbol{b}_n$, it follows from the definition that $\theta_1 \boldsymbol{c} = \eta^* \mathfrak{R}_{\theta_1}$. Since, for each p-simplex $(\sigma, \rho) \in X$, we have $k_{\theta_1}\eta(\sigma, \rho) = k_{\theta_1}(\sigma) = -\delta c_{p-1}((\sigma, \rho))$, where c_{p-1} is the basic cochain of X, then we have $\theta_1 \boldsymbol{c} = 0$. We choose an element $\mathfrak{y} \in \theta_2 \boldsymbol{c}$ and denote by ϕ_2 the cohomology operation defined by $\mathfrak{y}$. Let $f: (K, L) \rightarrow (X, D)$ and $g: (K, L) \rightarrow (K(B, p-1), D)$ be maps. It follows from the naturality of θ_1 that $\theta_1 \zeta = 0$, where $\zeta = (\eta f)^* \boldsymbol{b}_n = (\eta(\lambda(f \times g)))^* \boldsymbol{b}_n$. Furthermore, from the naturality of θ_2, we have

$$\theta_2 \zeta = \theta_2 (\eta f)^* \boldsymbol{b}_n = \theta_2 f^* \eta^* \boldsymbol{b}_n = \theta_2 f^* \boldsymbol{c} = f^* \theta_2 \boldsymbol{c} ,$$
$$\theta_2 \zeta = \lambda(f \times g)^* \theta_2 \boldsymbol{c} .$$

Since $f^* \mathfrak{y} \in f^* \theta_2 \boldsymbol{c}$ and $\lambda(f \times g)^* \mathfrak{y} \in \lambda(f \times g)^* \theta_2 \boldsymbol{c}$, it follows from Lemma 3 that $G_{\mathfrak{y}}(K, L) \subset G_{\theta_2}(K, L)$ and $\theta_2 = \tau \circ \phi_2$. Then the proof is complete from the following lemma.

LEMMA 6. *The cohomology operation* ψ_2 *defined by an element* $\mathfrak{y}$ *of* $H^q(X, C)$ *is minimal.*

PROOF. Since ψ_2 is the cohomology operation defined by $\mathfrak{y}$, we see that

$$\mathfrak{y} \in \psi_2 \boldsymbol{c} , \tag{3}$$

from the definition. Let ϕ_2 be the cohomology operation of the second kind satisfying the condition (2). As was shown in the proof of Theorem 1, the cohomology operation θ_2 defined by an element $\mathfrak{z} \in \phi_2 \boldsymbol{c}$ satisfies the condition

$$G_{\mathfrak{y}}(K, L) \subseteqq G_{\phi_2}(K, L) \quad \text{and} \quad \phi_2 = \tau \circ \theta_2 . \tag{4}$$

It follows from (3), (4) and $\mathfrak{z} \in \theta_2 \boldsymbol{c}$ that $\mathfrak{y} - \mathfrak{z} \in G_{\mathfrak{y}}(X)$. Then, from Lemmas 4 and 5, $\mathfrak{y}$ and $\mathfrak{z}$ are equivalent and $G_{\mathfrak{y}}(K, L) = G_{\mathfrak{z}}(K, L)$. This completes the proof.

The following theorem follows from Lemmas 5 and 6.

THEOREM 2. *There exists a* 1–1 *correspondence between the minimal cohomology operations relative to* $\{\theta_1(A, B, n, p), C, q\}$ *and the characteristic classes of elements of* $H^q(K(K(A, n), B, p-1\,;\, k_{\theta_1}), C)$.

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