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Local theory of rings of operators II.

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In the previous paper [1] we introduced a generalization $\sharp$ of the natural supporter $\natural$ of J. Dixmier [2] (cf. [1], Prop. 2.6–2.8) and studied the local theory concernining elements of AW^*-algebras (cf. [1], §§1–3; especially Prop. 3.7).

In this paper, we shall prove the following two theorems, as applications of these results.

THEOREM I. *Let R be a semi-finite AW^*-algebra acting on a Hilbert space $\mathfrak{H}$ satisfying the condition that every point of norm one of $\mathfrak{H}$ is p-normal in the sense of J. Feldman* [3]*) *considered as a state of R. Then R is a W^*-algebra acting on the same space $\mathfrak{H}$.*

This is a generalization of a theorem of J. Feldman [3], Theorem 1; we deal here with semi-finite AW^*-algebras, whereas J. Feldman dealt only with finite ones; it is not yet known whether the condition of semi-finiteness is also redundunt or not.

THEOREM II. *Let R_i $(i=1,2)$ be AW^*-algebras acting on a Hilbert space $\mathfrak{H}_i$ satisfying the conditions:* (1) *the unit element of R_i is the identity operator on $\mathfrak{H}_i$, and* (2) *every point of norm one of $\mathfrak{H}_i$ is p-normal considered as a state of R_i. Let φ be an algebraic $*$-isomorphism of R_1 onto R_2 in the sense of J.v. Neumann* [4]. *Suppose the commutants R_1', R_2' of R_1, R_2 respectively are normally infinite in the sense of* [1]. *Then φ is spacial (i.e. φ is written as $\varphi(c_1)=uc_1u^*$ for all $c_1\in R_1$, u being a linear isometry mapping $\mathfrak{H}_1$ onto $\mathfrak{H}_2$), if and only if there exists an algebraic $*$-isomorphism of R_1' onto R_2' whose restriction on the center R_{10} of R_1 coincides with that of φ on R_{10}.*

This is a generalization of a theorem of Y. Misonou [5]. We shall give a direct proof of this theorem and an alternative proof for the case where R_i are W^*-algebras. This latter proof is derived from (the local form of) a result of E.L. Griffin ([6] Theorem 9, [7] Theorem 3) giving a ecessary and sufficient condition for an algebraic $*$-isomorphism between essentially bounded W^*-algebras R_1, R_2 to be spacial. We shall give also a proof from

*) A point f of $\mathfrak{H}$ is called p-normal after [3], if for any orthogonal system $(e_\iota; \iota\in I)$ of projections of R we have $(\bigoplus(e_\iota; \iota\in I)f, f)=\sum((e_\iota f, f); \iota\in I)$,

our standpoint for this result of E. L. Griffin's, which we shall call Theorem III, as well as for a more general theorem (due to R. Kadison [16], dropping the condition of essential boundedness), which will be called Theorem IV. (Our Theorem II itself may be also obtained along the lines of E. L. Griffin's, but the proof would be more complicated to describe.)

In § 1, we introduce the notion of "mixed relative dimension". It is defined for two AW^*-algebras R_1, R_2 acting on Hilbert spaces $\mathfrak{H}_1, \mathfrak{H}_2$ respectively, when they are algebraically $*$-isomorphic, and it is determined by the algebraic $*$-isomorphism φ of R_1 onto R_2. Next we reestablish the theory of "qualitative comparison of $\mathfrak{M}_f^{M'}$ and $\mathfrak{M}_f^{M}$" due to F. J. Murray and J. v. Neumann [8], Chap. IX in making no use of infinite operator theoretical method and finally we prove Theorem I with the help of a result of J. Feldman [3], Theorem 1.

In § 2, we first give a direct proof of Theorem II. Next we reestablish the theory of the coupling operator due to E. L. Griffin [6], [7] with the aid of our local theory. Then we prove the local form (in the sense of [1]) Theorem III′ of Theorem III, from which Theorem III and the W^*-case of Theorem II follow. Finally we shall prove Theorem IV.

Throughout this paper we use terminologies in [1] without further reference.

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§ 1. Mixed relative dimension.

We introduce the following

Definition 1.1. *An* $\boldsymbol{AW^*}$*-algebra is a pair* $(R, \mathfrak{H})$ *formed by a Hilbert space* $\mathfrak{H}$ *and an* AW^**-algebra acting on* $\mathfrak{H}$ *satisfying the following conditions:*

(1.1) *the unit* 1 *of* R *is the identity operator of* $\mathfrak{H}$,

(1.2) *for any point* f *of* $\mathfrak{H}$ *and for any orthogonal system* $(e_\iota; \iota\in I)$ *of projections of* R *we have* $(\bigoplus(e_\iota; \iota\in I)f, f)=\sum((e_\iota f, f); \iota\in I)$,

where we denote by (f, g) *the inner product of points* f, g *of* $\mathfrak{H}$.

Let $\boldsymbol{R}$ be an $\boldsymbol{AW^*}$-algebra formed by $\mathfrak{H}$ and R. We call $\mathfrak{H}$, R the underlying Hilbert space, and the underlying AW^*-algebra of $\boldsymbol{R}$ respectively. We denote by R_0 the center of R and call $\boldsymbol{R}_0$ $(=((R_0, \mathfrak{H}))$ the center of $\boldsymbol{R}$. Denote by $\|f\|$ the norm of a point f of $\mathfrak{H}$ and by the same f the state of R defined by $f(a)=(af, f)$ for all $a\in R$.

Lemma 1.1. *Let* $\boldsymbol{R}$ *be an* $\boldsymbol{AW^*}$*-algebra and* $\mathfrak{H}$ *be its underlying Hilbert space. Then we have* $af=0$ *if and only if* $e_*(a)f=0$ $(a\in R, f\in\mathfrak{H})$.

Proof. Sufficiency: Since $a=ae_*(a)$, we have $af=a(e_*(a)f)=0$. Necessity: Denote by $(e_\alpha; 0\leqq\alpha<\infty)$ the resolution of the unit of a^*a. From $\alpha e_\alpha^c\leqq a^*a$ $(\alpha\geqq 0)$ it follows that $e_\alpha^c f=0$ for $\alpha>0$. As making $\alpha\downarrow 0$, we get $e_0^c f=0$. Since $e_0^c=e_*(a)$, we obtain our lemma. q. e. d.

Lemma 1.2. *Let $\boldsymbol{R}$ be an $\boldsymbol{AW}^*$-algebra and f be a point of $\mathfrak{H}$. If we denote by M_f the set of operators a's of R with $af=0$, then we have $M_f=Re_f$ for some (uniquely determined) projection e_f of R.*

Proof. It is easy to see that M_f is a closed left ideal of R. Denote by E_f a maximally orthogonal system of projections of M_f and by e_f the supremum of E_f. Then we can see $e_f\in M_f$ by (1.2). If $a\in M_f$, then we have $ae_f^c\in M_f$. Hence we have $e_*(ae_f^c)f=0$ by Lemma 1.1. By the maximality of E_f, we have $e_*(ae_f^c)=0$, that is, $ae_f^c=0$. Thus we get the desired equality $M_f=Re_f$. q. e. d.

The complement e_f^c of e_f is called the *supporter* of f in R and denoted by $e(f)$. Similarly the supporter f in the center R_0 of R is called the *central supporter* of f in R and denoted by $e_0(f)$. We say that a projection e of R fixes a point f of $\mathfrak{H}$ if $ef=f$.

Lemma 1.3. 1) *$e(f)$ is the minimal projection of R fixing a point f of $\mathfrak{H}$.*

2) *$e_0(f)$ is the minimal projection of R_0 fixing a point f of $\mathfrak{H}$.*

3) $e_0(f)=e(f)^\natural$.

4) $e(af)=e(ae(f))$.

Proof. Proof of 1). We have $e(f)f=f$, because $e(f)^c f=0$. On the other hand, from $ef=f$ it follows that $e^c f=0$, that is, $e^c e(f)=0$ by Lemma 1.2. Thus we get $e(f)\leqq e$.

Proof of 2) is similar.

Proof of 3). We have $e(f)^\natural f=f$, because $e(f)\leqq e(f)^\natural$. Hence it holds that $e_0(f)\leqq e(f)^\natural$. On the other hand, since $e_0(f)f=f$, we have $e(f)\leqq e_0(f)$. Hence we see $e(f)^\natural\leqq e_0(f)$. Thus we get 3).

Proof of 4). We have $e(af)\leqq e(ae(f))$, because $e(ae(f))af=e(ae(f))ae(f)f$ $=af$. Further, denoting $e(af)$ briefly by e, we have $e^c af=0$. Hence we have $e_*(e^c a)f=0$ by Lemma 1.1, that is, $e_*(e^c a)e(f)=0$ by Lemma 1.2. This means that $e^c ae(f)=0$, that is, $e(ae(f))\leqq e$ by the definition of $e(ae(f))$. Thus we get the desired equality 4). q. e. d.

Let $\boldsymbol{R}_i$ $(i=1,2)$ be $\boldsymbol{AW}^*$-algebras and φ be an algebraic $*$-isomorphism of R_1 onto R_2. We denote by R_i' the commutant of R_i on $\mathfrak{H}_i$ and by $\boldsymbol{R}_i'$ the $\boldsymbol{AW}^*$-algebra formed by $\mathfrak{H}_i$ and R_i'. We shall say that $\boldsymbol{R}_i'$ is the commutant of $\boldsymbol{R}_i$. Denote by A the set of operators a's mapping $\mathfrak{H}_1$ into $\mathfrak{H}_2$ satisfying $ac_1=\varphi(c_1)a$ for all $c_1\in R_1$. We use the same notations as in [1], § 2. (As AW^*-algebras R_i, R in [1], § 2, we take R_i', the full algebra of operators on $\mathfrak{H}_1\oplus\mathfrak{H}_2$ respectively.) Then it can be seen that $a^*b\in R_1'$, $ab^*\in R_2'$, and $R_2'aR_1'$

$\subseteqq A$ for $a, b \in A$. Denote by I_i the unit of R_i, by d_i the relative dimension of R_i, and by d_i' (or d_{ii}') that of R_i'. Moreover we denote by d_{12}' $(=d_{21}')$ the relative dimension of R_1' into R_2' and we write $d_{12}'(e_1') \leqq d_{12}'(e_2')$ (or $d_{12}'(e_1') \geqq d_{12}'(e_2')$) for projections e_i' of R_i' if there exists an operator a of A satisfying $e_*(a)=e_1'$ and $e(a) \leqq e_2'$ (or satisfying $e_*(a) \leqq e_1'$ and $e(a)=e_2'$). We write $d_{12}'(e_1')=d_{12}'(e_2')$ if $d_{12}'(e_1') \leqq d_{12}'(e_2')$ and $d_{12}'(e_1') \geqq d_{12}'(e_2')$. Then $d_{12}'(e_1')=d_{12}'(e_2')$ holds if and only if there exists an operator a of A satisfying $e_*(a)=e_1'$ and $e(a)=e_2'$. The relative dimensions d_{11}', d_{22}', and d_{12}' $(=d_{21}')$ satisfy the following properties: (1) three conditions $d_{11}'(e_1') \leqq d_{11}'(e_1^{(1)\prime})$, $d_{12}'(e_1^{(1)\prime}) \leqq d_{12}'(e_2^{(1)\prime})$, and $d_{22}'(e_2^{(1)\prime}) \leqq d_{22}'(e_2')$ imply $d_{12}'(e_1') \leqq d_{12}'(e_2')$, (2) three conditions $d_{11}'(e_1') \leqq d_{11}'(e_1^{(1)\prime})$, $d_{12}'(e_1^{(1)\prime}) \leqq d_{12}'(e_2^{(1)\prime})$, and $d_{21}'(e_2^{(1)\prime}) \leqq d_{21}'(e_1^{(2)\prime})$ imply $d_{11}'(e_1') \leqq d_{11}'(e_1^{(2)\prime})$, and the duals of these. (For any property depending on 1, 2, the suffixes of R_1', R_2', we call the property obtained by interchanging the role of 1, 2, its *dual.*) Denoting by E_i' the set of projections of R_i', we can find a mapping d', which carries $E_1' \cup E_2'$ (the union of E_1' and E_2' as point-sets) onto some semi-ordered set, satisfying the following condition:

(1.3) $d'(e_1') \leqq d'(e_2')$ holds if and only if $d_{11}'(e_1') \leqq d_{11}'(e_2')$ for $e_1', e_2' \in E_1'$, $d_{22}'(e_1') \leqq d_{22}'(e_2')$ for $e_1', e_2' \in E_2'$, $d_{12}'(e_1') \leqq d_{12}'(e_2')$ for $e_1' \in E_1', e_2' \in E_2'$, and $d_{21}'(e_1') \leqq d_{21}'(e_2')$ for $e_1' \in E_2', e_2' \in E_1'$.

Obviously the semi-ordered set $d'(E_1' \cup E_2')$ is uniquely determined except for isomorphism as a semi-ordered set. We call d' the *mixed relative dimension* of $\boldsymbol{R}_1'$ and $\boldsymbol{R}_2'$ determined by φ. Sometimes we denote $d'(e_1') \leqq d'(e_2')$ briefly by $e_1' \lesssim e_2'$.

We denote by $e'(f_i)$ the supporter of a point f_i of $\mathfrak{H}_i$ in R_i'.

Proposition 1.1. *$I_1^\# = I_2$ and $I_2^\# = I_1$.*

Proof. We shall prove $I_1 = I_2^\#$. Let us start by denying it. Then we can find a non-zero point f_1'' of $\mathfrak{H}_1$ with $f_1''(I_2^\#)=0$ and a non-zero point f_2' of $\mathfrak{H}_2$ with $f_2'((\varphi(e(f_1'')))^c)=0$. We put $e_2^{(1)}=e(f_2')$, $e_1^{(1)}=\varphi^{-1}(e_2^{(1)})$, and $f_1'=e_1^{(1)}f_1''$. Since $e_2^{(1)}=\varphi(e(f_1''))$, it holds that $e_1^{(1)}=e(f_1'')$. Hence we get $e_1^{(1)}=e(f_1')$ by Lemma 1.3. By a well known method, we may find a projection e_1 of R_1 with $e_1 \leqq e_1^{(1)}$ such that $f_2'(\varphi(e_1^{(2)}))=\theta f_1'(e_1^{(2)})$ for each projection $e_1^{(2)}$ of R_1 with $e_1^{(2)}=e_1$, where θ is a positive constant. We put $e_2=\varphi(e_1)$ and $f_i=e_i f_i'$. Then we have $e_i=e(f_i)$ by Lemma 1.3 and $\|\varphi(c_1)f_2\| \leqq \theta \|c_1 f_1\|$ for each operator c_1 of R_1. We denote by a^0 the module-isomorphism of $R_1 f_1$ onto $R_2 f_2$ defined by $a^0 c_1 f_1 = \varphi(c_1) f_2$ for each operator c_1 of R_1. Then a^0 can be extended to the operator a mapping $\mathfrak{H}_1$ into $\mathfrak{H}_2$ with $e_*(a) \leqq e'(f_1)$ and $e(a)=e'(f_2)$. It is easy to see that $0 \neq a \in A$. Hence we get $0 \neq e_*(a) \leqq I_2^\#$. On the other hand, from $f_1 = e_1 e_1^{(1)} f_1''$ it follows that $0 \neq e_*(a) \leqq e'(f_1'') \leqq I_1 - I_2^\#$. This leads to a contradiction. Thus we get $I_1 = I_2^\#$. Similarly, we obtain $I_2 = I_1^\#$. q. e. d.

Let $\boldsymbol{R}_i$ $(i=1, 2)$ be $\boldsymbol{AW}^*$-algebras and φ be an algebraic $*$-isomorphism of R_1 onto R_2. Denote by $\boldsymbol{R}_{i0}$ the center of $\boldsymbol{R}_i$, by $\boldsymbol{R}_{i0}'$ the center of the commutant $\boldsymbol{R}_i'$ of $\boldsymbol{R}_i$, and by φ_0 the restriction of φ on R_{10}. We notice that $R_{i0}\subseteqq R_{i0}'$, but I can not verify that $R_{i0}=R_{i0}'$. According to Prop. 1.1, $\sharp$ is extended to an algebraic $*$-isomorphism (denoted again by $\sharp$) of R_{10}' onto R_{20}' by [1], Prop. 2.8. Moreover, we have from the proof of [1], Prop. 2.8, $\varphi_0(e_{10})=e_{10}^{\sharp}$ for $e_{10}\in E_{10}$ $(=E_1\cap R_{10})$ and so $\varphi_0(c_{10})=c_{10}^{\sharp}$ for $c_{10}\in R_{10}$. The algebraic $*$-isomorphism $\sharp$ induces a homeomorphism ν' mapping the spectrum Ω_1' of R_{10}' onto the spectrum Ω_2' of R_{20}'. We identify a point λ_1' of Ω_1' with its image $\nu'(\nu_1')$ of Ω_2' by ν' and denote these $\lambda_1', \nu'(\lambda_1)$ by λ'. In this sense, we may consider the local relative dimension $d_{12}',_{\lambda'}$ of R_1' into R_2'. Namely, we say that $d_{12}',_{\lambda'}(e_1')\leqq d_{12}',_{\lambda'}(e_2')$ holds for $e_i'\in E_i'$ if and only if $d_{12}'(e_{10}(\lambda')e_1')\leqq d_{12}'(\varphi^0(e_{10}(\lambda'))e_2')$ holds for some projection $e_{10}(\lambda')$ of $E_{10}(\lambda')$. ($E_{10}(\lambda')$=the set of projections $e_{10}(\lambda')$'s of R_{10}' with $\lambda'(e_{10}(\lambda'))=1$.) By a similar argument as before, we may consider the local relative dimension $d_{\lambda'}'$ between R_1' and R_2'. It is composed of $d_{11,\lambda'}', d_{22,\lambda'}'$, and $d_{12,\lambda'}'$ $(=d_{21,\lambda'}')$. Sometimes we denote $d_{\lambda'}'(e_1')\leqq d_{\lambda'}'(e_2')$ briefly by $e_1'\precsim_{\lambda'}e_2'$.

Proposition 1.2. *The semi-order of $d_{\lambda'}'(E_1'\cup E_2')$ induced by $d_{\lambda'}'$ is linearly ordered.*

Proof. Let e_i' $(i=1,2)$ be an arbitrary projection of R_i'; Then we can find a maximal partial isometry u of A satisfying $e_*(u)\leqq e_1'$ and $e(u)\leqq e_2'$. Then we have $(e_2'-e(u))A(e_1'-e_*(u))=0$. Hence it holds that $(e_2'-e(u))^{\sharp}(e_1'-e_*(u))^{\natural}=0$ by [1], Prop. 2.6. If $\lambda((e_2'-e(u))^{\sharp})=0$, we have $e_2'\precsim_{\lambda'}e_1'$. And, if $\lambda((e_1'-e_*(u))^{\natural})=0$, we get $e_1'\precsim_{\lambda'}e_2'$. This completes the proof combining with [1], Prop. 3.7. q. e. d.

We say that a projection e of R is *cyclic* if it is the supporter of a point of $\mathfrak{H}$. It is not hard to see that every cyclic projection is countably decomposable. (Here we say that a projection e is *countably decomposable* if, for any decomposition $e=\bigoplus(e_\iota; \iota\in I)$, I must be at most countable.) Moreover we have the following

Lemma 1.4. *Let R be an AW^*-algebra and d be its relative dimension. If e_1, e_2 are normally infinite cyclic projections of R with $e_1^{\natural}=e_2^{\natural}$, then we have $d(e_1)=d(e_2)$.*

Proof. By [1], Prop. 3.7, we may assume that $e_1\leqq e_2$. Since e_1 is normally infinite, we have a decomposition $e_1=\bigoplus(e_1^{(n)}; 1\leqq n<\infty)$ with $d(e_1^{(1)})=d(e_1^{(n)}; 1\leqq n<\infty)$ by [1], Prop. 3.2. Then there exist a projection e_0 of R_0 and a decomposition $e_0e_2=\bigoplus(e_0e_1^{(\iota)}; \iota\in I)$ such that $d(e_0e_1^{(\iota)})=d(e_0e_1^{(1)})$ $(\iota\in I)$ and that I contains all natural numbers. Since e_2 is countably decomposable, I is a countable set. Hence we get $d(e_0e_1)=d(e_0e_2)$. This completes the proof, because the equivalence is normal as a property in the sense of [1], §1. q. e. d.

The following lemma is due to F. J. Murray and J. v. Neumann [8], but the present proof uses only bounded linear operators.

Lemma 1.5. *Let* $\boldsymbol{R}$ *be an* $\boldsymbol{AW}^*$*-algebra and* f, f_n $(1 \leqq n < \infty)$ *be points of* $\mathfrak{H}$. *Suppose that* (1) $f_n \to f$ *(strong) and* (2) $d(e(f_n)) \leqq d(e)$ *for some projection* e *of* R. *Then it holds that* (3) $d(e(f)) \leqq d(e)$.

Proof. If $e(f)=0$, then (3) is obvious. Hence we may assume that $e(f) \neq 0$. By taking $e(f)f_n$ instead of f_n, we may assume that $e(f_n) \leqq e(f)$. From the fact that $e(f_n)f \to e(f)f$ it follows that $e(f_n) \to e(f)$ (strong). In fact, it holds that $(bf;\, b \in R')$ is dense in $e(f)\mathfrak{H}$ $(=(e(f)g;\, g \in \mathfrak{H}))$, that $e(f_n)bf \to e(f)bf$, and that $\|e(f_n)\| \leqq 1$ $(1 \leqq n < \infty)$. We shall prove the lemma locally. Moreover we may assume that $e \leqq e(f)$ by [1], Prop. 3.7. First, if e is locally normally infinite, we have $d_\lambda(e) = d_\lambda(e(f))$ by Lemma 1.4, for the supporter of a point is countably decomposable. Next, if $e(f)$ is locally finite, from the fact that $e(f_n) \to e(f)$ (strong) follows that $t(e(f_n)) \to t(e(f))$ locally (strong) by a (similar) theorem of J. Dixmier [2], theorem 17, where we denote by t the trace of R defined locally, whose existstence was proved by Ti. Yen [9]. Hence, in view of (2), we get $t(e(f)) = t(e)$ locally, that is, $d_\lambda(e(f)) = d_\lambda(e)$. Finally, if e is locally finite and if $e(f)$ is locally normally infinite, by taking an arbitrary locally finite projection e' of R with $e' \leqq e(f)$ instead of $e(f)$ and by taking $e'f, e'f_n$ instead of f, f_n respectively, and by repeating the above argument, we can obtain $d_\lambda(e') \leqq d_\lambda(e)$. This is a contradiction. Since the property that $d(e(f)) \leqq d(e)$ is normal as a property in the sense of [1], § 1, we arrive at (3). q. e. d.

The following proposition is a generalization of [8], Lemma 9.3.3.

Proposition 1.3. *Let* $\boldsymbol{R}_i$ $(i=1,2)$ *be* $\boldsymbol{AW}^*$*-algebras and* φ *be an algebraic* $*$*-isomorphism of* R_1 *onto* R_2. *Then* $d(\varphi(e(f_1))) \leqq d(e(f_2))$ *holds if* $d'(e'(f_1)) \leqq d'(e'(f_2))$ *holds.*

Proof. Since $d'(e'(f_1)) \leqq d'(e'(f_2))$, we can find a partial isometry u of A with $e_*(u) = e'(f_1)$ and $e(u) \leqq e'(f_2)$. Put $f_2' = uf_1$. Then we have (1) $e'(f_2') = e(u)$ and (2) $e(f_2') = \varphi(e(f_1))$. In fact, $e_2'f_2' = 0$ $(e_2' \in E_2')$ holds if and only if $e_2'uf_1 = 0$, that is, $u^*e_2'uf_1 = 0$ and so $u^*e_2'ue'(f_1) = 0$ by Lemma 1.3 and then $e_2'ue'(f_1) = 0$. Since $e_*(u) = e'(f_1)$, this implies that $e_2'u = 0$ and so $e_2'e(u) = 0$. This shows (1). Similarly, $\varphi(e_1)f_2' = 0$ $(e_1 \in E_1)$ holds if and only $\varphi(e_1)uf_1 = 0$ and so $ue_1f_1 = 0$ and then $e_*(u)e_1f_1 = 0$ by Lemma 1.1. Since $e_*(u) = e'(f_1)$, this implies that $e_1f_1 = 0$. This shows (2). Since $e'(f_2') \leqq e'(f_2)$, we can find a sequence $(c_n;\, 1 \leqq n < \infty)$ of elements of R_2 such that $c_nf_2 \to f_2'$ (strong). Since $d(e(c_nf_2)) = d(e(c_ne(f_2))) \leqq d(e(f_2))$ by Lemma 1.3, it holds that $d(e(f_2')) \leqq d(e(f_2))$ by Lemma 1.5. From this and from (2) we get the assertion. q. e. d.

Corollary. *Let* $\boldsymbol{R}$ *be an* $\boldsymbol{AW}^*$*-algebra and* f_i $(i=1,2)$ *be points of* $\mathfrak{H}$. *Then* $d(e(f_1)) \leqq d(e(f_2))$ *holds if* $d'(e'(f_1)) \leqq d'(e'(f_2))$ *holds.*

Proof. We consider R, the identity mapping as R_i, φ in Prop. 1.3 respectively. Then we get readily the assertion. q. e. d.

Definition 1.2. *An* $\boldsymbol{AW}^*$*-algebra* $\boldsymbol{R}$ *is called* $\boldsymbol{W}^*$ *if its underlying* AW^**-algebra* R *is a* W^**-algebra acting on its underlying Hilbert space* $\mathfrak{H}$.

Let $\boldsymbol{R}$ be an $\boldsymbol{AW}^*$-algebra. Denote by $\bar{R}$ the weak closure of R on $\mathfrak{H}$. The $\boldsymbol{W}^*$-algebra formed by $\bar{R}$ and $\mathfrak{H}$ is called the weak closure of $\boldsymbol{R}$ and denoted by $\bar{\boldsymbol{R}}$. Denote by $\bar{e}(f)$ the supporter of a point f of $\mathfrak{H}$ in $\bar{R}$; denote by $\bar{d}$ the relative dimension of $\bar{R}$.

Now, we impose the following assumption:

(A) $\bar{d}(\bar{e}(f_1))=\bar{d}(\bar{e}(f_2))$ holds if $d(e(f_1))=d(e(f_2))$ holds $(f_1, f_2 \in \mathfrak{H})$.

It seems to me that this assumption is valid for any $\boldsymbol{AW}^*$-algebra, but I can not verify it. It is obvious that (A) holds for any $\boldsymbol{W}^*$-algebra.

From now on we shall denote by putting * the result which is valid under the assumption (A).

Lemma* 1.6. *Let* $\boldsymbol{R}$ *be an* $\boldsymbol{AW}^*$*-algebra with* (A) *and* f_i *be points of* $\mathfrak{H}$. *Then we have* $\bar{d}(\bar{e}(f_1)) \leqq \bar{d}(\bar{e}(f_2))$ *if* $d(e(f_1)) \leqq d(e(f_2))$ *holds.*

Proof. Since $d(e(f_1)) \leqq d(e(f_2))$, we can find a partial isometry u of R with $e_*(u)=e(f_1)$ and $e(u) \leqq e(f_2)$. Put $f_2'=e(u)f_2$. Then we have $e(f_2')=e(u)$ by Lemma 1.3 and so $d(e(f_2'))=d(e(f_1))$. Hence we have $d'(e'(f_2'))=d'(e'(f_1))$ by (A). On the other hand, it is easy to see that $e'(f_2') \leqq e'(f_2)$. From these it follows that $d'(e'(f_1)) \leqq d'(e'(f_2))$. By taking R' instead of R and by repeating the above argument, from $d'(e'(f_1)) \leqq d'(e'(f_2))$ it follows that $\bar{d}(\bar{e}(f_1)) \leqq \bar{d}(\bar{e}(f_2))$, for $\boldsymbol{R}$ is $\boldsymbol{W}^*$ and so satisfies (A). q. e. d.

Proposition* 1.3. *Let* $\boldsymbol{R}_i$ $(i=1,2)$ *be* $\boldsymbol{AW}^*$*-algebras with* (A)*; let* φ *be an algebraic* $*$*-isomorphism; let* f_i $(i=1,2)$ *be points of* $\mathfrak{H}_i$. *Then following statements are equivalent to each other:*

(1.4) $d(\varphi(e(f_1)))=d(e(f_2))$,

(1.5) $d'(e'(f_1))=d'(e'(f_2))$.

Proof. (1.5) implies (1.4). This implication has already shown in Prop. 1.3.

(1.4) implies (1.5). In order to see (1.4), we need only to see it locally with respect to a spectre λ' of R_i' by making use of [1], Prop. 1.1. Further, if $d_{\lambda'}'(e'(f_1)) \leqq d_{\lambda'}'(e'(f_2))$, we get (1.5) locally and so we may assume that $d_{\lambda'}'(e'(f_2)) \leqq d_{\lambda'}'(e'(f_1))$ by Prop. 1.2. Hence we can find two projections $e_{i0}'(\lambda')$ of $E_{i0}(\lambda')$ $(i=1,2)$ with $e_{10}'(\lambda')^{\#}=e_{20}'(\lambda')$ such that $d'(e_{20}'(\lambda')e'(f_2)) \leqq d'(e_{10}'(\lambda')e'(f_1))$. Write f_i^0 for $e_{i0}'(\lambda')f_i$ and denote by e_{i0} the minimal projection of R_i fixing $e_{i0}'(\lambda')$. Then we have $\varphi_0(e_{10})=e_{20}$, for φ_0 is the restriction of $\#$. Further we have $e(f_i^0)=e_{i0}e(f_i)$ and so $d(\varphi(e(f_1^0))) \leqq d(e(f_2^0))$. Moreover we have $e'(f_i^0)=e_{i0}'(\lambda')e'(f_i)$ and so $d'(e'(f_2^0)) \leqq d'(e'(f_1^0))$. Thus we have shown that, to see (1.5), we may assume without loss of generality that $d'(e'(f_2)) \leqq d'(e'(f_1))$. Therefore there exists a partial isometry u of A with $e_*(u) \leqq e'(f_1)$ and $e(u)$

$=e'(f_2)$. Put $f_1'=u^*f_2$. Then we have $e'(f_1')=e_*(u)$ and $\varphi(e(f_1'))=e(f_2)$ by a similar argument as in the proof of Prop. 1.3. Since $d(\varphi(e(f_1)))\leqq d(e(f_2))=d(\varphi(e(f_1')))$, we have $d(e(f_1))\leqq d(e(f_1'))$, for the relative dimension is an algebraical property in the sense of J. v. Neumann [4]. Hence we have $\bar{d}(\bar{e}(f_1))\leqq\bar{d}(\bar{e}(f_1'))$ by (A) and so $d'(e'(f_1))\leqq d'(e'(f_1'))$ by Prop. 1.3 (applying to $\bar{R}_1$). On the other hand, we have already had $e'(f_1')=e_*(u)\leqq e'(f_1)$. Hence we obtain $d'(e'(f_1))=d'(e'(f_1'))$ and so $=d'(e_*(u))=d'(e(u))=d'(e'(f_2))$. This shows (1.5). q. e. d.

Corollary*. *Let* $\boldsymbol{R}$ *be an* $\boldsymbol{AW}^*$*-algebra with* (A) *and* f_i $(i=1, 2)$ *be points of* $\mathfrak{H}$. *Then following statements are mutually equivalent:*

(1.6) $d(e(f_1))\leqq d(e(f_2))$,

(1.7) $d'(e'(f_1))\leqq d'(e'(f_2))$,

(1.8) $\bar{d}(\bar{e}(f_1))\leqq\bar{d}(\bar{e}(f_2))$.

Proof. The implications (1.7)→(1.8), (1.8)→(1.7), and (1.7)→(1.6) are consequences of Prop. 1.3 and so these are valid without the assumption (A). And the implication (1.6)→(1.8) is nothing but (A). q. e. d.

Proposition* 1.4. $R_0'=R_0$.

Proof. In order to see Prop*. 1.4, we need only to see that $e_0'(f)=e_0(f)$ for any point f of $\mathfrak{H}$, where we denote by $e_0'(f)$ the supporter of f in R_0'. So, let us denying the above assertion for some point f of $\mathfrak{H}$. We notice that $e_0'(f)\leqq e_0(f)$ and so we can find a non-zero point g of $\mathfrak{H}$ such as $(e_0(f)-e_0'(f))g=g$. Since $g\neq 0$, we have $0\neq e_0(g)\leqq e_0(f)$. Hence $e(g)^{\natural}e(f)^{\natural}\neq 0$ by Lemma 1.3. Therefore there exists a non-zero partial isometry u of R with $e_*(u)\leqq e(g)$ and $e(u)\leqq e(f)$ by [1], Prop. 2.7, (2.18)$'$. Put $g'=e_*(u)g$ and $f'=e(u)f$. Then we have $e(g')=e_*(u)$ and $e(f')=e(u)$ by Lemma 1.3. Thus we have $d(e(g))\geqq d(e(g'))=d(e(f'))\leqq d(e(f))$. Applying Corollary* of Prop*. 1.3 to these formula, we get $d'(e'(g))\geqq d'(e'(g'))=d'(e'(f'))\leqq d'(e'(f))$. Since $g'\neq 0$ and $f'\neq 0$ by the definition of supporter, we have from these $e'(g)^{\natural}e'(f)^{\natural}\neq 0$ by [1], Prop. 2.7, (2.18)$'$. Since $e'(g)^{\natural}=e_0'(g)$ and $e'(f)^{\natural}=e_0'(f)$ by Lemma 1.3, we get thus $e_0'(g)e_0'(f)\neq 0$. This is a contradiction, for $e_0'(f)g=e_0'(f)(e_0(f)-e_0'(f))g=0$ and so $e_0'(g)e_0'(f)=0$. q. e. d.

By virtue of Prop*. 1.4 every spectrum of R' is considered as a spectrum of R.

The following proposition is due to F. J. Murray-J. v. Neumann [8], I. E. Segal [10], and E. L. Griffin [6].

Proposition* 1.5. *Let* $\boldsymbol{R}$ *be an* $\boldsymbol{AW}^*$*-algebra with* (A). *Then we have*

(I) $e'(f)$ *is an irreducible projection of* R' *if and only if* $e(f)$ *is an irreducible projection of* R.

(II) $e'(f)$ *is a finite projection of* R' *if and only if* $e(f)$ *is a finite projection of* R.

(III) *$e'(f)$ is a purely infinite projection of R' if and only if $e(f)$ is a purely infinite projection of R, where we say that a projection e of R is purely infinite if eRe is purely infinite, that is, of type* (III).

Proof. Proof of (III). Let e_i' $(i=1,2)$ be projections of R' with $e_i'\leqq e'(f)$ and $e_i'^{\natural}=e'(f)^{\natural}$ $(i=1,2)$. Then we have $e_i'=e'(e_i'f)$ by Lemma 1.3 and $d(e(e_i'f))\leqq d(e(f))$ by Prop. 1.3. Since $e(f)$ is purely infinite, we have from the former $d(e(e_1'f))=d(e(f))=d(e(e_2'f))$. Applying Prop*. 1.3 to this fact, we get $d'(e_1')=d'(e_2')$. This means that $e'(f)$ is purely infinite.

Proof of (II). First we shall prove (II) locally. Since $e(f)$ is finite, it is locally finite (with respect to any spectre λ of R) by [1], Prop. 1.1. Hence, by the local form of (III) just proved, $e'(f)$ is not locally purely infinite and hence we can find a locally finite projection e' of R' satisfying that $\lambda(e'^{\natural})=\lambda(e'(f)^{\natural})$ and that $e'\leqq e'(f)$. We denote $e(e'f)$ briefly by e. Then, from the fact that $e(e'f)\leqq e(f)$ it follows that e is locally finite. Moreover, if e is locally non-singular projection of $e(f)Re(f)$, we may find a local decomposition $e(f)={}_\lambda\oplus(e_i\,;1\leqq i\leqq n)$ with $d_\lambda(e_i)\leqq d_\lambda(e)$ $(1\leqq i\leqq n)$. We denote $e'(e_if)$ briefly by e_i'. Since $d_\lambda(e_i)\leqq d_\lambda(e)$, we see $d_\lambda'(e_i')\leqq d_\lambda'(e')$ by the local form of Corollary* of Prop.* 1.3. We denote $\cup(e_i'\,;1\leqq i\leqq n)$ by $e^{(1)'}$. Then $e^{(1)'}$ is locally finite. On the other hand, we have $e^{(1)'}e_if={}_\lambda e_if$ $(1\leqq i\leqq n)$, that is $e^{(1)'}f={}_\lambda f$. This means that $e'(f)\leqq_\lambda e^{(1)'}$. (Here, we use the notation $f={}_\lambda g$ $(f,g\in\mathfrak{H})$, which means that $e_0(\lambda)f=e_0(\lambda)g$ for some $e_0(\lambda)\in E_0(\lambda)$.) Hence $e'(f)$ is locally finite. By [1], Lemma 4.2, it is easy to see that a spectre with respect to which e is locally singular, is a limiting spectre of spectres with respect to which e is locally non-singular. From this and from the normality of finiteness of a projection, we can conclude that $e'(f)$ is locally finite, even if we drop the assumption that e is locally non-singular. Thus we arrive at the assertion by [1], Prop. 1.1.

Proof of (I). First we notice that $e(e'f)=e'^{\natural}e(f)$ for $e'\leqq e'(f)$ if $e(f)$ is an irreducible projection of R. In fact, $e(e'f)=e(e'f)^{\natural}e(f)=e_0(e'f)e(f)=e'(e'f)^{\natural}e(f)=e'^{\natural}e(f)$. Now we take two projections e_1', e_2' of R' satisfying that $e_1'e_2'=0$ and that $e'(f)\geqq e_1'\sim e_2'\leqq e'(f)$. Since $(e_1'\oplus e_2')^{\natural}e(f)=e_1'^{\natural}e(f)$, we obtain $e_1'\oplus e_2'\sim e_1'$ by Corollary* of Prop.* 1.3. Since $e(f)$ is irreducible, it is finite and hence $e'(f)$ is finite by (II). Hence $e_1'\oplus e_2'$ is also finite. Then we must have $e_1'=e_2'=0$. This means that $e'(f)$ is irreducible. q. e. d.

Corollary*. *Let $\boldsymbol{R}$ be an $\boldsymbol{AW}$*-algebra with* (A). *Then we have*

(I) *R' (or $\bar{R}$) is of type* (I) *if and only if R is of type* (I).

(II) *R' (or $\bar{R}$) is of type* (II) *if and only if R is of type* (II).

(III) *R' (or $\bar{R}$) is of type* (III) *if and only if R is of type* (III).

Proof. If R is of type $(*)$ $(*=\text{I, II, III})$, there exists a projection $e(f)$ $(f\in\mathfrak{H})$ of R of the same type $(*)$ and so $e'(f)$ is of the same type $(*)$. This implies that R' is of the same type (*). Similarly, since $e(f)$ is of the same type (*) by applying Prop.* 1.5 to R', $\bar{R}$ is of the same type (*). By a similar argument, we obtain also the "only if" part of the assertion. q. e. d.

A triple $(R, \mathfrak{H}, f)$ formed by an $\boldsymbol{AW}^*$-algebra $\boldsymbol{R}$ $(=(R, \mathfrak{H}))$ and a point f of $\mathfrak{H}$ with $e(f)=1$ is called a *cyclic* $\boldsymbol{AW}^*$-algebra.

Let $\boldsymbol{R}$ be a finite, cyclic $\boldsymbol{AW}^*$-algebra formed by $\mathfrak{H}$, R, and f. By virtue of a theorem of Ti. Yen [9], R has a trace t (also cf. [1], Theorem 4.2). Denote by τ the p-normal state of R defined by $\tau(c)=f(t(c))$ $(c\in R)$. Then R is considered as a unitary space with an inner product (a, b) defined by $(a, b) =\tau(b^*a)$ $(a, b\in R)$. Denote by $\mathfrak{H}_\tau$ its completion and by η the injection of R into $\mathfrak{H}_\tau$. The Hilbert space $\mathfrak{H}_\tau$ is a representation space of R and its representation ϕ is faithful. J. Feldman [3] proved that this triple $\boldsymbol{R}_\tau$ $(=(\phi(R), \mathfrak{H}_\tau, \eta(1)))$ is a cyclic $\boldsymbol{W}^*$-algebra, where we denote the unit of R by 1. By making use of this result of J. Feldman, we prove the following

Lemma 1.7. *Every finite* $\boldsymbol{AW}^*$*-algebra is* $\boldsymbol{W}^*$.

Proof. Let $\boldsymbol{R}$ be a finite $\boldsymbol{AW}^*$-algebra and let f be a point of $\mathfrak{H}$. In order to prove Lemma 1.7, we need only to see that $\bar{e}(f)=e(f)$, for every operator of R is written as a uniform limit of linear combinations of projections of R and each projection of R is expressed as an orthogonal sum of such projections as $\bar{e}(f)$ for some $f\in\mathfrak{H}$. Hence we may assume that $\boldsymbol{R}$ is a finite, cyclic $\boldsymbol{AW}^*$-algebra formed by $\mathfrak{H}$, R, and f. By a well known method, we can find a non-zero projection e_1 of R such that $\theta_1\|ce_1f\|\leqq\|\phi(ce_1)\eta(1)\|\leqq \theta_2\|ce_1f\|$ $(c\in R)$, where θ_1, θ_2 are positive constants. From this it follows that $d'(e'(e_1f))=d'(e'(\phi(e_1)\eta(1)))$ by a similar argument as in the proof of Prop. 1.1. Since $\boldsymbol{R}_\tau$ is $\boldsymbol{W}^*$, $\phi(e_1)\boldsymbol{R}_\tau\phi(e_1)$ is also $\boldsymbol{W}^*$ and so e_1Re_1 is a W^*-algebra acting on $\mathfrak{H}$. Since the property that $\bar{e}(f)\in R$, is normal as a property in the sense of [1], § 1, we may assume without loss of generality that e_1 is simple of order n by [1], Lemma 4.2, Def. 4.2. Hence we can find a decomposition $1=\bigoplus(e_i; 1\leqq i\leqq n)$, with $d(e_i)=d(e_1)$ $(1\leqq i\leqq n)$. Since $d(e_i)=d(e_1)$, there exists a partial isometry u_i of R with $e_*(u_i)=e_i$ and $e(u_i)=e_1$.

Let $(c_\iota; \iota\in I)$ be an arbitrary weak Cauchy-hypersequence of operators of R. Denote its limit by $\bar{c}$. In order to see Lemma 1.7, we need only to show that $\bar{c}\in R$. Since e_1Re_1 is W^*, the weak limit $u_j\bar{c}u_i^*$ of $(u_jc_\iota u_i^*; \iota\in I)$ is contained in e_1Re_1 and hence in R. Therefore $e_j\bar{c}e_i$ $(=u_j^*u_j\bar{c}u_i^*u_i)$ is contained in R and so $\bar{c}$ $(=\sum_{i,j=1}^n e_j\bar{c}e_i)$ is contained in R. Thus we get $\bar{e}(f)\in R$ and $\bar{e}(f)=e(f)$. q. e. d.

Proof of Theorem I. Let $\boldsymbol{R}$ be a semi-finite $\boldsymbol{AW}^*$-algebra and f be a point of $\mathfrak{H}$. If $e(f)$ is finite, $(e(f)Re(f), e(f)\mathfrak{H})$ is $\boldsymbol{W}^*$ by Lemma 1.7. Hence $e(f)$ $(=\bar{e}(f))$ is finite in $\bar{R}$. This means that $\bar{R}$ is also semi-finite. Therefore, we need only to see that $e(f)$ is finite with $\bar{e}(f)$. Let us start by denying this fact. Hence we may assume that $e(f)$ is normally infinite in R.

Since $e(f)$ is semi-finite, there exists a finite projection e_1 such that $e_1 \leqq e(f)$. Then we can find a maximal decomposition $\oplus(e_\iota; \iota \in I) \leqq e(f)$ (the suffix $1 \in I$) with $d(e_\iota) = d(e_1)$ $(\iota \in I)$. Since $d(e_1) \leqq d(e - \oplus(e_\iota; \iota \in I))$ does not hold, we have $d_\lambda(e - \oplus(e_\iota; \iota \in I)) \leqq d_\lambda(e_1)$ for some spectre λ of R by [1], Prop. 3.7. In view of this fact, I must be infinite and further it is countable (say $I = (n; 1 \leqq n < \infty)$). Thus we get a new decomposition $e(f) = \oplus(e_n; 1 \leqq n < \infty)$ with $d(e_n) = d(e_1)$ $(1 \leqq n < \infty)$ by a well known tricke of [8]. We put $e^{(N)} = \oplus(e_n; 1 \leqq n \leqq N)$. Then $e^{(N)}$ is finite and so $(e^{(N)} Re^{(N)}, e^{(N)}\mathfrak{H})$ is $\boldsymbol{W}^*$ by Lemma 1.7 Since $e'(e^{(N)}f) \leqq e'(f)$, we have $\bar{d}(e^{(N)}) = \bar{d}(\bar{e}(e^{(N)}f)) \leqq \bar{d}(\bar{e}(f))$ by Lemma 1.3 and by Prop. 1.3. This is a contradiction. Thus we arrive at the assertion. q. e. d.

§ 2. Spacial isomorphism.

Let C_i $(i=1,2)$ be arbitrary systems of operators on Hilbert spaces $\mathfrak{H}_i$ $(i=1,2)$ and let φ be a one-to-one correspondence of C_1 onto C_2. Denote by B_i the commutant of C_i on $\mathfrak{H}_i$ and by D_i the W^*-algebra generated by C_i and B_i. The correspondence φ is called *spacial* (after F. J. Murray and J. v. Neumann [8]), if there exists a partial isometry u mapping $\mathfrak{H}_1$ onto $\mathfrak{H}_2$ satisfying $\varphi(c_1) = uc_1u^*$ $(c_1 \in C_1)$. Concerning this, we have the following generalization of theorems of K. Yosida [11], M. Eidelheit [12]-Y. Kawada [13], and I. E. Segal [10].

Theorem 2.1. *The correspondence φ is spacial if and only if it is extendable to an algebraic $*$-isomorphism of D_1 onto D_2.*

Proof. Necessity: The mapping $d_1 \rightarrow ud_1u^*$ is obviously an algebraic $*$-isomorphism of D_1 onto D_2 $(=uD_1u^*)$, which is the extension of φ in question.

Sufficiency: We notice that D_i has the commutative commutant D_i'. Denote by I_i the unit of D_i' and by the same φ the given extension of φ. Then we have $I_1^\# = I_2$ and $I_2^\# = I_1$ by Prop. 1.1. We shall see the sufficiency locally. In view of Prop. 1.2 we may assume without loss of generality that $I_1 \sim_\lambda e_{20}' \leqq I_2$ for some projection e_{20}' of D_2'. Since $\lambda(I_1)=1$, we have $\lambda(e_{20}')=1$ and so $\lambda(I_2 - e_{20}')=0$ Hence it holds that $e_{20}' = {}_\lambda I_2$. Thus we get $d_\lambda'(I_1) = d_\lambda'(I_2)$ and so $d'(I_1) = d'(I_2)$ by [1], Prop. 1.1. This shows the assertion. q. e. d.

We need the following lemma for the proof of Theorem II.

Lemma 2.1. *Let R be an AW^*-algebra and e be a projection of R. Then there exists a decomposition*

(2.1) $e^\natural = \oplus(e_{0\iota}; \iota \in {}_0I)$, *where each* $e_0 \in E_0$,

and, for each $\iota \in I$, there occurs one of following three cases:

(2.2) $e_{0\iota}e^\natural$ *is finite,*

(2.3) $e_{0\iota}e^\natural \sim e_{0\iota}e$,

(2.4) $e_{0\iota}e^{\natural}=\oplus(e_\kappa;\ \kappa\in K_\iota)$,
where $e_\kappa \sim ee_{0\iota}$ $(\kappa\in K_\iota)$ *and* $\aleph_0\leqq\bar{K}_\iota$ *(the cardinal number of* K_ι*).*

Proof. Since finiteness and equivalence are normal properties in our sense, we may assume without loss of generality that (2.2) and (2.3) do not occur. There exists a maximal orthogonal system $(e_{0\iota};\ \iota\in I)$ of projections of E_0, each of which satisfies (2.4). Denote $e^{\natural}-\oplus(e_{0\iota};\ \iota\in I)$ by e_0'. In view of [1], Prop. 3.7, it is easy to see that, if $e_0'\neq 0$, there exists a non-zero projection e_0 of E_0 with $e_0\leqq e_0'$ such that $e_0e=\oplus(e_\kappa;\ \kappa\in K_\iota)$ with $e_\kappa\sim e_0e$ $(\kappa\in K_\iota)$. If e_0e is (locally) finite, we have $\aleph_0\leqq\bar{K}_\iota$, because (2.2) does not occur and so $e_0e^{\natural}$ is normally infinite. On the other hand, if e_0e is (locally) normally infinite, we have also $\aleph_0\leqq\bar{K}_\iota$. For, otherwise, we must have $e_0e^{\natural}\sim e_0e$ by [1], Prop. 3.2. This is impossible, for (2.3) does not occur. From these we get always $\aleph_0\leqq\bar{K}_\iota$. This contradicts the property of e_0'. Therefore we must have $e_0'=0$ and so we get the desired assertion. q. e. d.

We are now in a position to prove Theorem II.

Proof of Theorem II. In view of Prop. 1.2, we may assume without loss of generality that $d'(I_1)=d'(e_2')$ for some projection e_2' of R_2'. Use now Lemma 2.1 with respect to e_2'. Since the spacial isomorphism is a normal property in our sense, we can moreover assume without loss of generality that one and only one of (2.2)–(2.4) occurs and $e_0^{\natural}=I_2$. But (2.2) can not occur. If (2.3) takes place, $e_2'\sim I_2$, and our theorem clearly holds. If (2.4) is the case, we still have $e_2'\sim I_2$, because, first we have $I_2=\oplus(e_\kappa';\ \kappa\in K)$ $(e_2'\sim e_\kappa'$ for any $\kappa\in K)$ by (2.4), and hence $I_1=\oplus(\varphi'^{-1}(e_\kappa');\ \kappa\in K)(\varphi'^{-1}(e_\kappa')$'s are mutually equivalent), and further $e_2'=\oplus(e_\kappa'';\ \kappa\in K)$ $(e_\kappa''$'s are mutually equivalent) by the fact that $e_2'\sim I_1$ with respect to the mixed relative dimension by φ, and thus we have $I_2=\oplus(e_{\kappa\kappa'};\ \kappa, \kappa'\in K)$ $(e_{\kappa\kappa'}$'s are mutually equivalent), and then we get finally $e_2'\sim I_2$ as $\bar{K}^2=\bar{K}$. q. e. d.

Corollary 1. *Let* $\boldsymbol{R}_i$ $(i=1,2)$ *be purely infinite* $\boldsymbol{W}^*$*-algebras and let* φ *be an algebraic *-isomorphism of* R_1 *onto* R_2. *Denote its commutant by* $\boldsymbol{R}_i'$. *Suppose that there is an algebraic *-isomorphism of* R_1' *onto* R_2', *which coincides with* φ *on the center* R_{10} *of* R_1. *Then* φ *is spacial.*

Proof. Since R_i is purely infinite, R_i' is also purely infinite by Corollary* of Prop.* 1.5 and hence normally infinite. Thus φ is spacial by Theorem II. q. e. d.

The following corollary involves the result of Y. Misonou [5].

Corollary 2. *Let* $\boldsymbol{R}_i$ $(i=1,2)$ *be* $\boldsymbol{AW}^*$*-algebras with the normally infinite commutant and with the underlying separable Hilbert space and let* φ *be an algebraic *-isomorphism of* R_1 *onto* R_2. *Then* φ *is spacial.*

Proof. In view of Prop. 1.2, we may assume without loss of generality that $d'(I_1)=d'(e_2')$ for some projection e_2' of R_2'. Since R_1' is normally infinite,

e_2' is normally infinite and satisfies $e_2'^{\natural}=I_2$. As we say in the proof of [1], Theorem 4.1, we can find a state f_2 of R_2' such that $f_2(e')=0$ if and only if $e'=0$ ($e'\in E_2'$). Hence e_2' and also I_2 are countably decomposable. From these it follows that $e_2'\sim I_2$ by virtue of the proof of Lemma 1.4. Thus we have $d'(I_1)=d'(I_2)$. q. e. d.

Corollary 3. *Let* $\boldsymbol{R}$ *be a purely infinite* (*or finite*) $\boldsymbol{W}^*$*-algebra and* φ *be an algebraic* $*$*-automorphism of* R, *which coincides with the identical mapping on its center. Then* φ *is spacial.*

Proof. According to Theorem II, we have only to show that our assertion holds, when R' is finite, in which case R also is finite from our assumption (see Corollary* of Prop.* 1.5). Let f_1' be a point of $\mathfrak{H}$ with $f_1'(I_1^c)=0$ and f_2 be a point of $\mathfrak{H}$ with $f_2((\varphi(e(f_1')))^c)=0$. Put $e_2=e(f_2)$, $e_1=\varphi^{-1}(e_2)$, and $f_1=e_1f_1'$. Since $e_2=\varphi(e(f_1'))$, we have $e_1=e(f_1')$. Hence we have $e_1=e(f_1)$ by Lemma 1.3. Denote the trace of R by t. By Prop.* 1.3, we have $e'(f_1)\sim e'(f_2)$ with respect to the mixed relative dimension determined by φ considered as an algebraic $*$-isomorphism of R_1 ($=R$) onto R_2 ($=R$). By the uniqueness of t, we get $t(c)=t(\varphi(c))$ for $c\in R$. Hence we have $t(e_1)=t(e_2)$, that is, $e_1\sim e_2$ by [1], (4.14). Therefore we have $e'(f_1)\sim e'(f_2)$ with respect to the relative dimension of R' by Prop*. 1.3. Hence $e'(f_1)\sim e'(f_1)$ with respect to the mixed relative dimension determined by φ.

Let E_1' be the set of projections e' of R' satisfying $e'\sim e'$ with respect to the mixed relative dimension determined by φ and let E_2' be a maximal orthogonal system of projections of E_1'. Denote the supremum of E_2' by e'. Then we have $e'\in E_1'$. Moreover we have $e'=I$. For, otherwise, taking the point f_1' such that $f_1'(e')=0$ in the above argument, we get at last a projection $e'(f_1)$ such that $e'(f_1)\sim e'(f_1)$, $e'(f_1)'=1-e'$, which is a contradiction. Hence we get $e'=I$ and thus arrive at the assertion. q. e. d.

Now, we reestablish the theory of the coupling operator of rings of operators due to E. L. Griffin [6], [7] by use of the local theory.

Let $\boldsymbol{R}$ be a $\boldsymbol{W}^*$-algebra. We say that a spectre λ of R is *cyclic* if there exists a point f of $\mathfrak{H}$ with $\lambda(e_0(f))=1$ satisfying following postulates (2.5)–(2.8):

(2.5) $t_\lambda(e(f))\neq 0$ if R is locally finite, where t_λ is the local trace of R,

(2.6) $e_0(f)=\bigoplus(e_\iota;\, \iota\in I)$, $e_\iota\sim e(f)$ ($\iota\in I$) if R is locally normally infinite,

(2.7) $t'_\lambda(e'(f))\neq 0$ if R' is locally finite, where t'_λ is the local trace of R',

(2.8) $e_0(f)=\bigoplus(e_\iota';\, \iota'\in I')$, $e_{\iota'}'\sim e'(f)$ ($\iota'\in I'$) if R' is locally normally infinite.

For a cyclic spectre λ of R, we write κ_λ (a) for $t_\lambda(e(f))^{-1}$, if R is locally finite and (b) for $\bar{I}$ (=the cardinal number of I), if R is locally normally infinite. Similarly, for a cyclic spectre λ of R, we write κ'_λ (a′) for $t'_\lambda(e'(f))^{-1}$, if R' is locally finite and (b′) for $\bar{I}'$ if R' is locally normally infinite. From now on we shall consider cyclic spectres only.

For the definition of the coupling operator of $\boldsymbol{R}$, we prepare following two lemmas.

LEMMA 2.2. *When R, R' are both locally finite, κ'/κ is independent of the choice of f.*

PROOF. Let f_i $(i=1,2)$ be points of $\mathfrak{H}$ satisfying $\lambda(e(f_i))=1$ and (2.5)–(2.8). In order to see Lemma 2.2, we need only to verify

(2.9) $t_\lambda(e(f_1))/t'_\lambda(e(f_1))=t_\lambda(e(f_2))/t'_\lambda(e'(f_2))$.

By Prop. 1.2, we may assume that $e(f_2)\sim e\leqq e(f_1)$ for some projection e of R. Put $f_2'=ef_1$. Then we have $e(f_2')=e$ by Lemma 1.3. Hence we have $e(f_2')\sim e(f_2)$ and then $e'(f_2')\sim e'(f_2)$ by Prop.* 1.3. To see (2.9), thus, we may assume without loss of generality that $e(f_2)\leqq e(f_1)$ and $f_2=e(f_2)f_1$.

Write briefly e, e', and f for $e(f_1), e'(f_1)$, and f_1. Denote by R_1 the cyclic $\boldsymbol{W}^*$-algebra formed by $ee'Hee'$, R_1 $(=ee'Ree')$, and f. Applying [8], Lemma 11.3.2, p. 186 twice, we get $R_1'=ee'R'ee'$. Hence $e(f_i), e'(f_i)$ in R_1, R_1' $(i=1,2)$ coincide with those in R, R' respectively, because $f_2=e(f_2)f_1$ and $e(f_2)\leqq e(f_1)$. Therefore we may take $\boldsymbol{R}_1$ in place of $\boldsymbol{R}$ without loss of generality.

Now construct the numerical trace τ and the finite cyclic W^*-algebra $\boldsymbol{R}_\tau$ formed by $\mathfrak{H}_\tau, R_\tau$ and $\eta(1)$ as in § 1. Then ϕ in § 1 is an algebraic $*$-isomorphism of R_1 onto R_τ $(=R_2)$. Since $\phi(e(f))=e(\eta(1))$, we have from Prop.* 1.3 $e'(f)\sim e'(\eta(1))$ with respect to the mixed relative dimension determined by ϕ. But $e'(f)$ and $e'(\eta(1))$ are both identical operators and so R_1 is spacially isomorphic to R_τ. Therefore, to see (2.9), we may take $\boldsymbol{R}_\tau$ in place of $\boldsymbol{R}_1$.

By J. Feldman [3], the commutant R_τ' of R_τ is dual-isomorphic to R_τ (in the sense of [14], § 2.2) under the mapping $\phi(a)\to\psi(a)$ $(a\in R_1)$ defined by $\psi(a)\eta(c)=\eta(ca)$ $(c\in R_1)$. Hence we have $t'_\lambda(\psi(a))=t_\lambda(\phi(a))$ $(a\in R_1)$, where t_λ is the local trace of R_τ and t'_λ is the local trace of R_τ'. Since $e'(f_2)=\psi(\phi^{-1}(e(f_2)))$, we get $t_\lambda(e(f_2))=t'_\lambda(e'(f_2))$. On the other hand, it is obvious that $t_\lambda(e(f_1))=t'_\lambda(e'(f_1))$ $(=1)$. This shows (2.9) and so completes the proof. q. e. d.

LEMMA 2.3. *If R is locally normally infinite, then κ_λ is independent of the choice of f except for κ_λ being $1\leqq\kappa_\lambda\leqq\aleph_0$. In this exceptional case, there exists a point f of $\mathfrak{H}$ such as $\kappa_\lambda=\aleph_0$.*

PROOF. Let f_i $(i=1,2)$ be points of $\mathfrak{H}$ satisfying $\lambda(e_0(f_i))=1$ and

(2.10) $e_0(f_i)=\bigoplus(e_\iota^{(i)0}\,;\,\iota^{(i)}\in I^{(i)}),\ e_\iota^{(i)0}\sim e(f_i)(\iota^{(i)}\in I^{(i)})\qquad (i=1,2)$.

Put $e_0=e_0(f_1)e_0(f_2)$. Since $\lambda(e_0)=\lambda(e_0(f_1))\lambda(e_0(f_2))=1$, we have $e_0\neq0$ and

(2.11) $e_0=\bigoplus(e_\iota^{(i)}\,;\,\iota^{(i)}\in I^{(i)})$ $(i=1,2)$,

where each $e_\iota^{(i)}$ is a cyclic projection of R, that is, $e_\iota^{(i)}=e(f_\iota^{(i)})$ for some $f_\iota^{(i)}$ of $\mathfrak{H}$. For each index $\iota^{(1)}$ of $I^{(1)}$, we denote by $K_\iota^{(1)}$ the set of indices $\iota^{(2)}$'s of $I^{(2)}$ such as $e_\iota^{(1)}e_\iota^{(2)}\neq0$. Since $\sum_{\iota^{(2)}}e_\iota^{(1)}e_\iota^{(2)}=e_\iota^{(1)}\neq0$, $K_\iota^{(1)}$ is non-empty (cf. [7], Lemma 1.2). Also, if $e_\iota^{(1)}e_\iota^{(2)}\neq0$, $e_\iota^{(2)}f_\iota^{(1)}\neq0$ and so $\overline{\overline{K_\iota(1)}}\leqq\aleph_0$ (cf. ibd.). Hence we get $I_\iota^{(2)}\subseteqq\bigcup(K_\iota^{(1)}\,;\,\iota^{(1)}\in I^{(1)})$ and so $I^{(1)}\leqq\aleph_0 I^{(2)}$. Therefore, if $\overline{I^{(2)}}\leqq\aleph_0$, $\overline{I^{(1)}}\leqq\aleph_0$. This implies that "$\kappa_\lambda\leqq\aleph_0$" is independent of the choice of f.

If $\aleph_0\neq\overline{I^{(1)}}$ and $\aleph_0\leqq\overline{I^{(1)}}$, we have $\aleph_0\neq\overline{I^{(2)}}$, $\aleph_0\leqq\overline{I^{(2)}}$, and $\overline{I^{(1)}}\leqq\overline{I^{(2)}}$. Similarly we have $\overline{I^{(2)}}\leqq\overline{I^{(1)}}$. Thus we get $\overline{I^{(1)}}=\overline{I^{(2)}}$ if $\aleph_0\neq\overline{I^{(1)}}$ and if $\aleph_0\leqq\overline{I^{(1)}}$. This shows that κ_λ is independent of the choice of f if $\aleph_0\neq\kappa_\lambda$ and if $\aleph_0=\kappa_\lambda$.

Suppose that $\bar{I}\leqq\aleph_0$ in (2.6) and use the notations in (2.6). If $e(f)$ is locally finite, we must have $\bar{I}=\aleph_0$. On the other hand, if $e(f)$ is locally normally infinite, we have from [1], Prop. 3.2,

(2.12) $e_0(\lambda)e(f)=\oplus(e_n\,;\,1\leqq n<\infty)$, $e_1\sim e_n$ $(1\leqq n<\infty)$ for some $e_0(\lambda)\in E_0(\lambda)$.

Since $e_1=e(f)$, e_1 is cyclic, say $e_1=e(g)$ for a suitable $g\in\mathfrak{H}$. Since $e_\iota\sim e(f)$, for each ι, we can get

(2.13) $e_0(\lambda)e=\oplus(e_{n,\iota}\,;\,1\leqq n<\infty)$, $e_1\sim e_{n,\iota}$ $(1\leqq n<\infty)$

and so we get

(2.14) $e_0(\lambda)e_0(f)=\oplus(e_{n,\iota}\,;\,1\leqq n<\infty,\ \iota\in I)$, $e_1\sim e_{n,\iota}$ $(1\leqq n<\infty,\ \iota\in I)$,

where the cardinal number of the set of indices of $e_{n,\iota}$'s is $\aleph_0 I$ and so $\aleph_0$. This completes the proof. q.e.d.

We say that R is locally countably decomposable if $I\leqq\aleph_0$. In order to fix our idea, we shall put $\kappa_\lambda=\aleph_0$ in this case. When R is locally normally infinite, we call κ_λ the local degree of R.

We are now in a position of introduce the following

Definition 2.1. *We call the following number or the pair of cardinal numbers* θ_λ *the local coupling operator of* R; *namely*

(2.15) $\theta_\lambda=\kappa'/\kappa$ *if* R *and* R' *are both locally finite,*

(2.16) $\theta_\lambda=(\kappa',1)$ *if* R *is locally finite and if* R' *is locally normally infinite,*

(2.17) $\theta_\lambda=(1,\kappa)$ *if* R *is locally normally infinite and if* R' *is locally finite,*

(2.18) $\theta_\lambda=(\kappa',\kappa)$ *if* R *and* R' *are both locally normally infinite.*

Let $\boldsymbol{R}_i$ $(i=1,2)$ be $\boldsymbol{W}^*$-algebras, and let φ be an algebraic $*$-isomorphism of R_1 onto R_2. Then φ induces a resptriction φ_0 on the center R_{10} of R_1 and φ_0 induces a homeomorphism ν of the spectrum Ω_1 of R_{10} onto the spectrum Ω_2 of R_{20}. We identify each point λ_1 of Ω_1 with its image λ_2 by ν and denote λ_1 and λ_2 by the same notation λ. Denote by $(\theta_\lambda)_i$ the local coupling operator of R_i with respect to λ. Now we introduce the following

Definition 2.2. *We say that* φ_0 *(or* φ*) takes* $(\theta_\lambda)_1$ *into* $(\theta_\lambda)_2$ *if* $(\theta_\lambda)_1=(\theta_\lambda)_2$.

Moreover we say that a $\boldsymbol{W}^*$-algebra $\boldsymbol{R}$ (or its local coupling operator) is *locally essentially bounded* if $\boldsymbol{R}$ is not locally normally infinite or if $\boldsymbol{R}'$ is not locally finite.

The following theorem is the local form of theorems of E. L. Griffin [6], [7] (cf. [6], Theorem 9, [7], Theorem 3).

Theorem III′. *Let* $\boldsymbol{R}_i$ $(i=1,2)$ *be* $\boldsymbol{W}^*$*-algebras with the locally essentially bounded local coupling operator* $(\theta_\lambda)_i$ *and let* φ *be a (locally) algebraic* $*$*-isomor-*

phism of R_1 *onto* R_2. *Then* φ *is spacial if and only if it takes* $(\theta_\lambda)_1$ *into* $(\theta_\lambda)_2$.

PROOF. The necessity is obvious and so we have only to verify the sufficiency. We divide the proof into two parts.

1) Let R_1 and R_1' be both locally finite. Then $(\theta_\lambda)_1$ is a scalar. Since $(\theta_\lambda)_1=(\theta_\lambda)_2$, $(\theta_\lambda)_2$ is also a scalar. Hence R_2 and R_2' are also both locally finite. Select an arbitrary non-zero point f_1'' of $\mathfrak{H}_1$ and then a non-zero point f_2' of $\mathfrak{H}_2$ such that $f_2'(\varphi(e(f_1''))^c)=0$. Put $f_1'=\varphi^{-1}(e(f_2'))f_1''$. Since $e(f_2')\leqq\varphi(e(f_1''))$, we get $e(f_1')=\varphi^{-1}(e(f_2'))$ by Lemma 1.3, that is, $e(f_2)=\varphi(e(f_1'))$. Since $t'_\lambda(e'(f_1'))\neq 0$ (t'_λ being the local trace of R_1'), we can find a natural number n such that $n^{-1}=t'_\lambda(e'(f_1'))$ and a projection e_1' of R_1' satisfying $e_1'=e'(f_1)$ and $t'_\lambda(e_1')=n^{-1}$. Put $f_1=e_1'f_1'$ and $f_2=\varphi(e(f_1))f_2'$. Then it is not hard to see that $e(f_2)=\varphi(e(f_1))$ and $t'_\lambda(e'(f_1))$ $(=t'_\lambda(e_1'))=n^{-1}$. Since R_1 is algebraically $*$-isomorphic to R_2, we have $t_\lambda(e(f_1))=t_\lambda(e(f_2))$, where t_λ is the local trace of R_i. From this and the assumption we have $t'_\lambda(e'(f_1))=t'_\lambda(e'(f_2))$, where t'_λ is the local trace of R_i'. Therefore we get $t'_\lambda(e'(f_2))=n^{-1}$. Hence we can find a local decomposition $I_i={}_\lambda\oplus(e_{i,j}'\,;1\leqq j\leqq n)$, $e_{1,j}'\sim_\lambda e_{2,j}'$ $(1\leqq j\leqq n)$ with respect to the local mixed relative demension determined by φ. Therefore we have $I_1\sim_\lambda I_2$.

2) If R_1' is locally normally infinite, R_2' is also locally normally infinite by the same reason as in 1). Then we can find a decomposition $e_0(\lambda)_iI_i=\oplus(e_{i,\iota}'\,;\iota\in I)$ for some projection $e_0(\lambda)_i\in E_0(\lambda)_i$ ($E_0(\lambda)_i$ being the set of projections $e_0(\lambda)_i$'s of R_{i0} such as $\lambda(e_0(\lambda)_i)=1$), where $e_{i,\iota}'\sim e_0(\lambda)_ie'(f_i')$ and f_i''s are points obtained in 1). Since $e'(f_1')\sim e'(f_2')$, we get readily $e_0(\lambda)_1I_1\sim e_0(\lambda)_2I_2$ and so $I_1\sim_\lambda I_2$ by making use of the complete additivity of the mixed relative dimension determined by φ. This completes the proof.

Let $\boldsymbol{R}_i$ $(i=1,2)$ be $\boldsymbol{W}^*$-algebras and φ be an algebraic $*$-isomorphism of R_1 onto R_2. We say that a spectre λ of R_i is *bicyclic* if it is cyclic both in R_1 and in R_2. Suppose that λ is a bicyclic spectre of R_i and that R_i is locally normally infinite with respect to this spectre λ. Then we denote by $(\kappa_\lambda)_i$ the degree of R_i.

LEMMA 2.4. $(\kappa_\lambda)_1=(\kappa_\lambda)_2$. (*The local degree of a locally normally infinite* $\boldsymbol{W}^*$*-algebra is an "algebraic invariant".*)

PROOF. Since λ is cyclic in R_1, there exist a point f_1' of $\mathfrak{H}_1$ and a decomposition $e_0(f_1')=\oplus(e_{1,\iota}\,;\iota\in I)$ satisfying $e(f_1')\sim e_{1,\iota}$ $(\iota\in I)$, $\aleph_0\leqq I=(\kappa_\lambda)_1$, and $\lambda(e_0(f_1'))=1$. Similarly, there exists a point f_2' of $\mathfrak{H}_2$ such that $\lambda(e_0(f_2'))=1$. Since $\lambda(e(f_i'))=1$ $(i=1,2)$, we can find a partial isometry u_2 of R_2 such that $e_*(u_2)\leqq e(f_2')$, $e(u_2)\leqq\varphi(e(f_1'))$, and $\lambda(e_*(u_2)^\natural)=1$ by [1], Prop. 2.6. Put $f_2=u_2f_2'$. Then we have $e(f_2)\leqq\varphi(e(f_1'))$ and $e(f_2)=e_*(u_2)$. Hence we have $\lambda(e_0(f_2))=1$ and so we may assume without loss of generality that $e_0(f_2)=1$. Put $f_1=\varphi^{-1}(e(f_2))f_1'$. Then we have $e(f_2)=\varphi(e(f_1))$ by Lemma 1.3. Further we have $e_0(f_1)=\varphi_0^{-1}(e_0(f_2))=1$. Since $e(f_1)\geqq e(f_1')$, there exists a maximally orthogonal system $(e_{\iota'}{}^0;\ \iota'\in I')$ of projections of R_1 such that $e_{\iota'}{}^0\sim e(f_1)$ $(\iota'\in I')$. Then, by [1], Prop. 3.7, there exists a spectre μ of R_1 such that $e_0(\mu)=\bigoplus(e_0(\mu)e_{\iota'}{}^0;\ \iota'\in I')$ for some $e_0(\mu)\in E_0(\mu)_1$ and we have $e_0(\mu)e_{\iota'}{}^0\sim e(e_0(\mu)f_1)$ $(\iota'\in I')$. Hence we have $\bar{I}'=\bar{I}$ by the proof of Lemma 2.3 and so $\bar{I}'=(\kappa_\lambda)_1$. By Zorn's Lemma, there exists a maximally orthogonal system $(e_{0\rho};\ \rho\in P)$ of projections of R_{10} such that, for each ρ, there is a decomposition $e_{0\rho}=\bigoplus(e_{\iota'}{}^{(\rho)};\ \iota'\in I^{(\rho)})$ with $e_{\iota'}{}^{(\rho)}\sim e(e_{0\rho}f_1)$ $(\iota'\in I^{(\rho)})$. It is easy to see that $1=\bigoplus(e_{0\rho};\ \rho\in P)$. In view of the above argument, we get $\overline{I^{(\rho)}}=\bar{I}$ and so me may identify $I^{(\rho)}$ with I. Put $e_\iota=\bigoplus(e_\iota{}^{(\rho)};\ \rho\in P)$. Then we have $1=\bigoplus(e_\iota;\ \iota\in I)$, $e_\iota\sim e(f_1)$ $(\iota\in I)$ by the complete additivity of the relative dimension. Since the relative dimension is an "algebraic invariant", we get from this a decomposition $1=\bigoplus(\varphi(e_\iota);\ \iota\in I)$, $\varphi(e_\iota)\sim e(f_2)$ $(\iota\in I)$. In view of Lemma 2.3, this shows that $(\kappa_\lambda)_1=(\kappa_\lambda)_2$. q. e. d.

With the aid of Lemma 2.4, Theorem II follows from Theorem III′. We see this as follows. First we notice that we need only to see it locally with respect to a bicyclic spectre λ. In fact, we can write I_i as an orthogonal sum $\bigoplus(e_{0i\iota^{(i)}};\ \iota^{(i)}\in I^{(i)})$ of projections of R_{i0} such that, for each $\iota^{(i)}$, every spectre λ with $\lambda(e_{0i,\iota^{(i)}})=1$ is cyclic and so for each $\iota^{(1)},\iota^{(2)}$, if $e_{01,\iota^{(1)}}e_{02,\iota^{(2)}}\neq 0$, every spectre μ with $\mu(e_{01,\iota^{(1)}}e_{02,\iota^{(2)}})=1$ is bicyclic, and moreover $1=\bigoplus(e_{01,\iota^{(1)}}e_{02,\iota^{(2)}};\ e_{01,\iota^{(1)}}e_{02,\iota^{(2)}}\neq 0,\ \iota^{(i)}\in I^{(i)}\ (i=1,2))$. Therefore, if Theorem 2 holds locally with respect to any bicyclic spectre, we have $e_{01,\iota^{(1)}}I_1\sim e_{02,\iota^{(2)}}I_2$ by [1], Prop. 1.1 and so $I_1\sim I_2$ by the complete additivity of the mixed relative dimension.

Local proof of Theorem II. Denote by $(\kappa'_\lambda)_i$ the local degree of R_i'. Then we have $(\kappa'_\lambda)_1=(\kappa'_\lambda)_2$ by Lemma 4.4, because R_1' is algebraically $*$-isomorphic to R_2' and R_i''s are both normally infinite. Denote by $(\kappa_\lambda)_i$ the local degree of R_i. If R_i is locally finite, there is no question. On the other hand, if R_i is normally infinite, we have $(\kappa_\lambda)_1=(\kappa_\lambda)_2$ by Lemma 2.4. Hence follows $(\theta_\lambda)_1=(\theta_\lambda)_2$. This shows that $I_1\sim_\lambda I_2$ by Theorem II′. q. e. d.

Let $\boldsymbol{R}$ be a $\boldsymbol{W}^*$-algebra. A projection e_1 of R is called *centrally orthogonal* to a projection e_2 of R if $e_1{}^\natural e_2{}^\natural=0$. A point f_1 of $\mathfrak{H}$ is called *centrally orthogonal* to a point f_2 of $\mathfrak{H}$ (with respect to R) if $e_0(f_1)e_0(f_2)=0$. We say that a projection e of R is *quasi-cyclic* (in R) if there exists a centrally orthogonal system F of points of $\mathfrak{H}$ such that $e=\bigoplus(e(f);\ f\in F)$. We denote e by $e(F)$. In this case, $\bigoplus(e'(f);\ f\in F)$ is also quasi-cyclic (in R'). We denote by $e'(F)$. We say that a spectre λ of R is *quasi-cyclic* if it satisfies following postulates:

(2.19) R is locally finite

or

(2.20) λ is the limiting spectre of spectres μ's, for which R is locally normally infinite of the local degree κ (κ being independent of μ)

and

(2.21) R' is locally finite
or
(2.22) λ is the limiting spectre of spectres μ's, for which R' is locally normally infinite of the local degree κ' (κ' being independent of μ).

Denote by e_{0f} the (uniquely determined) maximal projection of R_0 in the sense that e_0R and e_0R' are both finite. We write θ_0 for the function defined on the set of cyclic spectres λ's of R with $\lambda(e_{0f})=1$, whose value is θ_λ at λ.

If λ is a quasi-cyclic spectre of R with $\lambda(e_{0f})=0$, then λ is the limiting spectre of cyclic spectre μ's of R with the common local coupling operator θ by an easy computation. We write θ_λ for θ and call θ_λ the local coupling operator of R at λ. Then it is not hard to see that there exists a (uniquely determined) maximal projection $e_{0\theta}$ of R_0 in the sense that the local coupling operator is θ at every spectre λ of R with $\lambda(e_{0\theta})=1$. The spectre λ of R is a quasi-cyclic spectre of R with the local coupling operator θ if and only if $\lambda(e_{0\theta})=1$.

We write θ for the formal sum

$$\theta_0+\bigoplus(\theta e_0\,;\,\theta\in\Theta)\,,$$

where θ runs over $(1,\aleph),(\aleph',1),(\aleph',\aleph)$ $(\aleph_0=\aleph,\aleph')$ (or $\Theta=(\theta_\lambda\,;\,\lambda(e_{0f})=0)$) and call θ the coupling operator of R after E. L. Griffin [6], [7].

Remark. We can find a quasi-cyclic projection e of R with $e^\natural=e_{0f}$ such that $e=\bigoplus(e(f)\,;f\in F)$, F being a centrally orthogonal system of points of $\mathfrak{H}$. E. L. Griffin used $t(e(F))/t'(e'(F))$ instead of θ_0 in the definition of the coupled operator of R, where t is the trace of e_0R and t' is the trace of e_0R'. These function coincide with each other at every cyclic spectre λ of R with $\lambda(e_{0f})=1$ and as these are essentially the same.

We say that $\boldsymbol{R}$ is *essentially bounded* if it is locally essentially bounded with respect to every cyclic spectre of R. It is easy to see that this definition of essential boundedness coincides with that due to E. L. Griffin [6], [7].

Let $\boldsymbol{R}_i$ $(i=1,2)$ be $\boldsymbol{W}^*$-algebras and φ be an algebraic $*$-isomorphism of R_1 onto R_2. Denote by $(\theta)_i$ the coupling operator of R_i. We say that φ takes $(\theta)_1$ into $(\theta)_2$ if it takes $(\theta_\lambda)_1$ into $(\theta_\lambda)_2$ with respect to any bicyclic spectre λ of R_i. It is not hard to see that this definition coincides with that due to E. L. Griffin [6], [7].

We are now in a position to prove, as an application of Theorem III$'$, the following theorem of E. L. Griffin [6], Theorem 9 [7], Theorem 3.

Theorem III. *Let $\boldsymbol{R}_i$ $(i=1,2)$ be essentially brounded $\boldsymbol{W}^*$-algebras and φ be an algebraic $*$-isomorphism of R_1 onto R_2. Then φ is spacial if and only if it takes $(\theta)_1$ into $(\theta)_2$.*

Proof. The necessity is obvious and so we need only to see the sufficiency.

According to Theorem III$'$, φ is locally spacial with respect to any bicyclic spetre λ of R_i, because it takes $(\theta_\lambda)_1$ into $(\theta_\lambda)_2$. On the other hand, the set of bicyclic spectres of R_i is dense in the spectrum of R_{i0} and spacial isomorphism is a normal property in our sense. Hence φ is spacial by [1], Prop. 1.1. q. e. d.

Remark. Let $\boldsymbol{R}$ be a $\boldsymbol{W}^*$-algebra of type (II_∞) with (II_1) commutant. Denote by I the unit of the commutant R' of R. Then we have

Lemma 2.5. *I is quasi-cyclic.*

Proof. By the exhaustion method, we need only to see it locally with respect to any spectre λ of R, for which I is locally cyclic in R_0. Let λ be such a spectre of R. Since I is locally cyclic in R_0, we can find a projection e_0 of R_0 with $\lambda(e_0)=1$ such that e_0 is cyclic in R_0, that is, $e_0=e_0(f)$ for some f of $\mathfrak{H}$. For the sake of brevity, we may assume that $e_0=I$. Denote by t' the trace of R', by τ' the state $f\circ t'$ composed by f and t', and by $\mathfrak{H}_{\tau'}$ the completion of the unitary space $\eta'(R')$ with the inner product $<\eta'(a'),\eta'(b')>$ $=\tau'(b'^*a')$ for $a',b'\in R'$. For any a' of R', we define the bounded linear operator $\phi'(a')$ acting on $\mathfrak{H}_{\tau'}$ such that $\phi'(a')\eta(b')=\eta(a'b')$. Then, by [3], Theorem 1, the triple of $\mathfrak{H}_{\tau'},\phi'(R')$, and $\eta'(I)$ forms a $\boldsymbol{W}^*$-algebra and $\phi'(I)$ is cyclic in $\phi'(R')$. Since R is normally infinite and the commutant of $\phi'(R')$ is finite, we can easily see that $\phi'(I)\lesssim I$ with respect to the mixed relative dimension by ϕ'. Hence we can find a partial isometry u with $u^*u=\phi'(I)$ such that $u\phi'(a')=a'u$ for $a'\in R'$. Write f' for $u\eta(I)$. Then we have $I=e'(f')$. In fact, if $a'f'=0$, we have $a'u_\eta'(I)=0$ and so $u\phi'(a')\eta'(I)=0$, that is, $\eta'(a')=0$ and then $a'=0$. This means that $I=e'(f')$ and so I is locally cyclic. q. e. d.

By Lemma 2.5, there exists a centrally orthogonal system F' of points of $\mathfrak{H}$ such that $I=e'(F')$. We write e_1 for $e(F')$. Thereby the relative dimension $d(e_1)$ of e_1 is independent of a choice of F' within the condition that $I=e'(F')$. For a finite projection e_2 of R, we write $D(e_2/e_1)$ for $D(e_2)/D(e_1)$, where D is the relative dimension function of eRe and e is a finite projection of R with $e_1\leqq e$, $e_2\leqq e$, and $D(e_1\cup e_2)\geqq\varepsilon>0$ for some positive number ε. It is not hard to see that $D(e_2/e_1)$ is independent of the choice of e. Moreover $D(e_2/e_1)$ is considered as a continuous function on the spectrum of R_0, which may take ∞ as its value. We call this function $D(*/e_1)$ the relative dimension function of R with respect to e_1.

Let $\boldsymbol{R}_i$ $(i=1,2)$ be $\boldsymbol{W}^*$-algebras of type (II_∞) with (II_1) commutant and let φ be an algebraic $*$-isomorphism of R_1 onto R_2. Denote by I_i the unit of the commutant R_i' of R_i. By Lemma 2.5, we have $I_i=e'(F_i')$ for some centrally orthogonal system F' of points of $\mathfrak{H}_i$. We write e_i for $e_i(F')$. Denote by $D(*/e_i)$ the relative dimension function of R_i with respect to e_i. After R. Kadison [16], we can $D(\varphi(e_1)/e_2)$ the *linking operator* for φ and denote it by $\varDelta$. It is easy to see that $\varDelta$ depends only on φ.

The following lemma is due to R. Kadison [16].

LEMMA 2.6. *Let $\boldsymbol{R}_i$ $(i=1,2)$ be $\boldsymbol{W}^*$-algebras of type (II_∞) with (II_1) commutant and let φ be an algebraic $*$-isomorphism of R_1 onto R_2. Then φ is spacial if and only if $\Delta=1$.*

PROOF. The necessity is obvious and so we need only to see the sufficiency.

If $\Delta=1$, we have $d(\varphi(e_1))=d(e_2)$. Since $e_i=e(f_i')$, we have $d'(e'(f_1'))=d'(e'(f_2'))$ by Prop.* 1.3 and so $d'(I_1)=d'(I_2)$. Thus we get the assertion. q. e. d.

Combining Theorem III with Lemma 2.6, we have the following

THEOREM IV. *Let $\boldsymbol{R}_i$ $(i=1,2)$ be $\boldsymbol{W}^*$-algebras and φ be an algebraic $*$-isomorphism of R_1 onto R_2. Then φ is spacial if and only if the following conditions are satisfied:*

(a) *$\boldsymbol{R}_i$'s are both locally essentially bounded or both not locally essentially bounded with respect to any spectre λ of R_i,*

(b) *when $\boldsymbol{R}_i$'s are both locally essentially bounded, φ takes $(\theta_\lambda)_1$ into $(\theta_\lambda)_2$ and*

(c) *when $\boldsymbol{R}_i$'s are not both locally essentially bounded, the local linking operator Δ_λ for φ is locally equal to the identity operator.*

PROOF. The necessity is obvious and so we need only to see the sufficiency.

If (b) (or (c)) is the case, we may assume without loss of generality that R_i's are both essentially bounded (or both not essentially bounded) and the assertion is valid for this case by Theorem III (or Lemma 2.6). Hence φ is spacial by [1], Prop. 1.1. q. e. d.

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Addendum

After this paper had been prepared, Mr. J. Tomiyama has kindly sent me his recent paper:

[17] J. Tomiyama, A remark on the invariants of W^*-algebras, Tohoku Math. J., **10** (1958), 47-41,

which is closely related to this paper; especially Theorem II.

Also, after this paper had been prepared, the following paper had appeared.

[18] R. Kadison, Unitary invariants for representations of operator algebras, Ann. of Math., **66** (1957), 304-379.