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On measures invariant under given homeomorphism group of a uniform space. (A generalization of Haar measure.)

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Introduction. Let Ω be an abstract space. By an "outer measure" m^* in Ω, is meant a non-negative, real valued, countably subadditive set function defined on the class of all subsets of Ω, that is, a set function which satisfies the following conditions:

(1) $m^*(E)\geqq 0$ for every subset E of Ω, $m^*(\theta)=0$ where θ denotes the null-set.

(2) $E_1\subseteqq E_2$ implies $m^*(E_1)\leqq m^*(E_2)$.

(3) $m^*(\bigcup_{n=1}^{\infty} E_n)\leqq \sum_{n=1}^{\infty} m^*(E_n)$.

A set E is called to be m^*-measurable if, for every subset A of Ω,

(4) $m^*(A)=m^*(A\cap E)+m^*(A\cap E^c)$, where E^c denotes the complement of E.

It is well known that the class of all m^*-measurable subsets of Ω is a σ-additive (countably additive) class and m^* is a σ-additive measure on that class. For measurable subset E we shall write habitually $m(E)$ instead of $m^*(E)$.

If a group $\mathfrak{h}$ of transformations of Ω is given, then it is natural to consider, as a generalization of Haar measure, an $\mathfrak{h}$-invariant outer measure m^*, that is, an outer measure m^* such that

(5) $m^*(\sigma E)=m^*(E)$ for every subset $E\subseteqq\Omega$ and every $\sigma\in\mathfrak{h}$.

From this point of view, the Haar measure can be considered as follows:

(A) Let $\mathfrak{g}$ be a locally compact group. To each element $a\in\mathfrak{g}$ we make correspond a transformation φ_a of $\mathfrak{g}$ such that $\varphi_a(x)=ax$ for every $x\in\mathfrak{g}$. And we define $\varphi_a\varphi_b(x)=\varphi_a(\varphi_b(x))$. Then clearly we have $\varphi_a\varphi_b=\varphi_{ab}$. Hence the set $\{\varphi_a : a\in\mathfrak{g}\}$ can be regarded as a transformation group of $\mathfrak{g}$ by defining the group operation as above. We shall denote this transformation group by $\mathfrak{g}_1$. Of course $\mathfrak{g}_1$ is isomorphic with $\mathfrak{g}$ as an abstract group. If we set $\Omega=\mathfrak{g}$ and $\mathfrak{h}=\mathfrak{g}_1$, then our $\mathfrak{h}$-invariant outer measure in Ω is nothing but a left-invariant Haar (outer) measure in $\mathfrak{g}$.

Let $\mathfrak{g}$ be a locally compact and σ-compact group and m^* a left-invariant Haar measure. Then we have the following:

(i) Theorem of uniqueness. The left-invariant Haar measure is unque up to a multiplicative constant.

(ii) Theorem of decomposition-equivalence. Let A and B be two measurable subsets of $\mathfrak{g}$ having the same measure. Then there exist direct decompositions $A=M+\sum_{n=1}^{\infty} A_n$, $B=N+\sum_{n=1}^{\infty} B_n$ such that, $m(M)=m(N)=0$ and $B_n=g_nA_n$, $n=1,2,\cdots$, where g_n is an element of $\mathfrak{g}$ and every A_n is m^*-measurable.

In the proofs of these theorems, the fact that the Haar measure is a Weil measure (a measure m^* such that the measurability of $f(x)$ with respect to m^* implies that of $f(y^{-1}x)$ with respect to $m^*\times m^*$), plays an essential role. In our case, we introduce an $\mathfrak{h}$-invariant outer measure m^* in Ω and a left-invariant outer measure μ^* in $\mathfrak{h}$. And by discussing the product measure $m^*\times\mu^*$ in the product space $\Omega\times\mathfrak{h}$, we get the similar consequences as in the case of the Haar measure.

If Ω is a locally compact and σ-compact uniform space and $\mathfrak{h}$ satisfies some conditions, then an $\mathfrak{h}$-invariant outer measure is easily introduced in Ω and $\mathfrak{h}$ can be topologized in such a way that $\mathfrak{h}$ becomes a topological group. In this case, it is expected that the group $\mathfrak{g}_1$ (see (A)) which is topologized by our method is isomorphic with the original topological group $\mathfrak{g}$. This is proved in Theorem 3.8. In § 3, we shall show that "if the group $\mathfrak{h}$ which is topologized by our method becomes a locally compact and σ-compact group, then the corresponding theorems to the above assertions (i) and (ii) are also valid for our $\mathfrak{h}$-invariant measure m^*". In § 4, we shall examine properties of the topological group $\mathfrak{h}$ and show that under some assumptions $\mathfrak{h}$ becomes a locally compact and σ-compact group. (See Theorems 4.6, 4.7 and 4.8.) Incidentally we can prove a theorem (Theorem 4.9) which contains, as a special case, the fact that the Lebesgue measure is invariant under any rotation. This might be an interesting consequence.

Most of the results in the present paper was announced earlier in a note [12]. However the publication of the details has been delayed owing to the author's health. Recently, Prof. K. Yosida communicated to the author that the results are closely related to the "Mesure dans les espaces homogènes" in A. Weil's book [6] (see [6], 42–45), to which the author had not access when the note [12] was published. In fact, if the space Ω is compact, then it will be proved without difficulty that our $\mathfrak{h}$-invariant measure can be introduced in Ω by making use of Weil's results. But it seems to the author that such deduction will not be possible for non-compact spaces. The present paper was written following the advice of Prof. K. Yosida to whom I want to express my hearty thanks.

§ 1. Topological lemmas.

Definition 1.1. Let Ω be a Hausdorff space and $\mathfrak{B}$ the smallest countably additive class which contains all open subsets of Ω. A subset $B \subseteqq \Omega$ is called a Borel set if $B \in \mathfrak{B}$. A set B_0 is called a Baire set if its characteristic function $C_{B_0}(x)$ is a Baire function.

It is easily seen that the class of all Baire sets of Ω is a countable additive class. We shall denote this class by $\mathfrak{B}_0$. Then it is clear that $\mathfrak{B}_0 \subseteqq \mathfrak{B}$. But the converse is not always true.

Notation (N_1). In the rest of this paper, by "$\mathfrak{B}$" and "$\mathfrak{B}_0$" we shall denote the class of all Borel sets and the class of all Baire sets respectively.

Notation (N_2). Throughout this paper, for any subset E of a topological space Ω we shall denote the closure of E by E^a, the interior of E by E^i and the complement of E by E^c. Consequently, E^{ai} denotes the interior of the closure of E and E^{ic} denotes the complement of the interior of E, etc.

Definition 1.2. A subset $E \subseteqq \Omega$ is called an elementary closed set of Ω if E is expressible in the form $E = \{x; f(x) \geqq \lambda\}$, where $f(x)$ is a continuous function defined on Ω and λ is a real number.

The following theorems 1.1–1.5 are easy to prove.

Theorem 1.1. *The smallest countably additive class which contains all elementary closed sets of Ω coincides with $\mathfrak{B}_0$ (see (N_1)).*

Theorem 1.2. *If Ω is a metric space, then we have $\mathfrak{B} = \mathfrak{B}_0$.*

Theorem 1.3. *Let Ω be a Hausdorff space and A a Baire set of Ω. If σ is a homeomorphism of Ω, then the set σA is also a Baire set.*

Theorem 1.4. *Let Ω be a locally compact and σ-compact Hausdorff space. Then there exists a sequence $F_1, F_2, \cdots, F_n, \cdots$ of compact subsets of Ω, such that*

(B) $F_n \subseteqq F^i_{n+1}$ (*see* (N_2)), $i = 1, 2, \cdots$ and $\bigcup_{n=1}^{\infty} F_n = \Omega$.

Theorem 1.5. *If Ω is a locally compact and σ-compact Hausdorff space, then Ω is a normal space.*

Theorem 1.6. *In order that a locally compact and σ-compact Hausdorff space Ω be metrizable, it is necessary and sufficient that there exists a sequence $f_1(x), f_2(x), \cdots, f_n(x), \cdots$ of continuous functions which satisfies the following condition:*

(C) For any different two points p and q in Ω, there exists a function $f_n(x)$ such that $f_n(p) \neq f_n(q)$.

Proof. If Ω is metrizable, then Ω is clearly separable. So there exists an enumerable subset $M = \{a_1, a_2, \cdots, a_n, \cdots\}$ which is dense in Ω. If we define $f_n(x) = d(x, a_n)$, $n = 1, 2, \cdots$, then it is evident that the condition (C) is satisfied.

Next suppose that a sequence $f_1(x), f_2(x), \cdots, f_n(x), \cdots$ of continuous functions satisfies the condition (C). By Theorem 1.4 there exists a sequence $F_1, F_2, \cdots, F_n, \cdots$ of compact subsets of Ω, which satisfies the condition (B). Let $g_n(x)$ be a continuous function defined on Ω, such that

(D) $0 \leqq g_n(x) \leqq 1$ for every $x \in \Omega$, $g_n(x)=0$ for every $x \in F_n$, and $g_n(x)=1$ for every $x \in F_{n+1}^{ic}$ (see (N_2)).

Define

$$d(x, y)=\sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|f_n(x)-f_n(y)|}{1+|f_n(x)-f_n(y)|}+\sum_{n=1}^{\infty} \frac{1}{2^n} |g_n(x)-g_n(y)|.$$

Then it is not difficult to show that this metric $d(x, y)$ gives the original topology in Ω.

THEOREM 1.7. *Let B be an open Baire set of a locally compact and σ-compact Hausdorff space Ω. Then B is the sum of enumerable compact Baire sets.*

PROOF. Let $f(x)$ be the characteristic function of B. By using transfinite induction, we can easily see that the Baire function $f(x)$ is constructible from at most enumerable continuous functions $f_1(x), f_2(x), \cdots, f_n(x), \cdots$. On the other hand there exists a sequence $F_1, F_2, \cdots, F_n, \cdots$ of compact subsets which satisfies the condition (B). Let $g_n(x)$ be a continuous function which satisfies the condition (D). Identifying p and q when $f_n(p)=f_n(q)$ and $g_n(p)=g_n(q)$ hold for every n, we get a decomposition-space $\tilde{\Omega}$. It is easily seen that $\tilde{\Omega}$ is a locally compact and σ-compact Hausdorff space and B an open Baire set of $\tilde{\Omega}$. By the preceding theorem $\tilde{\Omega}$ is a metric space. Hence B is a F_σ set in $\tilde{\Omega}$. Since $\tilde{\Omega}$ is σ-compact, it is easily seen that B is the sum of enumerable compact Baire sets.

COROLLARY. *In order that a point p in Ω be a Baire set, it is necessary and sufficient that Ω satisfies the first axiom of countability at p.*

THEOREM 1.8. *Let Ω be a locally compact and σ-compact Hausdorff space and $\mathfrak{B}^*$ a countably additive class which contains an open basis. Then we have $\mathfrak{B}^* \supseteqq \mathfrak{B}_0$ (see (N_1)).*

PROOF. In order to prove our theorem, it is sufficient to show that for every continuous function $f(x)$ and every real number λ the open set $G=\{x; f(x)<\lambda\}$ is contained in $\mathfrak{B}^*$. If we set $A_n=\left\{x; f(x) \leqq \lambda-\frac{1}{n}\right\}$ then we have $G=\bigcup_{n=1}^{\infty} A_n$. Since A_n is the sum of enumerable compact sets, we can select from the class $\mathfrak{B}^*$ an enumerable system of open sets which cover the set A_n, such that the sum of these open sets is contained in G. For every A_n we select such an enumerable system of open sets. Then it is easily seen that G belongs to $\mathfrak{B}^*$.

COROLLARY 1. *Let Ω be a locally compact and σ-compact Hausdorff space. The smallest countably additive class which contains open basis of Ω coincides with $\mathfrak{B}_0$.*

COROLLARY 2. *Let Ω_1 and Ω_2 be two locally compact, σ-compact Hausdorff space and $\mathfrak{B}_0^1, \mathfrak{B}_0^2$ the countably additive classes of the Baire sets of Ω_1 and Ω_2 respectively. The smallest countably additive class which contains the sets of the form $B_1 \times B_2$, where $B_1 \in \mathfrak{B}_0^1$ and $B_2 \in \mathfrak{B}_0^2$, coincides with the class of all the Baire sets of the product space $\Omega_1 \times \Omega_2$.*

§ 2. Measure and topological outer measure.

DEFINITION 2.1. Let m^* be an outer measure in a Hausdorff space Ω. m^* is called a topological outer measure, if the following condition is satisfied:

(2.1) $A^a \cap B^a = \theta$ implies $m^*(A+B) = m^*(A) + m^*(B)$ (see (N_2) in § 1).

THEOREM 2.1. *Let m^* be a topological outer measure in a Hausdorff space Ω. Then the class $\mathfrak{B}_{m^*}$ of all m^*-measurable subsets of Ω contains $\mathfrak{B}_0$ (see* (N_1) *in* § 1).

PROOF. It is sufficient to show that every elementary closed set is m^*-measurable. But this can be proved quite similary as in the case of a metric space (see [8], p. 52).

THEOREM 2.2. *Let Ω be a locally compact, σ-compact Hausdorff space and m^* an outer measure in Ω. In order that m^* is a topological outer measure, it is necessary and sufficient that every Baire set of Ω is m^*-measurable.*

PROOF. The necessity of the condition is proved in the preceding theorem. We shall prove the sufficiency. Suppose that the condition is satisfied. Let A and B be two subsets of Ω such that $A^a \cap B^a = \theta$. By Theorem 1.5 Ω is a normal space, so there exists a continuous function $f(x)$, such that $0 \leqq f(x) \leqq 1$ for every $x \in \Omega$, $f(x)=1$ for every $x \in A^a$ and $f(x)=0$ for every $x \in B^a$. The set $M = \left\{x;\ f(x) \geqq \frac{1}{2}\right\}$ is an elementary closed set and hence a Baire set. Consequently M is an m^*-measurable set such that $A \subseteqq M$ and $B \subseteqq M^c$. Hence we have

$$m^*(A+B) = m^*((A+B) \cap M) + m^*((A+B) \cap M^c) = m^*(A) + m^*(B).$$

DEFINITION 2.2. Let Ω be a Hausdorff space, $\mathfrak{X}$ a countably additive class of subsets of Ω and m a countably additive measure on $\mathfrak{X}$. The measure space $(\Omega, \mathfrak{X}, m)$ is called to be normal if the following condition is satisfied:

(2.2) For every set $A \in \mathfrak{X}$ and every number $\varepsilon > 0$, there exists an open set $G \in \mathfrak{X}$ such that $A \subseteqq G$ and $m(G-A) < \varepsilon$.

DEFINITION 2.3. Let Ω be a Hausdorff space and m^* an outer measure in Ω. m^* is called to be normal if and only if the measure space $(\Omega, \mathfrak{B}_{m^*}, m)$ is normal, where $\mathfrak{B}_{m^*}$ is the class of all m^*-measurable subsets of Ω.

DEFINITION 2.4. Let E be an abstract space, $\mathfrak{X}$ a countably additive class of subsets of E and m a countably additive measure on $\mathfrak{X}$. For any subset

$A \subseteqq E$ we define

$$m^*(A)=\inf \sum_{n=1}^{\infty} m(X_n), \text{ where } A \subseteqq \bigcup_{n=1}^{\infty} X_n \text{ and } X_n \in \mathfrak{X},\ n=1,2,\cdots.$$

Then clearly m^* is an outer measure in E, and every set $A \in \mathfrak{X}$ is m^*-measurable and $m^*(A)=m(A)$. This outer measure m^* is called the outer measure in E induced by the measure space $(E, \mathfrak{X}, m)$.

THEOREM 2.3. *Let Ω be a Hausdorff space and $(\Omega, \mathfrak{X}, m)$ a measure space. If the measure space $(\Omega, \mathfrak{X}, m)$ is normal and Ω is the sum of enumerable sets of finite measure, then the outer measure m^* in Ω induced by the measure space $(\Omega, \mathfrak{X}, m)$ is also normal.*

The proof is simple and is omitted.

THEOREM 2.4. *Let Ω be a locally compact and σ-compact Hausdorff space and $(\Omega, \mathfrak{X}, m)$ a measure space. If $\mathfrak{X}$ contains the class $\mathfrak{B}_0$ (see* (N_1) *in* § 1), *then the outer measure m^* in Ω induced by the measure space $(\Omega, \mathfrak{X}, m)$ is a topological outer measure.*

PROOF. Let $\mathfrak{B}_{m^*}$ be the class of all m^*-measurable sets of Ω. Then it is easily seen that $\mathfrak{B}_{m^*} \supseteqq \mathfrak{X} \supseteqq \mathfrak{B}_0$. Hence by Theorem 2.2 we get our theorem.

THEOREM 2.5. *Let Ω be a locally compact and σ-compact Hausdorff space and $(\Omega, \mathfrak{X}, m)$ a measure space. If the measure space $(\Omega, \mathfrak{X}, m)$ is normal and $\mathfrak{X}$ contains the class $\mathfrak{B}_0$ (see* (N_1) *in* § 1), *then the following condition is satisfied:*
(2.3) For every set $A \in \mathfrak{X}$ of finite measure and any number $\varepsilon > 0$, there exists a compact set $F \in \mathfrak{X}$ such that $F \subseteqq A$ and $m(A-F) < \varepsilon$.

DEFINITION 2.5. Let Ω be a locally compact and σ-compact Hausdorff space and $\mathfrak{F}$ the family of all compact subsets of Ω. Suppose that to any $F \in \mathfrak{F}$ there corresponds a finite positive number $m(A)$ satisfying the following conditions:

1° $F_1 \subseteqq F_2$ implies $m(F_1) \leqq m(F_2)$.

2° $m(F_1 \cup F_2) \leqq m(F_1) + m(F_2)$.

3° $F_1 \cap F_2 = \theta$ implies $m(F_1 + F_2) = m(F_1) + m(F_2)$.

Then m is called a content on $\mathfrak{F}$.

THEOREM 2.6. *Let Ω be a locally compact and σ-compact Hausdorff space and $\mathfrak{F}$ the family of all compact subsets of Ω. And let m be a content on $\mathfrak{F}$.*

Define

$$m(G)=\sup\{m(F);\ G \supseteqq F \in \mathfrak{F}\}, \textit{ for every open set } G.$$

And we set

$$m^*(A)=\inf\{m(G);\ A \subseteqq G(\textit{open})\} \textit{ for every subset } A \subseteqq \Omega.$$

Then m^* is a topological outer measure in Ω and every Borel set of Ω is m^*-measurable and moreover m^* is normal (*see Definitions* 2.2 *and* 2.3).

The proof of this theorem is due to the following lemmas.

LEMMA 1. *For any two open sets G_1 and G_2 we have, using locally compactness of Ω,*

$$m(G_1 \cup G_2) \leqq m(G_1) + m(G_2)\,,$$

LEMMA 2. *For any sequence $G_1, G_2, \cdots, G_n, \cdots$ of open sets we have*

$$m(\bigcup_{n=1}^{\infty} G_n) \leqq \sum_{n=1}^{\infty} m(G_n)\,.$$

Thus

(i) $0 \leqq m^*(A) \leqq \infty$, $m^*(\theta)=0$,

(ii) $A \subseteqq B$ *implies* $m^*(A) \leqq m^*(B)$,

(iii) $m^*(\bigcup_{n=1}^{\infty} A_n) \leqq \sum_{n=1}^{\infty} m^*(A_n)$.

LEMMA 3. *For any two open sets G_1 and G_2 we have*

(iv) $m(G_1) \geqq m(G_1 \cap G_2) + m^*(G_1 \cap G_2^c)$.

PROOF. If $m(G_1)=\infty$, then the above inequality is trivial. Hence we may assume that $m(G_1 \cap G_2)<\infty$ and $m^*(G_1 \cap G_2^c)<\infty$. For any positive number ε, there exists a compact set F_1 such that $F_1 \subseteqq G_1 \cap G_2$ and $m(F_1)+\varepsilon > m(G_1 \cap G_2)$. On the other hand $G_1 \cap F_1^c$ is an open set, so there exists a compact set F_2 such that $F_2 \subseteqq G_1 \cap F_1^c$ and $m(F_2)+\varepsilon > m(G_1 \cap F_1^c)$. Hence we have

$$\begin{aligned} m(G_1) &\geqq m(F_1)+m(F_2) > m(G_1 \cap G_2)+m(G_1 \cap F_1^c)-2\varepsilon \\ &\geqq m(G_1 \cap G_2)+m^*(G_1 \cap G_2^c)-2\varepsilon\,. \end{aligned}$$

Thus we have (iv).

PROOF OF THEOREM 2.6. From Lemma 3 it is easily seen that any open set is m^*-measurable and consequently every Borel set is m^*-measurable. Hence by Theorem 2.2 m^* is a topological outer measure. Since Ω is σ-compact, it is evident that m^* is normal from the definition of m^*.

THEOREM 2.7. *Under the same assumptions of the preceding theorem, for any open Baire set G we set*

$$m(G)=\sup\{m(F)\,;\, G \supseteqq F \in \mathfrak{F}\}\,.$$

And for every subset A of Ω we define

$$m^*(A)=\inf\{m(G)\,;\, A \subseteqq G \text{ (open Baire set)}\}\,.$$

Then every Baire set is m^*-measurable and hence m^* is a topological outer measure in Ω. Moreover, m^* is normal, more precisely, for any m^*-measurable set A of Ω and every number $\varepsilon>0$ there exists an open Baire set G such that $A \subseteqq G$ and $m(G-A)<\varepsilon$.

DEFINITION 2.6. The outer measure m^* which is defined in Theorem 2.6 is called the outer measure of the first kind induced by the content m. On the other hand the outer measure m^* which is defined in Theorem 2.7 is called the outer measure of the second kind induced by the content m.

It is clear that these two outer measures are same on the class $\mathfrak{B}_0$ (see (N_1) in § 1).

REMARK. Let $\mathfrak{F}_0$ be the family of all compact Baire sets of Ω and m a content on $\mathfrak{F}_0$. Then we can define the outer measure m^* of the first kind or the second kind induced by a content m as in the above two theorems.

THEOREM 2.8. *Let Ω be a locally compact and σ-compact Hausdorff space and m a content on the family $\mathfrak{F}$ of all compact subsets of Ω. If m is invariant under a homeomorphism σ of Ω, then the outer measure m^* of the second kind (the first kind) induced by a content m is also invariant under σ.*

THEOREM 2.9. *Let Ω be a locally compact and σ-compact Hausdorff space and m a countably additive measure on $\mathfrak{B}_0$ (see (N_1) in § 1). If m is finite on any compact Baire set, then the measure space $(\Omega, \mathfrak{B}_0, m)$ is always normal.*

PROOF. The measure m can be regarded as a content on the family of all compact Baire sets of Ω. We introduce in Ω the outer measure m^* of the second kind induced by a content m (see the Remark of Definition 2.6). By Theorem 1.7 every open Baire set is the sum of enumerable compact Baire sets. Hence m^* coincides with m on every open Baire set and consequently on $\mathfrak{B}_0$ (see (N_1) in § 1). Thus by Theorem 2.7 we have our theorem.

THEOREM 2.10. *Let Ω be a locally compact and σ-compact Hausdorff space and m a countably additive measure on $\mathfrak{B}_0$. If m is finite on any compact Baire set, then there exists a countably additive measure $\bar{m}$ on $\mathfrak{B}$ (see (N_1) in § 1) such that $\bar{m}(A)=m(A)$ for every $A \in \mathfrak{B}_0$ and the measure space $(\Omega, \mathfrak{B}, \bar{m})$ is normal. Moreover such measure $\bar{m}$ is unique.*

PROOF. By the similar way as in the preceding theorem, we can introduce in Ω the outer measure m^* of the first kind induced by a content m on $\mathfrak{F}_0$. If we define $\bar{m}(A)=m^*(A)$ for every Borel set A of Ω, then it is easily seen that $\bar{m}$ satisfies the condition of the present theorem. The uniqueness is also easily proved.

THEOREM 2.11 (*Markoff*). *Let Ω be a compact Hausdorff space and C_Ω the Banach space of all real valued continuous functions defined on Ω. If L is a positive linear functional defined on $C_\Omega(L(f) \geqq 0$ for every $f(x) \geqq 0)$, then there exists a countably additive measure m on the class $\mathfrak{B}_0$ (see (N_1) in § 1) such that*

$$L(f)=\int_\Omega f(x)dm(x). \tag{2.4}$$

And such measure m is unique.

PROOF. Let F be a compact subset of Ω. We define $m(F)=\inf\{L(f); 0 \leqq f(x) \leqq 1$ for every $x \in \Omega$ and $f(x)=1$ for every $x \in F\}$. Then it is easily seen that m is a content on the family $\mathfrak{F}$ of all compact subsets of Ω. We introduce in Ω the outer measure m^* of the second kind induced by the content m. If we define $m(A)=m^*(A)$ for every Baire set A, then it is not difficult to show that the equality (2.4) holds.

THEOREM 2.12 (*Lusin*). *Let Ω be a locally compact and σ-compact Hausdorff space and m^* a topological outer measure in Ω. Assume that m^* is normal and finite on every compact set. If $f(x)$ is a m^*-measurable function such that $\int_\Omega |f(x)|dm(x)<\infty$, then for any positive number ε there exists a continuous function $g(x)$ defined on Ω satisfying*

$$\int_\Omega |f(x)-g(x)|\,dm(x)<\varepsilon\,.$$

The proof is easily deduced from the special case in which $f(x)$ is the characteristic function of an m^*-measurable set A of finite measure. For this special case there exists an open m^*-measurable set G and a compact m^*-measurable set F such that $F\subseteqq A\subseteqq G$ and $m(G-F)<\varepsilon$, and $g(x)$ is obtained as a continuous function defined on Ω such that $0\leqq g(x)\leqq 1$ for every $x\in\Omega$, $g(x)=1$ for every $x\in F$ and $g(x)=0$ for every $x\in G^c$.

§ 3. Invariant measure.

Let Ω be a uniform space and let $\{V_\alpha, \alpha\in\Theta\}$ be its complete system of symmetric uniform neighborhoods. Thus we have the conditions:

(3.1) For every $\alpha\in\Theta$ $p\in V_\alpha(p)$ and $\bigcap_{\alpha\in\Theta} V_\alpha(p)=\{p\}$.

(3.2) $q\in V_\alpha(p)$ implies $p\in V_\alpha(q)$ (condition of symmetricity).

(3.3) For every $\alpha,\beta\in\Theta$ there exists a $\gamma\in\Theta$ such that $V_\gamma(p)\subseteqq V_\alpha(p)\cap V_\beta(p)$.

(3.4) For every $\alpha\in\Theta$ there exists a $\beta\in\Theta$ such that $q\in V_\beta(p)$, $r\in V_\beta(q)$ imply $r\in V_\alpha(p)$.

In the rest of this §, by " Ω " we shall always mean a uniform space as above, unless the contrary is explicitly stated.

NOTATION 3.1. By " $\rho(p,q)<\alpha$ ", we shall mean that $q\in V_\alpha(p)$. From the condition (3.2) it is evident that $\rho(p,q)<\alpha$ implies $\rho(q,p)<\alpha$.

REMARK. If σ is a homeomorphism of Ω such that $\sigma V_\alpha(p)=V_\alpha(\sigma p)$ for every $p\in\Omega$, then it is easily seen that $\sigma^{-1}V_\alpha(q)=V_\alpha(\sigma^{-1}q)$ holds for every $q\in\Omega$. Hence $\rho(p,q)<\alpha$ implies $\rho(\sigma p,\sigma q)<\alpha$ and conversely $\rho(\sigma p,\sigma q)<\alpha$ implies $\rho(p,q)<\alpha$. So we shall write $\rho(p,q)=\rho(\sigma p,\sigma q)$.

DEFINITION 3.1. α-Net and α-Chain. Let A be a subset of Ω. A subset B of A is called an α-Net of A if the following condition is satisfied:

(3.5) For every point $p\in A$ there exists a point $q\in B$ such that $\rho(p,q)<\alpha$ (see Notation 3.1).

And B is called a finite or σ-finite α-Net, according as B is a finite or countably infinite set.

Let $p_0,p_1,\cdots,p_m$ be a finite system of points of Ω. The system $(p_0,p_1,\cdots,p_m)$

is called an α-Chain of order m if the following condition is satisfied:

(3.6) $$\rho(p_i, p_{i+1}) < \alpha, \quad i=0, 1, \cdots, m-1 .$$

DEFINITION 3.2. A subset A of Ω is called to be totally bounded if for every $\alpha \in \Theta$ there exists a finite α-Net of A. And A is called to be σ-bounded if for every $\alpha \in \Theta$ there exists a σ-finite α-Net of A.

NOTATION 3.2. By "$\rho(p, q) < [m]\alpha$", we shall mean that there exists an α-Chain $(p=p_0, p_1, \cdots, p_m=q)$ of order m (see Definition 3.1).

REMARK. If $\rho(p, q) < [m]\alpha$ and $\rho(q, r) < [n]\alpha$, then we have obviously $\rho(p, r) < [m+n]\alpha$. So we shall express this fact by the following inequality:

$$\rho(p, r) \leqq \rho(p, q) + \rho(q, r) < [m]\alpha + [n]\alpha = [m+n]\alpha .$$

NOTATION 3.3. Let A be a subset of Ω. By "$V_\alpha(A)$", we shall mean the set $\bigcup_{p\in A} V_\alpha(p)$. And we define $V_\alpha{}^2(A) = V_\alpha(V_\alpha(A)), \cdots, V_\alpha{}^n(A) = V_\alpha(V_\alpha{}^{n-1}(A)), \cdots$

It is easily seen that $V_\alpha{}^n(A) = \bigcup_{p\in A} \{q ; \rho(p, q) < [n]\alpha\}$.

THEOREM 3.1. *Let A be a subset of Ω. In order that A is compact it is necessary and sufficient that A is complete and totally bounded.*

THEOREM 3.2. *Suppose that Ω is connected. Then for every $\alpha \in \Theta$ and every point $p \in \Omega$ we have $\bigcup_{n=1}^{\infty} V_\alpha{}^n(p) = \Omega$.*

The theorems 3.1 and 3.2 are well known, so we shall omit the proofs.

LEMMA 3.1. *For every $\alpha \in \Theta$ and natural number n there exists a $\beta \in \Theta$ such that*

(3.7) $\rho(p, q) < [n]\beta$ *implies* $\rho(p, q) < \alpha$ (*see Notation* 3.1. *and* 3.2).

The proof is quite easy from condition (3.4).

THEOREM 3.3. *Suppose that $V_\alpha(p)$ is totally bounded for every point $p \in \Omega$. Using the above lemma we select a $\beta \in \Theta$ such that*

(3.8) $$\rho(p, q) < [2]\beta \text{ implies } \rho(p, q) < \alpha .$$

Then for every totally bounded set A the set $V_\beta(A)$ (see Notation 3.3) is also totally bounded and consequently $V_\beta{}^2(A), \cdots, V_\beta{}^n(A), \cdots$ are all totally bounded.

PROOF. Since A is totally bounded, there exists a finite β-Net $\{p_1, p_2, \cdots, p_n\}$ of A (see Definition 3.1). We shall show that $\bigcup_{i=1}^{n} V_\alpha(p_i) \supseteqq V_\beta(A)$. Let q be an arbitrary element in $V_\beta(A)$. Then there exists a point $p \in A$ such that $\rho(p, q) < \beta$. On the other hand there exists a p_i such that $\rho(p_i, p) < \beta$. Hence we have $\rho(p_i, q) < [2]\beta$. But this implies, from (3.8), $\rho(p_i, q) < \alpha$, that is, $q \in V_\alpha(p_i)$. So we have $\bigcup_{i=1}^{n} V_\alpha(p_i) \supseteqq V_\beta(A)$ and consequently $V_\beta(A)$ is totally bounded.

THEOREM 3.4. *Let Ω be a locally compact Hausdorff space whose topology is introduced by a system $\{V_\alpha, \alpha \in \Theta\}$ of symmetric uniform neighborhoods.*

Suppose that $\mathfrak{h}$ *is a group of homeomorphisms of* Ω *satisfying the following conditions:*

(C_I) $\sigma V_\alpha(p)=V_\alpha(\sigma p)$ *for every* $\sigma\in\mathfrak{h}$ *and* $p\in\Omega$.

(C_{II}) *For any two points* p *and* q *in* Ω, *there exists a homeomorphism* $\sigma\in\mathfrak{h}$ *such that* $\sigma p=q$.

Then we can introduce an outer measure m^* *in* Ω *which satisfies the following conditions:*

(i) $m^*(\sigma A)=m^*(A)$ *for every* $\sigma\in\mathfrak{h}$ *and every* $A\subseteqq\Omega$.

(ii) *Baire set is* m^**-measurable.*

(iii) *For any subset* A *of* Ω *there exists a Baire set* B *such that* $A\subseteqq B$ *and* $m^*(A)=m(B)$. (*For measurable set* B *we use* m *instead of* m^*.)

(iv) $m^*(G)>0$ *for every open set* G; $m^*(A)<\infty$ *for every totally bounded set* A.

Proof. We take a point p_0 in Ω. For every compact subset F and any $\alpha\in\Theta$, we can select a finite system $\sigma_1,\sigma_2,\cdots,\sigma_n\in\mathfrak{h}$ such that $F\subseteqq\bigcup_{i=1}^{n}\sigma_i V_\alpha(p_0)$. Let $l(F,\alpha)$ be the minimum of such number n's. Now we fix a compact set F_0 such that $F_0^i\neq\theta$ (see (N_2) in § 1). For any compact set F there exists a finite system $\sigma_1',\sigma_2',\cdots,\sigma_m'\in\mathfrak{h}$ such that $F\subseteqq\bigcup_{i=1}^{m}\sigma_i'F_0$. We select the smallest such number m and denote it by $l(F,F_0)$. And we define $h(F,\alpha)=l(F,\alpha)/l(F_0,\alpha)$. Then we have clearly $l(F,\alpha)\leqq l(F,F_0)\cdot l(F_0,\alpha)$ and consequently $0\leqq h(F,\alpha)\leqq l(F,F_0)$. Hence $\{h(F,\alpha),\alpha\in\Theta\}$ can be considered as a bounded generalized sequence of real numbers. (Defining $\alpha\leqq\beta$ if $V_\alpha(p)\supseteqq V_\beta(p)$, Θ becomes a directed set.) Let $m(F)$ be a generalized Banach limit of the generalized sequence $\{h(F,\alpha),\alpha\in\Theta\}$. Then it is easily seen that m is a content on the family $\mathfrak{F}$ of all compact subsets of Ω. Clearly m is invariant under $\mathfrak{h}$. We introduce in Ω the outer measure m^* of the second kind induced by the content m. Then it is easily proved that the outer measure m^* satisfies the conditions (i)–(iv) of the present theorem.

Corollary 1. *Let* $\mathfrak{g}$ *be a locally compact group, then a left* (*or right*) *invariant Haar measure can be introduced in* $\mathfrak{g}$.

Remark. Let R^n be the n-dimensional Euclidean space. Then there exists an outer measure m^* which is invariant under any isometric transformation of R^n onto itself. Consequently m^* is invariant under any orthogonal transformation of R^n. Let S be the surface of an n-dimensional cube Q with center at origin. Then an outer measure which is invariant under every rotations can be introduced in S.

Theorem 3.5. *In the preceding theorem, in order that* $m(\Omega)<\infty$ *it is necessary and sufficient that* Ω *is totally bounded.*

Proof. The sufficiency is evident. If Ω is not totally bounded, then there exists a $\alpha\in\Theta$ and a sequence $p_1, p_2, \cdots, p_n, \cdots$ of points in Ω such that $V_\alpha(p_i)\cap V_\alpha(p_j)=\theta$, for $i\neq j$. It is easily seen that $V_\alpha(p_1)$ contains an open Baire set U. On the other hand, from conditions (C_I) and (C_{II}) there exists a sequence $\sigma_1, \sigma_2, \cdots \sigma_n, \cdots$ of elements of $\mathfrak{h}$ such that $\sigma_i V_\alpha(p_1)=V_\alpha(p_i)$, $i=1, 2, \cdots$. Thus we have $m(\Omega)\geqq\sum_{i=1}^{\infty} m(\sigma_i U)=\infty$. Hence the condition is necessary.

THEOREM 3.6. *Let Ω be an abstract space and $\mathfrak{h}$ a group of transformations of Ω. Suppose that*

(1°) *An $\mathfrak{h}$-invariant σ-finite*[(*)] *outer measure m^* is introduced in Ω ((*) There exists a sequence $\{E_n, n=1, 2, \cdots\}$ of m^*-measurable sets of Ω such that $\Omega=\sum_{n=1}^{\infty} E_n$ and $m(E_n)<\infty$, $n=1, 2, \cdots$.),*

(2°) *A left-invariant, σ-finite outer measure μ^* is introduced in $\mathfrak{h}$,*

(3°) *For any two points p and q in Ω, there exists a transformation $\sigma\in\mathfrak{h}$ such that $\sigma p=q$,*

(4°) *If $f(x)$ is an m^*-measurable function defined on Ω, then $f(\sigma^{-1}x)$ is an $m^*\times\mu^*$-measurable function of two variables $x\in\Omega$ and $\sigma\in\mathfrak{h}$.*

Then for any two m^*-measurable sets A and B of positive measures there exists a $\sigma\in\mathfrak{h}$ such that $m(\sigma A\cap B)>0$. Consequently if $m(A)=m(B)$, then A is decomposition-equivalent to B, that is, there exist direct decompositions $A=M+A_1+A_2+\cdots+A_n+\cdots$, $B=N+B_1+B_2+\cdots+B_n+\cdots$ such that $m(M)=m(N)=0$, $\sigma_n A_n=B_n$, $\sigma_n\in\mathfrak{h}$, $n=1, 2, \cdots$.

PROOF. Let $C_A(x)$ be the characteristic function of A and $C_B(x)$ the characteristic function of B. If our theorem is not true, then we have

$$\int C_{\sigma A}(x)C_B(x)dm(x)=0 \qquad \text{for every } \sigma\in\mathfrak{h}.$$

Hence we have, using $C_{\sigma A}(x)=C_A(\sigma^{-1}x)$,

$$0=\int_{\mathfrak{h}}\Big(\int_{\Omega} C_{\sigma A}(x)C_B(x)dm(x)\Big)d\mu(\sigma)=\int_{\Omega}\Big(\int_{\mathfrak{h}} C_A(\sigma^{-1}x)C_B(x)d\mu(\sigma)\Big)dm(x). \tag{3.9}$$

We set $H_x=\{\sigma;\sigma^{-1}x\in A\}$. Then the set H_x is μ^*-measurable for almost all $x\in\Omega$. For any element y in Ω, there exists a $\sigma_0\in\mathfrak{h}$ such that $\sigma_0 x=y$. We can easily see that $\sigma_0 H_x=H_y$ and hence H_x is μ^*-measurable for all $x\in\Omega$ and $\mu(H_x)=\mu(H_y)$. Setting $\lambda=\mu(H_x)$, $x\in\Omega$, we have

$$\int_{\Omega}\Big(\int_{\mathfrak{h}} C_A(\sigma^{-1}x)d\mu(\sigma)\Big)dm(x)=\int\mu(H_x)dm(x)=\lambda m(\Omega).$$

On the other hand,

$$\int_{\Omega}\Big(\int_{\mathfrak{h}} C_A(\sigma^{-1}x)d\mu(\sigma)\Big)dm(x)=\int_{\mathfrak{h}}\Big(\int_{\Omega} C_A(\sigma^{-1}x)dm(x)\Big)d\mu(\sigma)$$
$$=\int_{\mathfrak{h}} m(\sigma A)d\mu(\sigma)=m(A)\mu(\mathfrak{h})>0$$

From the above two relations we see $\lambda > 0$. Hence we have

$$\int_{\mathfrak{h}}\Big(\int_{\Omega} C_{\sigma A}(x)C_B(x)dm(x)\Big)d\mu(\sigma)=\int_{\Omega}\Big(\int_{\mathfrak{h}} C_A(\sigma^{-1}x)C_B(x)d\mu(\sigma)\Big)dm(x)=\int_{\Omega}\mu(H_x)C_B(x)dm(x)$$

$$=\int_{\Omega}\lambda C_B(x)dm(x)=\lambda m(B)>0\,.$$

This contradicts (3.9). So there exists a $\sigma\in\mathfrak{h}$ such that $m(\sigma A\cap B)>0$. The last half of our theorem is easily proved.

Corollary 1. *In the above theorem, if A is a measurable set of positive measure, then there exists a sequence $\sigma_1, \sigma_2, \cdots\sigma_n, \cdots$ of elements of $\mathfrak{h}$ such that $m(\Omega-\bigcup_{i=1}^{\infty}\sigma_i A)=0$.*

Corollary 2. *Let m^* be a Weil measure in a group $\mathfrak{g}$. Then any two measurable sets having the same measure are mutually decomposition-equivalent.*

Theorem 3.7. *Assume that all hypotheses of Theorem 3.4 are satisfied. Then we can introduce a topology in $\mathfrak{h}$ as follows. For any index $\alpha\in\Theta$ and any compact subset $F\subseteqq\Omega$, we set*

$$U_{\alpha F}=\{\sigma\,;\ \textit{for every}\ p\in F,\ \sigma p\in V_\alpha(p)\}\,. \tag{3.10}$$

By "Σ" we denote the totality of all such $U_{\alpha F}$ (where α and F run over Θ and the class of all compact subsets of Ω respectively). If the system Σ is taken as a complete system of neighborhoods of the identity of $\mathfrak{h}$, then $\mathfrak{h}$ becomes a topological group.

Proof. a) The intersection of all $U_{\alpha F}$ of the system Σ is only the identity. This is quite evident.

b) For any $U_{\alpha F_1}$ and $U_{\beta F_2}$ there exists a $U_{\gamma F_3}$ such that $U_{\gamma F_3}\subseteqq U_{\alpha F_1}\cap U_{\beta F_2}$. We select a $\gamma\in\Theta$ such that $V_\gamma(p)\subseteqq V_\alpha(p)\cap V_\beta(p)$, and define $F_3=F_1\cup F_2$. Then clearly we have $U_{\gamma F_3}\subseteqq U_{\alpha F_1}\cap U_{\beta F_2}$.

c) For any $U_{\alpha F}$ there exists a $U_{\beta F_1}$ such that $U_{\beta F_1}\cdot U_{\beta F_1}^{-1}\subseteqq U_{\alpha F}$. By assumptions, the system $\{V_\alpha, \alpha\in\Theta\}$ of uniform neighborhoods of Ω is symmetric. Using this, it is easily seen that $U_{\beta F_1}^{-1}=U_{\beta F_1}$. Therefore in order to prove c) it is sufficient to show that there exists a $U_{\beta F_1}$ such that $U_{\beta F_1}^2\subseteqq U_{\alpha F}$. Without loss of generality we may assume that $V_\alpha(p)^a$ (see (N_2) in § 1) is compact for all $p\in\Omega$. We select a $\beta\in\Theta$ such that

(3.11) $\rho(p, q)<[2]\beta$ implies $\rho(p, q)<\alpha$ (see Theorem 3.3).

We chose a finite β-Net (see Definition 3.1) $\{p_1, p_2, \cdots, p_n\}$ of F and define $F_1=\bigcup_{i=1}^{n} V_\alpha(p_i)^a$. Then F_1 is compact and $F_1\supseteqq V_\beta(F)$ (see Notation 3.3). Let σ and σ' be arbitrary two elements of $U_{\beta F_1}$. Then for every point $p\in F$ we have $\sigma p\in V_\beta(p)\subseteqq V_\beta(F)\subseteqq F_1$. Consequently we have $\sigma'\sigma p\in V_\beta(\sigma p)$. Hence we have $\rho(p, \sigma'\sigma p)<[2]\beta$ and consequently $\rho(p, \sigma'\sigma p)<\alpha$. This shows $\sigma'\sigma\in U_{\alpha F}$

that is, $U_{\beta F_1}^2 \subseteqq U_{\alpha F}$.

d) For every $U_{\alpha F}$ and every $\sigma \in \mathfrak{h}$ there exists a $U_{\beta F_1}$ such that $\sigma^{-1} U_{\beta F_1} \sigma \subseteqq U_{\alpha F}$. We set $F_1 = \sigma F$ and $\beta = \alpha$. Then it is easily seen that $\sigma^{-1} U_{\beta F_1} \sigma \subseteqq U_{\alpha F}$.

From a), b), c) and d) we see that $\mathfrak{h}$ becomes a topological group.

THEOREM 3.8. *Let $\mathfrak{g}$ be a locally compact group. To any element $a \in \mathfrak{g}$ we make correspond a transformation φ_a of $\mathfrak{g}$ such that $\varphi_a(x) = ax, x \in \mathfrak{g}$. And we define $\varphi_a \varphi_b(x) = \varphi_a(\varphi_b(x))$. Then clearly we have $\varphi_a \varphi_b = \varphi_{ab}$. So the set $\{\varphi_a ; a \in \mathfrak{g}\}$ can be regarded as a transformation group of $\mathfrak{g}$. We shall denote this transformation group by $\mathfrak{g}_1$. We set $\Omega = \mathfrak{g}$ and $\mathfrak{h} = \mathfrak{g}_1$, and introduce a topology in $\mathfrak{g}_1$ by the method of the preceding theorem. Then the topological group $\mathfrak{g}_1$ is isomorphic with the original group $\mathfrak{g}$.*

The proof of this theorem is not difficult.

THEOREM 3.9. *Let m^* be an outer measure in Ω which is introduced in Theorem 3.4. Suppose that $\mathfrak{h}$ is topologized by the method of Theorem 3.7. Then m^* is a continuous measure, that is, for every measurable set A of finite measure and every $\varepsilon > 0$ there exists a neighborhood U of the identity of $\mathfrak{h}$ such that*

$$m(\sigma A \ominus A) < \varepsilon \qquad \text{for every } \sigma \in U,$$

where $\sigma A \ominus A$ denotes the symmetric difference of σA and A.

PROOF. There exist an open set G and a compact set F such that $F \subseteqq A \subseteqq G$ and $m(G-F) < \varepsilon/3$. It is easily seen that there exists an $\alpha \in \Theta$ such that $\bigcup_{p \in F} V_\alpha(p) \subseteqq G$. We set $U = U_{\alpha F}$ (see (3.10)). Then we have, for every $\sigma \in U$, $m(\sigma A \ominus A) \leqq m(\sigma A \ominus \sigma F) + m(\sigma F \ominus F) + m(F \ominus A) \leqq m(A-F) + m(G-F) + m(A-F) \leqq 3m(G-F) < \varepsilon$.

THEOREM 3.10. *To every point (x, σ) of the product space $\Omega \times \mathfrak{h}$ we make correspond an element $\sigma^{-1}x$ in Ω. Then we have a continuous mapping $\varphi(x, \sigma) = \sigma^{-1}x$ of $\Omega \times \mathfrak{h}$ into Ω.*

PROOF. Let $V_\alpha(\sigma^{-1}x)$ be a neighborhood of $\sigma^{-1}x$ in Ω. We select a $\beta \in \Theta$ such that $V_\beta(x)^a$ is compact and $\rho(p, q) < [2]\beta$ implies $\rho(p, q) < \alpha$. We set $F = V_\beta(x)^a$ and $U = U_{\beta F}$ (see (3.10)). Then for every $\tau \in U\sigma$ and every $y \in V_\beta(x)$, we have

$$\rho(\tau^{-1}y, \sigma^{-1}x) = \rho(y, \tau(\sigma^{-1}x)) \leqq \rho(y, x) + \rho(x, \tau\sigma^{-1}x) \leqq [2]\beta < \alpha .$$

(See Remarks in Notation 3.1 and 3.2.) This completes the proof.

COROLLARY. *If $f(x)$ is a continuous (Baire) function defined on Ω, then $f(\sigma^{-1}x)$ is a continuous (Baire) function of two variables $x \in \Omega$ and $\sigma \in \mathfrak{h}$.*

THEOREM 3.11. *Let Ω be a locally compact and σ-compact uniform space and $\mathfrak{h}$ a homeomorphism group of Ω satisfying the conditions* (C_{I}) *and* (C_{II}) *of Theorem 3.4. Let m^* be an outer measure introduced in Theorem 3.4. We assume that $\mathfrak{h}$ becomes a locally compact and σ-compact group, when we introduce a topology in $\mathfrak{h}$ as in Theorem 3.7. Then we can introduce a left-invariant*

Haar measure μ^* *in* $\mathfrak{h}$. *If* $f(x)$ *is an* m^**-measurable function defined on* Ω, *then* $f(\sigma^{-1}x)$ *is an* $m^*\times\mu^*$*-measurable function defined on the product space* $\Omega\times\mathfrak{h}$.

PROOF. In order to prove our theorem, it is sufficient to show that for any m^*-measurable set $A\subseteqq\Omega$ the set $\{(x,\sigma);\sigma^{-1}x\in A\}$ is $m^*\times\mu^*$-measurable. It is easily seen that every Baire set of $\Omega\times\mathfrak{h}$ is $m^*\times\mu^*$-measurable. Thus if A is a Baire set, then from the Corollary of the above theorem our assertion is evident. Hence it is sufficient to show that for any set A of m^*-measure 0 the set $\{(x,\sigma);\sigma^{-1}x\in A\}$ is a set of $m^*\times\mu^*$-measure 0. There exists a Baire set B such that $A\subseteqq B$ and $m(B)=0$. By using Fubini's theorem, it is easily seen that the set $\{(x,\sigma);\sigma^{-1}x\in B\}$ is a set of $m^*\times\mu^*$-measure 0. Consequently the set $\{(x,\sigma);\sigma^{-1}x\in A\}$ is clearly a set of $m^*\times\mu^*$-measure 0.

THEOREM 3.12. *Suppose that all hypotheses of the preceding theorem are satisfied. Let* $\Phi(A)$ *be a* σ*-additive set function defined on the class* $\mathfrak{B}_0$ *of all Baire sets of* Ω. *If* B *is a Baire set of* m^**-measure* 0 *in* Ω, *then we have* $\Phi(\sigma B)=0$ *for almost all* $\sigma\in\mathfrak{h}$.

PROOF. Without loss of generality we may assume that $\Phi(A)$ is non-negative. Then $\Phi(A)$ can be regarded as a σ-additive finite measure on $\mathfrak{B}_0$. We introduce two product measures $m^*\times\mu^*$ and $\Phi\times\mu^*$ in the product space $\Omega\times\mathfrak{h}$. Let $\Gamma=\{(x,\sigma);\sigma^{-1}x\in B\}$. By using Fubini's theorem it is easily seen that the set Γ is a set of $m^*\times\mu^*$-measure 0. Hence the set $H_x=\{\sigma;\sigma^{-1}x\in B\}$ is a set of μ^*-measure 0 for almost all $x\in\Omega$. As we have already remarked in the proof of Theorem 3.6, the set H_x is μ^*-measurable and has the same μ-measure for all $x\in\Omega$. Hence H_x is a set of μ^*-measure 0 for all $x\in\Omega$. This shows again that the set Γ is also a set of $\Phi\times\mu^*$-measure 0. From Fubini's theorem we see that the set $\sigma B=\{x;\sigma^{-1}x\in B\}$ is also a set of Φ-measure 0 for almost all $\sigma\in\mathfrak{h}$.

COROLLARY. *Let* $\bar{m}$ *be a measure defined on* $\mathfrak{B}_0$. *If* $\bar{m}$ *satisfies the condition* (i) *of Theorem* 3.4, *then from the above theorem it is easily seen that* $m(A)=0$ *implies* $\bar{m}(A)=0$ *for any Baire set* A. *Hence there exists an* m^**-measurable function* $f(x)$ *such that*

$$\bar{m}(A)=\int_A f(x)dm(x) \qquad \text{for every Baire set } A \text{ of } \Omega. \tag{3.11}$$

THEOREM 3.13. *Suppose that* Ω *and* $\mathfrak{h}$ *satisfy the hypotheses of Theorem* 3.4. *Let* m^* *be an outer measure in* Ω *which is introduced in Theorem* 3.4. *We introduce a topology in* $\mathfrak{h}$ *by the method of Theorem* 3.7. *We assume that* $\mathfrak{h}$ *is locally compact and* σ*-compact. In order that a* σ*-additive set function* $\Phi(A)$ *defined on the class* $\mathfrak{B}_0$ *of all Baire sets of* Ω *is absolutely continuous with respect to* m, *it is necessary and sufficient that one of the following conditions is satisfied:*

1) $\lim_{\sigma\to 0}\Phi(\sigma A)=\Phi(A)$ *for every totally bounded Baire set A of m^*-measure* 0.

2) $\lim_{\sigma\to 0}\Phi(\sigma A)=\Phi(A)$ *for every totally bounded Baire set A.*

3) $\lim_{\sigma\to 0}W(\Phi^\sigma-\Phi\,;F)=0$ *for every compact Baire set F of Ω, where Φ^σ denotes the set function such that $\Phi^\sigma(A)=\Phi(\sigma^{-1}A)$ and W denotes the absolute variation.*

PROOF. If $\Phi(A), A\in\mathfrak{B}_0$ is an absolutely continuous σ-additive set function with respect to m, then there exists an m^*-measurable function $f(x)$ such that $\Phi(A)=\int_A f(x)dm(x)$ for every $A\in\mathfrak{B}_0$. It is easily seen that $\Phi^\sigma(A)=\Phi(\sigma^{-1}A)=\int_{\sigma^{-1}A}f(x)dm(x)=\int_A f(\sigma^{-1}x)dm(x)$. Hence we have

$$W(\Phi^\sigma+\Phi\,;F)=\int_F|f(\sigma^{-1}x)-f(x)|\,dm(x)\,.$$

For every $\varepsilon>0$ there exists a continuous function $g(x)$ defined on Ω, such that $\int_\Omega|f(x)-g(x)|\,dm(x)<\varepsilon$ (see Theorem 2.12). Then we have

$$\begin{aligned}W(\Phi^\sigma-\Phi\,;F)&=\int_F|f(\sigma^{-1}x)-f(x)|\,dm(x)\\&\leqq\int_F|f(\sigma^{-1}x)-g(\sigma^{-1}x)|\,dm(x)+\int_F|g(\sigma^{-1}x)-g(x)|\,dm(x)+\int_F|g(x)-f(x)|\,dm(x)\\&<\int_F|g(\sigma^{-1}x)-g(x)|\,dm(x)+2\varepsilon\,.\end{aligned}$$

$g(x)$ being continuous, we have $\lim_{\sigma\to 0}W(\Phi^\sigma-\Phi\,;F)\leqq 2\varepsilon$. This shows that the condition 3) is satisfied. It is quite evident that 3) implies 2), and 2) implies 1). If the condition 1) is satisfied, then by the preceding theorem we have $\Phi(A)=0$ for every totally bounded Baire set A of m^*-measure 0. Hence Φ is absolutely continuous with respect to m.

THEOREM 3.14. *Under the assumptions of the preceding theorem, we have the following:*

(1) *For any two m^*-measurable sets A and B of positive measure there exists a $\sigma\in\mathfrak{h}$ such that $m(\sigma A\cap B)>0$. And consequently if $m(A)=m(B)$, then A is decomposition-equivalent to B with respect to m.*

(2) *Let $\bar{m}^*$ be an outer measure in Ω which satisfies the conditions* (i)-(iv) *of Theorem* 3.4. *Then there exists a number λ such that $\bar{m}^*(A)=\lambda m^*(A)$, that is, the $\mathfrak{h}$-invariant outer measure is unique up to a multiplicative constant.*

PROOF. (1) is evident from Theorem 3.6 and Theorem 3.11. We shall prove (2). By the Corollary of Theorem 3.12 we have

$$\bar{m}(A)=\int_A f(x)dm(x)\qquad\text{for every Baire set } A \text{ of } \Omega\,.$$

Hence in order to prove (2) it is sufficent to show that $f(x)\equiv\lambda$ on Ω. Suppose that $f(x)$ is not constant. Then there exist two real numbers r and R $(r<R)$ such that the sets $A=\{x; f(x)>R\}$, $B=\{x; f(x)<r\}$ have positive measures. From (1) there exists a $\sigma\in\mathfrak{h}$ such that $m(\sigma A\cap B)>0$. We set $M=A\cap\sigma^{-1}B$. Then we have

$$\int_M f(x)dm(x)=\overline{m}(M)=\overline{m}(\sigma M)=\int_{\sigma M} f(x)dm(x)\,.$$

On the other hand it is evident that $M\subseteqq A$ and $\sigma M\subseteqq B$. Hence the following inequalities hold.

$$\int_M f(x)dm(x)\geqq Rm(M)\,,\quad \int_{\sigma M} f(x)dm(x)\leqq rm(\sigma M)=rm(M)\,.$$

Without losing the generality we may assume that $0<m(M)<\infty$. So we have arrived at a contradiction.

Corollary 1. *The assumptions be the same as in Theorem* 3.13. *If τ is a homeomorphism of Ω such that $\tau\sigma=\sigma\tau$ for every $\sigma\in\mathfrak{h}$, then we have*

$$m^*(\tau A)=l(\tau)m^*(A)\,,$$

where $l(\tau)$ is a constant depending on τ.

Corollary 2. *Let m^* be a left-invariant Haar measure in a locally compact and σ-compact group $\mathfrak{g}$. Then we have*

$$m^*(Aa)=l(a)m^*(A) \qquad \textit{for every } A\subseteqq\mathfrak{g}\,,$$

where $l(a)$ is a constant depending on a.

Theorem 3.15. *Let Ω be a locally compact and σ-compact uniform space whose topology is introduced by a system $\{V_\alpha. \alpha\in\Theta\}$ of symmetric uniform neighborhoods. Suppose that $\mathfrak{h}$ is a group of homeomorphisms of Ω satisfying the following conditions:*

(C_I) *$\sigma V_\alpha(p)=V_\alpha(\sigma p)$ for every $\sigma\in\mathfrak{h}$ and every $p\in\Omega$.*

(C_{II}) *For any two points p and q in Ω there exists a homeomorphism $\sigma\in\mathfrak{h}$ such that $\sigma p=q$.*

We introduce a topology in $\mathfrak{h}$ as in Theorem 3.7. *We assume that $\mathfrak{h}$ is locally compact and σ-compact. Let $\mathfrak{h}_1$ be an abstract subgroup of $\mathfrak{h}$. If $\mathfrak{h}_1$ satisfies the condition* (C_{II}), *then we have the following:*

(1°) *We can introduce an outer measure m_1^* in Ω which satisfies the following conditions:*

(i) *$m_1^*(\sigma A)=m_1^*(A)$ for every $\sigma\in\mathfrak{h}_1$ and every subset $A\subseteqq\Omega$.*

(ii) *The Baire set is m_1^*-measurable.*

(iii) *For any subset A of Ω there exists a Baire set B such that $A\subseteqq B$ and $m_1^*(A)=m_1(B)$. (For measurable set B we shall use m_1 instead of m_1^*.)*

(iv) *$m_1^*(M)>0$ for every open set $M\neq 0$; $m_1^*(A)<\infty$ for every totally bounded set A.*

(2°) *The outer measure m_1^* in Ω which satisfies the above conditions* (i)–(iv) *is unique up to a multiplicative constant, and any two measurable sets A and B of the same measure are mutually decomposition-equivalent with respect to m_1^*.*

(3°) *The outer measure m_1^* is invariant under $\mathfrak{h}$.*

Proof. (1°) is already proved in Theorem 3.4. We shall prove (2°). Let $\bar{\mathfrak{h}}_1$ be the closure of $\mathfrak{h}_1$. Then $\bar{\mathfrak{h}}_1$ is a subgroup of the topological group $\mathfrak{h}$. Hence $\bar{\mathfrak{h}}_1$ is locally compact and σ-compact. The proof of the assertion (2°) is obtained by the following lemma:

Lemma. *The outer measure m_1^* is invariant under $\bar{\mathfrak{h}}_1$, that is, for every subset A of Ω and every $\sigma \in \bar{\mathfrak{h}}_1$, we have $m_1^*(\sigma A) = m_1^*(A)$.*

Proof of the Lemma. Let A be a Baire set of finite m_1-measure. From conditions (ii) and (iii) we see that m_1^* is normal (see Theorem 2.9, Definitions 2.2 and 2.3). Hence for every $\varepsilon > 0$ there exist a compact Baire set F and an open Baire set G such that $F \subseteqq A \subseteqq G$ and $m_1(G-F) < \varepsilon$. Let σ be any element of $\bar{\mathfrak{h}}_1$. We wish to show that $m_1(\sigma A) = m_1(A)$. It is easily seen that we can assume $m_1(\sigma G - \sigma F) < \varepsilon$. (Hint: Select G_1 and F_1 such that $F_1 \subseteqq \sigma A \subseteqq G_1$ and $m_1(G_1 - F_1) < \varepsilon$. And set $F_2 = F \cup \sigma^{-1} F_1$, $G_2 = G \cap \sigma^{-1} G_1$. Use these F_2 and G_2 instead of F and G.) Since $\bar{\mathfrak{h}}_1$ is dense in $\mathfrak{h}_1$, we can chose a $\sigma_1 \in \mathfrak{h}_1$ such that $\sigma_1^{-1}\sigma F \subseteqq G$ (see Theorem 3.7). Then we have

$$m_1(\sigma A) \leqq m_1(\sigma G) < m_1(\sigma F) + \varepsilon = m_1(\sigma_1^{-1}\sigma F) + \varepsilon \leqq m_1(G) + \varepsilon \leqq m_1(A) + 2\varepsilon .$$

This shows that $m_1(\sigma A) \leqq m_1(A)$. Similarly we have $m_1(\sigma^{-1}(\sigma A)) \leqq m_1(\sigma A)$, that is, $m_1(A) \leqq m_1(\sigma A)$, and thus $m_1(\sigma A) = m_1(A)$.

Now we shall prove (2°) by using the above Lemma. m_1^* is invariant under $\bar{\mathfrak{h}}_1$. On the other hand $\bar{\mathfrak{h}}_1$ is locally compact and σ-compact, so by Theorem 3.14 the $\bar{\mathfrak{h}}_1$-invariant measure is unique and consequently the $\mathfrak{h}_1$-invariant measure is unique (up to a multiplicative constant). Let A and B be two m_1^*-measurable sets of positive measures. Then there exists a $\sigma \in \bar{\mathfrak{h}}_1$ such that $m_1(\sigma A \cap B) > 0$. m_1 is a continuous measure, so it is easily seen that there exists a $\sigma_1 \in \mathfrak{h}_1$ such that $m_1(\sigma_1 A \cap B) > 0$.

Finally we shall prove (3°). Let m^* be an $\mathfrak{h}$-invariant outer measure in Ω which is introduced in Theorem 3.4. Of course m^* is invariant under $\mathfrak{h}_1$. Hence from (2°) of the present theorem we have $m(A) = \lambda m_1(A)$ for every Baire set A of Ω. Thus we get obviously the assertion (3°).

§ 4. Topological properties of $\mathfrak{h}$.

Assumption (A_1). Throughout this §, by " Ω " we shall always mean a locally compact and σ-compact uniform space whose topology is defined by a system $\{V_\alpha, \alpha \in \Theta\}$ of symmetric uniform neighborhoods, unless the contrary is explicitely stated.

Assumption (A_2). We assume that for any two points p and q in Ω there exists a homeomorphism σ of Ω such that $\sigma p=q$ and satisfies the condition (C_{II}) of Theorem 3.4.

THEOREM 4.1. *Ω is a complete space.*

The proof is easy and is omitted.

THEOREM 4.2. *Let $\mathfrak{h}$ be a group of homeomorphisms of Ω satisfying the conditions* (C_I) *and* (C_{II}) *of Theorem* 3.4. *Then $\mathfrak{h}$ is topologized as in Theorem* 3.7. *Let W be an open subset of $\mathfrak{h}$. In order that W is totally bounded it is necessary and sufficient that for every compact set $F\subseteqq\Omega$ the set $W(F)=\{\sigma p;\ \sigma\in W, p\in F\}$ is totally bounded.*

PROOF. First we shall prove the necessity of the condition. Suppose that W is a totally bounded open set of $\mathfrak{h}$ and F a compact set of Ω. We select an $\alpha\in\Theta$ and a $\beta\in\Theta$ such that

(4.1) $V_\alpha(p)^a$ is compact for every $p\in\Omega$, and $\rho(p,q)<[2]\beta$ implies $\rho(p,q)<\alpha$.
(See Theorem 3.3 and Notations 3.2 and 3.1.)
Let U be a neighborhood of the identity of $\mathfrak{h}$ such that $U=U_{\beta F}$ (see Theorem 3.7). Since W is totally bounded, there exist $\sigma_1,\sigma_2,\cdots\sigma_n\in\mathfrak{h}$ such that $\bigcup_{i=1}^{n}\sigma_i U\supseteqq W$. Hence we have $W(F)\subseteqq\bigcup_{i=1}^{n}\sigma_i U_{\beta F}(F)$. On the other hand from the definition of $U_{\beta F}$ (see Theorem 3.7) we see that $U_{\beta F}(F)\subseteqq V_\beta(F)$ (see Notation 3.3). So we have $W(F)\subseteqq\bigcup_{i=1}^{n}\sigma_i V_\beta(F)$. By Theorem 3.3 the set $V_\beta(F)$ is totally bounded, and therefore $W(F)$ is also a totally bounded set.

We shall prove the sufficiency of the condition. Suppose that $W(F_1)$ is always totally bounded for every compact subset $F_1\subseteqq\Omega$. Let U be a neighborhood of the identity of $\mathfrak{h}$ such that $U=U_{\alpha F}$ (see Theorem 3.7). We select a $\beta\in\Theta$ such that

$$\rho(p,q)<[4]\beta \quad \text{implies} \quad \rho(p,q)<\alpha\,. \tag{4.2}$$

Since $W(F)$ is totally bounded, there exists a β-Net (see Definition 3.1) $\{q_1,q_2,\cdots,q_m\}$ of $W(F)$. Similarly there exists a β-Net $\{p_1,p_2,\cdots,p_n\}$ of F. Let a finite system $\{\sigma p_1,\sigma p_2,\cdots,\sigma p_n\}$ corresponds to $\sigma\in W$. Then for every σp_i, $1\leqq i\leqq n$, there exists a q_j such that $\rho(\sigma p_i,q_j)<\beta$. So we set $m_i(\sigma)=j$, $1\leqq i\leqq n$. Thus we have a finite system $(m_1(\sigma),m_2(\sigma),\cdots,m_n(\sigma))$ of natural numbers corresponding to $(\sigma p_1,\sigma p_2,\cdots,\sigma p_n)$. It is clear that $1\leqq m_i(\sigma)\leqq m$, $i=1,2,\cdots,n$, and

$$\rho(\sigma p_i, q_{m_i(\sigma)})<\beta,\qquad i=1,2,\cdots,n\,. \tag{4.3}$$

If for two elements σ and τ of W the equality $(m_1(\sigma),m_2(\sigma),\cdots,m_n(\sigma))=(m_1(\tau),m_2(\tau),\cdots,m_n(\tau))$ holds, then σ and τ are called to be equivalent and denoted by $\sigma\sim\tau$. Thus W is divided into the classes of equivalent elements, and these classes are clearly finite. From each class we chose an element as a representative of that class. Then we get a finite system $\{\sigma_1, \sigma_2, \cdots, \sigma_t\}$ of elements of W. We shall show that $\bigcup_{i=1}^{t} \sigma_i U \supseteqq W$. Let σ be an arbitrary element of W. Then there exists a σ_s $(1 \leqq s \leqq t)$ such that

$$(m_1(\sigma), m_2(\sigma), \cdots, m_n(\sigma)) = (m_1(\sigma_s), m_2(\sigma_s), \cdots, m_n(\sigma_s)) .$$

Hence by (4.3)

$$\rho(\sigma p_i, q_{m_i(\sigma)}) < \beta, \quad \rho(\sigma_s p_i, q_{m_i(\sigma_s)}) < \beta , \qquad i = 1, 2, \cdots, n . \tag{4.4}$$

On the other hand $m_i(\sigma) = m_i(\sigma_s)$, $i = 1, 2, \cdots, n$ and so

$$\rho(\sigma p_i, \sigma_s p_i) < [2]\beta, \qquad i = 1, 2, \cdots, n . \tag{4.5}$$

Consequently we have

(4.6) $\rho(\sigma_s^{-1} \sigma p_i, p_i) < [2]\beta$, $i = 1, 2, \cdots, n$ (see the Remark in Notation 3.1).

Let p be an arbitrary point in F. Then there exists a p_i such that $\rho(p, p_i) < \beta$. Hence we have from (4.6)

$$\rho(p, \sigma_s^{-1}\sigma p) \leqq \rho(p, p_i) + \rho(p_i, \sigma_s^{-1}\sigma p_i) + \rho(\sigma_s^{-1}\sigma p_i, \sigma_s^{-1}\sigma p) < 4\beta < \alpha .$$

(See the remark in Notation 3.2.)

This shows that $\sigma_s^{-1}\sigma \in U_{\alpha F} = U$, that is, $\sigma \in \sigma_s U$. Hence W is totally bounded.

THEOREM 4.3. *Under the assumptions of the preceding theorem, in order that an open set W of $\mathfrak{h}$ is σ-bounded* (*see Definition* 3.2) *it is necessary and sufficient that for every compact set F of Ω the set $W(F) = \{\sigma p ; \sigma \in W, p \in F\}$ is σ-bounded.*

PROOF. This is proved quite similarly as in the preceding theorem.

THEOREM 4.4. *Let $\mathfrak{h}$ be a group of homeomorphisms of Ω satisfying the conditions* (C_I) *and* (C_{II}) *of Theorem* 3.4. *$\mathfrak{h}$ is topologized as in Theorem* 3.7. *In order that $\mathfrak{h}$ is totally bounded it is necessary and sufficient that Ω is compact.*

PROOF. If Ω is compact, then $\mathfrak{h}(F)$ is always totally bounded for every compact subset F of Ω. From Theorem 4.2 we see that $\mathfrak{h}$ is totally bounded. Conversely, suppose that $\mathfrak{h}$ is totally bounded. Then from Theorem 4.2 $\mathfrak{h}(p) = \Omega$ is totally bounded (see Assumption (A_2)). Hence Ω is compact (see Theorem 4.1).

THEOREM 4.5. *Under the assumptions of the preceding theorem $\mathfrak{h}$ is always σ-bounded, that is, for any open set U of $\mathfrak{h}$ there exists a sequence $\sigma_1, \sigma_2, \cdots \sigma, \cdots$ of elements of $\mathfrak{h}$ such that*

$$\bigcup_{i=1}^{\infty} \sigma_i U = \mathfrak{h} .$$

PROOF. This is evident from Theorem 4.3, as Ω is σ-bounded.

THEOREM 4.6. *Let $\mathfrak{h}$ be a group of homeomorphisms of Ω satisfying the conditions* (C_I) *and* (C_{II}) *of Theorem* 3.4. *$\mathfrak{h}$ is topologized as in Theorem* 3.7.

If Ω is connected, then $\mathfrak{h}$ is always locally totally bounded.

PROOF. Since Ω is locally compact, there exists an $\alpha \in \Theta$ such that

(4.7) $$V_\alpha(p)^a \text{ is compact for every } p \in \Omega .$$

We select a $\beta \in \Theta$ such that

(4.8) $$\rho(p, q) < [2]\beta \quad \text{implies} \quad \rho(p, q) < \alpha .$$

We take an arbitrary point p_0 in Ω. From Theorem 3.3 and 3.2 it is evident that

(4.9) $$V_\beta{}^n(p_0) \text{ is totally bounded for every } n \text{ and } \bigcup_{n=1}^{n} V_\beta{}^n(p_0) = \Omega .$$

Let U be a neighborhood of the identity of $\mathfrak{h}$ such that

(4.10) $$U = \{\sigma \,;\, \sigma p_0 \in V_\beta(p_0)\} .$$

We shall show that U is totally bounded. Let F be a compact subset of Ω. From (4.9) there exists $V_\beta{}^n(p_0)$ such that $F \subseteqq V_\beta{}^n(p_0)$ (notice that $V_\beta{}^m(p_0) \subseteqq V_\beta{}^{n+1}(p_0)^i$). Hence for every point $p \in F$ there exists a β-Chain (see Definition 3.1) $\{p_0, p_1, \cdots, p_n = p\}$ of order n. Then for every $\sigma \in U$ the system $\{p_0, \sigma p_0, \sigma p_1, \cdots, \sigma p_n = \sigma p\}$ is clearly a β-chain of order $n+1$. This shows that $\rho(p_0, \sigma p) < [n+1]\beta$, that is, $\sigma p \in V_\beta{}^{n+1}(p_0)$. So we have $U(F) \subseteqq V_\beta{}^{n+1}(p_0)$. Hence $U(F)$ is totally bounded. From Theorem 4.2 we see that U is totally bounded.

THEOREM 4.7. *Let Ω be a metric space whose bounded set is compact and $\mathfrak{h}$ a group of isometric transformations of Ω satisfying the condition* (C_{II}) *of Theorem* 3.4. *We introduce a topology in $\mathfrak{h}$ as in Theorem* 3.7. *Then $\mathfrak{h}$ is always locally totally bounded.*

PROOF. For any $\varepsilon > 0$ and any point $p_0 \in \Omega$ we set $U_{\varepsilon p_0} = \{\sigma \,;\, d(p_0, \sigma p_0) < \varepsilon\}$. Then $U_{\varepsilon p_0}$ is a neighborhood of the identity of $\mathfrak{h}$. We shall show that $U_{\varepsilon p_0}$ is totally bounded. Let F be a compact subset of Ω. Then there exists a number $R > 0$ such that

(4.11) $$d(p_0, p) < R \quad \text{for every} \quad p \in F .$$

For every $\sigma \in U_{\varepsilon p_0}$ and every $p \in F$, we have $d(p_0, \sigma p) \leqq d(p_0, \sigma p_0) + d(\sigma p_0, \sigma p) < \varepsilon + d(p_0, p) \leqq \varepsilon + R$. This shows that the set $U_{\varepsilon p_0}(F)$ is contained in the sphere of center p_0 and radius $\varepsilon + R$, and consequently is totally bounded. Hence by Theorem 4.2 we get our theorem.

THEOREM 4.8. *Let Ω be a locally compact and σ-compact uniform space whose topology is defined by a system $\{V_\alpha, \alpha \in \Theta\}$ of symmetric uniform neighborhoods. We assume that for any two points p_0 and q_0 in Ω there exists a homeomorphism σ such that $\sigma p_0 = q_0$ and*

(C_I) *$\sigma V_\alpha(p) = V_\alpha(\sigma p)$ for every $\alpha \in \Theta$ and every $p \in \Omega$.*

Let $\mathfrak{h}^*$ be a group of all the homeomorphisms σ's which satisfy the above condition (C_I). *$\mathfrak{h}^*$ is topologized as in Theorem* 3.7. *We assume that $\mathfrak{h}^*$ is* *locally totally bounded, then* $\mathfrak{h}^*$ *is complete if the following condition is satisfied:*

(C_{III}) *For any point* $q\in V_\alpha(p)$ *there exists a* $\beta\in\Theta$ *(depending on* q*) such that* $r\in V_\beta(p)$ *and* $s\in V_\beta(q)$ *imply* $s\in V_\alpha(r)$.

PROOF. Let $\{\sigma_\xi, \xi\in\Phi\}$ be a generalized Cauchy sequence of $\mathfrak{h}^*$, that is, a sequence such that for any neighborhood U of the identity of $\mathfrak{h}^*$ there exists a $\xi_0\in\Phi$ such that $\xi_0\leqq\eta$ and $\xi_0\leqq\zeta$ imply $\sigma_\zeta^{-1}\sigma_\eta\in U$. Then it is easily seen that for every point $p\in\Omega$ the sequence $\{\sigma_\xi p, \xi\in\Phi\}$ is also a generalized Cauchy sequence in Ω. Since Ω is complete (see Theorem 4.1), the generalized Cauchy sequence $\{\sigma_\xi p, \xi\in\Phi\}$ converges to a limit $\varphi(p)$. From the condition (C_{III}) we can easily see that

$$\rho(p,q)<\alpha \quad \text{implies} \quad \rho(\varphi(p),\varphi(q))<\alpha\,. \tag{4.12}$$

We shall show that the generalized sequence $\{\sigma_\xi^{-1}, \xi\in\Phi\}$ is also a generalized Cauchy sequence of $\mathfrak{h}^*$.

To prove this it is sufficient to show that for every neighborhood $U=U_{\alpha F}$ (see Theorem 3.7) of the identity of $\mathfrak{h}^*$ there exists a $\xi_0\in\Phi$ such that $\xi_0\leqq\eta$ and $\xi_0\leqq\zeta$ imply $\sigma_\zeta\sigma_\eta^{-1}\in U$. Let $U_0=U_0^{-1}$ be a totally bounded neighborhood of the identity of $\mathfrak{h}^*$. Then there exists a $\xi_0\in\Phi$ such that $\xi_0\leqq\eta$ and $\xi_0\leqq\zeta$ imply $\sigma_\zeta^{-1}\sigma_\eta\in U_0$. Hence we have

$$\sigma_\eta\in\sigma_{\xi_0}U_0 \quad \text{for every} \quad \eta\geqq\xi_0\,. \tag{4.13}$$

Define

$$F_0=(U_0^{-1}(\sigma_{\xi_0}^{-1}F))^a \tag{4.14}$$

Since $\sigma_{\xi_0}^{-1}F$ is compact and $U_0^{-1}(=U_0)$ is totally bounded, $U_0^{-1}(\sigma_{\xi_0}^{-1}F)$ is totally bounded (see Theorem 4.2) and consequently F_0 is compact. Let U^* be a neighborhood of the identity of $\mathfrak{h}^*$ such that

$$U^*=U_{\alpha F_0} \qquad \text{(see Theorem 3.7).} \tag{4.15}$$

Then there exists a $\xi^*\in\Phi$ such that $\xi^*\leqq\eta$ and $\xi^*\leqq\zeta$ imply $\sigma_\zeta^{-1}\sigma_\eta\in U^*$. We chose a $\bar{\xi}_0\in\Phi$ such that $\xi_0\leqq\bar{\xi}_0$ and $\xi^*\leqq\bar{\xi}_0$. Then for every $p\in F$, $\eta\geqq\bar{\xi}_0$ and $\zeta\geqq\bar{\xi}_0$, we have

$$\rho(\sigma_\zeta\sigma_\eta^{-1}p, p)=\rho(\sigma_\eta^{-1}p, \sigma_\zeta^{-1}p)=\rho(\sigma_\eta^{-1}p, \sigma\sigma_\eta^{-1}p), \text{ where } \sigma\in U^*$$

(using (4.13)),

$$=\rho(\sigma_0^{-1}\sigma_{\xi_0}^{-1}p, \sigma\sigma_0^{-1}\sigma_{\xi_0}^{-1}p), \text{ where } \sigma_0\in U_0\,,$$

(using (4.14) and (4.15)),

$$<\alpha\,.$$

This shows that $\sigma_\zeta\sigma_\eta^{-1}\in U=U_{\alpha F}$. Hence the generalized sequence $\{\sigma_\xi^{-1}, \xi\in\Phi\}$ is a Cauchy sequence of $\mathfrak{h}^*$. So the generalized sequence $\{\sigma_\xi^{-1}p, \xi\in\Phi\}$ of Ω converges to a limit $\psi(p)$. Then we have similarly

$$\rho(p,q)<\alpha \quad \text{implies} \quad \rho(\psi(p),\psi(q))<\alpha\,. \tag{4.16}$$

On the other hand it is easily seen that $\psi(\varphi(p))=p$ and $\varphi(\psi(q))=q$. This shows that φ is a one-to-one mapping of Ω onto itself. Moreover from (4.16) we have

$$(4.17) \qquad \rho(\varphi(p), \varphi(q))<\alpha \quad \text{implies} \quad \rho(\psi(\varphi(p)), \psi(\varphi(q)))=\rho(p, q)<\alpha .$$

From (4.12) and (4.17) we see that φ satisfies the condition (C_I). Hence φ belongs to $\mathfrak{h}^*$. It is easily seen that φ is the generalized limit of $\{\sigma_\xi, \xi\in\Phi\}$. So $\mathfrak{h}^*$ is complete.

From § 3 and the above discussions in this §, we have the following theorem.

THEOREM. *Let Ω be a locally compact and σ-compact uniform space whose topology is defined by a system $\{V_\alpha, \alpha\in\Theta\}$ of symmetric uniform neighborhoods. Let $\mathfrak{h}$ be a group of homeomorphisms of Ω satisfying the following conditions:*

(C_I) $\sigma V_\alpha(p)=V_\alpha(\sigma p)$ *for every* $\sigma\in\mathfrak{h}$ *and every point* $p\in\Omega$.

(C_{II}) *For any two points p and q in Ω, there exists a $\sigma\in\mathfrak{h}$ such that $\sigma p=q$.*

(1) *Then we can introduce an outer measure m^* in Ω, such that*

- (i) $m^*(\sigma A)=m^*(A)$ *for every* $\sigma\in\mathfrak{h}$ *and every subset A of Ω,*
- (ii) *Baire sets are m^*-measurable,*
- (iii) *For every subset A of Ω there exists a Baire set B such that $A\subseteq B$ and $m^*(A)=m(B)$,*
- (iv) $m^*(A)<\infty$ *for every totally bounded set* $A\subseteq\Omega$,
- (v) $m(G)>0$ *for every open Baire set* $G\neq 0$.

(2) *We can introduce a topology in $\mathfrak{h}$ as follows:*
For every $\alpha\in\Theta$ and every compact subset $F\subseteq\Omega$ we define

$$U_{\alpha F}=\{\sigma ;\ \text{for every } p\in F \ \sigma p\in V_\alpha(p)\} .$$

Then the system $\{U_{\alpha F}; \alpha\in\Theta,$ compact set $F\}$ can be taken as a complete system of neighborhoods of the identity of $\mathfrak{h}$. $\mathfrak{h}$ is always σ-bounded (*see Definition* 3.2). *In order that $\mathfrak{h}$ is totally bounded it is necessary and sufficient that Ω is compact.*

(3) *The outer measure m^* is continuous with respect to $\mathfrak{h}$, that is, for any $\varepsilon>0$ and any measurable set A of finite measure there exists a neighborhood U of $\mathfrak{h}$, such that*

$$m(\sigma A\ominus A)<\varepsilon \ \text{for every}\ \sigma\in U ,$$

where $\sigma A\ominus A$ denotes the symmetric difference of σA and A.

(4) *If $\mathfrak{h}$ is locally compact and σ-compact, then the outer measure m^* which satisfies the conditions* (i)–(v) *of the assertion* (1) *is unique* (*up to a multiplicative constant*), *and any two measurable sets A and B of the same measure are decomposition-equivalent to each other with respect to $\mathfrak{h}$.*

(5) *Suppose that $\mathfrak{h}$ is locally compact and σ-compact. If $\mathfrak{h}_1$ is an abstract*

subgroup of $\mathfrak{h}$ *satisfying the following condition:*

(C_{II}') *For any two points* p *and* q *in* Ω *there exists a* $\sigma \in \mathfrak{h}_1$ *such that* $\sigma p = q$.

Then any $\mathfrak{h}_1$*-invariant measure* m_1^* *is a constant multiple of the measure* m^* *which is introduced in* (1), *and consequently any* $\mathfrak{h}_1$*-invariant measure is invariant under* $\mathfrak{h}$. *Any two measurable sets* A *and* B *of the same measure are decomposition-equivalent to each other with respect to* $\mathfrak{h}_1$.

(6) *Here we assume the further condition* (C_{III}) *of Theorem* 4.8. *Let* $\mathfrak{h}^*$ *be a group of all the homeomorphisms* σ*'s which satisfy the condition* (C_I). (*Of course* $\mathfrak{h} \subseteq \mathfrak{h}^*$ *holds.*) *We introduce a topology in* $\mathfrak{h}^*$ *as in* (2). *If* $\mathfrak{h}^*$ *is locally totally bounded, then* $\mathfrak{h}^*$ *is complete and hence locally compact.* $\mathfrak{h}^*$ *is locally compact and* σ*-compact in the following cases:*

(α) Ω *is compact.* (*In this case* $\mathfrak{h}^*$ *is also compact.*)

(β) Ω *is locally compact,* σ*-compact and connected.*

(γ) Ω *is a metric space whose bounded set is compact.*

Hence in these cases, regarded as $\mathfrak{h} = \mathfrak{h}^*$, *the assertions* (4) *and* (5) *are satisfied.*

THEOREM 4.9. *Let* Ω *be a locally compact,* σ*-compact and connected Hausdorff space and* $\mathfrak{h}$ *a group of homeomorphisms of* Ω *satisfying the condition* (C_{II}) *of the preceding theorem. Suppose that* m^* *is an outer measure in* Ω *satisfying the conditions* (i)-(v) *of the above theorem. If we can define a system* $\{V_\alpha, \alpha \in \Theta\}$ *of symmetric uniform neighborhoods of* Ω *which satisfies the conditions*

(0) *the condition* (C_{III}) *of Theorem* 4.8 *is satisfied,*

(1) *the topology of* Ω *which is introduced by the system* $\{V_\alpha, \alpha \in \Theta\}$ *coincides with the original topology of* Ω,

(2) $\sigma V_\alpha(p) = V_\alpha(\sigma p)$ *for every* $\sigma \in \mathfrak{h}$ *and every* $p \in \Omega$,

then any $\mathfrak{h}$*-invariant measure coincides with* m^*, *and the outer measure* m^* *is invariant under any homeomorphism* σ *which satisfies the above condition* (2) (*even though* σ *does not belong to* $\mathfrak{h}$).

PROOF. We have our theorem from (4), (5) and (6) of the preceding theorem.

COROLLARY. *Let* Ω *be the* n*-dimensional Euclidean space and* $\mathfrak{h}$ *the group of all the translations of* Ω. *Let* $f(x_1, x_2, \cdots, x_n)$ *be a continuous function of* n*-variables satisfying the following conditions:*

1° $f(x_1, x_2, \cdots, x_n) \geqq 0, \quad f(x_1, x_2, \cdots, x_n) = 0 \rightleftarrows x_1 = x_2 = \cdots = x_n = 0.$

2° $f(x_1, x_2, \cdots, x_n) = f(-x_1, -x_2, \cdots, -x_n).$

3° $f(x_1, x_2, \cdots, x_n) + f(y_1, y_2, \cdots, y_n) \geqq f(x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n).$

4° $f(x_1, x_2, \cdots, x_n) \to 0$ implies $(x_1^2 + x_2^2 + \cdots + x_n^2) \to 0.$

We define

$d(x, y) = f(x_1 - y_1, x_2 - y_2, \cdots, x_n - y_n)$ *for any two points* $x = (x_1, x_2, \cdots, x_n)$ *and* $y = (y_1, y_2, \cdots, y_n)$ *in* Ω.

This metric $d(x, y)$ *is invariant under* $\mathfrak{h}$ *and obviously introduce the original* *topology in Ω. The Lebesgue measure is invariant under any d-isometric transformation σ. In particular, setting $f(x_1, x_2, \cdots, x_n)=\sqrt{x_1^2+x_2^2+\cdots+x_n^2}$, the Lebesgue measure is invariant under any rotation.*

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