

Conjugate classes of Cartan subalgebras in real semisimple Lie algebras.

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Introduction.

Cartan subalgebras play an important part in the theory of Lie algebras. Our main purpose in this paper is to find all Cartan subalgebras in real semisimple Lie algebras up to conjugacy under the adjoint groups or the full automorphism groups. The problem is very simple in complex semisimple Lie algebras, because all Cartan subalgebras in a Lie algebra over an algebraically closed field of characteristic zero are mutually conjugate. This conjugacy theorem does not hold for a Lie algebra over a field which is not algebraically closed. For example, let $\mathfrak{g} = \mathfrak{sl}(2, R)$ be the Lie algebra of all 2×2 real matrices with trace zero, then \mathfrak{g} has two Cartan subalgebras

$$\mathfrak{h}_1 = \left\{ \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}; t \in R \right\} \text{ and } \mathfrak{h}_2 = \left\{ \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}; \theta \in R \right\}.$$

\mathfrak{h}_1 and \mathfrak{h}_2 are not conjugate under an inner automorphism of \mathfrak{g} , because \mathfrak{h}_1 generates a non compact group

$$H_1 = \left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}; t \in R \right\},$$

while \mathfrak{h}_2 generates a compact group

$$H_2 = \left\{ \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}; \theta \in R \right\}.$$

The conjugate classes of Cartan subalgebras in a Lie algebra over a general field of characteristic zero were first treated by N. Iwahori and I. Satake [8]. They proved the conjugacy of Cartan subalgebras for solvable Lie algebras. Later, the conjugate classes of Cartan subalgebras (or subgroups) attracted the attention of mathematicians in connection with the theory of unitary representations. I. M. Gelfand and M. I. Graev remarked that the existence of $[n/2]+1$ different conjugate classes of Cartan subgroup in $SL(n, R)$ is connected with the existence of $[n/2]+1$ different principal non degenerate series of irreducible unitary representations of $SL(n, R)$. Harish-Chandra, interested also in this phenomenon, proved that

every Cartan subalgebra in a real semisimple Lie algebra \mathfrak{g} is conjugate to a standard one and there are only a finite number of conjugate classes of Cartan subalgebras in \mathfrak{g} (cf. [6]). A deeper study was made by B. Kostant [9]. He gave an invariant of the conjugate class of Cartan subalgebras, by means of which he could determine the conjugate classes of these subalgebras in real simple Lie algebras. (He announced to give a list containing all these informations, which is however not yet published.)

In the following, we shall first reestablish Harish-Chandra's results, generalizing a method of G. A. Hunt [7], and then give an invariant, which is simpler than that of Kostant's, of the conjugate class of Cartan subalgebras. In the final paragraphs, we shall give a complete list of conjugate classes of Cartan subalgebras under the adjoint groups or under the full automorphism groups for each type of classical or exceptional real simple Lie algebras, which will enable us to determine also the conjugate classes of Cartan subalgebras of real semisimple Lie algebras. More precisely, this paper is divided into five paragraphs as follows.

In §1, the problem of conjugacy under the adjoint group is reduced to the corresponding problem under the Weyl group. Our fundamental tool is Theorem 1 which is a generalization of a lemma used by G. A. Hunt in [7]. Theorem 1 is useful not only for compact Lie groups but also for non compact semisimple Lie groups, on account of the close connection between a semisimple Lie group G and the compact form G_u of G^c . By the repeated use of Theorem 1, we obtain Theorems 2, 3 and 4 which state each step of the reduction. Theorem 2 and Theorem 3 were obtained by Harish-Chandra [6]. Theorems 2, 3 and 4 were also announced by B. Kostant [9].

In §2, we shall prove that there is a one to one correspondence between conjugate classes of Cartan subalgebras in a real semisimple Lie algebra \mathfrak{g} under the adjoint group and conjugate classes of the sets of roots which satisfy certain conditions (admissible root systems, cf. Definition 9) under the Weyl group (Theorem 6). This is our main theorem.

In §3 and §4, we give the number of conjugate classes and a representative of each class for every real form of complex simple Lie algebras. The classical real simple Lie algebras are treated in §3, and the exceptional simple algebras are treated in §4. As an application of our results, we prove that the adjoint group G of a real semisimple Lie algebra \mathfrak{g} contains compact Cartan subgroups if and only if \mathfrak{g} is of the first category, i. e. \mathfrak{g} is defined by an involutive *inner* automorphism of the compact form \mathfrak{g}_u (Theorem 8).

In §5, we determine the conjugate classes of Cartan subalgebras in a real semisimple Lie algebra under the automorphism group \tilde{G} instead of the adjoint group G . There exists a one to one correspondence between the

conjugate classes of Cartan subalgebras under \tilde{G} and conjugate classes of the admissible root systems under the Cartan group \tilde{W} (Theorem 10). The conjugate classes under G are also conjugate classes under \tilde{G} in almost every real simple Lie algebra, except in one type of simple algebras. These simple Lie algebras are the Lie algebras of the type which we call (DIc). They are the Lie algebras of the orthogonal groups with respect to the quadratic forms with maximal indices over $4n$ -dimensional real vector spaces (Theorem 11). By the knowledge of conjugate classes in simple Lie algebras, we can easily find the conjugate classes of Cartan subalgebras in general semisimple Lie algebras (Theorem 12).

The conjugate classes of Cartan subalgebras in real Lie algebras with radicals can be determined by the method used in [8] and the results in this paper. In particular, the number of conjugate classes of Cartan subalgebras in a real Lie algebra is always finite. These subjects will be treated elsewhere.

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§ 1. Standard Cartan subalgebras.

DEFINITION 1. A subalgebra \mathfrak{h} of a semisimple Lie algebra \mathfrak{g} over a field of characteristic zero is called a *Cartan subalgebra*, if \mathfrak{h} satisfies the following two conditions:

- 1) \mathfrak{h} is a maximal abelian subalgebra of \mathfrak{g} .
- 2) For any $X \in \mathfrak{h}$, $\text{ad } X$ is a semisimple linear transformation.

DEFINITION 2. A real subalgebra \mathfrak{g}_0 of a complex Lie algebra \mathfrak{g} is called a *real form* of \mathfrak{g} , if the complexification $\mathfrak{g}_0^c = C \otimes \mathfrak{g}_0$ is isomorphic to \mathfrak{g} by the canonical mapping $a \otimes X \rightarrow aX$, i. e. if $\dim_R \mathfrak{g}_0 = \dim_C \mathfrak{g}$.

Notations. Throughout this paper we shall use the following notations. Lie groups are denoted by Latin capitals, and the Lie algebras are denoted by corresponding small German letters.

- \mathfrak{g} : A real semisimple Lie algebra.
- \mathfrak{g}^c : The complexification of \mathfrak{g} .
- θ : The conjugation of \mathfrak{g}^c with respect to \mathfrak{g} .

Every element X of \mathfrak{g}^c is uniquely expressed as

$$X = Y + \sqrt{-1} Z, \quad Y, Z \in \mathfrak{g},$$

then

$$\theta X = Y - \sqrt{-1} Z.$$

\mathfrak{g}_u : The compact real form of \mathfrak{g}^c .

G, G^c , and G_u : the adjoint group of $\mathfrak{g}, \mathfrak{g}^c$ and \mathfrak{g}_u respectively. Always we regard $G \subset G^c, G_u \subset G^c$.

$B(X, Y)$: The Killing form of G , i. e. $B(X, Y) = \text{Tr ad } X \text{ ad } Y$.

For every linear form λ on a Cartan subalgebra \mathfrak{h} of \mathfrak{g} we denote by H_λ the unique element of \mathfrak{h} satisfying

$$\lambda(H) = B(H_\lambda, H) \quad \text{for all } H \in \mathfrak{h}.$$

For every subset \mathfrak{a} of \mathfrak{g} , we denote the orthogonal complement of \mathfrak{a} by \mathfrak{a}^\perp , i. e.

$$\mathfrak{a}^\perp = \{X \in \mathfrak{g}; B(X, \mathfrak{a}) = 0\}.$$

Let σ be an involution, i. e. an automorphism of order 2, of a real semisimple Lie algebra \mathfrak{g} . We put

$$(1) \quad \mathfrak{k} = \{X \in \mathfrak{g}; \sigma X = X\}, \quad \mathfrak{p} = \{X \in \mathfrak{g}; \sigma X = -X\}.$$

then we have

$$(2) \quad \mathfrak{g} = \mathfrak{k} + \mathfrak{p} \quad (\text{direct sum})$$

and

$$(3) \quad [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}.$$

The last relation (3) shows that the subset

$$(4) \quad \mathfrak{g}_u = \mathfrak{k} + \sqrt{-1} \mathfrak{p}$$

is a real form of \mathfrak{g}^c .

In the following we shall always use the symbols $\mathfrak{k}, \mathfrak{p}$, and \mathfrak{g}_u in this sense. We shall denote the conjugation of \mathfrak{g}^c with respect to \mathfrak{g} (or \mathfrak{g}_u) by θ (or η). \mathfrak{g}_u is invariant by θ as a whole, and η is identical with σ on \mathfrak{g} . Conversely if we can find a real form \mathfrak{g}_u of \mathfrak{g}^c invariant by θ , then the conjugation η of \mathfrak{g}_u induces an involution σ on \mathfrak{g} .

DEFINITION 3. An involution σ is called *compact* if the real form \mathfrak{g}_u defined by (4) is a compact real form of \mathfrak{g}^c . The decomposition (2) associated with a compact involution σ is called a *Cartan decomposition* of \mathfrak{g} .

If \mathfrak{g} admits a complex structure, then $\mathfrak{k} = \mathfrak{g}_u$ and $\mathfrak{p} = \sqrt{-1} \mathfrak{g}_u$ give a Cartan decomposition of \mathfrak{g} , where \mathfrak{g}_u is a compact real form of \mathfrak{g} .

E. Cartan proved that every real semisimple Lie algebra always admits a Cartan decomposition (cf. E. Cartan [3], also G. D. Mostow [9]). He also proved that any two such decompositions are "conjugate to each other" (E. Cartan [3, p. 16]). This means that the following proposition holds. (We shall give here this proposition with proof, in view of its importance in what follows.)

PROPOSITION 1. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p} = \mathfrak{k}' + \mathfrak{p}'$ be two Cartan decompositions of \mathfrak{g} associated with involutions σ and σ' . Then there exists an element \mathfrak{p} in the adjoint

group G of \mathfrak{g} , which transforms \mathfrak{k} onto \mathfrak{k}' and \mathfrak{p} onto \mathfrak{p}' .

PROOF. Let $G_0 = O(B)$ be the orthogonal group with respect to the Killing form $B(X, Y)$ of \mathfrak{g} . Since σ belongs to G_0 , we can define the inner automorphism α of G_0 by means of $\sigma: \alpha(g) = \sigma g \sigma^{-1}$. Let $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ be the Cartan decomposition of \mathfrak{g}_0 associated with the involution $d\alpha = Ad\sigma$. Let K_0 be the analytic subgroup of G_0 generated by \mathfrak{k}_0 and $P_0 = \{\exp X \in G_0; X \in \mathfrak{p}_0\}$. Then every element $g \in G_0$ can be decomposed uniquely as $g = pk$; $p \in P_0$, $k \in K_0$. It is easily seen that for $p, k \in G_0$,

$$(5) \quad p \in P_0 \Leftrightarrow \alpha(p) = p^{-1} \Leftrightarrow p\sigma = \sigma p^{-1},$$

and

$$(6) \quad k \in K_0 \Leftrightarrow \alpha(k) = k \Leftrightarrow k\mathfrak{k} = \mathfrak{k} \text{ and } k\mathfrak{p} = \mathfrak{p}.$$

Now, owing to Sylvester's law of inertia there exists an element $g \in G_0$ such that

$$g\mathfrak{k} = \mathfrak{k}', \quad g\mathfrak{p} = \mathfrak{p}'.$$

If we decompose this element g as $g = pk$; $p \in P_0$, $k \in K_0$, then we have

$$(7) \quad p\mathfrak{k} = \mathfrak{k}', \quad p\mathfrak{p} = \mathfrak{p}'.$$

For any $X \in \mathfrak{k}$, $Y \in \mathfrak{p}$, pX belongs to $p\mathfrak{k} = \mathfrak{k}'$ and pY belongs to $p\mathfrak{p} = \mathfrak{p}'$, hence we have

$$\sigma'p(X+Y) = \sigma'(pX+pY) = pX-pY = p(X-Y) = p\sigma(X+Y), \text{ i. e.}$$

$$(8) \quad \sigma'p = p\sigma.$$

p is represented by a positive definite symmetric matrix with respect to an orthonormal base (i. e. a base (e_i) which satisfies $B(e_i, e_j) = \pm \delta_{ij}$) of \mathfrak{g} . Hence there exists a basis (U_i) of \mathfrak{g} consisting of the eigenvectors of p , i. e.

$$pU_i = c_i U_i \quad (c_i > 0).$$

Let γ_{ij}^k be the structure constants of \mathfrak{g} with respect to the basis (U_i) , then we have

$$(9) \quad [U_i, U_j] = \sum \gamma_{ij}^k U_k.$$

If we put $U_i' = pU_i$, then (U_i') is another basis of \mathfrak{g} . Let γ'_{ij}^k be the structure constant of \mathfrak{g} with respect to (U_i') , then

$$(10) \quad [U_i', U_j'] = \sum \gamma'_{ij}^k U_k'.$$

Replacing U_i' 's in the equality (10) by $U_i' = c_i U_i$ and comparing the so obtained relations with (9), we have the relation

$$(11) \quad c_i c_j \gamma_{ij}^k = c_k \gamma'_{ij}^k.$$

Let $V_i = \sigma U_i$, then applying σ to (9), we have

$$(12) \quad [V_i, V_j] = \sum \gamma_{ij}^k V_k.$$

Let $V_i' = pV_i$, then by the relation (8), we have

$$V_i' = p\sigma U_i = \sigma' pU_i = \sigma' U_i'.$$

Consequently, applying σ' to both sides of (10), we have

$$(13) \quad [V_i', V_j'] = \sum r'_{ij}{}^k V_k'.$$

By the last equality in (5), we have

$$(14) \quad V_i' = p\sigma U_i = \sigma p^{-1}U_i = c_i^{-1}V_i.$$

(12), (13) and (14) prove that

$$(15) \quad c_i c_j r'_{ij}{}^k = c_k r_{ij}{}^k.$$

Comparing (11) and (15), we see that $r_{ij}{}^k = 0 \Leftrightarrow r'_{ij}{}^k = 0$ and $c_i c_j = c_k$ if $r_{ij}{}^k \neq 0$. Hence

$$r'_{ij}{}^k = r_{ij}{}^k \quad \text{for all } i, j, k.$$

The last equalities prove that p is an automorphism of \mathfrak{g} . Let $p = \exp X$, $X \in \mathfrak{p}_0$, then the linear transformation $p^t = \exp tX$ has eigenvalues c_i^t with eigenvectors U_i . If t is real and $r_{ij}{}^k \neq 0$, then we have $c_i^t c_j^t = c_k^t$. Let $r_{ij}{}^k(t)$ be the structure constants of \mathfrak{g} with respect to the basis $U_i(t) = p^t U_i$, then by the same argument as was used to prove (11) we have

$$c_i^t c_j^t r_{ij}{}^k(t) = c_k^t r_{ij}{}^k(t).$$

Therefore we have $r_{ij}{}^k(t) = r_{ij}{}^k$ for any real parameter t and for all i, j, k . This fact implies that p^t ($0 \leq t \leq 1$) are automorphisms of \mathfrak{g} . Consequently p belongs to the connected component of the identity in the full automorphism group of \mathfrak{g} . However this component is nothing but the adjoint group of \mathfrak{g} . Thus we have proved the conjugacy of two Cartan decompositions under the action of the adjoint group.

For every Cartan subalgebra \mathfrak{h} of \mathfrak{g} , we have the following two subalgebras.

$$\mathfrak{h}^+ = \{X \in \mathfrak{h}; \text{ all eigenvalues of } \text{ad } X \text{ are purely imaginary}\},$$

$$\mathfrak{h}^- = \{X \in \mathfrak{h}; \text{ all eigenvalues of } \text{ad } X \text{ are real}\}.$$

The analytic subgroup of G generated by \mathfrak{h}^+ is a toroidal group, and the one generated by \mathfrak{h}^- is a vector group.

DEFINITION 4. We call \mathfrak{h}^+ the *toroidal part* of \mathfrak{h} and \mathfrak{h}^- the *vector part* of \mathfrak{h} .

When \mathfrak{g} admits a complex structure, that is, when \mathfrak{g} is obtained from a complex semisimple Lie algebra by the restriction of the base field, it is well known that

$$(16) \quad \mathfrak{h}^+ = \sqrt{-1} \mathfrak{h}^-$$

$$(17) \quad \mathfrak{h}^- = \sum_{\alpha: \text{root}} R H_\alpha.$$

and

$$(18) \quad \mathfrak{h} = \mathfrak{h}^+ + \mathfrak{h}^- \quad (\text{direct sum}).$$

PROPOSITION 2. *Let \mathfrak{h} be a Cartan subalgebra of a real semisimple Lie algebra \mathfrak{g} . Then we have the following two relations.*

- 1) $\mathfrak{h} = \mathfrak{h}^+ + \mathfrak{h}^-$ (direct sum),
- 2) $\mathfrak{h}^+ = (\mathfrak{h}^c)^+ \cap \mathfrak{h}$, $\mathfrak{h}^- = (\mathfrak{h}^c)^- \cap \mathfrak{h}$.

PROOF. It is clear that $\mathfrak{h}^+ \cap \mathfrak{h}^- = 0$, $\mathfrak{h}^+ \supset (\mathfrak{h}^c)^+ \cap \mathfrak{h}$ and $\mathfrak{h}^- \supset (\mathfrak{h}^c)^- \cap \mathfrak{h}$. Therefore, to prove the proposition 2, it is sufficient to show that $\mathfrak{h} = ((\mathfrak{h}^c)^+ \cap \mathfrak{h}) + ((\mathfrak{h}^c)^- \cap \mathfrak{h})$. The complexification \mathfrak{h}^c of \mathfrak{h} is a Cartan subalgebra of \mathfrak{g}^c . Let X be any element in \mathfrak{h} ($\subset \mathfrak{h}^c$). As \mathfrak{h}^c is decomposed into the direct sum of $(\mathfrak{h}^c)^+$ and $(\mathfrak{h}^c)^-$, X is uniquely decomposed as follows.

$$(19) \quad X = Y + Z, \quad Y \in (\mathfrak{h}^c)^+, \quad Z \in (\mathfrak{h}^c)^-.$$

We have

$$(20) \quad X = \theta X = \theta Y + \theta Z.$$

Let θ^* be the linear transformation of the dual space $(\mathfrak{h}^c)^*$ of \mathfrak{h}^c defined by $(\theta^* \lambda)(H) = \overline{\lambda(\theta H)}$, then θ^* induces a substitution of roots. Hence by (16) and (17), we have

$$(21) \quad \theta(\mathfrak{h}^c)^- = (\mathfrak{h}^c)^-, \quad \theta(\mathfrak{h}^c)^+ = (\mathfrak{h}^c)^+.$$

Therefore, comparing (19) and (20), we have

$$(22) \quad \theta Y = Y, \quad \theta Z = Z,$$

owing to the uniqueness of decomposition (19). (22) means

$$(23) \quad \begin{aligned} Y &\in \mathfrak{g} \cap (\mathfrak{h}^c)^+ = \mathfrak{h} \cap (\mathfrak{h}^c)^+, \\ Z &\in \mathfrak{g} \cap (\mathfrak{h}^c)^- = \mathfrak{h} \cap (\mathfrak{h}^c)^-. \end{aligned}$$

Consequently we have

$$(24) \quad \mathfrak{h} = (\mathfrak{h} \cap (\mathfrak{h}^c)^+) + (\mathfrak{h} \cap (\mathfrak{h}^c)^-).$$

(24) proves the proposition.

THEOREM 1. *Let \mathfrak{g} be a real semisimple Lie algebra and G the adjoint group of \mathfrak{g} . Suppose two subalgebras $\mathfrak{n}_1, \mathfrak{n}_2$ of \mathfrak{g} generate toroidal groups N_1, N_2 respectively in G . If there exists a compact subgroup L in G , such that*

$$(25) \quad [\mathfrak{h}\mathfrak{n}_1, \mathfrak{n}_2] \subset \mathfrak{l} \quad \text{for all } \mathfrak{l} \in L$$

(\mathfrak{l} is the Lie algebra of L), then there exists an element l_0 in L satisfying following two conditions:

- 1) Every element of $l_0 N_1 l_0^{-1}$ commutes with each element of N_2 .
- 2) $[l_0 \mathfrak{n}_1, \mathfrak{n}_2] = 0$.

PROOF. First, we prove that for any $X \in \mathfrak{n}_1$, $Y \in \mathfrak{n}_2$, there exists an l_0 in L such that $[l_0 X, Y] = 0$.

As $f(l) = B(lX, Y)$ is a continuous function on L , f attains its maximal value at some point l_0 in L . We define the function $g(t)$ of real variable t as $g(t) = B((\exp t \operatorname{ad} Z)l_0 X, Y)$ for any Z in \mathfrak{l} . Then $g(t)$ attains its maximum

at $t = 0$. So we have

$$(26) \quad 0 = g'(0) = B([Z, l_0 X], Y) = B(Z, [l_0 X, Y]).$$

The relations (25), (26) and the fact that B is negative definite on \mathfrak{l} , prove the equality $[l_0 X, Y] = 0$. Now, there exists an X_i in \mathfrak{n}_i ($i = 1, 2$) such that one parameter subgroup $\{\exp \operatorname{ad} tX_i; -\infty < t < \infty\}$ is everywhere dense in N_i .

For such X_1 and X_2 , we can apply the first part of this proof. There exists an element l_0 in L satisfying $[l_0 X_1, X_2] = 0$. Consequently, for any two real numbers s and t , $\exp \operatorname{ad} l_0 t X_1 = l_0 (\exp \operatorname{ad} t X_1) l_0^{-1}$ and $\exp \operatorname{ad} s X_2$ commute with each other. This proves 1), and 2) is a direct consequence of 1).

COROLLARY. Any two Cartan subalgebras in a compact or complex semisimple Lie algebra \mathfrak{g} are conjugate under the action of the adjoint group.

PROOF. Cartan subalgebra of a compact Lie algebra \mathfrak{g} is identical with a maximal abelian subalgebra in \mathfrak{g} . So, when \mathfrak{g} is compact semisimple, Theorem 1, 2) proves the conjugacy of any two Cartan subalgebras.

Next we shall consider the case when \mathfrak{g} is complex semisimple. Let $\mathfrak{h}_1, \mathfrak{h}_2$ be two Cartan subalgebras of \mathfrak{g} . We can construct a compact real form \mathfrak{g}_i ($i = 1, 2$) of \mathfrak{g} , containing $(\mathfrak{h}_i)^+$. By Proposition 1 there exists an element g in G such that

$$(27) \quad g \mathfrak{g}_1 = \mathfrak{g}_2.$$

Since $g \mathfrak{h}_1^+$ is a Cartan subalgebra of the compact semisimple Lie algebra \mathfrak{g}_2 , $g \mathfrak{h}_1^+$ is conjugate to \mathfrak{h}_2^+ . Therefore \mathfrak{h}_1 and \mathfrak{h}_2 are conjugate to each other.

DEFINITION 5. Let \mathfrak{p} be the subspace of \mathfrak{g} as defined (1). An abelian subalgebra \mathfrak{m} of \mathfrak{g} satisfying the following two conditions 1), 2) is called a *maximal abelian subalgebra in \mathfrak{p}* .

- 1) $\mathfrak{m} \subset \mathfrak{p}$.
- 2) There exists no abelian subalgebra \mathfrak{m}' of \mathfrak{g} , such that

$$\mathfrak{p} \supset \mathfrak{m}' \supsetneq \mathfrak{m}.$$

PROPOSITION 3. Let K be the analytic subgroup of G generated by \mathfrak{k} . Then, any two maximal abelian subalgebras $\mathfrak{m}_1, \mathfrak{m}_2$ in \mathfrak{p} are conjugate under the action of K .

PROOF. We regard both G and the adjoint group G_u of \mathfrak{g}_u as subgroups of the adjoint group G^c of \mathfrak{g}^c . Then we have $K = G \cap G_u$. Therefore it is sufficient to prove the conjugacy of two subalgebras $\mathfrak{n}_1 = \sqrt{-1} \mathfrak{m}_1$ and $\mathfrak{n}_2 = \sqrt{-1} \mathfrak{m}_2$ in \mathfrak{g}_u under K in order to prove Proposition 3. First we prove that \mathfrak{n}_i generates a toroidal group N_i in G_u . Let τ be the automorphism of G_u whose differential $d\tau$ is identical with θ on \mathfrak{g}_u .

Let \mathfrak{h}_i be a Cartan subalgebra containing \mathfrak{n}_i and H_i be the analytic subgroup of G_u generated by \mathfrak{h}_i . Then we have

$$(28) \quad N_i \subset \{h \in H_i; \tau(h) = h^{-1}\}.$$

Therefore the closure $L_i = \overline{N_i}$ is contained in the right hand side of (28):

$$(29) \quad L_i \subset \{h \in H_i; \tau(h) = h^{-1}\}.$$

Consequently $\tau(L_i) = L_i$ and the Lie algebra \mathfrak{l}_i of L_i is invariant by $\theta = d\tau$ and \mathfrak{l}_i is decomposed into direct sum $\mathfrak{l}_i = \mathfrak{l}_i^+ + \mathfrak{l}_i^-$ where

$$\mathfrak{l}_i^+ = \{X \in \mathfrak{l}_i; \theta(X) = X\}, \quad \mathfrak{l}_i^- = \{X \in \mathfrak{l}_i; \theta X = -X\}.$$

If $X \in \mathfrak{l}_i^+$, then

$$(30) \quad \tau(\exp X) = \exp \theta X = \exp X.$$

The relations (29) and (30) proves that the square of any element in the analytic subgroup L_i^+ of H_i generated by \mathfrak{l}_i^+ is the identity. On the other hand the subgroup $\{k \in H_i; k^2 = e\}$ in the toroidal group H_i is discrete. Consequently $\mathfrak{l}_i^+ = 0$ and $\mathfrak{l}_i = \mathfrak{l}_i^- \subset \sqrt{-1}\mathfrak{p}$. Therefore \mathfrak{l}_i is an abelian subalgebra in $\sqrt{-1}\mathfrak{p}$ and contains $\mathfrak{n}_i = \sqrt{-1}\mathfrak{m}_i$. \mathfrak{l}_i must coincide with \mathfrak{n}_i . This proves $L_i = \overline{N_i} = N_i$. Thus N_i is a connected, closed abelian subgroup of the compact group G_u . So N_i is toroidal.

The toroidal subgroups N_1, N_2 and the compact subgroup K of G_u satisfy the conditions of Theorem 1, because $[kn_1, n_2] \subset [\sqrt{-1}\mathfrak{p}, \sqrt{-1}\mathfrak{p}] \subset \mathfrak{k}$. Hence by Theorem 1, there exists an element k in K such that $[kn_1, n_2] = 0$. The last equality implies $kn_1 = n_2$ since kn_1 and n_2 are maximal abelian subalgebras in $\sqrt{-1}\mathfrak{p}$.

PROPOSITION 4. *Let \mathfrak{m} be a maximal abelian subalgebra in \mathfrak{p} . Then any maximal abelian subalgebra \mathfrak{h}_0 in \mathfrak{g} which contains \mathfrak{m} , is a Cartan subalgebra of \mathfrak{g} , and $\mathfrak{h}_0^- = \mathfrak{m} = \mathfrak{h}_0 \cap \mathfrak{p}$ and $\mathfrak{h}_0^+ = \mathfrak{h}_0 \cap \mathfrak{k}$.*

PROOF. Let η be the conjugation of $\mathfrak{g}^{\mathcal{C}}$ with respect to the compact real form \mathfrak{g}_u . Then for any $X \in \mathfrak{h}_0$ and $Y \in \mathfrak{m}$, we have

$$[\eta X, Y] = -[\eta X, \eta Y] = -\eta([X, Y]) = -\eta(0) = 0.$$

Therefore we see that

$$(31) \quad [X - \eta X, Y] = 0 \quad \text{for all } Y \in \mathfrak{m}.$$

The element $Z = X - \eta X$ is in \mathfrak{g} and satisfies $\eta Z = -Z$, hence Z is contained in \mathfrak{p} . Therefore (31) proves that $Z = X - \eta X$ belongs to \mathfrak{m} , since \mathfrak{m} is a maximal abelian subalgebra in \mathfrak{p} .

Consequently $X + \eta X = 2X - (X - \eta X)$ belongs to $\mathfrak{k} \cap \mathfrak{h}_0$. Thus any element X in \mathfrak{h}_0 can be represented as the sum $X = U + Z$, where $U = (X + \eta X)/2$ and $Z = (X - \eta X)/2$ belong to \mathfrak{k} and \mathfrak{p} respectively. Therefore $\text{ad } U$ and $\text{ad } Z$ are semisimple linear transformations and commute with each other. It follows that $\text{ad } X$ is semisimple. This proves that \mathfrak{h}_0 is a Cartan subalgebra of \mathfrak{g} . Moreover the fact that $U \in \mathfrak{k}, Z \in \mathfrak{m}$ proves $\mathfrak{h}_0^+ = \mathfrak{h}_0 \cap \mathfrak{k}, \mathfrak{h}_0^- = \mathfrak{h}_0 \cap \mathfrak{p} = \mathfrak{m}$.

DEFINITION 6. Let $(\mathfrak{f}, \mathfrak{p}, \mathfrak{m})$ be a triple consisting a subalgebra \mathfrak{f} , a subspace \mathfrak{p} and an abelian subalgebra \mathfrak{m} of \mathfrak{g} satisfying the following two conditions: 1) $\mathfrak{g} = \mathfrak{f} + \mathfrak{p}$ and this decomposition is a Cartan decomposition, 2) \mathfrak{m} is a maximal abelian subalgebra in \mathfrak{p} .

Such a triple is called a *standard triple*.

A Cartan subalgebra \mathfrak{h} of \mathfrak{g} is called *standard* with respect to a standard triple $(\mathfrak{f}, \mathfrak{p}, \mathfrak{m})$, if the conditions 1) $\mathfrak{h}^- \subset \mathfrak{m}$ and 2) $\mathfrak{h}^+ \subset \mathfrak{f}$ are satisfied.

THEOREM 2. 1) Any Cartan subalgebra \mathfrak{h} of a real semisimple Lie algebra \mathfrak{g} is standard with respect to a suitable standard triple $(\mathfrak{f}, \mathfrak{p}, \mathfrak{m})$.

2) Any Cartan subalgebra \mathfrak{h} of \mathfrak{g} is conjugate under the adjoint group G to one that is standard with respect to a fixed standard triple $(\mathfrak{f}, \mathfrak{p}, \mathfrak{m})$.

PROOF. 1) There exists a compact real form \mathfrak{g}_u containing $(\mathfrak{h}^c)^+$ and invariant by θ (conjugation of \mathfrak{g}^c with respect to \mathfrak{g}). The restriction σ of η (conjugation of \mathfrak{g}^c with respect to \mathfrak{g}_u) to \mathfrak{g} is a compact involution and defines a Cartan decomposition $\mathfrak{g} = \mathfrak{f} + \mathfrak{p}$. Then we have by Proposition 2

$$\begin{aligned} \mathfrak{h}^+ &= (\mathfrak{h}^c)^+ \cap \mathfrak{h} \subset \mathfrak{g}_u \cap \mathfrak{g} = \mathfrak{f} \\ \mathfrak{h}^- &= (\mathfrak{h}^c)^- \cap \mathfrak{h} \subset \sqrt{-1} \mathfrak{g}_u \cap \mathfrak{g} = \mathfrak{p}. \end{aligned}$$

Therefore \mathfrak{h} is standard with respect to a standard triple $(\mathfrak{f}, \mathfrak{p}, \mathfrak{m})$, where \mathfrak{m} is an arbitrary maximal abelian subalgebra in \mathfrak{p} containing \mathfrak{h}^- .

2) Let $(\mathfrak{f}, \mathfrak{p}, \mathfrak{m})$ be a fixed standard triple. A Cartan subalgebra \mathfrak{h} of \mathfrak{g} is standard with respect to a standard triple $(\mathfrak{f}', \mathfrak{p}', \mathfrak{m}')$. By Proposition 1 there exists a $g \in G$ such that

$$g\mathfrak{f}' = \mathfrak{f}, \quad g\mathfrak{p}' = \mathfrak{p}$$

and we have

$$\begin{aligned} g(\mathfrak{h}^+) &= g(\mathfrak{h} \cap \mathfrak{f}') \subset g\mathfrak{f}' = \mathfrak{f} \\ g(\mathfrak{h}^-) &= g(\mathfrak{h} \cap \mathfrak{p}') \subset g\mathfrak{p}' = \mathfrak{p}. \end{aligned}$$

Let \mathfrak{m}'' be a maximal abelian subalgebra in \mathfrak{p} containing $g(\mathfrak{h}^-)$, then by Proposition 3 there exists $k \in K$ (analytic subgroup of G generated by \mathfrak{f}) such that

$$k\mathfrak{m}'' = \mathfrak{m}.$$

Since k leaves invariant \mathfrak{f} and \mathfrak{p} as a whole, the element $s = kg$ transforms \mathfrak{h} to a Cartan subalgebra $s\mathfrak{h}$ which is standard with respect to $(\mathfrak{f}, \mathfrak{p}, \mathfrak{m})$.

Convention. 1) In the following we shall fix a standard triple $(\mathfrak{f}, \mathfrak{p}, \mathfrak{m})$ and use the word "standard" instead of "standard with respect to $(\mathfrak{f}, \mathfrak{p}, \mathfrak{m})$ ".

2) In the following we shall use the word "conjugate" always in the meaning of "conjugate under the action of the adjoint group", unless specifically noted.

PROPOSITION 5. Let \mathfrak{h}_1 and \mathfrak{h}_2 be two standard Cartan subalgebras of \mathfrak{g}

satisfying

$$\mathfrak{h}_1^- \subset \mathfrak{h}_2^-.$$

Then there exists an element $k \in K$ such that

$$\begin{aligned} kH &= H && \text{for all } H \in \mathfrak{h}_1^- \text{ and} \\ k\mathfrak{h}_2^+ &\subset \mathfrak{h}_1^+. \end{aligned}$$

Especially when $\mathfrak{h}_1^- = \mathfrak{h}_2^-$, this k satisfies $k\mathfrak{h}_2^+ = \mathfrak{h}_1^+$.

PROOF. Let

$$L = \{k \in K; kH = H \text{ for all } H \in \mathfrak{h}_1^-\}.$$

Then L is a compact subgroup of G .

\mathfrak{h}_1^+ , \mathfrak{h}_2^- and L satisfy the conditions of Theorem 1.

Let H_i^+ be the analytic subgroup generated by \mathfrak{h}_i^+ . Then $H_i^+ \subset K \cap H_i$ and $K \cap H_i$ is closed, so we have $\overline{H_i^+} \subset K \cap H_i$. The last inclusion relation and the equality $\mathfrak{h}_i^+ = \mathfrak{k} \cap \mathfrak{h}_i$ prove that $\overline{H_i^+} = H_i^+$. Therefore H_i^+ is a closed connected abelian subgroup of compact Lie group K , consequently H_i^+ is a toroidal group.

Next we prove

$$(32) \quad [\mathcal{L}\mathfrak{h}_2^+, \mathfrak{h}_1^+] \subset \mathfrak{l} \quad (\text{for all } l \in L).$$

Since $L \subset K$ and $\mathfrak{h}_i^+ \subset \mathfrak{k}$, we have by (3)

$$(33) \quad [\mathcal{L}\mathfrak{h}_2^+, \mathfrak{h}_1^+] \subset \mathfrak{k}.$$

And by Jacobi identity, we have

$$(34) \quad [\mathfrak{h}_1^-, [\mathcal{L}\mathfrak{h}_2^+, \mathfrak{h}_1^+]] = [[\mathfrak{h}_1^-, \mathcal{L}\mathfrak{h}_2^+], \mathfrak{h}_1^+] + [\mathcal{L}\mathfrak{h}_2^+, [\mathfrak{h}_1^-, \mathfrak{h}_1^+]].$$

The second term of right hand side of (34) is equal to zero, because \mathfrak{h}_1 is abelian. And the first term is also equal to zero, because

$$[\mathfrak{h}_1^-, \mathcal{L}\mathfrak{h}_2^+] = \mathcal{L}[\mathfrak{l}^{-1}\mathfrak{h}_1^-, \mathfrak{h}_2^+] = \mathcal{L}[\mathfrak{h}_1^-, \mathfrak{h}_2^+]$$

and $\mathfrak{h}_1^- \subset \mathfrak{h}_2^-$ and \mathfrak{h}_2 is abelian.

Therefore we have

$$(35) \quad [\mathfrak{h}_1^-, [\mathcal{L}\mathfrak{h}_2^+, \mathfrak{h}_1^+]] = 0.$$

The relations (33) and (35) prove (32) by the very definition of L .

Thus we can apply Theorem 1 to \mathfrak{h}_1^+ , \mathfrak{h}_2^+ and L . There exists an element $l \in L$ such that

$$(36) \quad [\mathcal{L}\mathfrak{h}_2^+, \mathfrak{h}_1^+] = 0.$$

On the other hand, by the assumption $\mathfrak{h}_1^- \subset \mathfrak{h}_2^-$, we have

$$(37) \quad [\mathcal{L}\mathfrak{h}_2^+, \mathfrak{h}_1^-] = \mathcal{L}[\mathfrak{h}_2^+, \mathfrak{l}^{-1}\mathfrak{h}_1^-] = \mathcal{L}[\mathfrak{h}_2^+, \mathfrak{h}_1^-] = 0.$$

(36) and (37) proves $[\mathcal{L}\mathfrak{h}_2^+, \mathfrak{h}_1] = 0$ consequently $\mathcal{L}\mathfrak{h}_2^+ \subset \mathfrak{h}_1$. Since $\mathcal{L}\mathfrak{h}_2^+ \subset \mathfrak{k}$, we have

$$\mathcal{L}\mathfrak{h}_2^+ \subset \mathfrak{k} \cap \mathfrak{h}_1 = \mathfrak{h}_1^+$$

which proves our Proposition 5.

We set now

$$N = \{k \in K; km = m\}.$$

Let $k \in N$, then the restriction $\varphi(k)$ of k to \mathfrak{m} is a linear transformation on \mathfrak{m} . We denote the totality of $\varphi(k)$ by W_s ; i. e.

$$W_s = \{\varphi(k); k \in N\}.$$

The homogeneous space $M = G/K$ is the symmetric Riemannian space of which the largest connected group of isometries is G . The group W_s coincides with the Weyl group (groupe (S)) of M defined by Cartan in [2, p. 356].

THEOREM 3. *Two standard Cartan subalgebras \mathfrak{h}_1 and \mathfrak{h}_2 are conjugate under the action of the adjoint group G if and only if their vector parts \mathfrak{h}_1^- and \mathfrak{h}_2^- are conjugate under the action of group W_s .*

PROOF. Sufficiency. Assume that there exists an element $k \in N$ such that $k\mathfrak{h}_1^- = \mathfrak{h}_2^-$. Then we can apply Proposition 5 to \mathfrak{h}_2 and $k\mathfrak{h}_1$, and there exists an element k_1 in K such that

$$k_1\mathfrak{h}_2^- = \mathfrak{h}_2^-, \quad k_1k\mathfrak{h}_1^+ = \mathfrak{h}_2^+.$$

Therefore let $k_2 = k_1k$ then $k_2 \in K$ and $k_2\mathfrak{h}_1 = \mathfrak{h}_2$.

Necessity. Let g be an element in G which transforms \mathfrak{h}_1 onto \mathfrak{h}_2 . Then g can be decomposed uniquely as the product

$$g = kp, \quad p \in \exp \text{ ad } \mathfrak{p}, \quad k \in K.$$

First of all, we shall show that $pH = H$ for all $H \in \mathfrak{h}_1^-$. Let $p = \exp \text{ ad } X$, ($X \in \mathfrak{p}$), $c = \cosh \text{ ad } X$ and $s = \sinh \text{ ad } X$. Then $p\mathfrak{h}_1^- = k^{-1}\mathfrak{h}_2^- \subset \mathfrak{p}$, $cp \subset \mathfrak{p}$ and $sp \subset \mathfrak{k}$. Therefore

$$(38) \quad s\mathfrak{h}_1^- = (p-c)\mathfrak{h}_1^- \in \mathfrak{k} \cap \mathfrak{p} = \{0\}.$$

Since s is a semisimple linear transformation and all the eigenvalues of $\text{ad } X$ are real, the kernel of s coincides with that of $\text{ad } X$. Therefore (38) implies that

$$(\text{ad } X)\mathfrak{h}_1^- = 0 \text{ i. e. } [X, \mathfrak{h}_1^-] = 0.$$

Consequently $pH = H$ for all $H \in \mathfrak{h}_1^-$ and we have

$$(39) \quad k\mathfrak{h}_1^- = g\mathfrak{h}_1^- = \mathfrak{h}_2^-.$$

Next we shall prove that two subalgebras, $\mathfrak{n}_1 = \sqrt{-1}\mathfrak{m}$ and $\mathfrak{n}_2 = \sqrt{-1}km$, and the subgroup

$$L = \{l \in K; lH = H \text{ for all } H \in \mathfrak{h}_2^-\}$$

of the adjoint group G_u of \mathfrak{g}_u satisfy the conditions of Theorem 1.

L is a compact subgroup of G_u . \mathfrak{n}_i is an abelian subalgebra of \mathfrak{g}_u , and \mathfrak{n}_i generates a toroidal group N_i in G_u . To prove the last fact it is sufficient

to prove that N_i is closed. This fact is proved as in the proof of Proposition 3. Now we shall prove the relation

$$(40) \quad [ln_i, n_2] \subset I \quad \text{for all } l \in L.$$

By the Jacobi identity and the fact that l is an automorphism of \mathfrak{g} , we have

$$(41) \quad [[ln_1, n_2], \mathfrak{h}_2^-] = [[lm, \mathfrak{h}_2^-], km] + [lm, [km, \mathfrak{h}_2^-]].$$

The first term of right hand side of (41) is equal to zero, because

$$[lm, \mathfrak{h}_2^-] = l[m, l^{-1}\mathfrak{h}_2^-] = l[m, \mathfrak{h}_2^-] \subset l[m, m] = 0.$$

The second term is also equal to zero, because

$$[km, \mathfrak{h}_2^-] = k[m, k^{-1}\mathfrak{h}_2^-] = k[m, \mathfrak{h}_1^-] \subset k[m, m] = 0.$$

Therefore (40) is proved.

Thus the conditions of Theorem 1 are satisfied. So we can find an element $l \in L$ such that

$$[ln_2, n_1] = [lkm, m] = 0.$$

Since lkm and m are both maximal abelian subalgebras in \mathfrak{p} , the last equality means that $lkm = m$.

Let $lk = a$, then $a \in K$ and we have

$$a\mathfrak{h}_1^- = lk\mathfrak{h}_1^- = l\mathfrak{h}_2^- = \mathfrak{h}_2^-,$$

$$am = lkm = m.$$

Therefore a belongs to the group N and \mathfrak{h}_1^- is transformed to \mathfrak{h}_2^- by a .

By Theorem 3 and Proposition 5, we have the following Corollary 1, 2 to Theorem 3.

COROLLARY 1. *Two standard Cartan subalgebras are conjugate under the adjoint group G if and only if they are conjugate under K .*

COROLLARY 2. *All Cartan subalgebras of which vector parts have maximal possible dimension are mutually conjugate under G .*

The following lemma, which is obvious, is used in the proof of Theorem 4.

LEMMA 1. *Let X be an element in $\sqrt{-1}\mathfrak{p}$, and $\mathfrak{g}(\lambda)$ be the eigenspace of $\text{ad } X$ with eigenvalue λ . Let $\lambda_0, \lambda_1, \dots, \lambda_n$ be the eigenvalues of $\text{ad } X$ and $\lambda_0 = 0$. Then we have*

$$\mathfrak{g}^c = \sum_{i=0}^n \mathfrak{g}(\lambda_i) \quad (\text{direct sum}).$$

Let $\mathfrak{f}(\lambda)$ (or $\mathfrak{p}(\lambda)$) be the projection of $\mathfrak{g}(\lambda)$ to \mathfrak{f}^c (or \mathfrak{p}^c) by the direct sum decomposition $\mathfrak{g}^c = \mathfrak{f}^c + \mathfrak{p}^c$. Then any element $Y \in \mathfrak{g}(\lambda)$ can be written as follows:

$$Y = U + V, \quad U \in \mathfrak{f}(\lambda), \quad V \in \mathfrak{p}(\lambda).$$

And U and V satisfy the following equalities:

$$[X, U] = \lambda V, \quad [X, V] = \lambda U.$$

Finally we have

$$\begin{aligned} \mathfrak{f}^c &= \mathfrak{f}(0) + \mathfrak{f}(\lambda_1) + \dots + \mathfrak{f}(\lambda_n) \quad (\text{direct sum}), \\ \mathfrak{p}^c &= \mathfrak{p}(0) + \mathfrak{p}(\lambda_1) + \dots + \mathfrak{p}(\lambda_n) \quad (\text{direct sum}) \end{aligned}$$

(cf. E. Cartan [2, Ch. I, 3]).

THEOREM 4. Let \mathfrak{h}_i ($i=1, 2$) be two standard Cartan subalgebras of \mathfrak{g} , and

$$\mathfrak{l}_i = (\mathfrak{h}_i^-)^+ \cap \mathfrak{m} = \{X \in \mathfrak{m}; B(X, \mathfrak{h}_i^-) = 0\}.$$

Then \mathfrak{h}_1 and \mathfrak{h}_2 are conjugate to each other if and only if \mathfrak{l}_1 and \mathfrak{l}_2 are conjugate under the action of the Weyl group W of \mathfrak{g}^c with respect to the Cartan subalgebra \mathfrak{h}_0^c . (\mathfrak{h}_0 is a fixed Cartan subalgebra of \mathfrak{g} containing \mathfrak{m} . cf. Proposition 4.)

PROOF. 1) Necessity of the condition. Let \mathfrak{h}_1 and \mathfrak{h}_2 be conjugate, then by Theorem 3, we can find an element $k \in K$ such that

$$k\mathfrak{h}_1^- = \mathfrak{h}_2^-, \quad k\mathfrak{m} = \mathfrak{m}.$$

Let L be the subgroup of G_u defined as follows:

$$L = \{l \in G_u; lH = H \text{ for all } H \in \mathfrak{m}\}$$

and let $\mathfrak{h}_u = \mathfrak{h}_0^+ + \sqrt{-1}\mathfrak{h}_0^-$, $\mathfrak{n}_1 = k\mathfrak{h}_u$ and $\mathfrak{n}_2 = \mathfrak{h}_u$. Then, L , \mathfrak{n}_1 and \mathfrak{n}_2 satisfy the conditions of Theorem 1. It is clear that L is a compact subgroup of G_u and that \mathfrak{n}_i generates a toroidal group in G_u . We have

$$[[l\mathfrak{n}_1, \mathfrak{n}_2], \mathfrak{m}] = 0 \quad \text{for all } l \in L,$$

as in the proof of Theorem 3. Therefore we have $[l\mathfrak{n}_1, \mathfrak{n}_2] \subset \mathfrak{l}$. By Theorem 1, we can find an element $l \in L$ such that $[l\mathfrak{n}_1, \mathfrak{n}_2] = 0$. Consequently we have

$$l\mathfrak{n}_1 = \mathfrak{n}_2.$$

Since $lk \in G_u$ keeps \mathfrak{h}_u invariant, the restriction s of lk to $(\mathfrak{h}_u)^c = (\mathfrak{h}_0)^c$ belongs to the Weyl group W of \mathfrak{g}^c with respect to $(\mathfrak{h}_0)^c$. Moreover by the definition of s , we have

$$s\mathfrak{m} = \mathfrak{m} \quad \text{and} \quad s\mathfrak{h}_1^- = \mathfrak{h}_2^-.$$

Consequently we have $s\mathfrak{l}_1 = \mathfrak{l}_2$.

2) Sufficiency of the condition. Let t be an element of W such that $t\mathfrak{l}_1 = \mathfrak{l}_2$. Then, there exists g in G_u , of which the restriction to \mathfrak{h}_u coincides with t . g satisfies the following conditions

$$g \in G_u, \quad g\mathfrak{h}_0^c = \mathfrak{h}_0^c, \quad g\mathfrak{l}_1 = \mathfrak{l}_2$$

g is decomposed as the product of a rotation $k \in K$ and a transvection $p \in \exp \operatorname{ad} \sqrt{-1}\mathfrak{p}$, i. e. $g = kp$.

Let

$$p = \exp \operatorname{ad} X, \quad X \in \sqrt{-1}\mathfrak{p}, \quad s = \sinh \operatorname{ad} X \quad \text{and} \quad c = \cosh \operatorname{ad} X.$$

Then, as in the proof of Theorem 3, we have

$$(42) \quad sI_1 = (p-c)I_1 \in \sqrt{-1}\mathfrak{f} \cap \mathfrak{p} = \{0\}.$$

Let $\mathfrak{g}(\lambda)$ be the eigenspace of $\text{ad } X$ with eigenvalue λ , then the eigenspace of s with eigenvalue ρ is the direct sum of $\mathfrak{g}(\lambda)$'s, where $\sinh \lambda = \rho$. Hence, (42) implies that

$$I_1 \subset \sum_n \mathfrak{g}(\sqrt{-1}n\pi).$$

Therefore any element Y of I_1 can be represented as follows:

$$(43) \quad Y = \sum_n Y_n, \quad Y_n \in \mathfrak{g}(\sqrt{-1}n\pi).$$

By Lemma 1, we have

$$(44) \quad Y_n = U_n + V_n, \quad U_n \in \mathfrak{f}(\sqrt{-1}n\pi), \quad V_n \in \mathfrak{p}(\sqrt{-1}n\pi),$$

and

$$(45) \quad [X, U_n] = \sqrt{-1}n\pi V_n.$$

By (43) and (44), we have

$$(46) \quad Y - \sum_n V_n = \sum_n U_n \in \mathfrak{p}^c \cap \mathfrak{f}^c = \{0\}.$$

By Lemma 1, $\sum_n \mathfrak{f}(\sqrt{-1}n\pi)$ is a direct sum, hence, by (46) we have

$$(47) \quad U_n = 0 \quad \text{for all } n.$$

(45) and (47) prove that

$$(48) \quad V_n = 0, \quad \text{if } n \neq 0.$$

(47) and (48) imply that

$$Y = V_0 \in \mathfrak{p}(0).$$

As Y is an arbitrary element of I_1 , we have $[X, I_1] = \{0\}$, and consequently

$$pY = Y \quad \text{for all } Y \in I_1.$$

Therefore we have

$$(49) \quad kI_1 = I_2, \quad k \in K.$$

Let

$$L = \{l \in G_u; lY = Y \text{ for all } Y \in I_2\}, \quad n_1 = km \text{ and } n_2 = m,$$

then L , n_1 and n_2 satisfy the conditions of Theorem 1. Therefore we can find an element $l \in L$ such that

$$lkm = m.$$

Then we have

$$lk \in N, \quad lkI_1 = I_2,$$

and consequently,

$$(50) \quad lk \in N, \quad lk\mathfrak{h}_1^- = \mathfrak{h}_2^-.$$

By Theorem 3, (50) implies that \mathfrak{h}_1 and \mathfrak{h}_2 are conjugate to each other.

§ 2. Vector parts of standard Cartan subalgebras.

In this section we give a necessary and sufficient condition that a subspace \mathfrak{n} of \mathfrak{m} is the vector part of a standard Cartan subalgebra \mathfrak{h} . Notations: $(\mathfrak{k}, \mathfrak{p}, \mathfrak{m})$ be a fixed standard triple of \mathfrak{g} . We choose a maximal abelian subalgebra \mathfrak{h}_0 in \mathfrak{g} containing \mathfrak{m} . Then, by Proposition 4, \mathfrak{h}_0 is a standard Cartan subalgebra of \mathfrak{g} , and $\mathfrak{h}_0^- = \mathfrak{m}$.

Let $\mathfrak{h}_0^{\mathcal{C}}$ be the complexification of \mathfrak{h}_0 in $\mathfrak{g}^{\mathcal{C}}$. Then $\mathfrak{h}_0^{\mathcal{C}}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathcal{C}}$. A (non zero) root of $\mathfrak{g}^{\mathcal{C}}$ with respect to $\mathfrak{h}_0^{\mathcal{C}}$ is simply called a root. The totality of roots is denoted by \mathbf{R} .

Let α be a root, then there exists an element $E_\alpha \neq 0$ in $\mathfrak{g}^{\mathcal{C}}$, such that

$$[H, E_\alpha] = \alpha(H)E_\alpha \quad \text{for all } H \in \mathfrak{h}_0^{\mathcal{C}}.$$

E_α is unique up to scalar factors. Moreover we have

$$(51) \quad [E_\alpha, E_\beta] = N_{\alpha, \beta} E_{\alpha + \beta},$$

where $N_{\alpha, \beta}$ is a complex number and we have

$$(52) \quad N_{\alpha, \beta} \begin{cases} = 0 & (\text{if } \alpha + \beta \notin \mathbf{R}) \\ \neq 0 & (\text{if } \alpha + \beta \in \mathbf{R}). \end{cases}$$

Let θ be the conjugation of $\mathfrak{g}^{\mathcal{C}}$ with respect to \mathfrak{g} , and let ${}^t\theta$ be the linear transformation on the dual space $(\mathfrak{h}_0^{\mathcal{C}})^{-*}$ of $(\mathfrak{h}_0^{\mathcal{C}})^-$ defined by

$$({}^t\theta\lambda)(H) = \lambda(\theta H) \quad (H \in (\mathfrak{h}_0^{\mathcal{C}})^-).$$

Since ${}^t\theta$ induces a substitution of roots, there exists a complex number κ_α for each roots α such that

$$(53) \quad \theta E_\alpha = \kappa_\alpha E_{{}^t\theta\alpha}.$$

It is well known that we can find a basis H_i ($1 \leq i \leq n$), E_α ($\alpha \in \mathbf{R}$) of $\mathfrak{g}^{\mathcal{C}}$ satisfying the following four conditions:

- (54) $H_i \in (\mathfrak{h}_0^{\mathcal{C}})^-$,
- (55) $B(E_\alpha, E_{-\alpha}) = -1$,
- (56) $N_{\alpha, \beta} = N_{-\alpha, -\beta} = \text{real number}$,
- (57) $|\kappa_\alpha| = 1$

(cf. H. Weyl [13] and G. D. Mostow [10]).

Let H_i and E_α be a basis satisfying the above four conditions, then

$$(58) \quad \mathfrak{g}_u = (\mathfrak{h}_0^{\mathcal{C}})^+ + \sum_{\alpha > 0} R(E_\alpha + E_{-\alpha}) + \sum_{\alpha > 0} R\sqrt{-1}(E_\alpha - E_{-\alpha})$$

is a compact real form of $\mathfrak{h}^{\mathcal{C}}$ and we have

$$(59) \quad \theta(\mathfrak{g}_u) = \mathfrak{g}_u.$$

We choose a basis H_i, E_α satisfying (54)–(57).

For any subspace \mathfrak{l} of \mathfrak{m} , we define $\mathbf{R}(\mathfrak{l})$ by

$$(60) \quad \mathbf{R}(\mathfrak{l}) = \{\alpha \in \mathbf{R}; H_\alpha \in \mathfrak{l}\},$$

and we define the following five subspaces which play important parts in the followings.

$$(61) \quad \begin{aligned} \mathfrak{f} &= \mathfrak{l}^+ \cap \mathfrak{h}_0 = \{X \in \mathfrak{h}_0; B(X, \mathfrak{l}) = 0\}, \\ Z^c(\mathfrak{f}) &= \{X \in \mathfrak{g}^c; [X, \mathfrak{f}] = 0\}, \quad Z(\mathfrak{f}) = \mathfrak{g} \cap Z^c(\mathfrak{f}), \\ \mathfrak{g}[\mathfrak{l}] &= \mathfrak{l}^c + \sum_{\alpha \in \mathbf{R}(\mathfrak{l})} CE_\alpha, \quad \text{and} \quad \mathfrak{g}(\mathfrak{l}) = \mathfrak{g} \cap \mathfrak{g}[\mathfrak{l}]. \end{aligned}$$

These five subspaces are subalgebras.

PROPOSITION 6. $Z^c(\mathfrak{f}) = \mathfrak{h}_0^c + \mathfrak{g}[\mathfrak{l}]$ ($\mathfrak{f} = \mathfrak{l}^+ \cap \mathfrak{h}_0$).

PROOF. Any element X of \mathfrak{g}^c is written as

$$X = H_0 + \sum_{\alpha \in \mathbf{R}} c_\alpha E_\alpha, \quad c_\alpha \in \mathbf{C}, H_0 \in \mathfrak{h}_0^c.$$

Since

$$[H, X] = \sum_{\alpha \in \mathbf{R}} c_\alpha B(H, H_\alpha) E_\alpha \quad \text{for all } H \in \mathfrak{f},$$

we have clearly $Z^c(\mathfrak{f}) = \mathfrak{h}_0^c + \sum_{\alpha \in \mathbf{R}(\mathfrak{l})} CE_\alpha$.

DEFINITION 7. A subspace \mathfrak{l} of \mathfrak{m} is called a *root space* if \mathfrak{l} is spanned by roots i. e. $\mathfrak{l} = \sum_{\alpha \in \mathbf{R}(\mathfrak{l})} RH_\alpha$.

PROPOSITION 7. If \mathfrak{h} is a standard Cartan subalgebra, then

$$\mathfrak{l} = \mathfrak{h}^{-+} \cap \mathfrak{m} = \{X \in \mathfrak{m}; B(X, \mathfrak{h}^-) = 0\}$$

is a root space.

PROOF. A standard Cartan subalgebra \mathfrak{h} is called *special* if $\mathfrak{h}^+ \supset \mathfrak{h}_0^+$. Proposition 5 proves that every standard Cartan subalgebra \mathfrak{h} is conjugate to a special standard Cartan subalgebra \mathfrak{h}_1 of which vector part \mathfrak{h}_1^- coincides with \mathfrak{h}^- . Hence we may assume that $\mathfrak{h}^+ \supset \mathfrak{h}_0^+$. In this case, we have $\mathfrak{f} = \mathfrak{h}_0 \cap \mathfrak{l}^+ = \mathfrak{h}_0^+ + \mathfrak{h}^-$ and $Z^c(\mathfrak{f})$ contains \mathfrak{h}^c . Let $\mathfrak{l}' = \sum_{\alpha \in \mathbf{R}(\mathfrak{l})} RH_\alpha$ and assume that $\mathfrak{l}' \neq \mathfrak{l}$, then there exists a non zero element H in \mathfrak{l} such that $B(H, H_\alpha) = 0$ for all $\alpha \in \mathbf{R}(\mathfrak{l})$. As we have by Proposition 6

$$[H, \mathfrak{h}] \subset [H, Z^c(\mathfrak{f})] = \{0\},$$

H belongs to $\mathfrak{h} \cap \mathfrak{p} = \mathfrak{h}^-$ which contradicts the facts that $H \in \mathfrak{l}$ and $H \neq 0$.

PROPOSITION 8. If \mathfrak{l} is a root space in \mathfrak{m} , then $\mathfrak{g}[\mathfrak{l}]$ is a semisimple subalgebra of \mathfrak{g}^c and \mathfrak{l}^c is a Cartan subalgebra of $\mathfrak{g}[\mathfrak{l}]$.

PROOF. Let \mathfrak{b} be the subspace of \mathfrak{g}^c defined by

$$\mathfrak{b} = \mathfrak{l}^c + \sum_{\alpha \in \mathbf{R}(\mathfrak{l})} CE_\alpha, \quad (\mathfrak{f} = \mathfrak{l}^+ \cap \mathfrak{h}_0),$$

then we have

(62) $\mathfrak{g}^\sigma = \mathfrak{g}[\mathfrak{l}] + \mathfrak{b}$ (direct sum)

and

(63) $[\mathfrak{g}[\mathfrak{l}], \mathfrak{b}] \subset \mathfrak{b}$,

because if $\alpha \in \mathbf{R}(\mathfrak{l})$ and $\beta \in \mathbf{R}(\mathfrak{l})$, then $\alpha + \beta \in \mathbf{R}(\mathfrak{l})$. By a theorem of N. Jacobson (cf. C. Chevalley [4, (b), p. 111]), we can conclude from (62) and (63) that $\mathfrak{g}[\mathfrak{l}]$ is a reductive Lie algebra. Next we show that the centre \mathfrak{z} of $\mathfrak{g}[\mathfrak{l}]$ consists of zero only.

Let

$$X = H_0 + \sum_{\alpha \in \mathbf{R}(\mathfrak{l})} a_\alpha E_\alpha, \quad H_0 \in \mathfrak{l}^\sigma$$

be an arbitrary element of \mathfrak{z} , then we have

$$0 = [H_\beta, X] = \sum_{\alpha \in \mathbf{R}(\mathfrak{l})} a_\alpha B(H_\beta, H_\alpha) E_\alpha \quad \text{for all } \beta \in \mathbf{R}(\mathfrak{l}).$$

Since $B(H_\alpha, H_\alpha) \neq 0$, we have $a_\alpha = 0$ for all $\alpha \in \mathbf{R}(\mathfrak{l})$, consequently $X = H_0 \in \mathfrak{l}^\sigma$. Therefore $\mathfrak{z} \subset \mathfrak{l}^\sigma$. However, since \mathfrak{l} is spanned by some H_α 's for $\alpha \in \mathbf{R}(\mathfrak{l})$, for any non zero element $H \in \mathfrak{l}^\sigma$, there exists a root $\alpha \in \mathbf{R}(\mathfrak{l})$ such that $B(H_\alpha, H) = \alpha(H) \neq 0$. Then we have

$$[H, E_\alpha] \neq 0 \quad \text{and} \quad H \notin \mathfrak{z}.$$

Thus we have proved that $\mathfrak{z} = \{0\}$ and $\mathfrak{g}[\mathfrak{l}]$ is semisimple. The above argument has also proved that \mathfrak{l}^σ is a maximal abelian subalgebra of $\mathfrak{g}[\mathfrak{l}]$. Moreover let ad be the adjoint representation of $\mathfrak{g}[\mathfrak{l}]$, then every $\text{ad}(X)$ ($X \in \mathfrak{l}^\sigma$) can be represented by a diagonal matrix. Therefore \mathfrak{l}^σ is a Cartan subalgebra of the semisimple Lie algebra $\mathfrak{g}[\mathfrak{l}]$.

REMARK. If \mathfrak{l} is a root space in \mathfrak{m} , then $\mathbf{R}(\mathfrak{l})$ is the root system of the semisimple Lie algebra $\mathfrak{g}[\mathfrak{l}]$ with respect to the Cartan subalgebra \mathfrak{l}^σ .

PROPOSITION 9. *Let \mathfrak{l} be a root space in \mathfrak{m} , then*

- 1) $\mathfrak{g}(\mathfrak{l}) = \mathfrak{g}[\mathfrak{l}] \cap \mathfrak{g}$ is a real form of $\mathfrak{g}[\mathfrak{l}]$.
- 2) $\mathfrak{g}(\mathfrak{l}) = (\mathfrak{g}(\mathfrak{l}) \cap \mathfrak{k}) + (\mathfrak{g}(\mathfrak{l}) \cap \mathfrak{p})$ is a Cartan decomposition of $\mathfrak{g}(\mathfrak{l})$.
- 3) $Z(\mathfrak{k}) \cap \mathfrak{k} = \mathfrak{h}_0^+ + (\mathfrak{g}(\mathfrak{l}) \cap \mathfrak{k})$, ($\mathfrak{k} = \mathfrak{l}^+ \cap \mathfrak{h}_0$).

PROOF. 1) To prove 1), it is sufficient to show that $\theta(\mathfrak{g}[\mathfrak{l}]) = \mathfrak{g}[\mathfrak{l}]$, where θ is the conjugation of \mathfrak{g}^σ with respect to \mathfrak{g} . As $H_\alpha \in \mathfrak{l} \subset \mathfrak{g}$ for all $\alpha \in \mathbf{R}(\mathfrak{l})$,

$$\theta H_\alpha = H_\alpha \quad \text{for all } \alpha \in \mathbf{R}(\mathfrak{l})$$

hence

$${}^t\theta\alpha = \alpha \quad \text{for all } \alpha \in \mathbf{R}(\mathfrak{l}).$$

Therefore we have $\theta(\mathfrak{g}[\mathfrak{l}]) = \mathfrak{g}[\mathfrak{l}]$ (cf. (53) and (61)).

2) Let \mathfrak{g}_u be the compact real form of \mathfrak{g}^σ defined by (58). Let η be the conjugation of \mathfrak{g}^σ with respect to \mathfrak{g}_u .

Both $\theta \circ \eta$ and $\eta \circ \theta$ are automorphisms of \mathfrak{g}^σ , and they coincide on the real form \mathfrak{g} , because if $X \in \mathfrak{k}$, $Y \in \mathfrak{p}$, then we have

$$(\eta \circ \theta)(X+Y) = \eta(X+Y) = X-Y = \theta X - \theta Y = (\theta \circ \eta)(X+Y).$$

Therefore we have

$$(64) \quad \eta \circ \theta = \theta \circ \eta \quad \text{on } \mathfrak{g}^{\mathcal{C}}.$$

Since $\eta H_{\alpha} = -H_{\alpha}$ for all $\alpha \in \mathbf{R}(l)$, we have $\eta(\mathfrak{g}[l]) = \mathfrak{g}[l]$ and $\eta(\mathfrak{g}(l)) = \mathfrak{g}(l)$. Since

$$\mathfrak{g}(l) \cap \mathfrak{k} = \{X \in \mathfrak{g}[l]; \theta X = X, \eta X = X\},$$

$$\mathfrak{g}(l) \cap \mathfrak{p} = \{X \in \mathfrak{g}[l]; \theta X = X, \eta X = -X\}$$

and

$$X = \frac{1}{2}(X + \eta X) + \frac{1}{2}(X - \eta X),$$

we have

$$\mathfrak{g}(l) = (\mathfrak{g}(l) \cap \mathfrak{k}) + (\mathfrak{g}(l) \cap \mathfrak{p}).$$

This decomposition is a Cartan decomposition of $\mathfrak{g}(l)$, because

$$(\mathfrak{g}(l) \cap \mathfrak{k}) + \sqrt{-1}(\mathfrak{g}(l) \cap \mathfrak{p}) \subset \mathfrak{k} + \sqrt{-1}\mathfrak{p} = \mathfrak{g}_{\mathfrak{u}}.$$

3) By Proposition 6 we have

$$Z^{\mathcal{C}}(\mathfrak{f}) = \mathfrak{f}^{\mathcal{C}} + \mathfrak{g}[l].$$

As $Z^{\mathcal{C}}(\mathfrak{f})$, $\mathfrak{f}^{\mathcal{C}}$ and $\mathfrak{g}[l]$ are invariant by θ , we have

$$Z(\mathfrak{f}) = Z^{\mathcal{C}}(\mathfrak{f}) \cap \mathfrak{g} = (\mathfrak{f}^{\mathcal{C}} \cap \mathfrak{g}) + (\mathfrak{g}[l] \cap \mathfrak{g}) = \mathfrak{f} + \mathfrak{g}(l).$$

Since $\mathfrak{f} = \mathfrak{h}_0^+ + (\mathfrak{f} \cap \mathfrak{p})$ (direct sum), we have

$$\mathfrak{h}_0^+ = \mathfrak{f} \cap \mathfrak{k}.$$

As η keeps \mathfrak{f} and $\mathfrak{g}(l)$ invariant as a whole we have

$$\begin{aligned} Z(\mathfrak{f}) \cap \mathfrak{k} &= (\mathfrak{f} \cap \mathfrak{k}) + (\mathfrak{g}(l) \cap \mathfrak{k}) \\ &= \mathfrak{h}_0^+ + (\mathfrak{g}(l) \cap \mathfrak{k}). \end{aligned}$$

Thus we have proved Proposition 9.

The above defined real semisimple Lie algebra $\mathfrak{g}(l)$ has a remarkable property that its Cartan subalgebra l has no toroidal part. As this property is essential for our later investigation, we shall give a special name to such a Lie algebra.

DEFINITION 8. A real semisimple Lie algebra \mathfrak{g} is called *normal* if \mathfrak{g} has a Cartan subalgebra \mathfrak{h} of which toroidal part is zero, i.e. $\mathfrak{h} = \mathfrak{h}^-$.

PROPOSITION 10. *The following two conditions for a real semisimple Lie algebra \mathfrak{g} are mutually equivalent.*

1) \mathfrak{g} is normal.

$$2) \quad \mathfrak{g} = \left\{ \sum_{i=1}^n a_i H_i + \sum_{\alpha \in \mathbf{R}} a_{\alpha} E_{\alpha}; a_i, a_{\alpha} \in \mathbf{R} \right\}$$

where (H_i, E_{α}) is a basis of $\mathfrak{g}^{\mathcal{C}}$ satisfying (54), (55) and (56).

PROOF. It is clear that 2) implies 1). Conversely let \mathfrak{g} be a normal real

form of \mathfrak{g}^c , and \mathfrak{h} be a Cartan subalgebra of \mathfrak{g} such that $\mathfrak{h}^- = \mathfrak{h}$. Then $\text{ad } H$ ($H \in \mathfrak{h}$) has only real eigenvalues. In view of this fact, we can easily obtain a basis (H_i, E_α) of \mathfrak{g}^c as required, following the argument of Weyl [13] concerning the construction of canonical base of a complex semisimple Lie algebra.

REMARK. 1) Proposition 10 proves that every complex semisimple Lie algebra has the unique normal real form up to isomorphisms.

2) E. Cartan called a real semisimple Lie algebra \mathfrak{g} normal when the character δ of \mathfrak{g} ($\delta = \dim \mathfrak{p} - \dim \mathfrak{k}$) is equal to the rank of \mathfrak{g} . Proposition 10 shows that our definition coincides with that of Cartan.

3) The normal real form of $\mathfrak{sl}(n, C)$ is $\mathfrak{sl}(n, R)$.

The normal real form of $\mathfrak{sp}(n, C)$ is $\mathfrak{sp}(n, R)$.

The normal real form of $\mathfrak{o}(n, C)$ is the Lie algebra of the orthogonal group of a quadratic form with the maximal index over an n -dimensional real vector space (cf. § 3).

4) Propositions 7, 8 and 9 imply that for each standard Cartan subalgebra \mathfrak{h} of \mathfrak{g} , there corresponds a normal semisimple Lie algebra $\mathfrak{g}(\mathfrak{l})$ where $\mathfrak{l} = \mathfrak{h}^- \cap \mathfrak{m}$. If \mathfrak{h}_1 and \mathfrak{h}_2 are two conjugate standard Cartan subalgebras, then $\mathfrak{g}(\mathfrak{l}_1)$ and $\mathfrak{g}(\mathfrak{l}_2)$ are isomorphic because their root systems are congruent by Theorem 4. The converse is in general not true.

For example, in the real simple Lie algebra (G, I) , there are two non conjugate standard Cartan subalgebras \mathfrak{h}_1 and \mathfrak{h}_2 (cf. § 4) of which toroidal parts are of dimension 1. However both $\mathfrak{g}(\mathfrak{l}_1)$ and $\mathfrak{g}(\mathfrak{l}_2)$ are isomorphic to $\mathfrak{sl}(2, R)$.

PROPOSITION 11. Let \mathfrak{g} be a normal real semisimple Lie algebra and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition of \mathfrak{g} .

Then the following two conditions are mutually equivalent.

- 1) $\text{rank } \mathfrak{k} = \text{rank } \mathfrak{g}$.
- 2) There exist n roots $\alpha_1, \dots, \alpha_n$ such that

$$\alpha_i \pm \alpha_j \in \mathbf{R} \text{ and } \alpha_i \neq \pm \alpha_j \text{ if } i \neq j,$$

where $n = \text{rank } \mathfrak{g}$.

PROOF. Let \mathfrak{h} be a Cartan subalgebra of \mathfrak{g} such that $\mathfrak{h}^- = \mathfrak{h}$. Using a basis H_i ($1 \leq i \leq n$), $E_\alpha (\alpha \in \mathbf{R})$ satisfying (54), (55) and (56), we have

$$\mathfrak{g} = \left\{ \sum_{i=1}^n a_i H_i + \sum_{\alpha \in \mathbf{R}} a_\alpha E_\alpha ; a_i, a_\alpha \in R \right\}$$

and

$$\mathfrak{g}_u = \left\{ \sqrt{-1} \sum_{i=1}^n a_i H_i + \sum_{\alpha > 0} (a_\alpha E_\alpha + \bar{a}_\alpha E_{-\alpha}) ; a_i \in R, a_\alpha \in C \right\}$$

where $\sum_{\alpha > 0}$ means the sum extended over all positive roots (we fix a lexicographic order in $(\mathfrak{h}^-)^*$).

Then we have

$$(65) \quad \mathfrak{k} = \mathfrak{g} \cap \mathfrak{g}_u = \left\{ \sum_{\alpha > 0} \alpha_\alpha (E_\alpha + E_{-\alpha}); \alpha_\alpha \in R \right\}.$$

Let

$$U_\alpha = E_\alpha + E_{-\alpha} \quad (\alpha > 0),$$

then we have

$$(66) \quad [U_\alpha, U_\beta] = N_{\alpha, \beta} U_{\alpha+\beta} + N_{\alpha, -\beta} U_{\alpha-\beta}.$$

Since \mathfrak{k} is isomorphic to a Lie algebra of a compact Lie group, every maximal abelian subalgebra of \mathfrak{k} is a Cartan subalgebra of \mathfrak{k} . Therefore if there exist n roots $\alpha_1, \dots, \alpha_n$ such that $\alpha_i \pm \alpha_j \notin \mathbf{R}$, $\neq 0$, then (66) proves that $\text{rank } \mathfrak{k} = n$.

Thus 2) implies 1).

Now we shall prove the converse.

We define a proposition (P_k) for a positive integer k as follows.

(P_k) : \mathfrak{g}^c has mutually different k positive roots $\alpha_1, \dots, \alpha_k$ such that

$$\alpha_i \pm \alpha_j \notin \mathbf{R} \quad (1 \leq i, j \leq k).$$

We shall prove that if $\text{rank } \mathfrak{k}$ is not smaller than k , then the proposition (P_k) is valid for \mathfrak{g} .

(P_1) is trivially valid. Next, we assume that (P_{k-1}) is valid for \mathfrak{g} . Then there exist $k-1$ positive roots $\alpha_1, \dots, \alpha_{k-1}$ such that $\alpha_i \pm \alpha_j \notin \mathbf{R}$. The equalities (52) and (66) prove that $\sum_{i=1}^{k-1} R U_{\alpha_i}$ is an abelian subalgebra of \mathfrak{k} .

Let \mathfrak{h}_1 be a Cartan subalgebra of \mathfrak{k} containing $\sum_{i=1}^{k-1} R \alpha_i$. An arbitrary element X in \mathfrak{h}_1 can be represented as

$$(67) \quad X = \sum_{\beta > 0} \alpha_\beta U_\beta.$$

Then we have

$$(68) \quad 0 = [U_{\alpha_i}, X] = \sum_{\beta > 0} \alpha_\beta (N_{\alpha_i, \beta} U_{\alpha_i + \beta} + N_{\alpha_i, -\beta} U_{\alpha_i - \beta}) \quad (1 \leq i \leq k-1).$$

If $\text{rank } \mathfrak{k} = \dim \mathfrak{h}_1 \geq k$, then there exists $X \in \mathfrak{h}_1$ such that $X \notin \sum_{i=1}^{k-1} R U_{\alpha_i}$. For such an X there exists at least one positive root $\beta \in \mathbf{R} - \{\alpha_1, \dots, \alpha_{k-1}\}$ such that the corresponding coefficient α_β in (67) does not vanish. The identity (68) proves that $N_{\alpha_i, \beta} = N_{\alpha_i, -\beta} = 0$ ($1 \leq i \leq k-1$). This implies $\alpha_i \pm \beta \notin \mathbf{R}$ ($1 \leq i \leq k-1$). If we put $\alpha_k = \beta$, then the proposition (P_k) is valid for \mathfrak{g} .

In this way we have proved that 1) implies 2).

REMARK. If two roots $\alpha_i \neq \pm \alpha_j$ satisfy the conditions $\alpha_i \pm \alpha_j \notin \mathbf{R}$, then α_i is orthogonal to α_j . Hence n roots $\alpha_1, \dots, \alpha_n$ in the condition 2) of Proposition 11 are linearly independent. Consequently, they form a basis of the Cartan subalgebra \mathfrak{h} of \mathfrak{g} , because $n = \text{rank } \mathfrak{g}$.

THEOREM 5. Let \mathfrak{g} be a real semisimple Lie algebra, and $(\mathfrak{k}, \mathfrak{p}, \mathfrak{m})$ be a

standard triple of \mathfrak{g} , and let \mathfrak{h}_0 be a Cartan subalgebra of \mathfrak{g} such that $\mathfrak{h}_0^- = \mathfrak{m}$. For any subspace \mathfrak{n} of \mathfrak{m} , we denote by \mathfrak{l} the subspace of \mathfrak{m} defined by

$$\mathfrak{l} = \mathfrak{n}^+ \cap \mathfrak{m} = \{X \in \mathfrak{m}; B(X, \mathfrak{n}) = 0\}.$$

Then a subspace \mathfrak{n} of \mathfrak{m} becomes the vector part \mathfrak{h}^- of a standard Cartan subalgebra \mathfrak{h} if and only if there exist l roots ($l = \dim \mathfrak{l}$) $\alpha_1, \dots, \alpha_l$ such that

- 1) $\alpha_i \pm \alpha_j \in \mathbf{R}$, $1 \leq i, j \leq l$ and $\alpha_i \pm \alpha_j \neq 0$ if $i \neq j$,
- 2) $\mathfrak{l} = \sum_{i=1}^l \mathbf{R}H_{\alpha_i}$.

PROOF. By Proposition 9 3), we have $Z(\mathfrak{f}) \cap \mathfrak{f} = \mathfrak{h}_0^+ + (\mathfrak{g}(\mathfrak{l}) \cap \mathfrak{f})$, ($\mathfrak{f} = \mathfrak{l}^+ \cap \mathfrak{h}_0$) and \mathfrak{h}_0^+ is contained in the centre of $Z(\mathfrak{f})$. We have following equivalences by Propositions 8, 9 and 11.

- \mathfrak{n} is the vector part of a standard Cartan subalgebra
- $\Leftrightarrow \mathfrak{n}$ is the vector part of a special standard Cartan subalgebra
- $\Leftrightarrow \text{rank } (Z(\mathfrak{f}) \cap \mathfrak{f}) = \dim \mathfrak{h}_0^+ + \dim \mathfrak{l}$
- $\Leftrightarrow \text{rank } (\mathfrak{g}(\mathfrak{l}) \cap \mathfrak{f}) = \dim \mathfrak{l} = \text{rank } \mathfrak{g}(\mathfrak{l})$
- \Leftrightarrow there exist l roots ($l = \dim \mathfrak{l}$) in $\mathbf{R}(\mathfrak{l})$ such that $\alpha_i \pm \alpha_j \in \mathbf{R}$ ($1 \leq i, j \leq l$).

We remark that if the conditions in Theorem 5 are satisfied, any Cartan subalgebra of $Z(\mathfrak{f}) \cap \mathfrak{f}$ gives the toroidal part of a standard Cartan subalgebra of which vector part \mathfrak{h}^- is equal to \mathfrak{n} .

COROLLARY. The number of conjugate classes of Cartan subalgebras in a real semisimple Lie algebra is always finite.

DEFINITION 9. A set of positive roots $\mathbf{F} = \{\alpha_1, \dots, \alpha_l\}$ satisfying the condition 1) in Theorem 5 is called an *admissible root system*.

Two admissible root systems \mathbf{F}_1 , and \mathbf{F}_2 are called *equivalent* and denoted by $\mathbf{F}_1 \equiv \mathbf{F}_2$ if \mathbf{F}_1 and \mathbf{F}_2 span the same subspace \mathfrak{l} of $(\mathfrak{h}_0^c)^-$; i. e.,

$$\sum_{\alpha \in \mathbf{F}_1} \mathbf{R}H_{\alpha} = \sum_{\beta \in \mathbf{F}_2} \mathbf{R}H_{\beta}.$$

Two admissible root systems \mathbf{F}_1 and \mathbf{F}_2 are called *conjugate* and denoted by $\mathbf{F}_1 \approx \mathbf{F}_2$ if there exists an element s in the Weyl group of \mathfrak{g}^c with respect to \mathfrak{h}_0^c such that $s\mathbf{F}_1 \equiv \mathbf{F}_2$.

The relations “ \equiv ” and “ \approx ” are equivalence relations.

Summing up the results of §1 and §2, we have the following Theorem 6 which is our main theorem.

THEOREM 6. There is a one to one correspondence between conjugate classes of Cartan subalgebras in a real semisimple Lie algebra \mathfrak{g} and the conjugate classes of admissible root systems contained in $\mathbf{R}(\mathfrak{m})$.

§ 3. Conjugate classes of Cartan subalgebras in classical real simple Lie algebras.

For the classification of real simple Lie algebras, we refer to E. Cartan [1], [3] and F. Gantmacher [5]. We denote real forms of complex simple Lie algebras by the same notations as was used by E. Cartan for the corresponding symmetric Riemannian spaces.

Before entering into the study of each type of simple Lie algebra, we give here some general remarks.

We use the classical linear Lie algebras as models of complex simple Lie algebra of classical types: i. e. we realize the complex simple Lie algebras of type A, B, C and D of rank n by $\mathfrak{sl}(n+1, C)$, $\mathfrak{o}(2n+1, C)$, $\mathfrak{sp}(n, C)$ and $\mathfrak{o}(2n, C)$ respectively. All of these Lie algebras \mathfrak{g}^c are self-adjoint, i. e. if X belongs to \mathfrak{g}^c , then ${}^t\bar{X}$ also belongs to \mathfrak{g}^c . Therefore a compact real form \mathfrak{g}_u of \mathfrak{g}^c is given as the intersection of \mathfrak{g}^c and the Lie algebra of the unitary group with the same degree.

If \mathfrak{g} is a self-adjoint real form of \mathfrak{g}^c , then $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is a Cartan decomposition of \mathfrak{g} , where $\mathfrak{k} = \{X \in \mathfrak{g}; {}^t\bar{X} = -X\}$ and $\mathfrak{p} = \{X \in \mathfrak{g}; {}^t\bar{X} = X\}$.

In order to prove that a subset \mathfrak{m} is a maximal abelian subalgebra in \mathfrak{p} , we shall use the following simple lemma.

LEMMA 2. *Let H be a diagonal matrix of which diagonal elements are h_1, \dots, h_n , and $A = (a_{ij})$ be a matrix of degree n . If $h_i \neq h_j$ and $[H, A] = 0$, then we have $a_{ij} = a_{ji} = 0$.*

Notations.

- E_m : Identity matrix of degree m .
- $S(m, C)$ ($S(m, R)$): The set of all complex (real) symmetric matrices of degree m .
- $\mathfrak{o}(m, C)$ ($\mathfrak{o}(m)$) : The set of all complex (real) skew symmetric matrices of degree m = the Lie algebra of $O(m, C)$ ($O(m)$).
- $H(m)$: The set of all Hermitian matrices of degree m .
- $\mathfrak{u}(m)$: The set of all skew Hermitian matrices of degree m = the Lie algebra of $U(m)$.
- $\mathfrak{su}(m)$: The set of all skew Hermitian matrices of degree m with the trace 0 = the Lie algebra of $SU(m)$.
- $\mathfrak{M}(n, m, C)$ ($\mathfrak{M}(n, m, R)$): The set of all complex (real) $n \times m$ matrices.
- $D(a_1, \dots, a_n)$: Diagonal matrix with the diagonal elements a_1, \dots, a_n .
- N : The number of conjugate classes of Cartan subalgebra in the given simple Lie algebra with rank n .
- F : Admissible root system (cf. Definition 9).

Type A

The real forms of $\mathfrak{sl}(n+1, \mathbb{C})$ are divided into three types (A I), (A II) and (A III).

There exists only one real form of type (A I) and (A II) respectively with the given rank up to isomorphisms. There exist $\lfloor \frac{n+1}{2} \rfloor + 1$ non isomorphic real forms of type (A III) with the rank n . The real form of type (A IV) in Cartan's notation is a particular real form contained in type (A III).

Type (A I).

$$\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{R}) = \{X \in \mathfrak{gl}(n+1, \mathbb{R}); \text{Tr} X = 0\}$$

is self-adjoint, and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is a Cartan decomposition of \mathfrak{g} , where

$$\mathfrak{k} = \{X \in \mathfrak{g}; {}^t X = -X\}, \text{ and } \mathfrak{p} = \{X \in \mathfrak{g}; {}^t X = X\}.$$

$$\mathfrak{h}_0 = \{D(h_1, \dots, h_{n+1}); h_i \in \mathbb{R}, \sum_{i=1}^{n+1} h_i = 0\}$$

is a Cartan subalgebra of \mathfrak{g} contained in \mathfrak{p} . Therefore \mathfrak{g} is the normal real form of $\mathfrak{sl}(n+1, \mathbb{C})$.

Let e_i ($1 \leq i \leq n+1$) be the linear form on $\mathfrak{h}_0^{\mathbb{C}}$ defined by

$$e_i(H) = h_i \quad (H = D(h_1, \dots, h_n) \in \mathfrak{h}_0^{\mathbb{C}}),$$

then the root system \mathbf{R} of $\mathfrak{g}^{\mathbb{C}}$ is expressed as follows:

$$(69) \quad \mathbf{R} = \{\pm(e_i - e_j); 1 \leq i < j \leq n+1\}.$$

The Weyl group W of $\mathfrak{g}^{\mathbb{C}}$ consists of all permutations of e_i 's.

Any admissible root system \mathbf{F} must be of the following type:

$$(70) \quad \mathbf{F} = \{(e_{i_1} - e_{i_2}), \dots, (e_{i_{2k-1}} - e_{i_{2k}})\}, i_1 < i_2, \dots, i_{2k-1} < i_{2k}.$$

where i_1, \dots, i_{2k} are different $2k$ integers contained in $\{1, 2, \dots, n+1\}$. The toroidal part of the corresponding Cartan subalgebra is of dimension k . Since the Weyl group W contains all permutations of e_i 's, every two k -dimensional \mathfrak{l} spanned by \mathbf{F} of type (70) are conjugate under W .

Consequently, by Theorem 4, every two Cartan subalgebras \mathfrak{h}_1 and \mathfrak{h}_2 of $\mathfrak{sl}(n+1, \mathbb{R})$ are conjugate if and only if $\dim \mathfrak{h}_1^+ = \dim \mathfrak{h}_2^+$.

Since the possible values of k are $0, 1, 2, \dots, \lfloor \frac{n+1}{2} \rfloor$, the number N of conjugate classes of Cartan subalgebras in $\mathfrak{sl}(n+1, \mathbb{R})$ is equal to $\lfloor \frac{n+1}{2} \rfloor + 1$:

$$N = \lfloor \frac{n+1}{2} \rfloor + 1.$$

A representative \mathfrak{h} of conjugate classes such that $\dim \mathfrak{h}^+ = k$ is given as follows:

Therefore we see that

$$(72) \quad \mathbf{R}(\mathfrak{m}) = \text{empty.}$$

(72) implies that the only possible Γ is $\{0\}$. Consequently, the number N of conjugate classes of Cartan subalgebras in \mathfrak{g} is equal to 1:

$$N = 1.$$

In other words, every Cartan subalgebra of \mathfrak{g} is conjugate to \mathfrak{h}_0 defined by (70).

\mathfrak{g} is an example of non compact real simple Lie algebra, of which Cartan subalgebras are mutually conjugate.

Type (A III). The Lie algebra of the type (A III) is the Lie algebra $\mathfrak{o}(H)$ of special unitary group with respect to a (not necessarily definite) Hermitian form H on a $(n+1)$ -dimensional complex vector space V . Let H and H' be two Hermitian forms on V , then $\mathfrak{o}(H) \cong \mathfrak{o}(H')$ if and only if index of $H = \text{index of } H'$.

Therefore, there are exactly $\left[\frac{n+1}{2}\right] + 1$ non isomorphic real forms of the type (A III) with the rank n .

Let H be of the index m ($0 \leq m \leq \left[\frac{n+1}{2}\right]$). Then $\mathfrak{g}_m = \mathfrak{o}(H)$ can be defined as follows. Let

$$H_m = \begin{pmatrix} E_m & 0 \\ 0 & -E_l \end{pmatrix}; \quad m+l = n+1, \quad \left(0 \leq m \leq \left[\frac{n+1}{2}\right]\right),$$

then we have

$$\begin{aligned} \mathfrak{g}_m &= \{X \in \mathfrak{sl}(n+1, C); {}^t \bar{X} H_m + H_m X = 0\} \\ &= \left\{ \begin{pmatrix} A & B \\ {}^t \bar{B} & D \end{pmatrix}; \begin{array}{l} A \in \mathfrak{u}(m), D \in \mathfrak{u}(l) \\ B \in \mathfrak{M}(m, l, C), \text{Tr} A + \text{Tr} D = 0 \end{array} \right\}. \end{aligned}$$

\mathfrak{g}_m is self-adjoint, hence $\mathfrak{g}_u = \mathfrak{su}(n+1)$ is a compact real form of \mathfrak{g}_m^C which is invariant under θ .

We have

$$\mathfrak{k} = \mathfrak{g}_m \cap \mathfrak{g}_u = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}; \begin{array}{l} A \in \mathfrak{u}(m), D \in \mathfrak{u}(l), \text{Tr} A + \text{Tr} D = 0 \end{array} \right\}$$

and

$$\mathfrak{p} = \mathfrak{g} \cap \sqrt{-1} \mathfrak{g}_u = \left\{ \begin{pmatrix} 0 & B \\ {}^t \bar{B} & 0 \end{pmatrix}; \begin{array}{l} B \in \mathfrak{M}(m, l, C) \end{array} \right\}.$$

Then

$$m = \left\{ \begin{pmatrix} & & h_1 & & \\ & & \dots & & \\ & & & h_m & \\ h_1 & & & & \\ \dots & & & & \\ & h_m & & & \end{pmatrix} ; h_i \in R \right\}$$

is a maximal abelian subalgebra in \mathfrak{p} , and

$$(73) \quad \mathfrak{h}_0 = \left\{ H = \begin{pmatrix} u_1 & & h_1 & & \\ & \dots & \dots & & \\ & & & h_m & \\ h_1 & & u_1 & & \\ \dots & & \dots & & \\ & h_m & & u_m & \\ & & & & u_{2m+1} \\ & & & & \dots \\ & & & & u_{n+1} \end{pmatrix} ; \begin{matrix} u_i \in \sqrt{-1} R, h_i \in R, \\ \text{Tr} H = 0 \end{matrix} \right\}$$

is a Cartan subalgebra containing m . $\mathfrak{h}_0^{\mathbb{C}}$ consists of matrices of the same type as H in (73) with complex elements h_i and u_i .

Let e_i be the linear function on $\mathfrak{h}_0^{\mathbb{C}}$ defined by

$$e_i(H) = \begin{cases} -\sqrt{-1} u_i + h_i & (1 \leq i \leq m), \\ -\sqrt{-1} u_{i-m} - h_{i-m} & (m+1 \leq i \leq 2m), \\ -\sqrt{-1} u_i & (2m+1 \leq i \leq n+1). \end{cases}$$

Then

$$R = \{ \pm(e_i - e_j); (1 \leq i < j \leq n+1) \}$$

is the root system of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{h}_0^{\mathbb{C}}$.

Therefore we have

$$R(m) = \{ \pm(e_i - e_{i+m}); (1 \leq i \leq m) \}.$$

The possible types of the admissible root systems F are as follows:

$$F = \{ (e_{i_k} - e_{i_k+m}); 1 \leq k \leq l \},$$

where i_1, \dots, i_l are different l integers from $\{1, 2, \dots, m\}$.

Since the Weyl group W contains all permutations of e_i 's, \mathfrak{h}_1 and \mathfrak{h}_2 are conjugate if and only if $\dim \mathfrak{h}_1^+ = \dim \mathfrak{h}_2^+$. The possible values of l are

0, 1, 2, ..., m. Consequently the number N of conjugate classes of Cartan subalgebras in \mathfrak{g}_m is equal to m+1:

$$N = m + 1.$$

It is remarkable that N does not depend on the rank n but depends on the index m only. A representative \mathfrak{h} of the conjugate class with $\dim \mathfrak{h}^- = k$ is given as follows:

$$\mathfrak{h} = \left\{ H = \begin{array}{|c|c|c|c|} \hline \begin{array}{c} u_1 \\ \dots \\ u_k \end{array} & & \begin{array}{c} h_1 \\ \dots \\ h_k \end{array} & \\ \hline & \begin{array}{c} u_{k+1} \\ \dots \\ u_m \end{array} & & \\ \hline \begin{array}{c} h_1 \\ \dots \\ h_k \end{array} & & \begin{array}{c} u_1 \\ \dots \\ u_k \end{array} & \\ \hline & & & \begin{array}{c} u_{m+k+1} \\ \dots \\ u_{2m} \end{array} \\ \hline & & & \begin{array}{c} u_{2m+1} \\ \dots \\ u_{n+1} \end{array} \\ \hline \end{array} \right. ; \left. \begin{array}{l} h_i \in R, u_i \in \sqrt{-1} R, \\ \text{Tr} H = 0 \end{array} \right\}.$$

Types B and D

$$\mathfrak{g}^C = \mathfrak{o}(s, C).$$

Type B_n : $\mathfrak{g}^C = \mathfrak{o}(2n+1, C)$ has only one kind of real form (B I) which contains n+1 non isomorphic real forms with the rank n.

Type D_n : The real forms of $\mathfrak{g}^C = \mathfrak{o}(2n, C)$ are divided into two types (D I) and (D III). Type (D I) contains n+1 non isomorphic real forms, type (D III) contains only one real form with the given rank. The real form of type (BD II) in Cartan's notation is a particular real form contained in type (BD I).

Type (BD I). \mathfrak{g} is the Lie algebra $\mathfrak{o}(Q)$ of the orthogonal group with respect to a quadratic form Q on a real vector space V. Let

$$s = \begin{cases} 2n+1 & \text{if } \mathfrak{g}^C \text{ is of type } B_n \\ 2n & \text{if } \mathfrak{g}^C \text{ is of type } D_n, \end{cases}$$

then $\dim V = s$.

$$\mathfrak{o}(Q) \cong \mathfrak{o}(Q') \quad \text{if and only if} \quad \text{index of } Q = \text{index of } Q'.$$

Therefore, there exist exactly $n+1$ non isomorphic real forms of the type (BDI) with the rank n .

Let Q be of the index m ($0 \leq m \leq n$), then $\mathfrak{g}_m = \mathfrak{o}(Q)$ can be defined as follows: let

$$B_m = \begin{pmatrix} 0 & E_m & 0 \\ E_m & 0 & 0 \\ 0 & 0 & -E_p \end{pmatrix}; \quad (p+2m=s),$$

then we have

$$\begin{aligned} \mathfrak{g}_m &= \{X \in \mathfrak{gl}(s, R); {}^tXB_m + B_mX = 0\} \\ &= \left\{ \begin{pmatrix} A & B & D \\ C & -{}^tA & F \\ {}^tF & {}^tD & L \end{pmatrix}; \begin{array}{l} A \in \mathfrak{gl}(m, R), B, C \in \mathfrak{o}(m), \\ D, F \in \mathfrak{M}(m, p, R), L \in \mathfrak{o}(p) \end{array} \right\}. \end{aligned}$$

Since \mathfrak{g}_m is self-adjoint, $\mathfrak{g}_m = \mathfrak{k} + \mathfrak{p}$ is a Cartan decomposition of \mathfrak{g}_m , where

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & B & D \\ B & -{}^tA & -D \\ -{}^tD & {}^tD & L \end{pmatrix}; \begin{array}{l} A, B \in \mathfrak{o}(m), L \in \mathfrak{o}(p), \\ D \in \mathfrak{M}(m, p, R) \end{array} \right\}$$

and

$$\mathfrak{p} = \left\{ \begin{pmatrix} A & B & D \\ -B & -A & D \\ {}^tD & {}^tD & 0 \end{pmatrix}; \begin{array}{l} A \in S(m, R), B \in \mathfrak{o}(m), \\ D \in \mathfrak{M}(m, p, R) \end{array} \right\}.$$

$$\mathfrak{m} = \{D(h_1, \dots, h_m, -h_1, \dots, -h_m, 0, \dots, 0); h_i \in R\}$$

is a maximal abelian subalgebra in \mathfrak{p} (cf. Lemma 2), and

$$\mathfrak{h}_0 = \left\{ H = \begin{array}{|c|c|c|} \hline \begin{array}{c} h_1 \\ \vdots \\ h_m \end{array} & & \\ \hline & \begin{array}{c} -h_1 \\ \vdots \\ -h_m \end{array} & \\ \hline & & \begin{array}{c} -u_1 \\ \vdots \\ -u_t \end{array} \\ \hline & \begin{array}{c} u_1 \\ \vdots \\ u_t \end{array} & \\ \hline \end{array} \right\}; \quad h_i \in R, u_i \in R$$

is a Cartan subalgebra of \mathfrak{g}_m containing \mathfrak{m} , where $t = \left[\frac{n-2m}{2} \right]$. The last row and column exist only for \mathfrak{g}_m of the type B.

Let e_i be the linear form on \mathfrak{h}_0 defined by

$$e_i(H) = \begin{cases} h_i & (1 \leq i \leq m), \\ \sqrt{-1} u_{i-m} & (m+1 \leq i \leq n), \end{cases}$$

then the root system of \mathfrak{g}_m^c is given as follows:

$$\mathbf{R} = \{ \pm(e_i \pm e_j), (1 \leq i < j \leq n) \}$$

for $\mathfrak{g}_m^c = D_n$, and

$$\mathbf{R} = \{ \pm(e_i \pm e_j), (1 \leq i < j \leq n); \pm e_i, (1 \leq i \leq n) \}$$

for $\mathfrak{g}_m^c = B_n$.

Therefore

$$\mathbf{R}(\mathfrak{m}) = \{ \pm(e_i \pm e_j), (1 \leq i < j \leq m) \}$$

for \mathfrak{g}_m of type (DI) and

$$\mathbf{R}(\mathfrak{m}) = \{ \pm(e_i \pm e_j), (1 \leq i < j \leq m); \pm e_i (1 \leq i \leq m) \}$$

for \mathfrak{g}_m of type (BI).

Now we investigate type B and type D separately.

Type (DI). The Weyl group W of \mathfrak{g}_m^c consists of all permutations of e_i 's and change of signatures of an even number of e_i 's. Therefore any admissible root system \mathbf{F} is conjugate to one of the followings:

$$(74) \quad \mathbf{F}(l, k) = \{ e_1 + e_2, e_1 - e_2, \dots, e_{2l-1} + e_{2l}, e_{2l-1} - e_{2l}, \\ e_{2l+1} - e_{2l+k+1}, \dots, e_{2l+k} - e_{2l+2k} \},$$

where two integers l, k satisfy the conditions:

$$(75) \quad 0 \leq k, 0 \leq l, 2(l+k) \leq m.$$

If $m = n$ and m is even, there exists another type of \mathbf{F} ; i. e.

$$(76) \quad \mathbf{F} = \{ \mathbf{F}(0, m-2), e_{m-1} + e_m \}$$

where $\mathbf{F}(0, m-2)$ is defined by (74).

Two $\mathbf{F}(l, k)$'s with different (l, k) are not conjugate under W .

The number m_p of lattice points on the straight line $x+y=p$ (p is a non negative integer) in the first quadrant is equal to $p+1$. Consequently,

$$N = \begin{cases} \sum_{p=0}^{\left[\frac{m}{2} \right]} m_p = \frac{1}{2} \left(\left[\frac{m}{2} \right] + 1 \right) \left(\left[\frac{m}{2} \right] + 2 \right), & m < n \text{ or } m = n = \text{odd}, \\ \sum_{p=0}^{m/2} m_p + 1 = \frac{1}{2} \left(\frac{m}{2} + 1 \right) \left(\frac{m}{2} + 2 \right) + 1, & m = n = \text{even}. \end{cases}$$

A representative \mathfrak{h} of the conjugate class characterized by (74) is given as follows:

(77) $\mathfrak{h} = \left\{ \begin{array}{|c|c|c|c|c|} \hline D_1 & & & D_2 & \\ \hline & D_3 & & & D_4 \\ \hline & & D_5 & & \\ \hline D_2 & & & D_1 & \\ \hline & D_4 & & & -D_3 \\ \hline & & & & -D_5 \\ \hline & & & & D_6 \\ \hline \end{array} \right. ; h_i \in R \},$

where

$$D_1 = \begin{pmatrix} & & h_1 & & \\ & & \dots & & \\ & & & & h_l \\ \hline h_1 & & & & \\ & & \dots & & \\ & & & & h_l \end{pmatrix}, D_2 = \begin{pmatrix} & & -h_{l+1} & & \\ & & \dots & & \\ & & & & -h_{2l} \\ \hline h_{l+1} & & & & \\ & & \dots & & \\ & & & & h_{2l} \end{pmatrix}, D_3 = \begin{pmatrix} h_{2l+1} & & & & \\ & \dots & & & \\ & & h_{2l+k} & & \\ \hline & & & & h_{2l+1} \\ & & & & \dots \\ & & & & h_{2l+k} \end{pmatrix},$$

(78)

$$D_4 = \begin{pmatrix} & & -h_{2l+k+1} & & \\ & & \dots & & \\ & & & & -h_{2l+2k} \\ \hline h_{2l+k+1} & & & & \\ & & \dots & & \\ & & & & h_{2l+2k} \end{pmatrix}, D_5 = \begin{pmatrix} h_{2l+2k+1} & & & & \\ & \dots & & & \\ & & & & \\ & & & & \\ & & & & h_m \end{pmatrix}, D_6 = \begin{pmatrix} & & -h_{m+1} & & \\ & & \dots & & \\ & & & & -h_n \\ \hline h_{m+1} & & & & \\ & & \dots & & \\ & & & & h_n \end{pmatrix}.$$

A representative of the conjugate class characterized by (75) is given also by (77), except that D_5 should be replaced by $D'_5 = D(h_{m-1}, -h_{m-1})$. (In this case, we have $l=0, h=m$, therefore D_1, D_2 and D_6 disappear.)

Type (BI). As

$$\begin{aligned} e_i \pm e_j \in \mathbf{R}, \quad (i \neq j); \quad (e_i \pm e_j) \mp (e_j \pm e_k) \in \mathbf{R}, \quad (i \neq k); \\ (e_i \pm e_j) \mp e_j \in \mathbf{R} \quad ; \quad (e_i + e_j) + (e_j - e_k) \in \mathbf{R}, \quad (i \neq k), \end{aligned}$$

the pairs $\{e_i, e_j\}$, $\{e_i \pm e_j, e_j \pm e_k\}$, $\{e_i \pm e_j, e_j\}$ and $\{e_i + e_j, e_j - e_k\}$ can not be contained in admissible root systems. On the other hand,

$$\begin{aligned} (e_i + e_j) \pm (e_i - e_j) \in \mathbf{R} \quad (i \neq j), \\ (e_i \pm e_j) \pm (e_k \pm e_l) \in \mathbf{R} \quad (i, j, k, l \text{ are mutually different}), \\ (e_i \pm e_j) \pm e_k \quad \in \mathbf{R} \end{aligned}$$

and the Weyl group W consists of all permutations of e_i 's and changes of signatures of e_i 's.

Therefore any admissible root system \mathbf{F} is conjugate to one of the systems listed in the following (79) and (79)'.

$$\begin{aligned} (79) \quad \mathbf{F}(l, k) = \{e_1 + e_2, e_1 - e_2, \dots, e_{2l-1} + e_{2l}, e_{2l-1} - e_{2l}, \\ e_{2l+1} - e_{2l+k+1}, \dots, e_{2l+k} - e_{2l+2j}\}, \\ 0 \leq k, 0 \leq l, \quad 2(k+l) \leq m. \end{aligned}$$

$$\begin{aligned} (79)' \quad \mathbf{F}'(l, k) = \{e_1 + e_2, e_1 - e_2, \dots, e_{2l-1} + e_{2l}, e_{2l-1} - e_{2l}, \\ e_{2l+1} - e_{2l+k+1}, \dots, e_{2l+k} - e_{2l+2k}, e_{2l+2k+1}\} \\ 0 \leq k, 0 \leq l, \quad 2(k+l)+1 \leq m. \end{aligned}$$

$\mathbf{F}(l, k)$ is not conjugate to $\mathbf{F}'(l', k')$ under W . And two $\mathbf{F}(l, k)$'s (or $\mathbf{F}'(l, k)$'s) with different (l, k) are not conjugate under W . Therefore the number N of conjugate classes of Cartan subalgebras is given as follows:

$$N = \begin{cases} 2 \sum_{p=0}^{\frac{m}{2}-1} m_p + m_{m/2} = \left(\frac{m}{2} + 1\right)^2 = \frac{(m+2)^2}{4}, & m = \text{even} \\ 2 \sum_{p=0}^{\lfloor \frac{m}{2} \rfloor} m_p = \left(\lfloor \frac{m}{2} \rfloor + 1\right) \left(\lfloor \frac{m}{2} \rfloor + 2\right) = \frac{(m+1)(m+3)}{4}, & m = \text{odd}. \end{cases}$$

(m_p is the number of lattice points on the line $x+y=p$ in the first quadrant.)

A representative \mathfrak{h} of the conjugate class associated with (79) is of the same type as (77). (The last row and column consisting of zero should be added.) A representative \mathfrak{h} of the conjugate class associated with (79)' is given as follows:

$$\mathfrak{h} = \left\{ \begin{array}{|c|c|c|c|c|c|c|c|} \hline & D_1 & & & D_2 & & & & \\ \hline & & D_3 & & & D_4 & & & \\ \hline & & & D_5' & & & & & -h_{2l+2k+1} \\ \hline D_2 & & & & D_1 & & & & \\ \hline & D_4 & & & & -D_3 & & & \\ \hline & & & & & & -D_5' & & h_{2l+2k+1} \\ \hline & & & & & & & D_6 & \\ \hline & & h_{2l+2k+1} & & & -h_{2l+2k+1} & & & \\ \hline \end{array} \right\}; \quad h_i \in R$$

where $D_5' = D(0, h_{2(l+k+1)}, \dots, h_m)$ and other D_i 's are same as in (78).

REMARK 1) The number N of conjugate classes in a Lie algebra \mathfrak{g} of type (BDI) does not depend upon the rank of \mathfrak{g} . N depends upon the index of the quadratic form Q only.

2) There exist non-conjugate Cartan subalgebras of which toroidal parts have the same dimension. For example, Cartan subalgebras which correspond to $F = \{e_1 - e_2\}$ and $F = \{e_1\}$ respectively are not conjugate.

3) When the quadratic form Q is of the maximal index, i. e. if $m = n$, then \mathfrak{m} coincides with \mathfrak{h}_0 . In other words, \mathfrak{g}_n is the normal real form of \mathfrak{g}^c .

4) We divide the Lie algebras of type (DI) into two classes (DI a) and (DI b) according to m is even or odd. We remark that a Lie algebra \mathfrak{g}_m of type (DI) has an m -dimensional admissible root system if and only if m is even. This fact implies that \mathfrak{g}_m has a Cartan subalgebra \mathfrak{h} which satisfies $\mathfrak{h}^+ = \mathfrak{h}$, if and only if \mathfrak{g}_m is of the type (DI a) (cf. Theorem 8 in §4).

Type (D III). \mathfrak{g} is the Lie algebra of the group G of linear transforma-

tions in $2n$ complex variables which keep invariant the quadratic form $z_1 z_{n+1} + \dots + z_n z_{2n}$ and Hermitian form $z_1 \bar{z}_1 + \dots + z_n \bar{z}_n - \dots - z_{2n} \bar{z}_{2n}$. If we perform the suitable transformation of variables, G is transformed to the group which keeps invariant $u_1 u_{k+1} + \dots + u_{3k} u_{4k} (+u_{2n-1} u_{2n})$ and $u_1 \bar{u}_{2k+1} + \dots + u_{2k} \bar{u}_{4k} + \bar{u}_1 u_{2k+1} + \dots + \bar{u}_{2k} u_{4k} (+u_{2n-1} \bar{u}_{2n-1} - u_{2n} \bar{u}_{2n})$, where $k = \lfloor \frac{n}{2} \rfloor$. The terms in parenthesis exist only if n is odd.

Let

$$K = \begin{array}{|c|c|c|c|c|} \hline & E_k & & & \\ \hline E_k & & & & \\ \hline & & & -E_k & \\ \hline & & -E_k & & \\ \hline & & & & \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \\ \hline \end{array} \quad \text{and} \quad L = \begin{array}{|c|c|c|} \hline & E_{2k} & \\ \hline E_{2k} & & \\ \hline & & \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \\ \hline \end{array}.$$

(The last two rows and columns in K and L exist only if n is odd. The same remark holds in the following.)

We have

$$\mathfrak{g} = \{X \in \mathfrak{gl}(2n, C); {}^t X K + K X = 0, {}^t \bar{X} L + L X = 0\}$$

$$= \left\{ \begin{array}{|c|c|c|c|} \hline A & B & & \\ \hline C & -{}^t A & & \\ \hline & & D & G \\ \hline F_k {}^t D F_k & & \begin{array}{|c|c|} \hline -{}^t \bar{A} & \bar{C} \\ \hline \bar{B} & \bar{A} \end{array} & F_k \bar{G} F_1 J \\ \hline -F_1 {}^t G F_k & & -J {}^t \bar{G} & \begin{array}{cc} a & 0 \\ 0 & -a \end{array} \\ \hline \end{array} \right\}; \begin{array}{l} A \in \mathfrak{gl}(k, C), B \in \mathfrak{o}(k, C), \\ C \in \mathfrak{o}(k, C), D \in \mathfrak{u}(2k), \\ a \in \sqrt{-1} R, \\ G \in \mathfrak{M}(2k, 2, C) \end{array}$$

where $F_k = \begin{pmatrix} 0 & E_k \\ E_k & 0 \end{pmatrix}$ and $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Since \mathfrak{g} is self-adjoint, we get a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ by

$$\mathfrak{k} = \{X \in \mathfrak{g}; {}^t\bar{X} = -X\}$$

}	A	B	D ₁	D ₂	G
	\bar{B}	\bar{A}	$-\bar{D}_2$	$-\bar{D}_1$	
	D ₁	D ₂	A	B	GJ
	$-\bar{D}_2$	$-\bar{D}_1$	\bar{B}	\bar{A}	
	${}^t\bar{G}$		$-J{}^t\bar{G}$		a 0

$$\left. \begin{array}{l} ; A \in \mathfrak{u}(k), B \in \mathfrak{o}(k, C), \\ D_1 \in \mathfrak{u}(k), D_2 \in \mathfrak{S}(k, C), \\ a \in \sqrt{-1}R, F_k\bar{G} - F_1G = 0, \\ G \in \mathfrak{M}(2k, 2, C) \end{array} \right\}$$

and

$$\mathfrak{p} = \{X \in \mathfrak{g}; {}^t\bar{X} = X\}$$

}	A	B	D ₁	D ₂	G
	$-\bar{B}$	$-\bar{A}$	\bar{D}_2	\bar{D}_1	
	$-D_1$	$-D_2$	$-A$	$-B$	$-GJ$
	$-\bar{D}_2$	$-\bar{D}_1$	\bar{B}	\bar{A}	
	${}^t\bar{G}$		$-J{}^t\bar{G}$		0

$$\left. \begin{array}{l} ; A \in H(k), B \in \mathfrak{o}(k, C), \\ D_1 \in \mathfrak{u}(k), D_2 \in \mathfrak{o}(k, C), \\ G \in \mathfrak{M}(2k, 2, C), \\ F_k\bar{G} + F_1G = 0 \end{array} \right\}$$

$\mathfrak{m} = \{D(h_1, \dots, h_k, -h_1, \dots, -h_k, -h_1, \dots, -h_k, h_1, \dots, h_k, (0, 0)); h_i \in R\}$ is a maximal abelian subalgebra in \mathfrak{p} (cf. Lemma 2).

A Cartan subalgebra \mathfrak{h}_0 containing \mathfrak{m} is given by

$$\mathfrak{h}_0 = \{D(h_1, \dots, h_k, -h_1, \dots, -h_k, -\bar{h}_1, \dots, -\bar{h}_k, \bar{h}_1, \dots, \bar{h}_k, (a, -a)); h_i \in C, a \in \sqrt{-1}R\}.$$

Then

$$\mathfrak{h}_0^\sigma = \{H = D(h_1, \dots, h_k, -h_1, \dots, -h_k, h_{k+1}, \dots, h_{2k}, -h_{k+1}, \dots, -h_{2k}, (h_{2k+1}, -h_{2k+1})); h_i \in C\}$$

is a Cartan subalgebra of $\mathfrak{g}^\sigma = \{X \in \mathfrak{gl}(2n, C); {}^tXK + KX = 0\}$.

Let e_i be the linear form on \mathfrak{h}_0^σ defined by

$$e_i(H) = h_i,$$

then we have

$$R = \{\pm(e_i \pm e_j), (1 \leq i < j \leq n)\}.$$

By the identity

$$B(X, Y) = \frac{1}{2(n-1)} \text{Tr } XY,$$

we have

$$R(m) = \{\pm(e_i - e_{i+k}), (1 \leq i \leq k)\}.$$

The Weyl group of \mathfrak{g}^c contains all permutations of e_i 's. Therefore, two Cartan subalgebras \mathfrak{h}_1 and \mathfrak{h}_2 of which vector parts have the same dimension are conjugate to each other.

The possible values of $\dim \mathfrak{h}^-$ are $k, k-1, \dots, 1, 0$. Consequently the number N of conjugate classes is equal to $k+1 = \left[\frac{n}{2}\right] + 1$:

$$N = \left[\frac{n}{2}\right] + 1.$$

A representative \mathfrak{h} of the conjugate class of which vector part has the dimension $k-l$ is given as follows:

$$\mathfrak{h} = \left\{ \begin{array}{cccccccc} \begin{array}{c} u_1 \\ \dots \\ u_l \end{array} & & & & \begin{array}{c} u_{l+1} \\ \dots \\ u_{2l} \end{array} & & & \\ & \begin{array}{c} h_{l+1} \\ \dots \\ h_k \end{array} & & & & & & \\ & & \begin{array}{c} -u_1 \\ \dots \\ -u_l \end{array} & & & \begin{array}{c} u_{l+1} \\ \dots \\ u_{2l} \end{array} & & \\ & & & \begin{array}{c} -h_{l+1} \\ \dots \\ -h_k \end{array} & & & & \\ \begin{array}{c} u_{l+1} \\ \dots \\ u_{2l} \end{array} & & & & \begin{array}{c} u_1 \\ \dots \\ u_l \end{array} & & & \\ & & & & & \begin{array}{c} -\bar{h}_{l+1} \\ \dots \\ -\bar{h}_k \end{array} & & \\ & & \begin{array}{c} u_{l+1} \\ \dots \\ u_{2l} \end{array} & & & \begin{array}{c} -u_1 \\ \dots \\ -u_l \end{array} & & \\ & & & & & & \begin{array}{c} \bar{h}_{l+1} \\ \dots \\ \bar{h}_k \end{array} & \\ & & & & & & & \begin{array}{c} u_{2k+1} \\ -u_{2k+1} \end{array} \end{array} \right\}; \left. \begin{array}{l} h_i \in \mathbb{C} \\ u_i \in \sqrt{-1}R \end{array} \right\}.$$

Type C

The real forms of $\mathfrak{g}^c = \mathfrak{sp}(n, C)$ are divided into two types: (C I) and (C II). The type (C I) consists of the only one real form and the type (C II) contains $\left[\frac{n}{2}\right]+1$ non isomorphic real forms with the rank n .

Type (C I).

$$\mathfrak{g} = \mathfrak{sp}(n, R) = \mathfrak{sp}(n, C) \cap \mathfrak{gl}(2n, R).$$

Let

$$J = \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix},$$

then

$$\begin{aligned} \mathfrak{g} &= \{X \in \mathfrak{gl}(2n, R); {}^tXJ + JX = 0\} \\ &= \left\{ \begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix}; A \in \mathfrak{gl}(n, R), B, C \in S(n, R) \right\}. \end{aligned}$$

\mathfrak{g} is self-adjoint, hence $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is a Cartan decomposition, where

$$\begin{aligned} \mathfrak{k} &= \{X \in \mathfrak{g}; {}^tX = -X\} \\ &= \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix}; A \in \mathfrak{o}(n), B \in S(n, R) \right\}, \\ \mathfrak{p} &= \left\{ \begin{pmatrix} A & B \\ B & -A \end{pmatrix}; A, B \in S(n, R) \right\}. \end{aligned}$$

$$\mathfrak{h}_0 = \{D(h_1, \dots, h_n, -h_1, \dots, -h_n); h_i \in R\}$$

is a Cartan subalgebra of \mathfrak{g} , and $\mathfrak{h}_0 \subset \mathfrak{p}$, therefore $\mathfrak{h}_0^- = \mathfrak{h}_0$. Hence $\mathfrak{g} = \mathfrak{sp}(n, R)$ is the normal real form of $\mathfrak{sp}(n, C)$.

Let e_i be the linear form on \mathfrak{h}_0^c defined by

$$e_i(D(h_1, \dots, h_n, -h_1, \dots, -h_n)) = h_i \quad (1 \leq i \leq n),$$

then

$$\mathbf{R}(\mathfrak{m}) = \mathbf{R} = \{\pm(e_i \pm e_j), (1 \leq i < j \leq n); \pm 2e_i (1 \leq i \leq n)\}.$$

The Weyl group W of \mathfrak{g}^c consists of all permutations of e_i 's and changes of the signatures of e_i 's.

Let \mathfrak{h} be a standard Cartan subalgebra of \mathfrak{g} , and $\mathfrak{l} = \mathfrak{h}^{-+} \cap \mathfrak{m}$. By Theorem 4, any admissible root system \mathbf{F} is conjugate to one of the followings:

$$(80) \quad \mathbf{F}(k, l) = (2e_1, \dots, 2e_k, e_{k+1} - e_{k+2}, \dots, e_{k+2l-1} - e_{k+2l}),$$

where k and l satisfy

$$(81) \quad 0 \leq k, \quad 0 \leq l, \quad k+2l \leq n.$$

Since $(2e_i, 2e_i) \neq (e_j - e_k, e_j - e_k)$, the elements of the Weyl group W can not transform $2e_i$ to $e_j - e_k$. Consequently each conjugate class is characterized by two integers (k, l) satisfying the condition (81). The number m_p of lattice points on the line $k+2l=p$ in the first quadrant is given as follows:

$$m_p = \begin{cases} \frac{p}{2} + 1 & p = \text{even}, \\ \frac{p+1}{2} & p = \text{odd}. \end{cases}$$

Therefore the number N of conjugate classes is equal to

$$N = \sum_{p=0}^n m_p = \begin{cases} \left(\frac{n}{2} + 1\right)^2 = \frac{(n+2)^2}{4}, & n = \text{even}, \\ \left(\left[\frac{n}{2}\right] + 1\right)\left(\left[\frac{n}{2}\right] + 2\right) = \frac{(n+1)(n+3)}{4}, & n = \text{odd}. \end{cases}$$

A representative \mathfrak{h} of the conjugate class with the characteristic (k, l) is given as follows:

$$\mathfrak{h} = \left\{ \begin{array}{cccccc} & & & -h_1 & & \\ & & & \dots & & \\ & & & -h_k & & \\ & A_1 & & & & \\ & \dots & & & & \\ & A_l & & & & \\ & & h_{k+2l+1} & & & \\ & & \dots & & & \\ & & h_n & & & \\ h_1 & & & & & \\ \dots & & & & & \\ h_k & & & & & \\ & & & & -A_1 & \\ & & & & \dots & \\ & & & & -A_l & \\ & & & & & -h_{k+2l+1} \\ & & & & & \dots \\ & & & & & -h_n \end{array} \right\}; \quad \left. \begin{array}{l} A_i = \begin{pmatrix} h_{k+l+i} & -h_{k+i} \\ h_{k+i} & h_{k+l+i} \end{pmatrix}, \\ h_i \in \mathbb{R} \end{array} \right\}.$$

Type (C II). \mathfrak{g} is the Lie algebra \mathfrak{g}_m of the group of linear transformations in $2n$ complex variables which keep invariant the skew symmetric bilinear form $\sum_{i=1}^n (z_i z_{i+n}' - z_{i+n} z_i')$ and the Hermitian form $\sum_{i=1}^m (z_i \bar{z}_i + z_{i+n} \bar{z}_{i+n}) -$

$\sum_{j=1}^{n-m} (z_{j+m}\bar{z}_{j+m} + z_{j+m+n}\bar{z}_{j+m+n})$ with index m ($0 \leq m \leq [\frac{n}{2}]$). As $\mathfrak{g}_{m_1} \cong \mathfrak{g}_{m_2}$ if and only if $m_1 = m_2$, there are $[\frac{n}{2}] + 1$ non isomorphic real forms of the type (C II) with the rank n . Let

$$J = \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix} E_m & & & \\ & -E_{n-m} & & \\ & & E_m & \\ & & & -E_{n-m} \end{pmatrix},$$

then

$$\mathfrak{g} = \{X \in \mathfrak{gl}(2n, \mathbb{C}); {}^tXJ + JX = 0, {}^t\bar{X}H + HX = 0\}$$

$$= \left\{ \begin{pmatrix} A & B & D & F \\ {}^t\bar{B} & C & {}^tF & G \\ -\bar{D} & \bar{F} & \bar{A} & -\bar{B} \\ {}^t\bar{F} & -\bar{G} & -{}^tB & \bar{C} \end{pmatrix}; \begin{array}{l} A \in \mathfrak{u}(m), C \in \mathfrak{u}(n-m), \\ B, F \in \mathfrak{M}(m, n-m, \mathbb{C}), \\ D \in \mathfrak{S}(m, \mathbb{C}), G \in \mathfrak{S}(n-m, \mathbb{C}) \end{array} \right\}.$$

\mathfrak{g} is self-adjoint, so we have a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, where

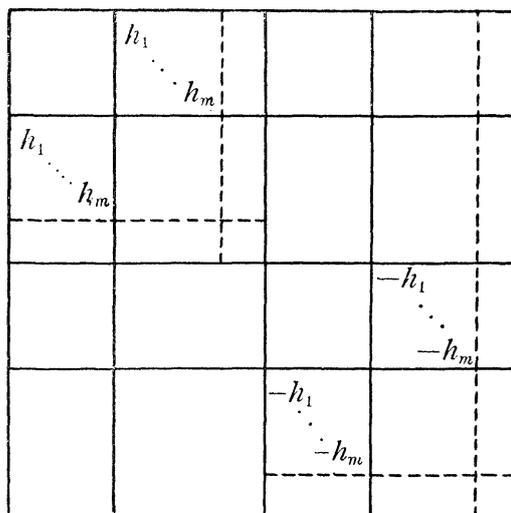
$$\mathfrak{k} = \{X \in \mathfrak{g}; {}^t\bar{X} = -X\}$$

$$= \left\{ \begin{pmatrix} A & 0 & D & 0 \\ 0 & C & 0 & G \\ -\bar{D} & 0 & \bar{A} & 0 \\ 0 & -\bar{G} & 0 & \bar{C} \end{pmatrix}; \begin{array}{l} A \in \mathfrak{u}(m), C \in \mathfrak{u}(n-m), \\ D \in \mathfrak{S}(m, \mathbb{C}), G \in \mathfrak{S}(n-m, \mathbb{C}) \end{array} \right\},$$

and

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & B & 0 & F \\ {}^t\bar{B} & 0 & {}^tF & 0 \\ 0 & \bar{F} & 0 & -\bar{B} \\ {}^t\bar{F} & 0 & -{}^tB & 0 \end{pmatrix}; B, F \in \mathfrak{M}(m, n-m, \mathbb{C}) \right\}.$$

$$\mathfrak{m} = \{H^- = H^-(h_1, \dots, h_m) =$$



$$; h_i \in \mathbb{R} \}$$

is a maximal abelian subalgebra in \mathfrak{p} .

Let

$$\mathfrak{h} = \{H^+ = D(u_1, \dots, u_m, u_1, \dots, u_m, u_{2m+1}, \dots, u_n, -u_1, \dots, -u_m, -u_1, \dots, -u_m, -u_{2m+1}, \dots, -u_n); u_i \in \sqrt{-1}R\}$$

then $\mathfrak{h}_0 = \mathfrak{h} + \mathfrak{m}$ is a Cartan subalgebra of \mathfrak{g} and $\mathfrak{h}_0^+ = \mathfrak{h}$, $\mathfrak{h}_0^- = \mathfrak{m}$.

If we define the linear form e_i on \mathfrak{h}_0 as

$$e_i(H^+ + H^-) = \begin{cases} -\sqrt{-1} u_i + h_i & 1 \leq i \leq m, \\ -\sqrt{-1} u_{i-m} - h_{i-m} & m+1 \leq i \leq 2m, \\ -\sqrt{-1} u_i & 2m+1 \leq i \leq n, \end{cases}$$

and extension e_i as the linear form on $\mathfrak{h}_0^{\mathcal{C}}$, then

$$\mathbf{R} = \{\pm(e_i \pm e_j), (1 \leq i < j \leq n); \pm 2e_i, (1 \leq i \leq n)\}$$

is the root system of $\mathfrak{g}^{\mathcal{C}}$ with respect to $\mathfrak{h}_0^{\mathcal{C}}$.

Therefore we have

$$\mathbf{R}(\mathfrak{m}) = \{\pm(e_i - e_{i+m}), (1 \leq i \leq m)\}.$$

The Weyl group of $\mathfrak{g}^{\mathcal{C}}$ contains all permutations of e_i 's. Therefore two Cartan subalgebras \mathfrak{h}_1 and \mathfrak{h}_2 are conjugate if and only if $\dim \mathfrak{h}_1^- = \dim \mathfrak{h}_2^-$. The possible values of $\dim \mathfrak{h}^-$ are $m, m-1, \dots, 1, 0$. Consequently the number N of conjugate classes is equal to $m+1$:

$$N = m+1.$$

A representative \mathfrak{h} of the conjugate class of which vector part has the dimension k is given as follows:

$$\mathfrak{h} = \{H^-(h_1, \dots, h_k, 0, \dots, 0) + D(u_1, \dots, u_m, u_1, \dots, u_k, u_{m+k+1}, \dots, u_n, -u_1, \dots, -u_m, -u_1, \dots, -u_k, -u_{m+k+1}, \dots, -u_n); h_i \in R, u_i \in \sqrt{-1}R\}.$$

§ 4. Conjugate classes of Cartan subalgebras in exceptional real simple Lie algebras.

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition of real semisimple Lie algebra \mathfrak{g} , and $\mathfrak{g}_u = \mathfrak{k} + \sqrt{-1}\mathfrak{p}$. Then the conjugation θ of $\mathfrak{g}^{\mathcal{C}}$ with respect to \mathfrak{g} induces an involution τ of \mathfrak{g}_u . Real semisimple Lie algebra \mathfrak{g} is called of the *first category* if τ is an inner automorphism of \mathfrak{g}_u , i. e. τ belongs to the component of the identity G_u of the full automorphism group of \mathfrak{g}_u , otherwise \mathfrak{g} is called of the *second category*.

Complex semisimple Lie algebras regarded as Lie algebra over reals

are always of the second category. Among the real forms of complex simple Lie algebras, the algebras of type (A_n I) ($n \geq 2$), (A II), (D_n Ib) ($n \geq 3$), (E I) and (E IV) are of the second category, other real forms are of the first category.

Let \mathfrak{g} be of the first category, then τ belongs to a Cartan subgroup (maximal torus) H of G_u . Therefore τ can be expressed as

$$\tau = \exp \operatorname{ad} H_0, \quad H_0 \in \mathfrak{h},$$

where \mathfrak{h} is the Lie algebra of H . Hence $\tau H = H$ for all $H \in \mathfrak{h}$. So we have

$$\mathfrak{h} \subset \mathfrak{k} = \{X \in \mathfrak{g}_u; \tau X = X\}.$$

Now we shall call a Cartan subalgebra \mathfrak{h} of a real semisimple Lie algebra \mathfrak{g} *compact* if $\mathfrak{h}^+ = \mathfrak{h}$. Then by the above argument we have the following proposition which is important in what follows.

PROPOSITION 12. *A real semisimple Lie algebra \mathfrak{g} has compact Cartan subalgebras if \mathfrak{g} is of the first category.*

REMARK. We shall see that the converse of this proposition is also valid (cf. Theorem 8).

Now let \mathbf{R} be the root system of $\mathfrak{g}^c = \mathfrak{g}_u^c$ with respect to the Cartan subalgebra \mathfrak{h}^c contained in \mathfrak{k}^c , and let E_α ($\alpha \in \mathbf{R}$) be non zero elements of \mathfrak{g}^c satisfying (51). As $[\mathfrak{h}^c, \mathfrak{k}^c] \subset \mathfrak{k}^c$, $[\mathfrak{h}^c, \mathfrak{p}^c] \subset \mathfrak{p}^c$, E_α belongs either to \mathfrak{k}^c or to \mathfrak{p}^c . We put

$$\mathbf{R}_t = \{\alpha \in \mathbf{R}; E_\alpha \in \mathfrak{k}^c\} \quad \text{and} \quad \mathbf{R}_p = \{\alpha \in \mathbf{R}; E_\alpha \in \mathfrak{p}^c\}.$$

A root α in \mathbf{R} is called *compact* or *non compact* according to $\alpha \in \mathbf{R}_t$ or $\alpha \in \mathbf{R}_p$.

Let (H_i, E_α) be a basis of \mathfrak{g}^c satisfying the conditions (54), (55), (56) and (57), then we may assume that \mathfrak{g}_u is defined by (58). We introduce a fixed lexicographic order in \mathbf{R} , and we put $\mathbf{R}^+ = \{\alpha > 0; \alpha \in \mathbf{R}\}$, $\mathbf{R}_t^+ = \mathbf{R}^+ \cap \mathbf{R}_t$ and $\mathbf{R}_p^+ = \mathbf{R}^+ \cap \mathbf{R}_p$. Let

$$(82) \quad U_\alpha = (E_\alpha + E_{-\alpha}) / (2(\alpha, \alpha))^{1/2} \quad \text{and} \quad V_\alpha = \sqrt{-1} (E_\alpha - E_{-\alpha}) / (2(\alpha, \alpha))^{1/2},$$

then we have

$$\mathfrak{k} = \mathfrak{h} + \sum_{\alpha \in \mathbf{R}_t^+} (RU_\alpha + RV_\alpha),$$

$$\mathfrak{p} = \sqrt{-1} \sum_{\alpha \in \mathbf{R}_p^+} (RU_\alpha + RV_\alpha).$$

For each root α , we denote by H_α and H_α^0 the unique elements which satisfy the condition (83) and (84) respectively.

$$(83) \quad H_\alpha \in \mathfrak{h}^c, \quad B(H_\alpha, H) = \alpha(H) \quad \text{for all } H \in \mathfrak{h}^c.$$

$$(84) \quad H_\alpha^0 = \sqrt{-1} H_\alpha / (\alpha, \alpha).$$

Let ρ_α be the automorphism $\rho_\alpha = \exp \operatorname{ad} V_\alpha$ of \mathfrak{g}^c , then we have

$$(85) \quad \rho_{\alpha}(H_{\alpha}^0) = U_{\alpha} \quad \text{and} \quad \rho_{\alpha}(U_{\alpha}) = -H_{\alpha}^0.$$

Let $\mathbf{F} = \{\alpha_1, \dots, \alpha_l\}$ be a maximal admissible root system in \mathbf{R}_p^+ , i.e. a set of α_i 's which satisfy the following three conditions.

- a) $\alpha_i \in \mathbf{R}_p^+$ ($1 \leq i \leq l$), b) $\alpha_i \pm \alpha_j \notin \mathbf{R}$,
- c) \mathbf{F} is a maximal set having the properties a) and b).

Then

$$\mathfrak{m} = \sqrt{-1} \sum_{i=1}^l R U_{\alpha_i}$$

is a maximal abelian subalgebra in \mathfrak{p} . Let

$$\mathfrak{b} = \{X \in \mathfrak{h}; [X, \mathfrak{m}] = 0\},$$

then $\mathfrak{h}_0 = \mathfrak{b} + \mathfrak{m}$ is a Cartan subalgebra of \mathfrak{g} , and $\mathfrak{h}_0^+ = \mathfrak{b}$ and $\mathfrak{h}_0^- = \mathfrak{m}$. We put

$$\rho = \rho_{\alpha_1} \rho_{\alpha_2} \cdots \rho_{\alpha_l}$$

then ρ is an automorphism of \mathfrak{g}^c , and we have

$$\rho(H) = H \quad \text{for all } H \in \mathfrak{b}.$$

As $\alpha_i \pm \alpha_j \notin \mathbf{R}$ and $(\alpha_i, \alpha_j) = 0$ if $i \neq j$, we have $\rho_{\alpha_i}(U_{\alpha_j}) = U_{\alpha_j}$ and $\rho_{\alpha_i}(H_{\alpha_j}) = H_{\alpha_j}$, if $i \neq j$. Therefore we have by (85)

$$\rho(H_{\alpha_i}) = -\sqrt{-1} (\alpha_i, \alpha_i) U_{\alpha_i} \quad (1 \leq i \leq l).$$

Consequently the automorphism ρ of \mathfrak{g}^c transforms $\sum_{i=1}^l R H_{\alpha_i}$ onto \mathfrak{m} , and \mathfrak{h}^c onto \mathfrak{h}_0^c .

Thus we have proved the following theorem.

THEOREM 7. *Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition of a real semisimple Lie algebra \mathfrak{g} of the first category and \mathfrak{h} a Cartan subalgebra of \mathfrak{g} contained in \mathfrak{k} . Let $\mathbf{F} = \{\alpha_1, \dots, \alpha_l\}$ be a maximal admissible root system in \mathbf{R}_p . Then there exist an automorphism ρ of \mathfrak{g}^c and a Cartan subalgebra \mathfrak{h}_0 of \mathfrak{g} which satisfy the following conditions: 1) ρ transforms \mathfrak{h}^c onto \mathfrak{h}_0^c , 2) $\mathfrak{h}_0^- = \mathfrak{m}$ is a maximal abelian subalgebra in \mathfrak{p} , 3) $\mathbf{R}' = \{\alpha' = {}^t\rho(\alpha); \alpha \in \mathbf{R}\}$ is the root system with respect to \mathfrak{h}_0^c and $\{\alpha'_1, \dots, \alpha'_l\}$ spans \mathfrak{h}_0^- .*

Finally we give a proposition which is interesting in comparison with Corollary 2 to Theorem 3. The proposition will be used in the followings.

PROPOSITION 13. *In a real semisimple Lie algebra, all Cartan subalgebras of which toroidal parts have maximal possible dimension are mutually conjugate.*

PROOF. Let \mathfrak{b} be a maximal abelian subalgebra of \mathfrak{k} , then it is known that there exists the unique Cartan subalgebra \mathfrak{h} which contains \mathfrak{b} and $\mathfrak{h}^+ = \mathfrak{b}$ (cf. Murakami [11, p. 107]). Hence if \mathfrak{h} is a Cartan subalgebra of \mathfrak{g} of which toroidal part \mathfrak{h}^+ has maximal dimension, then $\dim \mathfrak{h}^+ = \text{rank } \mathfrak{k}$.

Let \mathfrak{h}_1 and \mathfrak{h}_2 be two standard Cartan subalgebras which satisfy

$\dim \mathfrak{h}_1^+ = \dim \mathfrak{h}_2^+ = \text{rank } \mathfrak{k}$, then \mathfrak{h}_1^+ and \mathfrak{h}_2^+ are two Cartan subalgebras of \mathfrak{k} . Hence there exists an element $k \in K$, which transforms \mathfrak{h}_1^+ onto \mathfrak{h}_2^+ . By the uniqueness of Cartan subalgebra \mathfrak{h}_i of \mathfrak{g} containing \mathfrak{h}_i^+ , we have $k\mathfrak{h}_1 = \mathfrak{h}_2$.

Notations: We make now the following conventions for the sake of brevity. We denote a root $\alpha' = {}^t\rho(\alpha)$ on \mathfrak{h}_0^c by the same letter α which is a root on \mathfrak{h}^c . Hence H_α and $H_{\alpha'}$ is defined by the conditions.

$$\begin{aligned} H_\alpha &\in \mathfrak{h}^c, & B(H_\alpha, H) &= \alpha(H) & \text{for all } H \in \mathfrak{h}^c, \\ H_{\alpha'} &\in \mathfrak{h}_0^c, & B(H_{\alpha'}, H) &= \alpha(H) & \text{for all } H \in \mathfrak{h}_0^c. \end{aligned}$$

We identify the dual space $(\mathfrak{h}_0^c)^*$ of \mathfrak{h}_0^c with \mathfrak{h}_0^c by the use of Killing form $B(X, Y)$. So we often denote $H_{\alpha'}$ by α .

We denote by e_1, e_2, \dots orthogonal vectors with the same length \sqrt{c} in a Euclidean vector space, and we define $e^{(r)}$ by

$$e^{(r)} = e_1 + e_2 + \dots + e_r.$$

We introduce a lexicographic order so as to $\sum m_i e_i > 0$ if $m_1 = \dots = m_{r-1} = 0$ and $m_r > 0$. We denote the reflection with respect to a root α by S_α i. e.

$$S_\alpha(x) = x - 2[(\alpha, x)/(\alpha, \alpha)]\alpha.$$

Type E_6

The root system R of complex simple Lie algebra E_6 is

$$(86) \quad R = \{ \pm(e_i - e_j), (1 \leq i < j \leq 6), \pm 2^{1/2}e_7, \pm(2^{-1/2}e_7 + 2^{-1}e^{(6)} - e_i - e_j - e_k) \} \\ (1 \leq i < j < k \leq 6).$$

Cf. Pontrjagin [12, p. 504],

$\pm(2^{-1/2}e_7 + 2^{-1}e^{(6)} - e_i - e_j - e_k)$ is represented as $\pm 2^{-1/2}e_7 + 2^{-1} \sum_{i=1}^6 \varepsilon_i e_i$. We denote this root by $(\varepsilon_1, \dots, \varepsilon_6, \pm 1)$ or $(\varepsilon, \pm 1)$.

The operations of the generators of the Weyl group W are as follows:

$$(87) \quad S_{e_i - e_j} = \text{substitution of } e_i \text{ for } e_j.$$

$$(88) \quad S_{2^{1/2}e_7} = \text{change of signature of } e_7.$$

$$(89) \quad S_{(\varepsilon, 1)}(e_k - e_l) = \begin{cases} e_k - e_l & (\varepsilon_k = \varepsilon_l), \\ (\varepsilon', -1) & (\varepsilon_k = 1, \varepsilon_l = -1), \\ (\varepsilon', 1) & (\varepsilon_k = -1, \varepsilon_l = 1), \end{cases}$$

where $\varepsilon_i' = -\varepsilon_i$ ($i \neq k, l$) and $\varepsilon_j' = \varepsilon_j$ ($j = k, l$).

$$(90) \quad S_{(\varepsilon, 1)}(2^{1/2}e_7) = (\varepsilon, -1), \quad S_{(\varepsilon, -1)}(2^{1/2}e_7) = (\varepsilon, 1).$$

$$(91) \quad S_{(\varepsilon, 1)}(\delta, 1) = \begin{cases} (-\delta, -1) & (\varepsilon = \delta), \\ 2^{1/2}e_7 & (\varepsilon = -\delta), \\ e_k - e_l & (\varepsilon_i \neq \delta_i, i = k, l; \varepsilon_j = \delta_j, j \neq k, l), \\ (\delta, 1) & ((\varepsilon, \delta) = 0). \end{cases}$$

We denote by $\alpha_1, \alpha_2, \alpha_3$ and α_4 the following four positive roots.

$$(92) \quad \begin{cases} \alpha_1 = (1, 1, 1, -1, -1, -1, 1), & \alpha_2 = (1, -1, -1, -1, 1, 1, 1), \\ \alpha_3 = (1, -1, 1, -1, 1, -1, -1), & \alpha_4 = (1, 1, -1, -1, -1, 1, -1). \end{cases}$$

Complex simple Lie algebra E_6 has five types of real forms (E I), (E II), (E III), (E IV) and (E X).

(E I) and (E IV) are of the second category, other real forms are of the first category.

Type (E I). (E I) is the normal real form of E_6 , hence $\mathbf{R}(\mathfrak{m})$ is equal to \mathbf{R} defined by (86). It is easily verified that $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ defined by (92) is a maximal admissible root system in \mathbf{R} . Therefore the possible values of dimension of troidal parts of Cartan subalgebras are 0, 1, 2, 3 and 4. We shall prove that two Cartan subalgebras of (E I) which satisfy $\dim \mathfrak{h}_1^+ = \dim \mathfrak{h}_2^+$ are mutually conjugate. In order to prove this fact, it is sufficient to show that any admissible root system in \mathbf{R} is conjugate to one of the following five systems $\mathbf{F}_0, \mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3$ and \mathbf{F}_4 under the action of the Weyl group W .

$$(93) \quad \begin{cases} \mathbf{F}_0 = \phi \text{ (empty set), } \mathbf{F}_1 = \{\alpha_1\}, \mathbf{F}_2 = \{\alpha_1, \alpha_2\}, \\ \mathbf{F}_3 = \{\alpha_1, \alpha_2, \alpha_3\}, \mathbf{F}_4 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}. \end{cases}$$

1° All one dimensional \mathbf{F} 's are mutually conjugate, because every root is transformed to α_1 by an element of W . The last fact is easily seen from (87), (88), (89) and (90).

2° Every 2-dimensional admissible root system is equivalent to one of the followings.

- (1) $\{(\varepsilon, 1), (\delta, 1)\} \quad (\varepsilon, \delta) = 0,$
- (2) $\{e_i - e_j, e_k - e_l\} \quad i, j, k, l \text{ are mutually different,}$
- (3) $\{e_i - e_j, 2^{1/2}e_7\}.$

Every \mathbf{F} 's of type (2) are conjugate to $\mathbf{F}_5 = \{e_1 - e_2, e_3 - e_4\}$, because all substitutions of e_i 's are contained in W (cf. (87)). \mathbf{F}_5 is transformed to one of the type (1) by $S_{(-1, 1, -1, 1, -1, 1, 1)}$ (cf. (89)).

Every \mathbf{F} 's of type (3) are conjugate to $\mathbf{F}_6 = \{e_1 - e_2, 2^{1/2}e_7\}$ (cf. (87)). \mathbf{F}_6 is transformed to one of the type (1) by $S_{(-1, 1, -1, -1, 1, 1, 1)}$ (cf. (90)). As all permutations of e_i 's are contained in W , we have

$$\{(\varepsilon, 1), (\delta, 1)\} \approx \{(\varepsilon, 1), (\varepsilon', 1)\},$$

where $\varepsilon_i' = \varepsilon_i$, ($i = 2, 6$) and $\varepsilon_j' = -\varepsilon_j$ ($j \neq 2, 6$) and $(\varepsilon_2, \varepsilon_6)$ must be either $(1, -1)$ or $(-1, 1)$, because if $(\varepsilon_2, \varepsilon_6) = (1, 1)$ or $(-1, -1)$, the number of ε_i 's (or ε_i' 's) which are equal to 1 can not be three. We may assume $(\varepsilon_2, \varepsilon_6) = (1, -1)$, because we can substitute e_2 for e_6 if necessary. In this case the possible $(\varepsilon_1, \varepsilon_3, \varepsilon_4, \varepsilon_5)$ are as follows:

$$(1, 1, -1, -1), (1, -1, 1, -1), (1, -1, -1, 1), \\ (-1, 1, 1, -1), (-1, 1, -1, 1), (-1, -1, 1, 1).$$

The six \mathbf{F} 's so obtained are transformed to $\{\alpha_1, \alpha_2\}$ by suitable permutations of e_i 's.

Thus we have proved that every two dimensional admissible root system is conjugate to \mathbf{F}_2 .

3°) By the result in 2°), every 3-dimensional \mathbf{F} is conjugate to $\{\alpha_1, \alpha_2, \beta\}$. We may assume that β is of the type $(\varepsilon, 1)$, because if $\beta = e_i - e_j$, a suitable element S_δ of W transforms $\{\alpha_1, \alpha_2, e_i - e_j\}$ to $\{\alpha_1, \alpha_2, (\varepsilon, 1)\}$. As β is orthogonal to α_1 and α_2 , the possible positive roots β are one of the followings: $\alpha_3, \alpha_4, \beta_1 = (1, 1, -1, -1, 1, -1, -1), \beta_2 = (1, -1, 1, -1, -1, 1, -1)$. $\{\alpha_1, \alpha_2, \alpha_4\}, \{\alpha_1, \alpha_2, \beta_1\}$ and $\{\alpha_1, \alpha_2, \beta_2\}$ are transformed to $\{\alpha_1, \alpha_2, \alpha_3\}$ by $S_{e_2 - e_3} S_{e_5 - e_6}, S_{e_3 - e_3}$ and $S_{e_5 - e_6}$ respectively.

Thus we have proved that every 3-dimensional \mathbf{F} is conjugate to $\mathbf{F}_3 = \{\alpha_1, \alpha_2, \alpha_3\}$.

4°) Every 4-dimensional \mathbf{F} is conjugate to \mathbf{F}_4 . This fact is proved as in 3°). However this fact is also a direct consequence of Proposition 13.

Thus we have proved that two Cartan subalgebras \mathfrak{h}_1 and \mathfrak{h}_2 in (E I) are conjugate if and only if $\dim \mathfrak{h}_1^+ = \dim \mathfrak{h}_2^+$ and every admissible root system of (E I) is conjugate to one of $\mathbf{F}_0, \mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3$, and \mathbf{F}_4 . Consequently the number N of conjugate classes of Cartan subalgebras in (E I) is equal to five:

$$N = 5.$$

Type (E II). The real form (E II) is defined by the involution

$$\tau = \exp \operatorname{ad}(\pi \sqrt{-1} H_{2^{1/2} e_7} / c).$$

The set $\mathbf{R}_\mathfrak{p}$ of non compact roots is

$$\mathbf{R}_\mathfrak{p} = \{ \pm(2^{-1/2} e_7 + 2^{-1} e^{(6)} - e_i - e_j - e_k); 1 \leq i < j < k \leq 6 \}.$$

The number of non compact roots is 40 and the character $\delta = 2 \dim \mathfrak{p} - \dim \mathfrak{g} = 2 \times 40 - 78 = 2$. The roots $\alpha_1, \alpha_2, \alpha_3$ and α_4 defined in (92) are all non compact. Hence a maximal abelian subalgebra \mathfrak{m} in \mathfrak{p} is defined by

$$\mathfrak{m} = RH_{\alpha_1'} + RH_{\alpha_2'} + RH_{\alpha_3'} + RH_{\alpha_4'}.$$

Therefore F_0, F_1, F_2, F_3 and F_4 defined in (93) are all contained in $R(m)$ and every admissible root system in $R(m)$ is conjugate to one of these five systems. Hence \mathfrak{h}_1 and \mathfrak{h}_2 are conjugate if and only if $\dim \mathfrak{h}_1^+ = \dim \mathfrak{h}_2^+$, and the number of conjugate classes are equal to five:

$$N = 5.$$

Type (E III). The real form (E III) is defined by the involution

$$\tau = \exp \operatorname{ad}[\pi\sqrt{-1}(H_{e_5} - H_{e_6} - H_{e_7})/c].$$

The set R_p of non compact roots is

$$R_p = \{ \pm(e_i - e_5), \pm(e_i - e_6), \pm(2^{-1/2}e_7 + 2^{-1}e^{(6)} - e_i - e_j - e_k), \\ \pm(2^{-1/2}e_7 + 2^{-1}e^{(6)} - e_i - e_5 - e_6) \}$$

where i, j, k are mutually different integers from 1 to 4. The number of non compact roots is equal to 32 and the character $\delta = 2 \times 32 - 78 = -14$.

Let α_1 be the root defined in (92) and $\beta_1 = e_1 - e_6$, then $\{\alpha_1, \beta_1\}$ is a maximal admissible root system in R_p^+ . And we have

$$R(m) = \{ \pm\alpha_1, \pm\beta_1 \}$$

by Theorem 7. As $\{\beta_1\}$ is conjugate to $\{\alpha_1\}$ under W (cf. 1°) in Type (E I)), every admissible root system is conjugate to one of ϕ (empty set), $\{\alpha_1\}$ and $\{\alpha_1, \beta_1\}$. Therefore \mathfrak{h}_1 is conjugate to \mathfrak{h}_2 if and only if $\dim \mathfrak{h}_1^+ = \dim \mathfrak{h}_2^+$ and the number N of conjugate classes is equal to three:

$$N = 3.$$

Type (E IV). The Lie algebra (E IV) is the real form of E_6 with the character $\delta = -26$. It is known that $\dim m = 2$ and $\mathfrak{k} \cong F_4$ (cf. E. Cartan [2, p. 422]). Hence every Cartan subalgebra \mathfrak{h} of (E IV) must have 4-dimensional toroidal part \mathfrak{h}^+ . By Proposition 13 (or by Corollary 2 to Theorem 3), all Cartan subalgebras of (E IV) are mutually conjugate. Thus, the number N of conjugate classes is equal to one:

$$N = 1.$$

Type (E X). The Lie algebra (E X) is the compact real form of E_6 . Therefore all Cartan subalgebras of (E X) are mutually conjugate, and

$$N = 1.$$

Type E_7

The root system R of complex simple Lie algebra E_7 is

$$(94) \quad R = \{e_i - e_j, 2^{-1}e^{(8)} - e_i - e_j - e_k - e_l\}$$

where i, j, k, l are mutually different integers from 1 to 8 (cf. [12]). We denote $2^{-1}e^{(8)} - e_i - e_j - e_k - e_l = 2^{-1} \sum_{i=1}^8 \varepsilon_i e_i$ by $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_8)$ or ε .

The operations of Weyl group W are as follows.

(95) $S_{e_i - e_j}$ = the substitution of e_i for e_j .

(96) $S_\varepsilon(e_k - e_l) = \begin{cases} \varepsilon' & \varepsilon_k = 1, \quad \varepsilon_l = -1, \\ -\varepsilon' & \varepsilon_k = -1, \quad \varepsilon_l = 1, \\ e_k - e_l & \varepsilon_k = \varepsilon_l, \end{cases}$

where

(97) $\varepsilon'_i = -\varepsilon_i$ ($i \neq k, l$) and $\varepsilon'_j = \varepsilon_j$ ($j = k, l$).

(98) $S_\varepsilon(\delta) = \begin{cases} -\delta & \delta = \pm \varepsilon, \\ \delta - \varepsilon = \pm(e_k - e_l) & \delta = -\varepsilon', \\ \delta + \varepsilon = \pm(e_k - e_l) & \delta = \varepsilon', \\ \delta & (\varepsilon, \delta) = 0, \end{cases}$

where ε' is defined by (97).

The complex simple Lie algebra E_7 has four different real forms (E V), (E VI), (E VII) and (E XI).

Type (EV). The Lie algebra (E V) is the normal real form of E_7 . Hence $R(m)$ is identical with R defined by (94).

Let α_i ($1 \leq i \leq 7$) be seven roots defined as follows:

(99) $\begin{cases} \alpha_1 = (1, 1, 1, -1, -1, -1, 1, -1), & \alpha_2 = (1, -1, -1, 1, 1, -1, 1, -1), \\ \alpha_3 = (1, -1, 1, -1, 1, -1, -1, 1), & \alpha_4 = (1, 1, -1, 1, -1, -1, -1, 1), \\ \alpha_5 = (1, -1, -1, -1, -1, 1, 1, 1), & \alpha_6 = (1, -1, 1, 1, -1, 1, -1, -1), \\ \alpha_7 = (1, 1, -1, -1, 1, 1, -1, -1). \end{cases}$

Let

(100) $F_i = \{\alpha_1, \alpha_2, \dots, \alpha_i\}$ ($0 \leq i \leq 7$), $F_8 = \{\alpha_1, \alpha_2, \alpha_5\}$, $F_9 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_5\}$,

(F_0 = empty set), then every admissible root system is conjugate to one of these ten F_i 's.

1°) A root $e_k - e_l$ is transformed by S_ε to a root $\varepsilon' = 2^{-1} \sum \varepsilon'_i e_i$, (cf. (96)) and a root $\varepsilon = 2^{-1} \sum \varepsilon_i e_i$ is transformed to α_1 by an element of W (cf. (95)). Hence every 1-dimensional F is conjugate to $\{\alpha_1\}$.

2°) By the result in 1°), every 2-dimensional F is conjugate to $\{\varepsilon, \beta\}$. β may be assumed to be of the type $\varepsilon' = 2^{-1} \sum \varepsilon'_i e_i$, because if $\beta = e_i - e_j$, $S_{\varepsilon''}$ for a suitable $\varepsilon'' = 2^{-1} \sum \varepsilon''_i e_i$ transforms $\{\varepsilon, e_i - e_j\}$ to $\{\varepsilon, \delta\}$ (cf. (96), (97)).

Let

$$\begin{aligned} \varepsilon_{i_1} = \varepsilon_{i_2} = \varepsilon_{i_3} = \varepsilon_{i_4} = -\varepsilon_{i_5} = -\varepsilon_{i_6} = -\varepsilon_{i_7} = -\varepsilon_{i_8} = 1, \\ \delta_{j_1} = \delta_{j_2} = \delta_{j_3} = \delta_{j_4} = -\delta_{j_5} = -\delta_{j_6} = -\delta_{j_7} = -\delta_{j_8} = 1, \end{aligned}$$

then every one of $(i_1, i_2, i_3, i_4) \cap (j_1, j_2, j_3, j_4)$, $(i_1, i_2, i_3, i_4) \cap (j_5, j_6, j_7, j_8)$, $(i_5, i_6, i_7, i_8) \cap (j_1, j_2, j_3, j_4)$ and $(i_5, i_6, i_7, i_8) \cap (j_5, j_6, j_7, j_8)$ consists of two elements. Let these four set be $\{k_1, k_2\}$, $\{k_3, k_4\}$, $\{k_5, k_6\}$ and $\{k_7, k_8\}$ respectively, then the permutation

$$\begin{pmatrix} k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8 \\ 1 \ 7 \ 2 \ 3 \ 4 \ 5 \ 6 \ 8 \end{pmatrix}$$

transforms $\{\varepsilon, \delta\}$ to $\{\alpha_1, \alpha_2\}$.

Hence every 2-dimensional F is conjugate to $\{\alpha_1, \alpha_2\}$.

3°) By the result in 2°), every 3-dimensional F is conjugate to $\{\alpha_1, \alpha_2, \beta\}$. β may be assumed to be of the type $\varepsilon = 2^{-1} \sum \varepsilon_i e_i$, because if $\beta = e_i - e_j$, S_δ for a suitable δ transforms $\{\alpha_1, \alpha_2, e_i - e_j\}$ to $\{\alpha_1, \alpha_2, \varepsilon\}$ (cf. (96), (97)). The positive root ε must be orthogonal to α_1 and α_2 . Hence merely the following nine vectors in (101) are possible as ε .

$$(101) \quad \begin{cases} \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \beta_1 = (1, 1, -1, 1, -1, 1, -1, -1), \\ \beta_2 = (1, -1, 1, -1, 1, 1, -1, -1), \beta_3 = (1, 1, -1, -1, 1, -1, -1, 1), \\ \beta_4 = (1, -1, 1, 1, -1, -1, -1, 1). \end{cases}$$

Now

$$\{\alpha_1, \alpha_2, \alpha_4\}, \{\alpha_1, \alpha_2, \alpha_6\}, \{\alpha_1, \alpha_2, \alpha_7\}, \\ \{\alpha_1, \alpha_2, \beta_1\}, \{\alpha_1, \alpha_2, \beta_2\}, \{\alpha_1, \alpha_2, \beta_3\} \text{ and } \{\alpha_1, \alpha_2, \beta_4\}$$

are transformed to $\{\alpha_1, \alpha_2, \alpha_3\}$ by (23)(45), (45)(68), (23)(68), (23)(45)(68), (68), (23) and (45) respectively, where (ij) represents the substitution of e_i for e_j , i. e. $(ij) = S_{e_i - e_j}$.

Thus every 3-dimensional F are conjugate either to $F_3 = \{\alpha_1, \alpha_2, \alpha_3\}$ or to $F_8 = \{\alpha_1, \alpha_2, \alpha_6\}$ and F_3 and F_8 are not conjugate.

4°) Every 4-dimensional F is conjugate to $\{\alpha_1, \alpha_2, \alpha_3, \beta\}$ or to $\{\alpha_1, \alpha_2, \alpha_5, \gamma\}$ by the result in 3°). β must be a positive root of type $2^{-1} \sum \varepsilon_i e_i$ and orthogonal to $\alpha_1, \alpha_2, \alpha_3$, hence merely the following four vectors in (102) are possible as β :

$$(102) \quad \alpha_4, \alpha_5, \alpha_6, \alpha_7.$$

$\{\alpha_1, \alpha_2, \alpha_3, \alpha_6\}, \{\alpha_1, \alpha_2, \alpha_3, \alpha_7\}$ are transformed to $F_9 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_5\}$ by (37)(48) and (28)(57) respectively.

γ must by one of $\alpha_3, \alpha_4, \alpha_6, \alpha_7$. $\gamma = \alpha_3$ gives F_9 . $\gamma = \alpha_4, \alpha_6$ and α_7 give systems which are transformed to F_9 by (23)(45), (45)(68) and (23)(68) respectively. Thus every 4-dimensional F is conjugate either to F_4 or to F_9 .

The space $[F_4]$ spanned by the roots in F_4 contains 24 roots, but $[F_9]$ contains 8 roots. Therefore F_4 is not conjugate to F_9 .

5°) The possible types of 5-dimensional F 's are $\{\varepsilon^{(1)}, \varepsilon^{(2)}, \varepsilon^{(3)}, \varepsilon^{(4)}, \varepsilon^{(5)}\}$ where $\varepsilon^{(i)}$'s are mutually orthogonal roots of type $2^{-1} \sum \varepsilon_i e_i$.

By the results in 4°), F is conjugate to one of $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta\}$ and $\{\alpha_1, \alpha_2, \alpha_3, \alpha_5, \gamma\}$. As β , merely one of α_5, α_6 and α_7 are possible.

$$\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_6\} \quad \text{and} \quad \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_7\}$$

are transformed to $F_5 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ by (37)(48) and (27)(58). γ must be one of α_4, α_6 and α_7 . $\gamma = \alpha_4$ gives F_5 . $\gamma = \alpha_6$ and α_7 give systems which are transformed to F_5 by (26)(38) and (46)(58) respectively.

Thus every 5-dimensional F is conjugate to F_5 .

6°) The possible types of 6-dimensional F 's are $\{\varepsilon^{(1)}, \dots, \varepsilon^{(6)}\}$. By the results in 5°), $\{\varepsilon^{(1)}, \dots, \varepsilon^{(6)}\}$ is conjugate to $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \varepsilon\}$. As ε , merely α_6 and α_7 are possible. $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_7\}$ is transformed to $F_6 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$ by (23)(54).

Thus every 6-dimensional F is conjugate to F_6 .

7°) Every 7-dimensional F is equivalent to F_7 defined in (100), because the vectors in F span \mathfrak{h} .

Thus we have proved that every admissible root system is conjugate to one of the ten systems defined in (100). And any two of these ten systems are not conjugate. Therefore the number N of conjugate classes is equal to ten:

$$N = 10.$$

Type (E VI). The real form (E VI) is defined by the involution

$$\tau = \exp \operatorname{ad}(\pi\sqrt{-1}(H_{e_7} - H_{e_8})/c).$$

The set R_p of non compact roots is

$$R_p = \{ \pm(e_i - e_7), \pm(e_i - e_8), 2^{-1}e^{(8)} - e_i - e_j - e_k - e_7, \\ 2^{-1}e^{(8)} - e_i - e_j - e_k - e_8 \}$$

where i, j, k are mutually different integers from 1 to 6. The number of non compact roots is 64 and the character $\delta = 2 \times 64 - 133 = -5$.

Let $\alpha_1, \alpha_2, \alpha_3$ and α_4 be the roots defined in (99), then $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is a maximal admissible root system in R_p . Hence by Theorem 7,

$$m = RH_{\alpha_1}' + RH_{\alpha_2}' + RH_{\alpha_3}' + RH_{\alpha_4}' \\ = \{xe_1 + ye_2 + ze_3 - ze_4 - ye_5 - xe_6 + we_7 - we_8; x, y, z, w \in R\}$$

is a maximal abelian subalgebra in \mathfrak{p} and we have

$$R(m) = \{ \pm\alpha_1, \pm\alpha_2, \pm\alpha_3, \pm\alpha_4, \pm(e_1 - e_6), \pm(e_2 - e_5), \pm(e_3 - e_4), \pm(e_7 - e_8) \}.$$

By the results on the admissible root systems in the algebra of type (E V), every admissible root system in $R(m)$ is conjugate to one of F_0, F_1, F_2, F_3 ,

and F_4 defined in (100). Therefore \mathfrak{h}_1 is conjugate to \mathfrak{h}_2 if and only if $\dim \mathfrak{h}_1^+ = \dim \mathfrak{h}_2^+$. The number N of conjugate classes is equal to five:

$$N = 5.$$

Type (E VII). The real form (E VII) of E_7 is defined by the involution

$$\tau = \exp \operatorname{ad}(\pi\sqrt{-1}(He_1 + He_2 + He_3 + He_4 + He_5 + He_6 - 3He_7 - 3He_8)/4c).$$

The set R_p of non compact roots is

$$R_p = \{\pm(e_i - e_7), \pm(e_i - e_8), 2^{-1}e^{(8)} - e_i - e_j - e_7 - e_8, \\ 2^{-1}e^{(8)} - e_i - e_j - e_k - e_l\}$$

where i, j, k, l are mutually different integers from 1 to 6. The number of non compact roots is equal to 54 and the character $\delta = 2 \times 54 - 133 = -25$.

Let α_5, α_6 and α_7 be the roots defined in (99), then $\{\alpha_5, \alpha_6, \alpha_7\}$ is a maximal admissible root system in R_p . Hence by Theorem 7,

$$m = RH_{\alpha_5}' + RH_{\alpha_6}' + RH_{\alpha_7}' \\ = \{x(e_1 + e_8) + y(e_2 + e_5) + z(e_3 + e_4) - (x + y + z)(e_7 + e_8); x, y, z \in R\}$$

is a maximal abelian subalgebra in \mathfrak{p} . Therefore we have

$$R(m) = \{\pm\alpha_5, \pm\alpha_6, \pm\alpha_7\}.$$

By the results on the admissible root system in R (cf. type (E V)) every fundamental root system in $R(m)$ is conjugate to one of $\phi, \{\alpha_5\}, \{\alpha_5, \alpha_6\}, \{\alpha_5, \alpha_6, \alpha_7\}$. Hence \mathfrak{h}_1 is conjugate to \mathfrak{h}_2 if and only if $\dim \mathfrak{h}_1^+ = \dim \mathfrak{h}_2^+$. The number N of conjugate classes is equal to four:

$$N = 4.$$

Type (E XI). The algebra (E XI) is the compact real form of E_7 . Hence all Cartan subalgebras in (E XI) are mutually conjugate. Thus, the number N of conjugate classes is equal to one:

$$N = 1.$$

Type E_8

Complex simple Lie algebra E_8 is of rank 8 and of dimension 248. E_8 has three non isomorphic real forms (E VIII), (E IX) and (E XII).

The root system of E_8 is

$$(103) \quad R = \{\pm e_i \pm e_j, \pm(2^{-1}e^{(8)} - e_i), \pm(2^{-1}e^{(8)} - e_i - e_j - e_k)\},$$

where i, j, k are mutually different integers from 1 to 8. We denote a root $\alpha = 2^{-1} \sum_{i=1}^8 \varepsilon_i e_i$ by $(\varepsilon_1, \dots, \varepsilon_8)$ or simply by ε .

The operations of the Weyl group W are as follows:

(104) $S_{e_i-e_j}$ = the substitution of e_i for e_j .

(105) $S_{e_i-e_j}S_{e_i+e_j}$ = the changes of signatures of e_i and e_j .

$$(106) \quad S_\varepsilon(e_i-e_j) = \begin{cases} \varepsilon' & (\varepsilon_i = 1, \varepsilon_j = -1), \\ -\varepsilon' & (\varepsilon_i = -1, \varepsilon_j = 1), \\ e_i - e_j & (\varepsilon_i = \varepsilon_j). \end{cases}$$

$$(107) \quad S_\varepsilon(e_i+e_j) = \begin{cases} \varepsilon' & (\varepsilon_i = \varepsilon_j = 1), \\ -\varepsilon' & (\varepsilon_i = \varepsilon_j = -1), \\ e_i - e_j & (\varepsilon_i \neq \varepsilon_j). \end{cases}$$

In (106) and (107) $\varepsilon_k' = \varepsilon_k$ ($k = i, j$), $\varepsilon_k' = -\varepsilon_k$ ($k \neq i, j$).

$$(108) \quad S_\varepsilon(\delta) = \begin{cases} -\delta & (\delta = \pm \varepsilon) \\ \delta_i e_i + \delta_j e_j & ((\varepsilon, \delta) = 1 \text{ or } -1) \\ \delta & ((\varepsilon, \delta) = 0). \end{cases}$$

Type (E VIII). The Lie algebra (E VIII) is the normal real form of E_8 . Hence $\mathbf{R}(\mathfrak{m})$ is identical with \mathbf{R} defined by (103).

Now consider the following eight roots:

$$(109) \quad \begin{aligned} \alpha_1 &= e_1 + e_7, \alpha_2 = e_1 - e_7, \alpha_3 = e_2 + e_8, \alpha_4 = e_2 - e_8, \\ \alpha_5 &= e_3 + e_4, \alpha_6 = e_3 - e_4, \alpha_7 = e_5 + e_6, \alpha_8 = e_5 - e_6. \end{aligned}$$

We shall prove that every admissible root system in \mathbf{R} is conjugate to one of the following ten systems from \mathbf{F}_0 to \mathbf{F}_9 .

$$(110) \quad \begin{aligned} \mathbf{F}_0 &= \phi, \mathbf{F}_1 = \{\alpha_1\}, \mathbf{F}_2 = \{\alpha_1, \alpha_2\}, \\ \mathbf{F}_3 &= \{\alpha_1, \alpha_2, \alpha_3\}, \mathbf{F}_4 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}, \mathbf{F}_5 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_5\}, \\ \mathbf{F}_6 &= \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}, \mathbf{F}_7 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}, \\ \mathbf{F}_8 &= \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}, \mathbf{F}_9 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8\}. \end{aligned}$$

We shall use the following two systems \mathbf{F}_{10} and \mathbf{F}_{11} which are conjugate to \mathbf{F}_6 and \mathbf{F}_7 respectively:

$$\mathbf{F}_{10} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_7\}, \mathbf{F}_{11} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_7\}.$$

1° $\{\varepsilon\}$ is conjugate to $\{e_i \pm e_j\}$ by (108) and $\{e_i \pm e_j\}$ is conjugate to $\{\alpha_1\}$ or $\{\alpha_2\}$ by (104). $\{\alpha_1\}$ is transformed to $\{\alpha_2\}$ by $S_{e_6-e_7}S_{e_6+e_7}$ (cf. (105)).

Thus every 1-dimensional \mathbf{F} is conjugate to $\{\alpha_1\}$.

2° The possible types of 2-dimensional \mathbf{F} are as follows:

- (1) $\{e_i + e_j, e_k + e_l\}$, (2) $\{e_i - e_j, e_k - e_l\}$, (3) $\{\varepsilon, \delta\}$, $(\varepsilon, \delta) = 0$,
- (4) $\{e_i + e_j, e_k - e_l\}$, (5) $\{e_i + e_j, \varepsilon\}$, $\varepsilon_i = -\varepsilon_j$, (6) $\{e_i - e_j, \varepsilon\}$, $\varepsilon_i = \varepsilon_j$,
- (7) $\{e_i + e_j, e_i - e_j\}$.

(2) \approx (1) and (4) \approx (1) by (105). (6) is conjugate to one of the type (5) by (105). (1) is transformed to one of the type (3) by S_{ε} , ($\varepsilon_i = \varepsilon_j$, $\varepsilon_k = \varepsilon_l$, cf. (107)). (1) is transformed to one of the type (5) by S_{δ} ($\delta_i = -\delta_j$, $\delta_k = \delta_l$, cf. (106) and (107)). Every F of type (1) is conjugate to $F_{12} = \{\alpha_1, \alpha_3\}$ by (104). Every F of type (7) is conjugate to F_2 . $S_{(-1,1,1,1,1,1,1,1,1)}S_{(1,-1,1,1,1,1,-1,-1)}$ transforms F_2 to F_{12} . Thus we have proved that every two dimensional F is conjugate to F_2 .

3°) By the result in 2°), every 3-dimensional F is conjugate to a system of type $\{\alpha_1, \alpha_2, \beta\}$, where β is a positive root which is orthogonal to α_1 and α_2 . Therefore β must be one of the following roots: $e_i \pm e_j$, ($i, j = 2, 3, 4, 5, 6, 8, i < j$). $\{\alpha_1, \alpha_2, e_i + e_j\} \approx \{\alpha_1, \alpha_2, e_i - e_j\}$ by (105), $\{\alpha_1, \alpha_2, e_i + e_j\} \approx F_3$ by (104). Therefore every 3-dimensional F is conjugate to F_3 .

4°) By the result in 3°), every 4-dimensional F is conjugate to $\{\alpha_1, \alpha_2, \alpha_3, \beta\}$. β must be one of $e_2 - e_8, e_i \pm e_j$ ($i, j = 3, 4, 5, 6, 8, i < j$). $\beta = e_2 - e_8$ and $\beta = e_3 + e_4$ give F_4 and F_5 respectively. Any other β 's give the systems conjugate to F_5 . Thus every 3-dimensional F is conjugate either to F_4 or to F_5 . And F_4 can not be conjugate to F_5 , because the space $[F_4]$ spanned by the vectors in F_4 contains 24 roots, but $[F_5]$ contains only 8 roots.

5°) By the results in 4°), every 5-dimensional F is conjugate to one of the following systems:

$$(8) \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta\}, \quad (9) \{\alpha_1, \alpha_2, \alpha_3, \alpha_5, \gamma\}.$$

As β in (8), 4 roots, $e_3 \pm e_4, e_5 \pm e_6$, are possible. All 4 systems so obtained are conjugate to F_6 . As γ in (9), 4 roots, $e_2 - e_8, e_3 - e_4, e_5 \pm e_6$, are possible. $\gamma = e_2 - e_8$ gives F_6 . $\gamma = e_3 - e_4$ gives a system conjugate to F_6 . $\gamma = e_5 + e_6$ gives F_{10} . $\gamma = e_5 - e_6$ gives a system conjugate to F_{10} . F_6 is transformed to F_{10} by $S_{(-1,1,1,1,-1,-1,-1,-1)}S_{e_2-e_7}S_{(1,1,-1,-1,-1,-1,-1,-1)}S_{e_2+e_8}S_{e_2-e_8}$. Therefore every 5-dimensional F is conjugate to F_6 .

6°) By the result in 5°), every 6-dimensional F is conjugate to one of the following systems:

$$(10) \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \beta\}.$$

As β in (10), $e_3 - e_4, e_5 \pm e_6$ are possible. $\beta = e_3 - e_4$ gives F_7 . $\beta = e_5 + e_6$ gives F_{11} . $\beta = e_5 - e_6$ gives a system conjugate to F_{11} . F_{11} is conjugate to F_7 . Hence every 6-dimensional F is conjugate to F_7 .

7°) By the results in 6°), every 7-dimensional F is conjugate to one of the following systems

$$(11) \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \beta\}.$$

The possible β 's in (11) are $e_5 + e_6$ and $e_5 - e_6$. $\beta = e_5 + e_6$ gives F_8 . $\beta = e_5 - e_6$ gives a system which is transformed to F_8 by $S_{e_6-e_7}S_{e_6+e_7}$.

Hence every 7-dimensional F is conjugate to F_8 .

8°) Every 8-dimensional F spans $\mathfrak{h} = \mathfrak{h}^-$, hence F is equivalent to F_9 .

Thus we have proved that every fundamental root system in R is conjugate to one of the ten systems defined in (110), and any two of these 10 systems are not conjugate.

Therefore the number N of conjugate classes of Cartan subalgebras in (E VIII) is equal to ten:

$$N = 10.$$

Type (E IX). (E IX) is the real form of E_8 defined by the involution

$$\tau = \exp \operatorname{ad}(\pi\sqrt{-1}(H_{e_7} + H_{e_8})/c).$$

The set R_p of non compact roots is

$$(111) \quad R_p = \{ \pm(e_i \pm e_7), \pm(e_i \pm e_8), \pm(2^{-1}e^{(8)} - e_i), \\ \pm(2^{-1}e^{(8)} - e_i - e_7 - e_8), \pm(2^{-1}e^{(8)} - e_i - e_j - e_k) \},$$

where i, j, k are different integers from 1 to 6.

The number of non compact roots is equal to 112, and the character $\delta = 2 \times 112 - 248 = -24$.

Let $\alpha_1, \alpha_2, \alpha_3$, and α_4 be the four roots defined in (109), then $F_4 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is a maximal fundamental root system in R_p . Hence by Theorem 7,

$$m = RH_{\alpha_1}' + RH_{\alpha_2}' + RH_{\alpha_3}' + RH_{\alpha_4}' \\ = \{xe_1 + ye_2 + ze_3 + we_8; x, y, z, w \in R\}$$

is a maximal abelian subalgebra in \mathfrak{p} . Therefore we have

$$(112) \quad R(m) = \{ \pm e_i \pm e_j; i, j = 1, 2, 7, 8 \}.$$

Every 4-dimensional F in $R(m)$ spans m , and is equivalent to F_4 . By the results on (E VIII), every i -dimensional F in $R(m)$ is conjugate to F_i defined in (110) ($0 \leq i \leq 3$). Therefore the number N of conjugate classes of Cartan subalgebras in (E IX) is equal to five:

$$N = 5.$$

Type (E XII). (E XII) is the compact real form of E_8 . Hence all Cartan subalgebras in (E XII) are mutually conjugate and we have

$$N = 1.$$

Type F

Complex simple Lie algebra F_4 is of rank 4 and of dimension 52. The root system R of complex simple Lie algebra F_4 is

$$(113) \quad R = \{ \pm e_i, \pm e_i \pm e_j, 2^{-1}(\pm e_1 \pm e_2 \pm e_3 \pm e_4) \},$$

where i and j are different integers from 1 to 4. We denote the root $2^{-1} \sum_{i=1}^4 \varepsilon_i e_i$ by $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$ or simply by ε .

The operations of the Weyl group W are as follows:

(114) S_{e_i} = the change of signature of e_i .

(115) $S_{e_i - e_j}$ = the substitution of e_i for e_j .

(116) $S_{e_i - e_j} S_{e_i + e_j}$ = the changes of signatures of e_i and e_j .

(117) $S_\varepsilon(e_i - e_j) = \begin{cases} -(\varepsilon_k e_k + \varepsilon_l e_l) & \varepsilon_i = -1, \varepsilon_j = -1, \\ \varepsilon_k e_k + \varepsilon_l e_l & \varepsilon_i = -1, \varepsilon_j = 1, \\ e_i - e_j & \varepsilon_i = \varepsilon_j. \end{cases}$

(118) $S_\varepsilon(e_i + e_j) = \begin{cases} -(\varepsilon_k e_k + \varepsilon_l e_l) & \varepsilon_i = \varepsilon_j = 1, \\ \varepsilon_k e_k + \varepsilon_l e_l & \varepsilon_i = \varepsilon_j = -1, \\ e_i + e_j & \varepsilon_i \neq \varepsilon_j. \end{cases}$

In (117) and (118), $\{i, j, k, l\} = \{1, 2, 3, 4\}$.

(119) $S_\varepsilon(\delta) = \begin{cases} -\delta & \varepsilon = \pm \delta, \\ \pm e_i & (\varepsilon, \delta) = 2^{-1}, \\ \delta & (\varepsilon, \delta) = 0. \end{cases}$

(120) $S_\varepsilon(e_i) = \varepsilon^{(i)},$

where $\varepsilon_j^{(i)} = -\varepsilon_j$ ($j \neq i$) and $\varepsilon_i^{(i)} = 1$.

The complex simple Lie algebra F_4 has three different real forms (F I) (F II) and (F III).

Type (F I). (F I) is the normal real form of F_4 , hence $\mathbf{R}(\mathfrak{m})$ is equal to \mathbf{R} defined in (113).

A complete system of representatives of conjugate classes of admissible root system is given by (121).

(121) $\mathbf{F}_0 = \phi, \mathbf{F}_1 = \{e_1\}, \mathbf{F}_2 = \{e_1 + e_2\},$
 $\mathbf{F}_3 = \{e_1 + e_2, e_1 - e_2\}, \mathbf{F}_4 = \{e_1 + e_2, e_3\},$
 $\mathbf{F}_5 = \{e_1 + e_2, e_1 - e_2, e_3 + e_4\}, \mathbf{F}_6 = \{e_1 + e_2, e_1 - e_2, e_3\},$
 $\mathbf{F}_7 = \{e_1 + e_2, e_1 - e_2, e_3 + e_4, e_3 - e_4\}.$

1° The possible types of 1-dimensional \mathbf{F} are as follows:

(1) $\{e_i\},$ (2) $\{e_i + e_j\},$ (3) $\{e_i - e_j\},$ (4) $\{\varepsilon\}.$

(3) is conjugate to (2) by (114) and (4) is conjugate to (1) by (120). Every \mathbf{F} of type (1) ((2)) is conjugate to \mathbf{F}_1 (\mathbf{F}_2) by (115). \mathbf{F}_1 can not be conjugate to \mathbf{F}_2 , because $2(e_1, e_1) = (e_1 + e_2, e_1 + e_2)$.

2° The possible types of 2-dimensional \mathbf{F} are as follows:

- (5) $\{e_i, e_j\}$, (6) $\{e_i+e_j, e_k+e_l\}$, (7) $\{e_i-e_j, e_k-e_l\}$,
 (8) $\{\varepsilon, \delta\}$, (9) $\{e_i, e_k+e_l\}$, (10) $\{e_i, e_k-e_l\}$,
 (11) $\{e_i+e_j, e_i-e_j\}$, (12) $\{e_i+e_j, e_k-e_l\}$, (13) $\{e_i+e_j, \varepsilon\}$,
 (14) $\{e_i-e_j, \varepsilon\}$.

There are certain relations among these systems. (5) \equiv (11), (6) \approx (7), (9) \approx (10), (6) \approx (12) and (13) is conjugate to one of the type (14) by (114). (8) \approx (5) by (120). (9) is conjugate to one of the type (13) by (118) and (120). Every \mathbf{F} of type (7) is conjugate to $\mathbf{F}_{11} = \{e_1-e_2, e_3-e_4\}$. \mathbf{F}_{11} is transformed to \mathbf{F}_3 by $S_{(1,1,-1,1)}$ (cf. (118)). Every \mathbf{F} of type (9) ((11)) is conjugate to \mathbf{F}_4 (\mathbf{F}_3). Thus we have proved that every two dimensional \mathbf{F} is conjugate either to \mathbf{F}_3 or to \mathbf{F}_4 . \mathbf{F}_3 and \mathbf{F}_4 can not be conjugate, because the space $[\mathbf{F}_3]$ spanned by the vectors in \mathbf{F}_3 contains eight roots $\{\pm e_1, \pm e_2, \pm(e_1 \pm e_2)\}$ and $[\mathbf{F}_4]$ contains only four roots $\{\pm(e_1+e_2), \pm e_3\}$.

3°) By the result in 2°) every 3-dimensional \mathbf{F} is conjugate to one of the following:

$$(15) \{\mathbf{F}_3, \alpha\}, \quad (16) \{\mathbf{F}_4, \beta\}.$$

The positive root α in (15) is orthogonal to $e_1 \pm e_2$, hence α is one of $e_3 \pm e_4, e_3, e_4$. Therefore every \mathbf{F} of type (15) is conjugate either to \mathbf{F}_5 or to \mathbf{F}_6 . The possible positive root β in (16) is $e_1 - e_2$. $\beta = e_1 - e_2$ gives \mathbf{F}_6 . Thus we have proved that every 3-dimensional \mathbf{F} is conjugate to one of \mathbf{F}_5 and \mathbf{F}_6 . The root system $\mathbf{R}([\mathbf{F}_5])$ (where $[\mathbf{F}_5]$ is the space spanned by the vectors in \mathbf{F}_5) is of the type C_3 but $\mathbf{R}([\mathbf{F}_6])$ is of the type B_3 . Hence \mathbf{F}_5 and \mathbf{F}_6 can never be conjugate.

4°) All 4-dimensional \mathbf{F} span $\mathfrak{h} = \mathfrak{h}^-$ and they are mutually conjugate. Thus we have proved that the eight admissible root systems in (121) make a complete representative system of conjugate classes of the admissible root systems in \mathbf{R} . Consequently, the number of conjugate classes of Cartan subalgebras in (F I) is equal to eight:

$$N = 8.$$

Type (F II). The real form (F II) is defined by the involution

$$\tau = \exp \operatorname{ad}(2\pi\sqrt{-1} H_{e_1}/c).$$

The set \mathbf{R}_p of non compact roots is

$$\mathbf{R}_p = \{2^{-1}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)\}.$$

The number of non compact roots is equal to 16, and the character $\delta = 2 \times 16 - 52 = -20$. For any two roots α_1 and α_2 ($\alpha_1 \neq \pm \alpha_2$) in \mathbf{R}_p , either $\alpha_1 + \alpha_2$ or $\alpha_1 - \alpha_2$ is again a root in \mathbf{R} . Hence maximal admissible root systems are

1-dimensional. It is clear by (104) that every 1-dimensional admissible root system is conjugate to $\{2^{-1}(e_1+e_2+e_3+e_4)\}$. Therefore the number N of conjugate classes of Cartan subalgebras in (F II) are equal to two:

$$N=2.$$

\mathfrak{h}_1 and \mathfrak{h}_2 are conjugate if and only if $\dim \mathfrak{h}_1^+ = \dim \mathfrak{h}_2^+$.

Type (F III). (FIII) is the compact real form of F_4 . All Cartan subalgebras are mutually conjugate:

$$N=1;$$

Type G

Complex simple Lie algebra \mathfrak{g}^c of type G_2 has two non-isomorphic real forms. The one (type (G I)) is the normal real form of \mathfrak{g}^c , and the other (type (G II)) is the compact real form of \mathfrak{g}^c .

Type (G I). Since \mathfrak{g} is normal, there exists a Cartan subalgebra \mathfrak{h}_0 of \mathfrak{g} of which toroidal part \mathfrak{h}_0^+ is $\{0\}$. Then we have $\mathfrak{h}_0 = \mathfrak{m}$ and $\mathbf{R}(\mathfrak{m}) = \mathbf{R}$. \mathbf{R} consists of twelve vectors shown in Fig. 1. Fig. 1 shows that two roots with the same length can be transformed each other by the action of the Weyl group W . Therefore every admissible root system in \mathbf{R} is conjugate to one of the following four systems under the action of W .

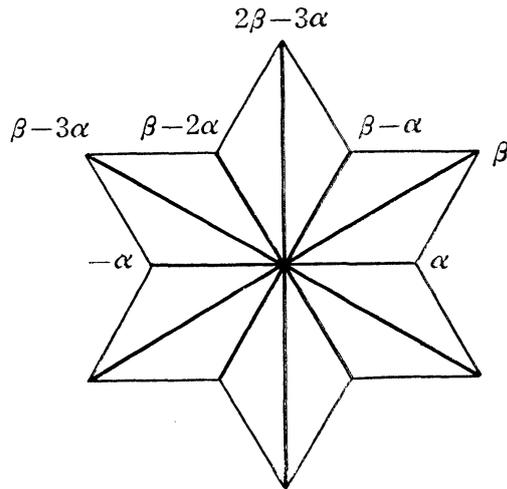


Fig. 1

- (1) $\mathbf{F}_0 = \{\phi\}$, (2) $\mathbf{F}_1 = \{\alpha\}$,
- (3) $\mathbf{F}_2 = \{\beta\}$, (4) $\mathbf{F}_3 = \{\alpha, \beta\}$.

$\{\alpha\}$ and $\{\beta\}$ are not conjugate under W , because $(\beta, \beta) = 3(\alpha, \alpha)$.

Consequently, the number N of conjugate classes is equal to four:

$$N=4.$$

Type (G II). (G II) is the compact real form of G_2 , therefore every two Cartan subalgebras of (G II) are conjugate to each other, i. e.

$$N=1.$$

Finally we give a theorem.

THEOREM 8. *Real semisimple Lie algebra \mathfrak{g} has a compact Cartan subalgebra if and only if \mathfrak{g} is of the first category.*

PROOF. The "If" part of the theorem is proved in Proposition 13. To prove the converse, it is sufficient to show that every real simple Lie algebra of the second category has no compact Cartan subalgebra. A real simple Lie algebra is either a real form of a complex simple Lie algebra or a complex simple Lie algebra regarded as Lie algebra over reals. The simple Lie algebras of the latter type are always of the second category, and they have not compact Cartan subalgebras by (16) in §1. The real forms of the second category of complex simple Lie algebras are algebras of types $(A_n I)$ ($n \geq 2$), $(A II)$, $(D_n Ib)$ ($n \geq 3$), $(E I)$ and $(E IV)$. The fact that Lie algebras of these five types have no compact Cartan subalgebra is verified by the results in §3 and §4.

§ 5. Conjugate classes under the automorphism group.

In this section, we study the conjugate classes of Cartan subalgebras under the action of the automorphism group instead of the adjoint group.

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a fixed Cartan decomposition of a real semisimple Lie algebra \mathfrak{g} , and η be the conjugation of \mathfrak{g}^c with respect to the compact real form $\mathfrak{g}_u = \mathfrak{k} + \sqrt{-1}\mathfrak{p}$, then

$$\langle X, Y \rangle = -B(X, \eta Y)$$

is a positive definite Hermitian form on \mathfrak{g}^c .

We denote the full automorphism group of \mathfrak{g} , \mathfrak{g}_u and \mathfrak{g}^c by \tilde{G} , \tilde{G}_u and \tilde{G}^c respectively. \tilde{G} and \tilde{G}_u are regarded as subgroups of \tilde{G}^c . \tilde{G}^c is an algebraic group which is self-adjoint with respect to $\langle X, Y \rangle$. By a lemma due to C. Chevalley [4, (a), p. 201], every element g of \tilde{G}^c is decomposed uniquely to the product

$$g = kp, \quad k \in \tilde{G}_u, \quad p \in \exp \sqrt{-1} \mathfrak{g}_u.$$

If g belongs to \tilde{G} , then k and p also belong to \tilde{G} . Hence every element g of \tilde{G} is decomposed to the product of an element k of the maximal compact subgroup $\tilde{K} = \tilde{G} \cap \tilde{G}_u$ and an element $p = \exp \operatorname{ad} X$, $X \in \mathfrak{p} = \mathfrak{g} \cap \sqrt{-1} \mathfrak{g}_u$. We remark that

$$\tilde{K} = \{k \in \tilde{G}; k\mathfrak{k} = \mathfrak{k}, k\mathfrak{p} = \mathfrak{p}\}.$$

Let \mathfrak{m} be a maximal abelian subalgebra in \mathfrak{p} , and

$$\tilde{N} = \{k \in \tilde{K}; k\mathfrak{m} = \mathfrak{m}\} \quad \text{and} \quad \tilde{W}_s = \{\varphi(k); k \in \tilde{N}\},$$

where $\varphi(k)$ is the restriction of k to \mathfrak{m} .

Let \mathfrak{h}_0 be a Cartan subalgebra of \mathfrak{g} containing \mathfrak{m} , and \tilde{W} be the Cartan

group of \mathfrak{g}^c with respect to \mathfrak{h}_0^c . \tilde{W} is the group of all linear transformations on \mathfrak{h}_0^c which transform every root to another root. It is well known that \tilde{W} consists of all the restrictions of the automorphisms of \mathfrak{g}^c which keep invariant \mathfrak{h}_0^c .

THEOREM 9. 1) Two standard Cartan subalgebras \mathfrak{h}_1 and \mathfrak{h}_2 are conjugate under the automorphism group \tilde{G} if and only if their vector parts \mathfrak{h}_1^- and \mathfrak{h}_2^- are conjugate under the group \tilde{W}_s . 2) Two standard Cartan subalgebras \mathfrak{h}_1 and \mathfrak{h}_2 are conjugate if and only if \mathfrak{l}_1 and \mathfrak{l}_2 are conjugate under the Cartan group \tilde{W} , where $\mathfrak{l}_i = \{X \in \mathfrak{m}; B(X, \mathfrak{h}_i^-) = 0\}$.

PROOF. The proofs of Theorem 3 and Theorem 4 are valid if we replace K, N, W_s and W with $\tilde{K}, \tilde{N}, \tilde{W}_s$ and \tilde{W} .

DEFINITION 10. Two admissible root systems F_1 and F_2 are called conjugate under the Cartan group \tilde{W} or \tilde{W} -conjugate if there exists an element s in \tilde{W} such that $sF_1 \equiv F_2$.

By Theorem 5 and Theorem 9, 2), we have the following Theorem 10 which is fundamental for the problem of conjugacy under the automorphism group \tilde{G} .

THEOREM 10. There is a one to one correspondence between the conjugate classes of Cartan subalgebras in a real semisimple Lie algebra \mathfrak{g} under the full automorphism group \tilde{G} and the \tilde{W} -conjugate classes of admissible root systems in $\mathbf{R}(\mathfrak{m})$.

Now, we determine explicitly the conjugate classes of Cartan subalgebras under the automorphism group \tilde{G} .

The normal real form of complex semisimple Lie algebra D_{2n} is denoted by $(D_{2n}Ic)$. Lie algebra $(D_{2n}Ic)$ for a certain integer n is called of the type (DIc) . Type (DIc) is a particular type of the type (DIa) . Lie algebras of the type (DIa) other than algebras of the type (DIc) are called of the type $(DIId)$. There are five non isomorphic algebras of the type (D_4I) . One of them is (D_4Ic) . Other four algebras are called of the type (D_4Ie) .

The number of conjugate classes of Cartan subalgebras in a Lie algebra \mathfrak{g} under the adjoint group G and under the full automorphism group \tilde{G} are denoted by $N(\mathfrak{g})$ and $\tilde{N}(\mathfrak{g})$ respectively.

THEOREM 11. 1) In all real simple Lie algebras except algebras of type (DIc) , two Cartan subalgebras are conjugate under the automorphism group \tilde{G} if and only if they are conjugate under the adjoint group G . 2) In the Lie algebra $(D_{2n}Ic)$, $n \geq 3$, the n -dimensional admissible root system defined by (76) is conjugate to one of the systems defined in (74) under the Cartan group \tilde{W} . All other conjugate classes under W are also conjugate classes under \tilde{W} . Hence we have

$$\tilde{N}(D_{2n}Ic) = N(D_{2n}Ic) - 1 \quad (n \geq 3).$$

3) In the Lie algebra (D_4Ic) , there are three 2-dimensional conjugate classes of admissible root systems under W . All systems in these three classes are mutually conjugate under \tilde{W} . All other conjugate classes under W are also conjugate classes under \tilde{W} . Hence we have

$$\tilde{N}(D_4Ic) = N(D_4Ic) - 2 = 5.$$

PROOF. 1), 2) Any real simple Lie algebra is either complex simple or a real form of a complex simple Lie algebra. In every complex simple Lie algebra, all Cartan subalgebras are mutually conjugate under G , hence they are also conjugate under \tilde{G} . The Cartan group \tilde{W} coincides with the Weyl group W in all complex simple Lie algebras except A_n, D_n and E_6 . Hence in all real forms of complex simple Lie algebras of types B_n, C_n, E_7, E_8, F_4 , and G_2 , the conjugacy of Cartan subalgebras under \tilde{G} coincides with the conjugacy under G . In all real forms of A_n and E_6 and the simple Lie algebras of types (D III) and (D_4Ie) , two Cartan subalgebras \mathfrak{h}_1 and \mathfrak{h}_2 are conjugate under G if and only if $\dim \mathfrak{h}_1^+ = \dim \mathfrak{h}_2^+$ (cf. § 3 and § 4), hence in each real forms of A_n and E_6 and the Lie algebras of types (DIII) and (D_4Ie) conjugacy under \tilde{G} coincides with the conjugacy under G .

The Cartan group \tilde{W} of D_n ($n > 4$) consists of all substitutions of e_i 's and changes of signatures of arbitrary number of e_i 's, hence any two different systems in (74) are not conjugate under \tilde{W} and the admissible root system (76) is conjugate to a system in (74) under \tilde{W} . Thus we have proved that $\tilde{N}(D Ib) = N(D Ib)$, $\tilde{N}(D Id) = N(D Id)$ and $\tilde{N}(D_{2n} Ic) = N(D_{2n} Ic) - 1$ if $2n \geq 6$.

3) In the Lie algebra (D_4Ic) , there are three 2-dimensional admissible root systems defined as follows:

$$F_1 = \{e_1 + e_2, e_1 - e_2\}, F_2 = \{e_1 + e_2, e_3 + e_4\}, F_3 = \{e_1 + e_2, e_3 - e_4\}.$$

Every 2-dimensional F is conjugate to one of F_1, F_2 and F_3 , and any two of F_1, F_2 and F_3 are not conjugate under the Weyl group W .

However F_1, F_2 , and F_3 are mutually conjugate under the Cartan group \tilde{W} . The Cartan group contains the subgroup P consisting of the linear transformations which keep invariant the set S of all simple roots. An S in D_4 is indicated in Fig. 2. Hence $P \subset \tilde{W}$ contains an element T defined as follows:

$$\begin{aligned} T(e_1 - e_2) &= e_3 + e_4, & T(e_3 + e_4) &= e_3 - e_4, \\ T(e_3 - e_4) &= e_1 - e_2, & T(e_2 - e_3) &= e_2 - e_3. \end{aligned}$$

T transforms F_1 and F_2 to F_2 and F_3 respectively.

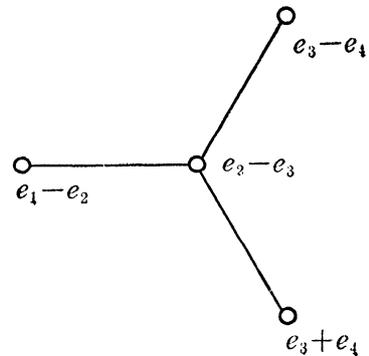


Fig. 2

All four conjugate classes under W other than $\{F_1\}$, $\{F_2\}$ and $\{F_3\}$ are also conjugate classes under \tilde{W} . Thus we have completed the proof of Theorem 11.

Now we have a complete knowledge on the conjugate classes of Cartan subalgebras in real simple Lie algebras under the adjoint groups or under the full automorphism groups. This suffices to determine the conjugate classes of Cartan subalgebras in general real semisimple Lie algebras.

Indeed, let $\mathfrak{g} = \mathfrak{g}_1 + \dots + \mathfrak{g}_r$ be the decomposition of a semisimple Lie algebra \mathfrak{g} into the simple ideals \mathfrak{g}_i , then every Cartan subalgebra \mathfrak{h} of \mathfrak{g} is the direct sum of the Cartan subalgebras \mathfrak{h}_i of \mathfrak{g}_i . Let $\mathfrak{h} = \mathfrak{h}_1 + \dots + \mathfrak{h}_r$ and $\mathfrak{h}' = \mathfrak{h}'_1 + \dots + \mathfrak{h}'_r$; $\mathfrak{h}_i, \mathfrak{h}'_i \subset \mathfrak{g}_i$ be two Cartan subalgebras in \mathfrak{g} , then we have clearly the following theorem.

THEOREM 12. 1) \mathfrak{h} and \mathfrak{h}' are conjugate under the adjoint group G of \mathfrak{g} if and only if \mathfrak{h}_i and \mathfrak{h}'_i are conjugate under the adjoint group G_i of \mathfrak{g}_i . Hence the number $N(\mathfrak{g})$ of conjugate classes of Cartan subalgebras in \mathfrak{g} under the adjoint group G is the product of that of \mathfrak{g}_i , i. e.

$$N(\mathfrak{g}) = \prod_{i=1}^r N(\mathfrak{g}_i),$$

where $N(\mathfrak{g}_i)$ is the number of conjugate classes of Cartan subalgebras in simple Lie algebras \mathfrak{g}_i under G_i . 2) Mutually isomorphic simple components \mathfrak{g}_i are identified to a fixed one among them. We divide the set $I = \{1, 2, \dots, r\}$ of indices of \mathfrak{g}_i 's into I_k ($1 \leq k \leq s$) so that we have $I = \bigcup_{k=1}^s I_k, I_k \cap I_l = 0$ ($k \neq l$), and $\mathfrak{g}_i = \mathfrak{g}_j$ if i and j belong to the same I_k and $\mathfrak{g}_i \not\cong \mathfrak{g}_j$ if $i \in I_k, j \in I_l$ and $k \neq l$. Then, \mathfrak{h} is conjugate to \mathfrak{h}' under the automorphism group \tilde{G} of \mathfrak{g} if and only if there exists a permutation σ of $\{1, 2, \dots, r\}$ which keeps each I_k invariant and each $\mathfrak{h}_{\sigma(i)}$ is conjugate to \mathfrak{h}'_i under the automorphism group \tilde{G}_i of \mathfrak{g}_i . Hence the number $\tilde{N}(\mathfrak{g})$ of conjugate classes of Cartan subalgebras in \mathfrak{g} under the automorphism group \tilde{G} of \mathfrak{g} is given as follows:

$$\tilde{N}(\mathfrak{g}) = \prod_{k=1}^s \binom{n_k + r_k - 1}{r_k}$$

where $n_k = \tilde{N}(\mathfrak{g}_i), i \in I_k$ i. e. n_k is the number of conjugate classes of Cartan subalgebras in \mathfrak{g}_i under \tilde{G}_i ($i \in I_k$), and r_k is the number of integers contained in I_k .

EXAMPLE. Let $\mathfrak{g} = \mathfrak{sl}(2, R) + \mathfrak{sl}(2, R)$. As $N(\mathfrak{sl}(2, R)) = 2$ (cf. § 3), $\tilde{N}(\mathfrak{sl}(2, R)) = 2$ by Theorem 11. Hence we have $\tilde{N}(\mathfrak{g}) = \binom{2+2-1}{2} = 3$ by Theorem 12. As is well known \mathfrak{g} is isomorphic to the Lie algebras \mathfrak{g}_1 of the orthogonal group of the quadratic form $x_1^2 + x_2^2 - x_3^2 - x_4^2$ (cf. [1, p. 353]). We know that $N(\mathfrak{g}_1) = 4$ in § 3, therefore $\tilde{N}(\mathfrak{g}_1) = 3$ by Theorem 11.

This result on \mathfrak{g}_1 is in accordance with the above result on \mathfrak{g} .

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References

- [1] E. Cartan, Les groupes réels simple finis et continus, *Ann. Éc. Norm.*, **31** (1914), 263-355.
 - [2] E. Cartan, Sur certaines formes riemanniennes remarquables des géométries à groupe fondamental simple, *Ann. Éc. Norm.*, **44** (1927), 345-467.
 - [3] E. Cartan, Groupes simples clos et ouverts et géométrie riemannienne, *J. Math. pures et appliquées*, **8** (1929), 133.
 - [4] C. Chevalley, (a) *Theory of Lie groups*, I. Princeton, 1946. (b) *Théorie des groupes de Lie*, t. III, Hermann, Paris, 1955.
 - [5] F. Gantmacher, On the classification of real simple Lie groups, *Mat. Sbornik*, **5** (1939), 217-250.
 - [6] Harish-Chandra, The characters of semisimple Lie groups, *Trans. Amer. Math. Soc.*, **83** (1956), 98-163.
 - [7] G. A. Hunt, A theorem of E. Cartan, *Proc. Amer. Math. Soc.*, **7** (1956), 307-308.
 - [8] N. Iwahori and I. Satake, On Cartan subalgebras of a Lie algebra, *Kōdai Math. Sem. Rep.*, **3** (1950), 57-60.
 - [9] B. Kostant, Conjugacy of real Cartan subalgebras, *Proc. Nat. Acad. Sci. U. S. A.*, **41** (1955), 967-970.
 - [10] G. D. Mostow, A new proof of E. Cartan's theorem on the topology of semi-simple groups, *Bull. Amer. Math. Soc.*, **55** (1949), 969-980.
 - [11] S. Murakami, Supplements and corrections to my paper: On the automorphisms of a real semisimple Lie algebra, *J. Math. Soc. Japan*, **5** (1953), 105-112.
 - [12] L. S. Pontrjagin, *Continuous groups* (in Russian), second edition, Moscow, 1954.
 - [13] H. Weyl, *The structure and representations of continuous groups*. The Institute for Advanced Study, Princeton, 1935.
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