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On projective transformations of Riemannian manifolds.

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The main purpose of this paper is to prove the following theorems:

THEOREM 1. *Let M and M' be n-dimensional Riemannian manifolds and suppose that M is locally reducible (but M' is not necessarily locally reducible). If there exists a projective transformation of M to M', then*

1) *the transformation preserves the curvature tensor, or*

2) *the local homogeneous holonomy group at any point of M' is the proper orthogonal group $O^+(n)$.*[1]

THEOREM 2. *Let M and M' be complete Riemannian manifolds. In order that there exist a non-affine projective transformation of M to M', it is necessary that both M and M' be irreducible.*

By Theorem 2 and a theorem due to T. Y. Thomas [**8**] and A. Lichnerowicz [**3**], we have

COROLLARY.[2] *If a complete Riemannian manifold with parallel Ricci tensor admits a non-affine projective transformation, then the manifold is an irreducible Einstein manifold.*

Recently, several authors [**4**], [**7**] investigated the manifolds with parallel Ricci tensor and T. Nagano [**4**] proved that a complete Einstein manifold admitting a non-affine projective transformation is the only manifold whose universal covering space is a sphere.

The author expresses his hearty thanks to his colleague T. Nagano who gave him many valuable suggestions in the course of the preparation of this paper.

§ 1. Preliminaries.

Let M and M' be n-dimensional Riemannian manifolds and f a diffeomorphism of M to M'. For a geometric object Ω' on M', we denote by $'\Omega$ the geometric object on M induced from Ω' by f. For instance we denote by

1) In this paper we suppose that the class of differentiability of manifolds and of transformations is not less than 4. If the class is of C^∞, then, in Case 2), the infinitesimal holonomy group of M' is also $O^+(n)$. As to these holonomy groups, see A. Nijenhuis [**5**].

2) T. Nagano has proved this corollary by a different method.

$'g_{\mu\lambda}$ the induced metric tensor on M from the metric tensor of M' [3].

A transformation f is said to be projective if it carries geodesics in M to geodesics in M'. As is well-known, a necessary and sufficient condition for a transformation f to be projective is that there exist a vector field p_λ on M such that

(1.1) $$'\{{}_{\mu}{}^{\kappa}{}_{\lambda}\} = \{{}_{\mu}{}^{\kappa}{}_{\lambda}\} + p_\mu A_\lambda^\kappa + p_\lambda A_\mu^\kappa ,$$

A_λ^κ being the unity tensor. The vector field p_λ should be a gradient vector field:

(1.2) $$p_\lambda = \partial_\lambda p ,$$

p being a scalar field. If p_λ is the null vector field, the transformation is affine.

It is well known that, for a projective transformation, we have

(1.3) $$'K_{\nu\mu\lambda}{}^{\kappa} = K_{\nu\mu\lambda}{}^{\kappa} + A_\nu^\kappa p_{\mu\lambda} - A_\mu^\kappa p_{\nu\lambda}$$

and

(1.4) $$\nabla_\nu {}'g_{\mu\lambda} = 2p_\nu {}'g_{\mu\lambda} + p_\mu {}'g_{\nu\lambda} + p_\lambda {}'g_{\nu\mu} ,$$

where ∇ indicates the covariant differentiation with respect to $\{{}_{\mu}{}^{\kappa}{}_{\lambda}\}$, $K_{\nu\mu\lambda}{}^{\kappa}$ is the curvature tensor of M and we have put

(1.5) $$p_{\mu\nu} = -\nabla_\mu p_\lambda + p_\mu p_\lambda .$$

From the integrability condition of (1.4), we have

(1.6) $$K_{\nu\mu\lambda}{}^{\alpha} {}'g_{\alpha\kappa} + K_{\nu\mu\kappa}{}^{\alpha} {}'g_{\alpha\lambda} = p_{\nu\lambda} {}'g_{\mu\kappa} + p_{\nu\kappa} {}'g_{\mu\lambda} - p_{\mu\lambda} {}'g_{\nu\kappa} - p_{\mu\kappa} {}'g_{\nu\lambda} .$$

On the other hand, a Riemannian manifold M is said to be locally reducible at a point of M, if the local holonomy group at the point is reducible; in other words, in a neighborhood of the point, the manifold is locally a product of a number of Riemannian manifolds:

(1.7) $$M_1 \times M_2 \times \cdots \times M_r ,$$

that is, there is a coordinate system in which the metric tensor field $g_{\mu\lambda}$ is given by a reduced matrix[4]

(1.8) $$\begin{pmatrix} g_{j_1 i_1}(x^{h_1}) & & & \\ & g_{j_2 i_2}(x^{h_2}) & & \\ & & \cdots\cdots\cdots & \\ & & & g_{j_r i_r}(x^{h_r}) \end{pmatrix}$$

3) In this paper we follow the notations of J. A. Schouten [**6**], especially, pp. 287-296.

4) Supposing that the dimension of each part M_t is n_t $(t=1, 2, \cdots, r)$, $n_1+n_2+\cdots+n_r=n$, the indices h_t, i_t, j_t, $\cdots$ run from $n_1+\cdots+n_{t-1}+1$ to $n_1+\cdots+n_t$.

each $g_{j_t i_t}$ depending only on x^{h_t}'s. Such a coordinate system will be called a separating coordinate system. If M is locally reducible at every point, then M is said to be locally reducible; otherwise M is said to be irreducible.

If a Riemannian manifold is complete and reducible, that is, the restricted homogeneous holonomy group is reducible, then the product (1.7) has the global meaning of de Rham decomposition [1]: The universal covering space of the manifold is a direct product of a number of irreducible Riemannain manifolds in the large.

In a separating coordinate system, the non-vanishing components of $\{{}_{\mu}{}^{\kappa}{}_{\lambda}\}$ and of $K_{\nu\mu\lambda}{}^{\kappa}$ are respectively only $\{{}_{j_t}{}^{h_t}{}_{i_t}\}$ and $K_{k_t j_t i_t}{}^{h_t}$, which are dependent only on x^{h_t} $(t=1,2,\cdots,r)$. These properties still hold when we decompose locally M into two parts $M_1\times M_2$, where M_1 and M_2 may be reducible or not. In the proofs of lemmas in § 2, we shall use this decomposition.

§ 2. Lemmas.

LEMMA 1. *Let M be locally reducible at a point x and f a projective transformation of M to M'. If the induced metric tensor field $'g_{\mu\lambda}$ has the property*

$$'g_{j_2 i_1}=0 \tag{2.1}$$

in a separating coordinate neighborhood of x in M, then M' is also reducible at $x'=f(x)$ and the transformation f is affine at x.

PROOF. By assumption, we have

$$'g^{i_2 h_1}=0\,. \tag{2.2}$$

Putting $\kappa=h_2$, $\lambda=i_1$, $\mu=j_1$ in (1.1), we have

$$'\{{}_{j_1}{}^{h_2}{}_{i_1}\}=-\frac{1}{2}\,'g^{h_2 a_2}\partial_{a_2}{}'g_{j_1 i_1}=0\,. \tag{2.3}$$

Hence $'g_{j_1 i_1}$ are independent of x^{h_2}'s. Similarly $'g_{j_2 i_2}$ are independent of x^{h_1}'s. Thus M' is locally reducible. Putting again $\kappa=h_2$, $\lambda=i_1$, $\mu=j_2$ in (1.1), we have $p_{i_1}=0$ and similarly $p_{i_2}=0$. Thus the vector field p_λ of the transformation vanishes.

LEMMA 2. *Let M be a locally reducible Riemannian manifold, but M' not necessarily locally reducible. For a projective transformation of M to M', we have an equation*

$$p_{\mu\lambda}=\sigma' g_{\mu\lambda}\,, \tag{2.4}$$

σ being a proportional factor.

PROOF. In a separating coordinate system at an arbitrary point of M, we prove first that we have

$$p_{\mu i_1}=\sigma_{i_1}{}'g_{\mu i_1}\,,\qquad p_{\mu i_2}=\sigma_{i_2}{}'g_{\mu i_2} \qquad \text{(not summed in } i_1 \text{ and } i_2) \tag{2.5}$$

for every i_1 and every i_2, where σ_{i_1} and σ_{i_2} are proportional factors. We suffice to prove the first equation of (2.5) for a fixed i_1. Putting $\kappa=\lambda=i_1$ and $\nu=k_2$ in (1.6), we have

$$p_{k_2 i_1}{}'g_{\mu i_1}=p_{\mu i_1}{}'g_{k_2 i_1} \qquad \text{(not summed in } i_1). \tag{2.6}$$

If $'g_{k_2 i_1}\neq 0$ for some k_2, then the envisaged equation follows directly from (2.6). If $'g_{k_2 i_1}=0$ for all k_2, we have $p_{k_2 i_1}=0$ for all k_2. Then, putting $\lambda=i_1$, $\mu=j_1$, $\nu=k_2$ in (1.6), we have

$$p_{k_2\kappa}{}'g_{j_1 i_1}=p_{j_1 i_1}{}'g_{k_2\kappa}\,. \tag{2.7}$$

Since not all of $'g_{k_2\kappa}$ vanish, we have

$$p_{j_1 i_1}=\sigma_{i_1}{}'g_{j_1 i_1} \qquad \text{(not summed in } i_1) \tag{2.8}$$

and hence we obtain (2.5).

Next, if $'g_{j_2 i_1}\neq 0$, then we have clearly $\sigma_{i_1}=\sigma_{j_2}$. On the other hand, if $'g_{j_2 i_1}=0$, then, putting $\kappa=\mu=j_2$ and $\lambda=\nu=i_1$ in (1.6), we have

$$(\sigma_{j_2}-\sigma_{i_1})'g_{i_1 i_1}{}'g_{j_2 j_2}=0 \qquad \text{(not summed in } i_1 \text{ and } j_2). \tag{2.9}$$

Since any principal minor matrix of a positive definite matrix is also positive definite, we have

$$\sigma_{i_1}=\sigma_{j_2}=\sigma \tag{2.10}$$

for all i_1 and for all j_2. Q. E. D.

LEMMA 3. *Under the hypothesis of Lemma* 2, *we have an equation*

$$K_{\nu\mu\lambda}{}^{\kappa}p_{\kappa}=0\,. \tag{2.11}$$

PROOF. Substituting (2.4) into (1.6), we have

$$K_{\nu\mu\lambda}{}^{\alpha\prime}g_{\alpha\kappa}+K_{\nu\mu\kappa}{}^{\alpha\prime}g_{\alpha\lambda}=0\,, \tag{2.12}$$

and, in a separating coordinate system,

$$K_{k_1 j_1 i_1}{}^{a_1\prime}g_{a_1 h_2}=0\,. \tag{2.13}$$

Differentiating covariantly this equation with respect to x^{l_2} and substituting (1.4), we obtain

$$K_{k_1 j_1 i_1}{}^{a_1}p_{a_1}{}'g_{l_2 h_2}=0\,. \tag{214}$$

Since not all of $'g_{l_2 h_2}$ vanish, we have

$$K_{k_1 j_1 i_1}{}^{a_1}p_{a_1}=0\,. \tag{2.15}$$

Similarly we have $K_{k_2 j_2 i_2}{}^{a_2}p_{a_2}=0$. Q. E. D.

From (2.13) and Lemma 1, we can state

THEOREM 3. *Let a Riemannian manifold M be locally decomposed into $M_1\times\cdots\times M_r$. If the components $K_{j_t i_t}$ of the Ricci tensor $K_{\mu\lambda}$ corresponding to one of the parts M_t's form a regular matrix in a separating coordinate neighborhood, then a projective transformation of M to a Riemannian manifold M' is affine in the neighborhood.*

From the equations (1.3), (2.4) and (2.11), we obtain an important equation

$$'K_{\nu\mu\lambda}{}^{\kappa}p_{\kappa}=\sigma(p_{\nu}{}'g_{\mu\lambda}-p_{\mu}{}'g_{\nu\lambda})\,. \tag{2.16}$$

LEMMA 4. *The proportional factor σ is a constant.*

PROOF.[5] The equation (1.5) can be written in the form

$$-{}'\nabla_{\mu}p_{\lambda}-p_{\mu}p_{\lambda}=\sigma'g_{\mu\lambda} \tag{2.17}$$

by means of $'\{{}_{\mu}{}^{\kappa}{}_{\lambda}\}$. Differentiating covariantly this equation with respect to $'\{{}_{\mu}{}^{\kappa}{}_{\lambda}\}$, and applying the Ricci formula, we obtain

$$(\partial_{\nu}\sigma)'g_{\mu\lambda}-(\partial_{\mu}\sigma)'g_{\nu\lambda}=0\,, \tag{2.18}$$

and thus $\partial_{\nu}\sigma=0$. Q. E. D.

PROOF OF THEOREM 1. 1) If and only if σ is equal to zero, the transformation preserves the curvature tensor. 2) If σ is not equal to zero, the equation (2.16) shows that the local homogeneous holonomy group (or the infinitesimal holonomy group) is the proper orthogonal group $O^{+}(n)$.

§ 3. Local structure of the manifold M'.

Since we shall deal only with M' in this paragraph, we shall drop 'prime,' distinguishing quantities of M' from those of M, for brevity.

Now we consider hypersurfaces V''s in M' defined by equations

$$p=\text{constant}. \tag{3.1}$$

Transvection of (2.17) with p^{μ} $(='g^{\mu\lambda}p_{\lambda})$ leads to

$$p^{\mu}\nabla_{\mu}p^{\lambda}=-(\sigma+p_{\mu}p^{\mu})p^{\lambda}\,, \tag{3.2}$$

and hence the integral curves of the vector field p^{κ} are geodesics and the congruence C' of these geodesics is normal to the hypersurfaces V''s. If i^{κ} is the unit vector field of p^{κ} on M' and we put

$$p^{\kappa}=qi^{\kappa}\,, \tag{3.3}$$

then we have from (2.17)

$$-(\nabla_{\mu}q)i_{\lambda}-q\nabla_{\mu}i_{\lambda}-q^{2}i_{\mu}i_{\lambda}=\sigma g_{\mu\lambda} \tag{3.4}$$

and

$$\nabla_{\mu}q=-(\sigma+q^{2})i_{\mu}\,. \tag{3.5}$$

5) This simple proof is due to Prof. K. Yano.

These imply that

$$\nabla_\mu i_\lambda = -h(g_{\mu\lambda} - i_\mu i_\lambda)\,, \tag{3.6}$$

where

$$h = \sigma/q\,. \tag{3.7}$$

We choose a local coordinate system u^a $(a, b, c, d = 1, 2, \cdots, n-1)$ in one of the hypersurfaces V''s and give the same coordinates to the points of other hypersurfaces which lie on the same geodesic of C'. On each hypersurface V', the second fundamental tensor is

$$h_{cb} = -B_{c\ b}^{\mu\lambda} \nabla_\mu i_\lambda = h\bar{g}_{cb}\,, \tag{3.8}$$

where $B_b{}^\lambda = \partial_b x^\lambda$ and $\bar{g}_{cb}$ is the induced metric tensor of V'. Therefore the hypersurfaces are totally geodesic if $\sigma = 0$ or totally umbilical if $\sigma \neq 0$.

From the Codazzi equation

$$\nabla_d h_{cb} - \nabla_c h_{db} = B_{d\ c\ b}^{\nu\mu\lambda} K_{\nu\mu\lambda\kappa} i^\kappa \tag{3.9}$$

and (2.16), we obtain

$$(\nabla_d h)\bar{g}_{cb} - (\nabla_c h)\bar{g}_{db} = 0\,. \tag{3.10}$$

By transvection with $\bar{g}^{cb}$, we see that h is constant on each of the hypersurfaces and consequently so is q.

Along a geodesic $x^\kappa = x^\kappa(s)$ of the congruence C', where s is the arc-length in M', we have

$$i^\kappa = dx^\kappa/ds\,, \tag{3.11}$$

$$q = p_\lambda(dx^\lambda/ds) = dp/ds\,, \tag{3.12}$$

and (2.17) becomes an ordinary differential equation

$$\frac{d^2p}{ds^2} + \left(\frac{dp}{ds}\right)^2 + \sigma = 0\,. \tag{3.13}$$

According to the sign of σ, we put

$$\sigma = \begin{cases} \text{I)} & 0\,, \\ \text{II)} & +c^2\,, \\ \text{III)} & -c^2\,, \end{cases} \tag{3.14}$$

c being a positive constant. Then, solving (3.13), q is given by

$$q = \begin{cases} \text{I)} & 1/(s-a)\,, \\ \text{II)} & -c\tan c(s-a)\,, \\ \text{III, i)} & c\coth c(s-a) \quad \text{for} \ |q| > c\,, \\ \text{III, ii)} & c\tanh c(s-a) \quad \text{for} \ |b| < c\,, \end{cases} \tag{3.15}$$

and the general solution p is given by

$$(3.16)\qquad p=\begin{cases} \text{I)} & \log|s-a|+b\,, \\ \text{II)} & \log|\cos c(s-a)|+b\,, \\ \text{III, i)} & \log|\sinh c(s-a)|+b\,, \\ \text{III, ii)} & \log\cosh c(s-a)+b\,, \end{cases}$$

a and b being arbitrary constants.

Since p and q are constant on each V', we can assume, by a suitable choice of a and b, that the points on all geodesics of C' corresponding to the same value of s are on the same hypersurface, which we denote by $V'(s)$. Moreover, without loss of generality, we may put $a=0$ and $b=0$.

If we regard the parameter s of the geodesics of C' as the n-th coordinate of the manifold M', then (u^a, s) constitute a local coordinate system of M'. In this coordinate system, the metric tensor has components

$$(3.17)\qquad \begin{aligned} &g_{cb}(u^a, s)=\bar{g}_{cb}(u) \qquad \text{on} \quad V'(s)\,, \\ &g_{nb}=0, \quad g_{nn}=1\,, \end{aligned}$$

and the vector of the projective transformation has components

$$(3.18)\qquad p_b=0, \quad p_n=q\,.$$

If we put $\lambda=b$, $\mu=c$ in (2.17), we have an equation

$$(3.19)\qquad \frac{1}{2}\,q\,\frac{\partial g_{cb}}{\partial s}+\sigma g_{cb}=0\,,$$

whose solution is given by

$$(3.20)\qquad g_{cb}(u, s)=\exp\Big(-2\sigma\int\frac{1}{q}\,ds\Big)\,f_{cb}(u)\,,$$

that is,

$$(3.21)\qquad g_{cb}(u, s)=\begin{cases} \text{I)} & f_{cb}(u)\,, \\ \text{II)} & (\sin cs)^2 f_{cb}(u)\,, \\ \text{III, i)} & 4(\cosh cs)^2 f_{cb}(u)\,, \\ \text{III, ii)} & 4(\sinh cs)^2 f_{cb}(u)\,, \end{cases}$$

where $f_{cb}(u)$ depend only on $u^1,\cdots, u^{n-1}$. Thus the local structure of M' has been determined.

Theorem 1 tells us that, if both M and M' are locally reducible, then a projective transformation of M to M' preserves the curvature tensor, that is, $\sigma=0$. Conversely, if there is a non-affine projective transformation preserving the curvature tensor, then we have

(3.22) $$p_{\mu\lambda}=0\,,$$

and therefore M and M' are locally reducible. Therefore we can conclude

THEOREM 4. *A necessary and sufficient condition that Riemannian manifolds, which are locally related to each other by a non-affine projective transformation, be both locally reducible, is that the transformation preserve the curvature tensor.*

§4. Proof of Theorem 2.

It is sufficient to prove that, if one of the manifolds, say M, is locally reducible, then there exists no non-affine projective transformation. We use again 'prime' to distinguish quantities of M' from those of M. We see from (3.16) that, if M' is complete, the function p has singularities on a hypersurface $V'(0)$ in Case I)[6] or in Case III, i) or on $V'(\pi/2c)$ in Case II).

In Case III, ii), we consider the inverse image of a geodesic $x^\kappa=x^\kappa('s)$ of C'. The image is also a geodesic in $M: x^\kappa=x^\kappa(s)$, s being the arc-length in M. Along the geodesic in M, the equation (1.4) becomes

(4.1) $$\frac{d}{ds}\left(\frac{d\,'s}{ds}\right)^2=4\frac{dp}{ds}\left(\frac{d\,'s}{ds}\right)^2.$$

Then, we have

(4.2) $$d\,'s/ds=A\cosh^2 c\,'s\,,$$

A being an arbitrary constant, and the solution of (4.1) with initial condition $'s=0$ for $s=0$ is given by

(4.3) $$s=(\tanh c\,'s)/Ac\,,$$

or

(4.4) $$\exp 2c\,'s=(1+Acs)/(1-Acs)\,.$$

Substituting (4.4) into (3.16, III, ii), we have

(4.5) $$p=-(\log(1-(Acs)^2))/2\,.$$

Therefore, if M is complete, p has singularities on a hypersurface $V(1/Ac)$ in M. Q. E. D.

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6) In case I), the theorem has been proved by S. Ishihara [**2**].

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