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On some partition functions.

Dedicated to Professor Z. Suetuna on his completion of sixty years.

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Introduction. Let $p_{\kappa}(n; a, M)$ be the number of partitions of a positive integer n into positive summands of the form $(Ml\pm a)^{\kappa}$ $(l=0,1,2,\cdots)$, where M, a and κ are integers satisfying $M\geqq 2$, $0<a<M$, $(a, M)=1$ and $\kappa\geqq 1$.

The first object of the present paper is to derive a suitable transformation formula for the generating function of $p_{\kappa}(n; a, M)$ and to determine the asymptotic behavior of the generating function in the neighborhood of its singularity at each rational point of the unit circle. A precise (not asymptotic) transformation equation will be obtained in § 1 of this paper.

Secondly, we shall give, in § 2, an asymptotic formula for the partition function $p_{\kappa}(n; a, M)$ for large values of n.

The special case $\kappa=1$ of our partition problem has been discussed in [**3**].

It should be noted that the case $M=2$ is equivalent to the case $M=4$, since we clearly have $p_{\kappa}(n; 1, 2)=p_{\kappa}(n; 1, 4)$. Therefore we may assume that $M\geqq 3$ in the sequel.

1. The transformation equation. The generating function of $p_{\kappa}(n; a, M)$ is given by

$$F_{\kappa}(x; a, M)=1+\sum_{n=1}^{\infty}p_{\kappa}(n; a, M)x^{n}=\prod_{\substack{\nu>0\\ \nu\equiv\pm a(M)}}(1-x^{\nu^{\kappa}})^{-1},$$

where x is a complex variable with $|x|<1$.

Now let h, k be coprime integers with $k\geqq 1$. We set

$(k, M)=D$ (the greatest common divisor of k and M),
$\{k, M\}=K$ (the least common multiple of k and M);

and put $k=k_1 D$. Further we write

$$x=\exp(2\pi ih/k-2\pi z),$$

where z is a complex variable with $\Re(z)>0$. Define

$$\phi_\kappa(k, a, M) = \begin{cases} 1 & (D>1), \\ \{2\sin(\pi\xi/M)\}^{-\kappa} & (D=1), \end{cases} \tag{1}$$

where ξ is an integer defined by $\xi m_k \equiv a \pmod{M}$ $(0<\xi<M)$, the number m_k being the least positive integer such that $m_k^{\kappa} \equiv 0 \pmod{k}$. Next we define

$$\sigma_\kappa(h, k) = \sum_{\substack{0<\nu<K \\ \nu\equiv \pm a(M)}} ((\nu/K))((h\nu^\kappa/k))$$

with the abbreviation

$$((t)) = \begin{cases} 0, & \text{if } t \text{ is an integer,} \\ t-[t]-1/2, & \text{otherwise,} \end{cases}$$

$[t]$ denoting the greatest integer not exceeding t. We remark here that

$$\sigma_\kappa(h, k) = 0 \qquad \text{for } \kappa \text{ even,}$$

since we have

$$\sum_{\substack{0<\nu<K \\ \nu\equiv \pm a(M)}} \left(\left(\frac{\nu}{K}\right)\right)\left(\left(\frac{h\nu^\kappa}{k}\right)\right) = \sum_{\substack{0<\nu<K \\ \nu\equiv \pm a(M)}} \left(\left(\frac{K-\nu}{K}\right)\right)\left(\left(\frac{h(K-\nu)^\kappa}{k}\right)\right)$$

$$= -\sum_{\substack{0<\nu<K \\ \nu\equiv \pm a(M)}} \left(\left(\frac{\nu}{K}\right)\right)\left(\left(\frac{h\nu^\kappa}{k}\right)\right)$$

for κ even. Further notations are as follows:

$$B_\kappa(a, M) = \begin{cases} \dfrac{M^\kappa}{\kappa+1} B_{\kappa+1}(a/M) & (\kappa \text{ odd}), \\ 0 & (\kappa \text{ even}), \end{cases}$$

where $B_{\kappa+1}(t)$ is the Bernoulli polynomial of order $\kappa+1$;

$$\varepsilon_{\kappa,s} = -ie^{\pi i(s-1/2)/\kappa},$$

$$\mu_{\nu,s} = \begin{cases} 1 & (\nu^* = 0), \\ \nu^*/k & (\nu^* \neq 0,\ s \text{ odd}), \\ 1-\nu^*/k & (\nu^* \neq 0,\ s \text{ even}), \end{cases} \qquad (s = 1, 2, \cdots, \kappa), \tag{2}$$

with

$$\nu^* = h\nu^\kappa - k[h\nu^\kappa/k]. \tag{3}$$

Then our transformation formula may be stated in the following theorem.

THEOREM 1. *We have*

$$F_\kappa(x; a, M) = \phi_\kappa(k, a, M) \exp\Big\{-2\pi z B_\kappa(a, M)$$

$$+\frac{\Gamma(1+1/\kappa)}{K(2\pi z)^{1/\kappa}} \sum_{m=1}^{\infty} m^{-1-1/\kappa} \sum_{\substack{0<\nu<K \\ \nu\equiv \pm a(M)}} \exp(2\pi i m h\nu^\kappa/k) + \pi i \sigma_\kappa(h, k)\Big\} \tag{4}$$

$$\times \prod_{\substack{0<\nu<K \\ \nu\equiv\pm a(M)}} \prod_{s=1}^{\kappa} \prod_{l=0}^{\infty} \{1-\exp(-2\pi(\varepsilon_{\kappa,s}/K)\{(l+\mu_{\nu,s})/z\}^{1/\kappa}-2\pi i\nu/K)\}^{-1},$$

where $t^{1/\kappa}$ is always to be taken as the principal value.

PROOF. Recently the author [4] has obtained the following functional equations:

$$\sum_{l=0}^{\infty}\{\lambda((l+\alpha)^{\kappa}z-i\beta)+\lambda((l+1-\alpha)^{\kappa}z+i\beta)\}+\frac{2\pi z}{\kappa+1}B_{\kappa+1}(\alpha)$$

$$(5)\qquad =\sum_{s=1}^{\kappa}\sum_{l=0}^{\infty}\{\lambda(\varepsilon_{\kappa,s}\{(l+\beta_s)/z\}^{1/\kappa}+i\alpha)+\lambda(\varepsilon_{\kappa,s}\{(l+1-\beta_s)/z\}^{1/\kappa}-i\alpha)\}$$

$$+\frac{2\Gamma(1+1/\kappa)}{(2\pi z)^{1/\kappa}}\sum_{m=1}^{\infty}\frac{\cos(2\pi m\beta)}{m^{1+1/\kappa}}+2\pi i\left(\alpha-\frac{1}{2}\right)\left(\beta-\frac{1}{2}\right)\qquad(\kappa\text{ odd}),$$

$$\sum_{l=0}^{\infty}\{\lambda((l+\alpha)^{\kappa}z-i\beta)+\lambda((l+1-\alpha)^{\kappa}z-i\beta)\}$$

$$(6)\qquad =\sum_{s=1}^{\kappa}\sum_{l=0}^{\infty}\{\lambda(\varepsilon_{\kappa,s}\{(l+\beta_s)/z\}^{1/\kappa}+i\alpha)+\lambda(\varepsilon_{\kappa,s}\{(l+\beta_s)/z\}^{1/\kappa}-i\alpha)\}$$

$$+\frac{2\Gamma(1+1/\kappa)}{(2\pi z)^{1/\kappa}}\sum_{m=1}^{\infty}\frac{e^{2\pi i m\beta}}{m^{1+1/\kappa}}\qquad(\kappa\text{ even}),$$

where $0\leqq\alpha\leqq1$, $0<\beta<1$ (or $0<\alpha<1$, $0\leqq\beta\leqq1$), $\Re(z)>0$,

$$(7)\qquad \beta_s=\begin{cases}\beta & (s\text{ odd}),\\ 1-\beta & (s\text{ even}),\end{cases}$$

and

$$\lambda(t)=-\log(1-e^{-2\pi t})\qquad(\text{the principal value}).$$

It may be mentioned that the particular case $\kappa=1$ of (5) has been proved in [2].

We shall now prove Theorem 1 in the case κ odd, by using the above equation (5). We first notice that

$$\nu^*\equiv h\nu^{\kappa}\pmod{k},\qquad 0\leqq\nu^*<k$$

by the definition (3). Let us put

$$(8)\qquad \alpha=\nu/K,\quad \beta=\nu^*/k,$$

so that we have $0<\alpha<1$, $0\leqq\beta<1$. Substituting (8) into (5) and replacing z by $K^{\kappa}z$, the left member of (5) becomes

$$\sum_{l=0}^{\infty}\{\lambda((l+\nu/K)^{\kappa}K^{\kappa}z-i\nu^*/k)+\lambda((l+1-\nu/K)^{\kappa}K^{\kappa}z+i\nu^*/k)\}$$

$$+\frac{2\pi K^{\kappa}z}{\kappa+1}B_{\kappa+1}\left(\frac{\nu}{K}\right),$$

and this is equal to

$$\sum_{l=0}^{\infty}\{\lambda((Kl+\nu)^{\kappa}(z-ih/k))+\lambda((Kl+K-\nu)^{\kappa}(z-ih/k))\}$$

$$(9)\qquad +\frac{2\pi K^{\kappa}z}{\kappa+1}B_{\kappa+1}\Big(\frac{\nu}{K}\Big);$$

while the right member of (5) becomes

$$\begin{aligned}&\sum_{s=1}^{\kappa}\sum_{l=0}^{\infty}\{\lambda((\varepsilon_{\kappa,s}/K)\{(l+(\nu^*/k)_s/z\}^{1/\kappa}+i\nu/K)\\ (10)\qquad &\quad+\lambda((\varepsilon_{\kappa,s}/K)\{(l+1-(\nu^*/k)_s/z\}^{1/\kappa}-i\nu/K)\}\\ &\quad+\frac{2\Gamma(1+1/\kappa)}{K(2\pi z)^{1/\kappa}}\sum_{m=1}^{\infty}\frac{\cos(2\pi m\nu^*/k)}{m^{1+1/\kappa}}+2\pi i\Big(\frac{\nu}{K}-\frac{1}{2}\Big)\Big(\frac{\nu^*}{k}-\frac{1}{2}\Big).\end{aligned}$$

Next, we see that

$$(K-\nu)^*=k-\nu^*\qquad(\nu^*\neq 0),$$

and hence

$$(11)\qquad 1-(\nu^*/k)_s=(1-\nu^*/k)_s=((K-\nu)^*/k)_s\qquad(\nu^*\neq 0)$$

by the definition (7) of β_s.

We[1)] also note that the values of ν satisfying $\nu^*=0$, $0<\nu<K$ and $\nu\equiv a$ (mod M) are given by

$$(12)\qquad \xi m_k,\ (\xi+M)m_k,\ (\xi+2M)m_k,\ \cdots,(\xi+(r-1)M)m_k\qquad(r=k/m_k),$$

where ξ and m_k are defined in the lines following (1); and further that the case $\nu^*=0$ may occur if and only if $D=1$.

Now, using (11), we see that (10) becomes, for $\nu^*\neq 0$,

$$\begin{aligned}&\sum_{s=1}^{\kappa}\sum_{l=0}^{\infty}\{\lambda((\varepsilon_{\kappa,s}/K)\{(l+(\nu^*/k)_s)/z\}^{1/\kappa}+i\nu/K)\\ (13)\qquad &\quad+\lambda((\varepsilon_{\kappa,s}/K)\{(l+((K-\nu)^*/k)_s)/z\}^{1/\kappa}+i(K-\nu)/K)\}\\ &\quad+\frac{2\Gamma(1+1/\kappa)}{K(2\pi z)^{1/\kappa}}\sum_{m=1}^{\infty}\frac{\cos(2\pi mh\nu^{\kappa}/k)}{m^{1+1/\kappa}}+2\pi i\Big(\frac{\nu}{K}-\frac{1}{2}\Big)\Big(\frac{\nu^*}{k}-\frac{1}{2}\Big);\end{aligned}$$

while, for $\nu^*=0$, (10) is written in the form

$$\begin{aligned}&\sum_{s=1}^{\kappa}\sum_{l=0}^{\infty}\{\lambda((\varepsilon_{\kappa,s}/K)\{(l+1)/z\}^{1/\kappa}+i\nu/K)\\ (14)\qquad &\quad+\lambda((\varepsilon_{\kappa,s}/K)\{(l+1)/z\}^{1/\kappa}+i(K-\nu)/K)\}\\ &\quad+\frac{2\Gamma(1+1/\kappa)}{K(2\pi z)^{1/\kappa}}\sum_{m=1}^{\infty}m^{-1-1/\kappa}+2\pi i\Big(\frac{\nu}{K}-\frac{1}{2}\Big)\Big(-\frac{1}{2}\Big)\end{aligned}$$

1) Concerning the assertions in this paragraph see Schoenfeld [**6**, p. 882].

$$+\sum_{s:\text{ odd}}\lambda(i\nu/K)+\sum_{s:\text{ even}}\lambda(-i\nu/K).$$

We separate our discussion into two cases according as $D>1$ or $D=1$.

Case (i): $D>1$. By a remark following (12), we have $\nu^*\neq 0$. Equating (9) with (13) and summing up the result over $\nu=a, M+a, \cdots, (k_1-1)M+a$, we obtain

$$\sum_{j=0}^{\infty}\{\lambda((jM+a)^{\kappa}(z-ih/k))+\lambda((jM+M-a)^{\kappa}(z-ih/k))\}$$

$$+\frac{2\pi K^{\kappa}z}{\kappa+1}\sum_{\nu}B_{\kappa+1}\left(\frac{\nu}{K}\right)$$

(15)

$$=\sum_{\substack{0<\nu<K\\ \nu\equiv\pm a(M)}}\sum_{s=1}^{\kappa}\sum_{l=0}^{\infty}\lambda((\varepsilon_{\kappa,s}/K)\{(l+(\nu^*/k)_s)/z\}^{1/\kappa}+i\nu/K)$$

$$+\frac{2\Gamma(1+1/\kappa)}{K(2\pi z)^{1/\kappa}}\sum_{m=1}^{\infty}m^{-1-1/\kappa}\sum_{\nu}\cos(2\pi mh\nu^{\kappa}/k)+2\pi i\sum_{\nu}\left(\frac{\nu}{K}-\frac{1}{2}\right)\left(\frac{\nu^*}{k}-\frac{1}{2}\right)$$

$$\left(\textstyle\sum_{\nu}=\sum_{0<\nu<K,\,\nu\equiv a(M)}\right).$$

Here we can write

(16) $$2\sum_{\nu}\cos(2\pi mh\nu^{\kappa}/k)=\sum_{\substack{0<\nu<K\\ \nu\equiv\pm a(M)}}\exp(2\pi imh\nu^{\kappa}/k),$$

and

$$\sum_{\nu}B_{\kappa+1}\left(\frac{\nu}{K}\right)=\frac{1}{k_1^{\kappa}}B_{\kappa+1}\left(\frac{a}{M}\right)^{2)},$$

so that

(17) $$\frac{2\pi K^{\kappa}z}{\kappa+1}\sum_{\nu}B_{\kappa+1}\left(\frac{\nu}{K}\right)=\frac{2\pi M^{\kappa}z}{\kappa+1}B_{\kappa+1}\left(\frac{a}{M}\right)=2\pi zB_{\kappa}(a, M).$$

Moreover we have

(18) $$(\nu^*/k)_s=\mu_{\nu,s}$$

by (7) and (2). Finally it is easy to see that

$$2\sum_{\nu}\left(\frac{\nu}{K}-\frac{1}{2}\right)\left(\frac{\nu^*}{k}-\frac{1}{2}\right)=\sum_{\substack{0<\nu<K\\ \nu\equiv\pm a(M)}}\left(\frac{\nu}{K}-\frac{1}{2}\right)\left(\frac{\nu^*}{k}-\frac{1}{2}\right)$$

(19)

$$=\sum_{\substack{0<\nu<K\\ \nu\equiv\pm a(M)}}\left(\left(\frac{\nu}{K}\right)\right)\left(\left(\frac{h\nu^{\kappa}}{k}\right)\right)=\sigma_{\kappa}(h, k).$$

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2) This result will be verified by using the Fourier expansion of the Bernoulli polynomial.

Inserting (16)-(19) into (15), we are led to the desired formula (4) with κ odd and $D>1$.

Case (ii): $D=1$. In this case, we must distinguish between the two cases $\nu^*\neq 0$ and $\nu^*=0$. Noticing (13), (14), and using the values given by (12), the equation which is corresponding to (15) is found to be

$$\sum_{j=0}^{\infty}\{\lambda((jM+a)^{\kappa}(z-ih/k))+\lambda((jM+M-a)^{\kappa}(z-ih/k))\}+\frac{2\pi K^{\kappa}z}{\kappa+1}\sum_{\nu}B_{\kappa+1}\Big(\frac{\nu}{K}\Big)$$

$$=\sum_{\substack{0<\nu<K\\ \nu\equiv\pm a(M),\nu^*\neq 0}}\sum_{s=1}^{\kappa}\sum_{l=0}^{\infty}\lambda((\varepsilon_{\kappa,s}/K)\{(l+(\nu^*/k)_s)/z\}^{1/\kappa}+i\nu/K)$$

$$+\sum_{\substack{0<\nu<K\\ \nu\equiv\pm a(M),\nu^*=0}}\sum_{s=1}^{\kappa}\sum_{l=0}^{\infty}\lambda((\varepsilon_{\kappa,s}/K)\{(l+1)/z\}^{1/\kappa}+i\nu/K)$$

(20)

$$+\frac{2\Gamma(1+1/\kappa)}{K(2\pi z)^{1/\kappa}}\sum_{m=1}^{\infty}m^{-1-1/\kappa}\sum_{\nu}\cos(2\pi mh\nu^{\kappa}/k)+2\pi i\sum_{\substack{\nu\\ \nu^*\neq 0}}\Big(\frac{\nu}{K}-\frac{1}{2}\Big)\Big(\frac{\nu^*}{k}-\frac{1}{2}\Big)$$

$$+2\pi i\sum_{q=0}^{r-1}\Big\{\frac{(\xi+qM)m_k}{K}-\frac{1}{2}\Big\}\Big(-\frac{1}{2}\Big)+\frac{\kappa+1}{2}\sum_{q=0}^{r-1}\lambda(i(\xi+qM)m_k/K)$$

$$+\frac{\kappa-1}{2}\sum_{q=0}^{r-1}\lambda(-i(\xi+qM)m_k/K)\,.$$

Here observing that $m_k/K=m_k/(Mk)=1/(Mr)$, we have

$$2\pi i\sum_{q=0}^{r-1}\Big\{\frac{(\xi+qM)m_k}{K}-\frac{1}{2}\Big\}\Big(-\frac{1}{2}\Big)=-\pi i\sum_{q=0}^{r-1}\Big(\frac{\xi+qM}{Mr}-\frac{1}{2}\Big),\tag{21}$$

and

$$\frac{\kappa+1}{2}\sum_{q=0}^{r-1}\lambda(i(\xi+qM)m_k/K)+\frac{\kappa-1}{2}\sum_{q=0}^{r-1}\lambda(-i(\xi+qM)m_k/K)$$

(22)

$$=\frac{\kappa}{2}\sum_{q=0}^{r-1}\Big\{-2\log\Big(2\sin\frac{\pi(\xi+qM)}{Mr}\Big)\Big\}+\frac{1}{2}\sum_{q=0}^{r-1}2\pi i\Big(\frac{\xi+qM}{Mr}-\frac{1}{2}\Big)$$

since

$$\lambda(i\alpha)+\lambda(-i\alpha)=-2\log(2\sin\pi\alpha)\,,$$

$$\lambda(i\alpha)-\lambda(-i\alpha)=2\pi i\Big(\alpha-\frac{1}{2}\Big)\qquad(0<\alpha<1)\,.$$

Using the identity

$$\prod_{q=0}^{r-1}2\sin\frac{\pi(\beta+q)}{r}=2\sin(\pi\beta)\qquad(0<\beta<1)\,,$$

the first sum on the right of (22) reduces to $-\kappa\log(2\sin(\pi\xi/M))$, and the second sum is cancelled by the right member of (21). Furthermore we can use again (16), (17); and the relations (18), (19) still hold with $\nu^*\neq 0$. We thus obtain our formula (4) from (20) in the case $D=1$.

This completes the proof of Theorem 1 for κ odd.

When κ is even, we may start with the equation (6), and the proof will proceed as in the case of odd κ.

The special case $k=1$ of Theorem 1 perhaps deserves explicit mention, since it exhibits the behavior of $F_\kappa(x; a, M)$ near $x=1$. We have

$$F_\kappa(x; a, M)$$
$$=\{2\sin(\pi a/M)\}^{-\kappa}\exp\{-2\pi z B_\kappa(a, M)+2\Gamma(1+1/\kappa)\zeta(1+1/\kappa)/M(2\pi z)^{1/\kappa}\}$$
$$\times\prod_{\nu=a, M-a}\prod_{s=1}^{\kappa}\prod_{l=0}^{\infty}\{1-\exp(-2\pi(\varepsilon_{\kappa,s}/M)\{(l+1)/z\}^{1/\kappa}-2\pi i\nu/M)\}^{-1},$$

where $x=e^{-2\pi z}$ $(\Re(z)>0)$, and $\zeta(t)$ denotes the Riemann zeta-function.

2. Asymptotic properties of the partition function. In a recent paper E. Grosswald [**1**] has treated a certain type of partition functions and derived their asymptotic formulas. The method there used is essentially a saddle point method and is based on the following lemma:

LEMMA ([**1**, p. 121]). *Let* $f(x)=\sum_{n=0}^{\infty}a_n x^n$ *be analytic inside the unit circle, and define the functions* $a(r)=r\cdot d(\log f(r))/dr$ *and* $b(r)=r\cdot da(r)/dr$ *of* $r=|x|$. *Denote by* $\rho=\rho_n$ *the unique root of* $a(\rho)=n$, *and assume that for* $r_0<r<1$, *functions* $\delta(r)>0$ *and* $u(r)$ *exist, with the following properties: As* $n\to\infty$, *one has, for some* $\alpha>0$:

(a) $\displaystyle\int_{|\theta|\geqq\delta(\rho)}|f(\rho e^{i\theta})|\,d\theta=O(n^{-\alpha}f(\rho)b(\rho)^{-1/2})$;

(b) $$\int_{-\delta(\rho)}^{+\delta(\rho)}\Big(f(\rho e^{i\theta})-f(\rho)\exp\Big\{i\theta a(\rho)-\frac{1}{2}\theta^2 b(\rho)\Big\}\Big)e^{-in\theta}d\theta$$
$$=(2\pi)^{1/2}f(\rho)b(\rho)^{-1/2}(u(\rho)+O(n^{-\alpha}));$$

(c) $\delta(\rho)^2 b(\rho)\geqq 2\alpha\log n$.

Then

$$a_n=\rho^{-n}(2\pi b(\rho))^{-1/2}f(\rho)\ (1+u(\rho)+O(n^{-\alpha})).$$

This lemma can also be applied to our generating function $F_\kappa(x; a, M)$ by utilizing Theorem 1 and employing a method similar to that of Grosswald[3].

We can thus conclude the following result, though we omit a detailed

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3) See Grosswald [**1**, pp. 121-124].

derivation.

THEOREM 2. *We have, as* $n\to\infty$,

$$p_1(n;a,M)=\frac{1}{4}\csc(\pi a/M)\cdot(3M)^{-1/4}n^{-3/4}\exp(2\pi(n/3M)^{1/2})$$
$$\times\Big\{1-(M/3)^{1/2}\Big(\frac{9}{16\pi}+\frac{\pi}{2}B_2(a/M)\Big)n^{-1/2}+O(n^{-1})\Big\}, \tag{23}$$

and, for $\kappa\geqq 2$,

$$p_\kappa(n;a,M)=(2\pi)^{-1/2}\{2\sin(\pi a/M)\}^{-\kappa}(1+1/\kappa)^{-1/2}C_\kappa(M)^{1/2}n^{-(2\kappa+1)/(2\kappa+2)}$$
$$\times\exp((1+\kappa)C_\kappa(M)n^{1/(\kappa+1)})\cdot\Big\{1-\frac{(1+2\kappa)(2+\kappa)}{24\kappa(1+\kappa)}C_\kappa(M)^{-1}n^{-1/(\kappa+1)}+O(n^{-2/(\kappa+1)})\Big\}$$

with the abbreviation

$$C_\kappa(M)=\Big(\frac{2\Gamma(1+1/\kappa)\zeta(1+1/\kappa)}{\kappa M}\Big)^{\kappa/(\kappa+1)}.$$

Formula (23) may be obtained as a particular case of Grosswald's formula [**1**, p. 124, formula (17)] if M is a prime number. In fact, we can express $p_1(n;a,M)$ exactly as a convergent infinite series (see [**5**]).

As a direct consequence of Theorem 2, we infer the following

COROLLARY. *For fixed two values* a_1, a_2 *of* a, *we have*

$$\lim_{n\to\infty}(p_\kappa(n;a_1,M):p_\kappa(n;a_2,M))=\{\sin(\pi a_1/M)\}^{-\kappa}:\{\sin(\pi a_2/M)\}^{-\kappa}.$$

Defense Academy.

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