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On fundamental exact sequences.

To Professor Z. Suetuna on his 60th birthday.

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Introduction.

Let $\mathfrak{g}$ be a group, $\mathfrak{h}$ its normal subgroup, and M a $\mathfrak{g}$-module. If $H^1(\mathfrak{h}, M)=0$, then the sequence

$$0 \rightarrow H^2(\mathfrak{g}/\mathfrak{h}, M^{\mathfrak{h}}) \xrightarrow{\lambda} H^2(\mathfrak{g}, M) \xrightarrow{\rho} H^2(\mathfrak{h}, M)$$

is exact, where λ is the lift, ρ the restriction, and $M^{\mathfrak{h}}$ the submodule of M consisting of $\mathfrak{h}$-invariant elements. This is the so-called *fundamental exact sequence*, the importance of which is recognized in connection with the theory of simple algebras and the theory of algebraic number fields. Many interesting and useful generalizations have been made of it: Using the mechanism of spectral sequences Hochschild and Serre obtained in [8] an exact sequence involving the transgression, and in [9] its analogue in the cohomology of Lie algebras. Adamson [1], who initiated the relative cohomology theory of groups with respect to not necessarily normal subgroups, generalized it to the case of non-normal subgroups but missing the transgression, and a recent work of Nakayama [10] proved it in general form with transgression.

Our purpose in the present paper is to prove a quite general proposition of such nature in the framework of the relative homological algebra due to Hochschild [6], and to show that the known results cited above appear as its special cases. Our result is proved in Theorem 1 of §1 in the form without transgression, and in Theorem 2 of §3 in the form with transgression. The proof of Theorem 2 is immediate using Theorem 1, once the subgroup $[\mathrm{Ext}_S(A, B)]^R$ of $\mathrm{Ext}_S(A, B)$ (S is a subring of R) and the notion of transgression are appropriately defined. The introduction of such notions is done in §§2, 3, while §§4, 5 treat explicitly the case of groups and Lie algebras respectively.

In dealing with Ext, we use mainly projective resolutions of contravariant variables, taking account of the adaptation to the usual treatment of cohomology groups of groups and Lie algebras. All our arguments can be dualized to obtain exact sequences of Tor, and specifically of the homology groups of groups, Lie algebras, etc. It is quite plausible that our exact sequences will be deduced from an appropriate spectral sequence.

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§ 1. Lift and restriction.

1.1. Throughout the paper R is a ring with an identity, and S a subring containing the identity. Let A and B be R-modules. Let Y be an R-projective resolution of A and Z an (R, S)-projective resolution of A. Then by definition

$$\mathrm{Ext}_R^n(A, B) = H^n(\mathrm{Hom}_R(Y, B)),$$
$$\mathrm{Ext}_{R,S}^n(A, B) = H^n(\mathrm{Hom}_R(Z, B)).$$

Since Z is acyclic, there exists a map $Y \rightarrow Z$ over the 1_A, the identity homomorphism of A, and it induces canonically homomorphisms of homology groups:

$$\lambda_{R,S}^n: \ \mathrm{Ext}_{R,S}^n(A, B) \rightarrow \mathrm{Ext}_R^n(A, B), \qquad n \geqq 0,$$

which will be called the *lift* homomorphisms (or the *inflation* homomorphisms). λ^0 reduces to the identity of $\mathrm{Hom}_R(A, B)$. Let $0 \rightarrow B_1 \rightarrow B_2 \rightarrow B_3 \rightarrow 0$ be an exact sequence of R-modules. Then the diagram

$$\begin{array}{ccccccccc}
\cdots \rightarrow & \mathrm{Ext}_{R,S}^n(A, B_2) & \rightarrow & \mathrm{Ext}_{R,S}^n(A, B_3) & \xrightarrow{\Delta_{R,S}^n} & \mathrm{Ext}_{R,S}^{n+1}(A, B_1) & \rightarrow & \mathrm{Ext}_{R,S}^{n+1}(A, B_2) & \rightarrow \cdots \\
& \downarrow \lambda & & \downarrow \lambda & & \downarrow \lambda & & \downarrow \lambda & \\
\cdots \rightarrow & \mathrm{Ext}_R^n(A, B_2) & \rightarrow & \mathrm{Ext}_R^n(A, B_3) & \xrightarrow{\Delta_R^n} & \mathrm{Ext}_R^{n+1}(A, B_1) & \rightarrow & \mathrm{Ext}_R^{n+1}(A, B_2) & \rightarrow \cdots
\end{array}$$

is commutative, where Δ in both lines denote the respective connecting homomorphisms.

The *restriction* homomorphisms

$$\rho_{R,S}^n: \ \mathrm{Ext}_R^n(A, B) \rightarrow \mathrm{Ext}_S^n(A, B), \qquad n \geqq 0,$$

are defined similarly. Namely, if X is an S-projective resolution of A, there is a map of S-complexes $X \rightarrow Y$ over 1_A, and it induces ρ. ρ^0 is the natural injection $\mathrm{Hom}_R(A, B) \rightarrow \mathrm{Hom}_S(A, B)$, and ρ commutes with the connecting homomorphism etc., similarly as λ.

By definition, the (R, S)-resolution Z is S-trivial, and it follows that

$$\rho^n \cdot \lambda^n = 0, \qquad n \geqq 1. \tag{1}$$

Put $A^\circ = R \otimes_S A$, the tensor product being taken with respect to the natural right S-module structure of R, and its R-module structure being defined with respect to the left R-module structure of R (the covariant extension of A); let π be the natural epimorphism $R \otimes_S A \to R \otimes_R A = A$, A_* the kernel of π, and κ the natural injection $A_* \to A^\circ$. The resulting exact sequence

$$0 \to A_* \xrightarrow{\kappa} A^\circ \xrightarrow{\pi} A \to 0$$

is S-trivial, since we have an S-homomorphism ι: $A \to A^\circ$ satisfying $\pi \cdot \iota = 1_A$; take for example $\iota(a) = 1 \otimes a$. Since A° is (R, S)-projective, we have

$$\mathrm{Ext}^n_{R,S}(A^\circ, B) = 0, \qquad n \geqq 1,$$

for every B. We denote generally any natural isomorphism of type $\mathrm{Hom}_Q(M, \mathrm{Hom}_P(L, N)) \to \mathrm{Hom}_P(L \otimes_Q M, N)$, where P and Q are rings, N a left P-module, M a left Q-module and L a left P-and right Q-module, by s. For example

$$s^{-1}: \mathrm{Hom}_R(X^\circ, B) \cong \mathrm{Hom}_S(X, \mathrm{Hom}_R(R, B)) = \mathrm{Hom}_S(X, B)\,.$$

Since X° is an R-projective complex over A°, we have a canonical homomorphism $\mathrm{Ext}_R(A^\circ, B) \to H(\mathrm{Hom}_R(X^\circ, B))$. This, followed by the homomorphism induced by the above s^{-1} yields homomorphisms

$$\mu^n: \ \mathrm{Ext}^n_R(A^\circ, B) \to \mathrm{Ext}^n_S(A, B), \qquad n \geqq 0\,.$$

μ^1 is always a monomorphism, since there exists an R-projective resolution $\tilde{Y}$ of A° such that $\tilde{Y}_0 = R \otimes_S X_0$, $\tilde{Y}_1 = R \otimes_S X_1$. Since $H(X^\circ) = \mathrm{Tor}^S(R, A)$, all μ^n $(n \geqq 0)$ are isomorphisms if $\mathrm{Tor}^S_p(R, A) = 0$ $(p \geqq 1)$.

Dually, put $B' = \mathrm{Hom}_S(R, B)$ considered as an R-module by $(rb')(r_1) = b'(r_1 r)$ $(b' \in B', r, r_1 \in R)$ (the contravariant extension of B), denote the natural injection $B = \mathrm{Hom}_R(R, B) \to \mathrm{Hom}_S(R, B)$ by α, the residue class module $B'/\alpha B$ by B^*, and the natural epimorphism $B' \to B^*$ by β. The resulting exact sequence

$$0 \to B \xrightarrow{\alpha} B' \xrightarrow{\beta} B^* \to 0$$

is also S-trivial, an S-homomorphism γ: $B' \to B$ with $\gamma \cdot \alpha = 1_B$ being given by $\gamma(b') = b'(1)$ $(b' \in B')$. Since B' is (R, S)-injective, we have

$$\mathrm{Ext}^n_{R,S}(A, B') = 0, \qquad n \geqq 1,$$

for every A. Now we have a canonical homomorphism $H(\mathrm{Hom}_S(Y, B)) \to \mathrm{Ext}_S(A, B)$ since Y is acyclic. This, combined with the homomorphism induced by the natural isomorphism

$$s: \ \mathrm{Hom}_R(Y, B') \to \mathrm{Hom}_S(R \otimes_R Y, B) = \mathrm{Hom}_S(Y, B)\,,$$

yields homomorphisms

$$\nu^n: \quad \mathrm{Ext}_R^n(A, B') \to \mathrm{Ext}_S^n(A, B), \qquad n \geqq 0.$$

ν is also defined as the composition of homomorphisms

$$\mathrm{Ext}_R(A, B') \xrightarrow{\rho} \mathrm{Ext}_S(A, B') \xrightarrow{\gamma} \mathrm{Ext}_S(A, B)$$

as is easily verified. (A similar fact holds for μ). If R is left S-projective, Y is also an S-projective resolution of B, and ν^n are isomorphisms (In reality we have only to assume $\mathrm{Ext}_S^p(R, B) = 0$ $(p \geqq 0)$).

It is also easily verified that the following diagram is commutative (Cartan-Eilenberg [2, Chap. VI, § 4]):

$$\begin{tikzcd} & \mathrm{Ext}_R^n(A^\circ, B) \arrow[dr, "\mu"] & \\ \mathrm{Ext}_R^n(A, B) \arrow[ur, "\pi"] \arrow[rr, "\rho"] \arrow[dr, "\alpha"'] & & \mathrm{Ext}_S^n(A, B) \\ & \mathrm{Ext}_R^n(A, B') \arrow[ur, "\nu"'] & \end{tikzcd}.$$

1.2. The interpretation of Ext as the equivalence classes of module extensions naturally suggests:

PROPOSITION 1 (Hirata [5]). *The following sequence is exact:*

$$0 \to \mathrm{Ext}_{R,S}^1(A, B) \xrightarrow{\lambda} \mathrm{Ext}_R^1(A, B) \xrightarrow{\rho} \mathrm{Ext}_S^1(A, B).$$

PROOF. With respect to $0 \to A_* \to A^\circ \to A \to 0$, the exact sequence of $\mathrm{Ext}_{R,S}$ is mapped to the exact sequence of Ext_R by the lift homomorphisms. Since λ^0 is always an isomorphism and $\mathrm{Ext}_{R,S}^1(A^\circ, B) = 0$, our λ^1 is a monomorphism by the five lemma. On the other hand, we have, since μ^1 is a monomorphism,

$$\mathrm{Ker}\,\rho^1 = \mathrm{Ker}\,\mu^1\pi^1 = \mathrm{Ker}\,\pi^1 = \mathrm{Im}\,\Delta_R^0 = \mathrm{Im}\,\Delta_R^0\lambda^0 = \mathrm{Im}\,\lambda^1\Delta_{R,S}^0 = \mathrm{Im}\,\lambda^1.$$

Let us define R-modules $B^{(n)}$ $(n \geqq 0)$ recursively as follows:

$$B^{(0)} = B, \qquad B^{(n)} = (B^{(n-1)})' = (B')^{(n-1)} \qquad (n \geqq 1).$$

As the sequence $0 \to B \to B' \to B^* \to 0$ S-splits, the sequence $0 \to B^{(n-1)} \to B^{(n)} \to (B^*)^{(n-1)} \to 0$ also S-splits for every $n \geqq 1$. Hence

$$\mathrm{Ext}_S^p(A, B^{(n)}) \cong \mathrm{Ext}_S^p(A, B^{(n-1)}) + \mathrm{Ext}_S^p(A, (B^*)^{(n-1)}), \quad (p \geqq 0, n \geqq 1). \tag{2}$$

THEOREM 1. *Let R be left S-projective. Let $n \geqq 1$ and assume that* $\mathrm{Ext}_S^p(A, B^{(n-p)}) = 0$ $(0 < p < n)$. *Then the following sequence is exact:*

$$0 \to \mathrm{Ext}_{R,S}^n(A, B) \xrightarrow{\lambda} \mathrm{Ext}_R^n(A, B) \xrightarrow{\rho} \mathrm{Ext}_S^n(A, B).$$

PROOF by induction on n. The case $n = 1$ is proved above. Let $n > 1$.

It follows from the assumptions that $\mathrm{Ext}_S^p(A, B^{*(n-1-p)})=0$ $(0<p<n-1)$, $\mathrm{Ext}_S^{n-1}(A, B^*)=0$ and $\mathrm{Ext}_S^{n-1}(A, B)=0$. By the first of these, the induction assumption applies to B^*: the sequence

$$0 \to \mathrm{Ext}_{R,S}^{n-1}(A, B^*) \xrightarrow{\lambda^*} \mathrm{Ext}_R^{n-1}(A, B^*) \to \mathrm{Ext}_S^{n-1}(A, B^*)=0$$

is exact, namely λ^* is an isomorphism. Now the kernel of λ: $\mathrm{Ext}_{R,S}^n(A, B) \to \mathrm{Ext}_R^n(A, B)$ is $\Delta_{R,S}(\mathrm{Ker}(\Delta_R\lambda^*))$, and this reduces to 0, since Δ_R has the kernel $\beta\mathrm{Ext}_R^{n-1}(A, B')=\beta\nu^{-1}\mathrm{Ext}_S^{n-1}(A, B)=0$ (remark that ν is an isomorphism by the S-projectivity of R). On the other hand, the kernel of $\rho=\nu\alpha$: $\mathrm{Ext}_R^n(A, B) \to \mathrm{Ext}_S^n(A, B)$ coincides with

$$\mathrm{Ker}\,\alpha=\mathrm{Im}\,\Delta_R=\mathrm{Im}(\Delta_R\lambda^*)=\mathrm{Im}(\lambda\Delta_{R,S})=\mathrm{Im}\,\lambda\,.$$

REMARK. Similarly we can prove the exactness of the above sequence under the asssumptions that R is right S-flat and $\mathrm{Ext}_S^p(A_{(n-p)}, B)=0$ $(0<p<n)$, where $A_{(n)}$ is defined recursively by $A_{(0)}=A$, $A_{(n)}=(A_{(n-1)})^\circ$ $(n\geqq 1)$.

AN APPLICATION (Inequalities of Hochschild [**7**]). a) Assume that R is right S-flat, and that the left global dimension of S is finite, say $\mathrm{l.gl.dim}\,S=n$. For an R-module B let

$$0 \to B \to Q_1 \to \cdots \to Q_n \to C \to 0 \tag{3}$$

be an exact sequence of R-modules where Q_i are R-injective. Then Q_i are also S-injective since R is S-flat, hence C is also S-injective by $\mathrm{l.gl.dim}\,S=n$. Hence, for every R-module A,

$$\mathrm{Ext}_R^{n+k}(A, B) \cong \mathrm{Ext}_R^k(A, C) \cong \mathrm{Ext}_{R,S}^k(A, C), \qquad (k\geqq 1)\,,$$

and we have

$$\dim_R A \leqq \dim_{R,S} A + \mathrm{l.gl.dim}\,S\,.$$

b) Assume furthermore that R is left S-projective. Then an R-projective module is S-projective, so that $\dim_S A \leqq \dim_R A$ for every R-module A. Now, assume that $\dim_S A=n<\infty$. Then $\mathrm{Ext}_S^p(A, C) \cong \mathrm{Ext}_S^{n+p}(A, B)=0$ for $p>0$. Hence the above theorem applies to (A, C), and we have similarly as in the case a)

$$\dim_R A \leqq \dim_{R,S} A + \dim_S A\,.$$

1.3. Let R be a *supplemented algebra* over a commutative ring K with an identity, and ε be its supplementation epimorphism $R \to K$. Then the cohomology groups $H^n(R, B)$ of R with coefficients in an R-module B are defined as $\mathrm{Ext}_R^n(K, B)$, K being considered as an R-module via ε (Cartan-Eilenberg [**2**, Chap. X]). An R-projective resolution of K is provided by the so-called (normalized) standard complex $Y=N(R, \varepsilon)$: Y_n consists of R-linear combinations of K-mu ti-linear symbols $[r_1, \cdots, r_n]$, $r_i \in R$. (If one of r_i is in

K, $[r_1, \cdots, r_n]=0$.) The augmentation $\eta: Y_0 \to K$ is defined by $\eta[\ ]=1$, and the differentiation by

$$d_n[r_1, \cdots, r_n] = r_1[r_2, \cdots, r_n] + \sum_{i=1}^{n-1} (-1)^i [r_1, \cdots, r_i r_{i+1}, \cdots, r_n] + (-1)^n [r_1, \cdots, r_{n-1}](\varepsilon r_n).$$

Let S be a subalgebra of R containing K. Then S itself is a supplemented algebra over K, and a projective resolution of the S-module K is given by $X=N(S, \varepsilon)$. It is clear that the correspondence $\rho: [s_1, \cdots, s_n] \to [s_1, \cdots, s_n]$ $(s_i \in S)$ is a map of S-complexes $X \to Y$ over 1_K.

Now, the standard (R, S)-projective resolution Z of K is defined as follows (Hochschild [**6**, § 2]): Define A_{*^n} recursively by $A_{*^0}=A$, $A_{*^n}=(A_{*^{n-1}})_*$ $(n>0)$ for any R-module A, and put $Z_n = R \otimes_S K_{*^n}$; then $K_{*^{n+1}}$ is the kernel of the natural epimorphism $\pi: Z_n \to K_{*^n}$. The augmentation η is the natural epimorphism $\pi: R \otimes_S K \to K$, and the differentiation d_n is the composition $R \otimes_S K_{*^n} \to K_{*^n} \to R \otimes_S K_{*^{n-1}}$. We define operators $q_r: A \to A_*(r \in R)$ by

$$q_r(a) = r \otimes a - 1 \otimes ra, \qquad a \in A.$$

Then $q_{r_1} \cdots q_{r_n}(1) \in K_{*^n}$ for $r_1, \cdots, r_n \in R$. Using this operator, we define an R-homomorphism $\lambda: Y \to Z$ by

$$\lambda[r_1, \cdots, r_n] = 1 \otimes q_{r_1} \cdots q_{r_n}(1).$$

We shall show that λ is a map of R-complexes over 1_K. Clearly,

$$\eta\lambda[\ ] = \eta(1 \otimes 1) = 1 = \eta[\ ].$$

Since $d_n \lambda[r_1, \cdots, r_n] = q_{r_1} \cdots q_{r_n}(1)$, we have to show

$$\lambda d_n[r_1, \cdots, r_n] = q_{r_1} \cdots q_{r_n}(1) \qquad (n \geqq 1).$$

This is true for $n=1$, since

$$\lambda d_1[r] = \lambda((r-\varepsilon(r))[\ ]) = (r-\varepsilon(r)) \otimes 1 = q_r(1).$$

Let $n>1$ and assume it is true for $n-1$. We have

$$\begin{aligned} &\lambda d_n[r, \cdots, r_n] - q_{r_1} \cdots q_{r_n}(1) \\ &= 1 \otimes [r_1 q_{r_2} \cdots q_{r_n}(1) + \sum_{i=1}^{n-1} (-1)^i q_{r_1} \cdots q_{r_i r_{i+1}} \cdots q_{r_n}(1) \\ &\qquad + (-1)^n q_{r_1} \cdots q_{r_{n-1}}(1) \varepsilon(r_n)]. \end{aligned}$$

But by the induction assumption, the right hand side is $1 \otimes \lambda d_{n-1} d_n [r_1, \cdots, r_n] = 0$. Hence λ is a map $Y \to Z$ over 1_K. Since $q_s(a)=0$ for $s \in S$, we have

$$\lambda_n \cdot \rho_n = 0 \qquad (n \geqq 1).$$

Namely, *using standard complexes,* (1) *holds not only for cohomology classes, but also for individual cocycles.*

REMARK. Define operators p_r $(r \in R)$ in the complex Y, by

$$p_r(1) = (r - \varepsilon(r))[\ ],$$

$$p_r(r_0[r_1, \cdots, r_n]) = r[r_0, r_1, \cdots, r_n] - [rr_0, r_1, \cdots, r_n],$$

then, we see easily

$$[r_1, \cdots, r_n] = (-1)^n p_{r_1} \cdots p_{r_n}[\ ],$$

$$d[r_1, \cdots, r_n] = p_{r_1} \cdots p_{r_n}(1),$$

and the fact that λ is a map is formulated in the form

$$\lambda p_{r_1} \cdots p_{r_n}(1) = q_{r_1} \cdots q_{r_n}(1).$$

Now, we assume that S is a *normal* subalgebra of R in the sense of Cartan-Eilenberg [**2**, Chap. XVI, § 6]. Let I_S be the kernel of the supplementation $S \to K$, and denote by B^S the submodule of B consisting of S-*invariant* elements, namely the elements annihilated by I_S. Then Hochschild [**6**, § 6] proves that

$$\mathrm{Ext}_{R,S}(K, B) \cong \mathrm{Ext}_{R'}(K, B^S),$$

provided the ring $R' = R \otimes_S K = R/RI_S$ is K-projective. In this case, it is clear that the above-defined λ coincides with the usual lift homomorphism.

§ 2. $[\mathrm{Ext}_S^n(A, B)]^R$.

Let S and T be subrings of R, and assume that R is left T-projective. Let A and B be R-modules and assume that $\mathrm{Tor}_n^S(R, A) = 0$ $(n \geqq 1)$.

The projection $\pi: R \otimes_S A \to A$ and the injection $\alpha: B \to \mathrm{Hom}_T(R, B)$ induce respectively

$$\pi: \mathrm{Ext}_T(A, B) \to \mathrm{Ext}_T(R \otimes_S A, B),$$

$$\alpha: \mathrm{Ext}_S(A, B) \to \mathrm{Ext}_S(A, \mathrm{Hom}_T(R, B)).$$

By assumptions, $R \otimes_S X$ is a T-projective resolution of $R \otimes_S A$, if X is an S-projective resolution of A, and the natural isomorphism

$$s: \mathrm{Hom}_S(X, \mathrm{Hom}_T(R, B)) \to \mathrm{Hom}_T(R \otimes_S X, B)$$

induces an isomorphism of Ext:

$$s: \mathrm{Ext}_S(A, \mathrm{Hom}_T(R, B)) \to \mathrm{Ext}_T(R \otimes_S A, B).$$

$\mathrm{Ext}_T(A, B)$ and $\mathrm{Ext}_S(A, B)$ are mapped in these isomorphic groups by π and α respectively.

Now, let $S = T$. We shall say that $F \in \mathrm{Ext}_S(A, B)$ is *stable* with respect to R, if $\pi F = s\alpha F$. For $f \in \mathrm{Hom}_S(X, B)$, $fd = 0$, we have

$$\text{(4)} \qquad \begin{aligned} f\pi(r \otimes x) &= f(rx), \\ s\alpha f(r \otimes x) &= rf(x), \end{aligned} \qquad (r \in R,\ x \in X).$$

The stable elements form a subgroup of $\mathrm{Ext}_S(A, B)$, which we shall denote by $[\mathrm{Ext}_S(A, B)]^R$. Since R is assumed to be left S-projective, we can take as X an R-projective resolution of A, and it follows that $\rho G (G \in \mathrm{Ext}_R^n(A, B))$ is always stable, since $f\pi = s\alpha f$ if and only if $f \in \mathrm{Hom}_R(X, B)$.

Assuming that R is right S-flat and $\mathrm{Ext}_R^n(R, B) = 0$ $(n \geqq 0)$, we can define similarly a stability of elements of $\mathrm{Ext}_S(A, B)$. If R is left S-projective and also right S-flat, then these two notions coincide, as is seen using double complexes.

Since R is left S-projective, ν^n are isomorphisms, and it is immediately verified that $\rho\nu^{-1}F = s^{-1}\pi F$, where ρ denotes the restriction $\mathrm{Ext}_R^n(A, B') \to \mathrm{Ext}_S^n(A, B')$. Hence *$F$ is stable if and only if*

$$\alpha F = \rho\nu^{-1}F. \tag{5}$$

(We could define the stability of F by the equality (5), only assuming that ν^n are isomorphisms.) The condition (5) is equivalent to $\rho\nu^{-1}F \in \mathrm{Im}(\alpha)$. Indeed, if $\rho\nu^{-1}F = \alpha F_1$, we have $F = \gamma\rho\nu^{-1}F = \gamma\alpha F_1 = F_1$. Since $\mathrm{Im}(\alpha) = \mathrm{Ker}(\beta)$, and $\beta\rho = \rho\beta$, this condition is also written in the form

$$\rho\beta\nu^{-1}F = 0, \tag{5'}$$

ρ denoting here $\mathrm{Ext}_R^n(A, B^*) \to \mathrm{Ext}_S^n(A, B^*)$.

§ 3. Transgression.

We assume for a moment that there are given S-, R-, (R, S)-projective resolutions of an R-module A, say X, Y, Z respectively, together with the S-map $\rho: X \to Y$ and R-map $\lambda: Y \to Z$, both over 1_A, satisfying $\lambda \cdot \rho = 0$, as in the case of supplemented algebras (§ 1.3).

Let H be an element in the kernel of the lift

$$\lambda^{n+1}: \mathrm{Ext}_{R,S}^{n+1}(A, B) \to \mathrm{Ext}_R^{n+1}(A, B), \qquad n \geqq 1,$$

and let H be represented by $h \in \mathrm{Hom}_R(Z_{n+1}, B)$, $hd = 0$. Put

$$h\lambda = -gd, \qquad g \in \mathrm{Hom}_R(Y_n, B).$$

Then $g\rho d = gd\rho = -h\lambda\rho = 0$. Hence

$$f = g \cdot \rho$$

determines an element F of $\mathrm{Ext}_S^n(A, B)$. We shall show that *F is stable*, assuming that ν^n is an isomorphism. Since the connecting homomorphism $\Delta_{R,S}^n$ is an isomorphism, there exist $h^* \in \mathrm{Hom}_R(Z_n, B^*)$ and $h' \in \mathrm{Hom}_R(Z_n, B')$ such that

$$h^*d = 0, \qquad \beta h' = h^*, \qquad h'd = \alpha h.$$

Put $g' = h'\lambda + \alpha g \in \mathrm{Hom}_R(Y_n, B')$. Then we have

$$g'd = h'\lambda d + \alpha g d = \alpha h \lambda - \alpha h \lambda = 0 .$$

Let G' be the class of the cocycle g'. Then $\alpha F = \rho G'$ since

$$g'\rho = h'\lambda\rho + \alpha g \rho = \alpha f ,$$

and $F = \gamma\alpha F = \gamma\rho G' = \nu G'$. Hence F is stable as the condition (5) holds.

Any other possible choice of h and g determines an element congruent to F mod $\rho\,\mathrm{Ext}_R^n(A, B)$. Hence we have a homomorphism

$$\mathrm{Ker}\,\lambda^{n+1} \to [\mathrm{Ext}_S^n(A, B)]^R / \rho\,\mathrm{Ext}_R^n(A, B) . \tag{6}$$

One essential point in establishing the fundamental exact sequence is the observation that in certain circumstances this homomorphism is in fact an isomorphism.

Return to the general case, and assume that ν^n is an isomorphism and also that the sequence

$$0 \to \mathrm{Ext}_{R,S}^n(A, B^*) \xrightarrow{\lambda} \mathrm{Ext}_R^n(A, B^*) \xrightarrow{\rho} \mathrm{Ext}_S^n(A, B^*)$$

is exact. Let F be a stable element of $\mathrm{Ext}_S^n(A, B)$; then $\rho\beta\nu^{-1}F = 0$ by (5′). By the exactness of the above sequence, there exists a unique $H^* \in \mathrm{Ext}_{R,S}^n(A, B^*)$ such that

$$\beta\nu^{-1}F = \lambda H^*$$

and H^* determines an element $H = \Delta_{R,S}^n H^* \in \mathrm{Ext}_{R,S}^{n+1}(A, B)$. We shall call the map $F \to H$ the *transgression* homomorphism, denoting it by τ^n:

$$\tau^n F = \Delta_{R,S}^n (\lambda^n)^{-1} \beta^n (\nu^n)^{-1} F, \qquad F \in [\mathrm{Ext}_S^n(A, B)]^R . \tag{7}$$

It follows immediately from the definition the exactness of the sequence

$$\mathrm{Ext}_R^n(A, B) \xrightarrow{\rho} [\mathrm{Ext}_S^n(A, B)]^R \xrightarrow{\tau} \mathrm{Ext}_{R,S}^{n+1}(A, B) \xrightarrow{\lambda} \mathrm{Ext}_R^{n+1}(A, B) . \tag{8}$$

Indeed, $\tau F = 0$ is equivalent to $\beta\nu^{-1}F = 0$, namely to $\nu^{-1}F \in \mathrm{Im}(\alpha)$. Applying ν, this is equivalent to $F \in \mathrm{Im}(\nu\alpha) = \mathrm{Im}(\rho)$. On the other hand $\lambda H = 0$ is equivalent to $\Delta_R \lambda \Delta_{R,S}^{-1} H = 0$, namely to $\lambda\Delta_{R,S}^{-1} H \in \mathrm{Im}(\beta) = \mathrm{Im}(\beta\nu^{-1})$. Put $\lambda\Delta_{R,S}^{-1}H = \beta\nu^{-1}F$, and apply ρ, then we have $0 = \rho\beta\nu^{-1}F$. Hence F is stable and $H = \tau F$.

Now we prove

THEOREM 2. *Let R be left S-projective. Let $n \geqq 1$ and assume that* $\mathrm{Ext}_S^p(A, B^{(n+1-p)}) = 0$ $(0 < p < n)$. *Then the transgression homomorphism is defined, and the following sequence is exact:*

$$0 \to \mathrm{Ext}_{R,S}^n(A, B) \xrightarrow{\lambda} \mathrm{Ext}_R^n(A, B) \xrightarrow{\rho} [\mathrm{Ext}_S^n(A, B)]^R$$

$$\xrightarrow{\tau} \mathrm{Ext}_{R,S}^{n+1}(A, B) \xrightarrow{\lambda} \mathrm{Ext}_R^{n+1}(A, B) .$$

PROOF. By assumptions together with (2), we have $\mathrm{Ext}_S^p(A, B^{(n-p)})=0$ $(0<p<n)$ and $\mathrm{Ext}_S^p(A, B^{*(n-p)})=0$ $(0<p<n)$. By Theorem 1, the following two sequences are exact:

$$0 \to \mathrm{Ext}^n_{R,S}(A, B) \xrightarrow{\lambda} \mathrm{Ext}^n_R(A, B) \xrightarrow{\rho} \mathrm{Ext}^n_S(A, B),$$

$$0 \to \mathrm{Ext}^n_{R,S}(A, B^*) \xrightarrow{\lambda} \mathrm{Ext}^n_R(A, B^*) \xrightarrow{\rho} \mathrm{Ext}^n_S(A, B^*).$$

It follows from the exactness of the latter sequence, that τ can be defined and that (8) is exact. q. e. d.

To this theorem similar remarks apply as to Theorem 1.

In the presence of such resolutions X, Y, Z and maps ρ, λ as at the beginning of this paragraph, *τ is in fact the inverse of the homomorphism* (6). To see this, let $G'=\nu^{-1}F$ be represented by $g' \in \mathrm{Hom}_R(Y_n, B')$, $g'd=0$. Then H^* is represented by h^* with the property $h^*\lambda \sim \beta g'$, and then $H=\tau F$ by h, where $\alpha h=h'd$, $\beta h'=h^*$. Let $h^*\lambda=\beta g'+g^*d$ with $g^* \in \mathrm{Hom}_R(Y_{n-1}, B^*)$, and take $g_0' \in \mathrm{Hom}_R(Y_{n-1}, B')$ such that $g^*=\beta g_0'$. Then $\beta h'\lambda=\beta g'+\beta g_0'd$, so there exists $g \in \mathrm{Hom}_R(Y_n, B)$ such that

$$h'\lambda=g'+g_0'd-\alpha g,$$

and we have

$$\alpha h\lambda=h'd\lambda=h'\lambda d=-\alpha g d,$$

whence $h\lambda=-gd$. Now, $F=\nu G'$ is represented by f such that

$$f(x)=(g'+g_0'd)\rho(x)(1)=(h'\lambda+\alpha g)\rho(x)(1)=\alpha g\rho(x)(1)=g\rho(x).$$

namely $f=g\cdot\rho$. Thus our assertion is proved.

REMARK. Following an idea of Massey, Takasu [**11**] developed another type of relative theory for supplemented algebras: Using the absolute Ext, he put

$$H^n(R, S: B)=\mathrm{Ext}^{n-1}_R(K_*, B), \qquad (n \geqq 1)$$

where R, S, B, K_* are the same as in § 1, 1.3. It arises the question of the relationship of two relative theories. We have always the following homomorphism

$$\mathrm{Ext}^{n+1}_{R,S}(A, B) \xrightarrow{\Delta^{-1}} \mathrm{Ext}^n_{R,S}(A_*, B) \xrightarrow{\lambda} \mathrm{Ext}^n_R(A_*, B). \tag{9}$$

If we assume $\mathrm{Tor}^S_p(R, A)=0$ $(p \geqq 1)$, μ^n are isomorphisms, and the exact sequence of Ext belonging to the sequence $0 \to A_* \to A^\circ \to A \to 0$ can be modified to the form

$$\cdots \to \mathrm{Ext}^n_R(A, B) \xrightarrow{\rho} \mathrm{Ext}^n_S(A, B) \to \mathrm{Ext}^n_R(A_*, B) \xrightarrow{\Delta} \mathrm{Ext}^{n+1}_R(A, B) \to \cdots \tag{10}$$

Now, when the above homomorphism (9) is monomorphic, a new exact sequence will be deduced from (10) if we replace $\mathrm{Ext}_R^n(A_*, B)$ by $\mathrm{Ext}_{R,S}^{n+1}(A, B)$, and accordingly $\mathrm{Ext}_S^n(A, B)$ by its appropriate subgroup. It is easy to see that this new sequence is precisely the sequence of Theorem 2.

§ 4. Groups.

Let $\mathfrak{g}$ be a group, $\mathfrak{h}$ a subgroup of $\mathfrak{g}$, and let $R = Z(\mathfrak{g})$, $S = Z(\mathfrak{h})$ be their groupalgebras over the ring of integers. R is both left and right S-free.

For a (left) $\mathfrak{h}$-module M and an element $\tau \in \mathfrak{g}$, we denote by M_τ the same M considered as an $\mathfrak{h}^\tau$-module ($\mathfrak{h}^\tau = \tau\mathfrak{h}\tau^{-1}$) by putting $\eta^\tau(m) = \eta m$ ($\eta \in \mathfrak{h}, m \in M$). For an $\mathfrak{h}$-module X, we have $Z(\mathfrak{h})_\tau \otimes_{\mathfrak{h}} X \cong X_\tau$, where the tensor product is taken with respect to the natural right $\mathfrak{h}$-module structure of $Z(\mathfrak{h})$. If M is given by the restriction of operator domain from a $\mathfrak{g}$-module M, then the $\mathfrak{h}^\tau$-module M_τ is $\mathfrak{h}^\tau$-isomorphic with the $\mathfrak{h}^\tau$-module M obtained by the restriction of operators, by the correspondence $a_\tau \rightleftarrows \tau a$.

Let $\mathfrak{k}$ be another subgroup of $\mathfrak{g}$, and let

$$\mathfrak{g} = \bigcup_i \mathfrak{k}\tau_i\mathfrak{h}$$

be the double coset decomposition of $\mathfrak{g}$ with respect to $\mathfrak{k}$ and $\mathfrak{h}$. Then we have

$$Z(\mathfrak{g}) \cong \sum_i Z(\mathfrak{k}) \otimes_{\mathfrak{k}\cap\mathfrak{h}^{\tau_i}} Z(\mathfrak{h})_{\tau_i}, \qquad ((\mathfrak{k}, \mathfrak{h})\text{-isomorphism})$$

by the correspondence

$$\kappa\tau_i\eta \rightleftarrows \kappa \otimes_i \eta, \qquad (\kappa \in \mathfrak{k},\ \eta \in \mathfrak{h}).$$

It follows that for an $\mathfrak{h}$-module A and a $\mathfrak{k}$-module B

$$\begin{aligned} \mathrm{Hom}_{\mathfrak{k}}(Z(\mathfrak{g}) \otimes_{\mathfrak{h}} X, B) &\cong \prod_i \mathrm{Hom}_{\mathfrak{k}}(Z(\mathfrak{k}) \otimes_{\mathfrak{k}\cap\mathfrak{h}^{\tau_i}} X_{\tau_i}, B) \\ &\cong \prod_i \mathrm{Hom}_{\mathfrak{k}\cap\mathfrak{h}^{\tau_i}}(X_{\tau_i}, B), \end{aligned}$$

$\prod$ denoting the Cartesian product, where X is an $\mathfrak{h}$-projective resolution of A. If furthermore A is a $\mathfrak{g}$-module, we take as X a $\mathfrak{g}$-projective resolution of A. Then X_{τ_i} is isomorphic to X as $\mathfrak{h}^{\tau_i}$-resolution of A. Hence *we have the following direct product decomposition*

(11) $$\mathrm{Ext}_{\mathfrak{k}}(Z(\mathfrak{g}) \otimes_{\mathfrak{h}} A, B) \cong \prod_i \mathrm{Ext}_{\mathfrak{k}\cap\mathfrak{h}^{\tau_i}}(A, B),$$

for a $\mathfrak{g}$*-module* A *and a* $\mathfrak{k}$*-module* B.

Let both A and B be $\mathfrak{g}$-modules. Then the τ_i-component of πF, $F \in \mathrm{Ext}_{\mathfrak{k}}(A, B)$, is $\rho_{\mathfrak{k},\mathfrak{k}\cap\mathfrak{h}^{\tau_i}}F$:

$$\pi = \prod_i \rho_{\mathfrak{k},\mathfrak{k}\cap\mathfrak{h}^{\tau_i}},$$

since for $f \in \mathrm{Hom}_{\mathfrak{k}}(X, B), fd = 0$, the τ_i-component $(f\pi)_i \in \mathrm{Hom}_{\mathfrak{k}\cap\mathfrak{h}^{\tau_i}}(X, B)$ of $f\pi$ is given by

$$(f\pi)_i(x) = f\pi(\tau_i \otimes \tau_i^{-1}x) = f(x)\,.$$

While the τ_i-component of $s\alpha f$, for $f \in \mathrm{Hom}_{\mathfrak{h}}(X, B), fd = 0$, is

$$(s\alpha f)_i(x) = s\alpha f(\tau_i \otimes \tau_i^{-1}x) = \tau_i f(\tau_i^{-1}x)\,.$$

Since $f \to \tau_i f \tau_i^{-1}$ induces the isomorphism $I_{\tau_i}: \mathrm{Ext}_{\mathfrak{h}}(A, B) \to \mathrm{Ext}_{\mathfrak{h}^{\tau_i}}(A, B)$, the τ_i-component of $s\alpha F$ is $\rho_{\mathfrak{h}^{\tau_i},\mathfrak{k}\cap\mathfrak{h}^{\tau_i}} I_{\tau_i} F$:

$$s\alpha = \prod_i \rho_{\mathfrak{h}^{\tau_i},\mathfrak{k}\cap\mathfrak{h}^{\tau_i}} I_{\tau_i}\,.$$

Therefore, $F \in \mathrm{Ext}_{\mathfrak{h}}(A, B)$ *is stable if and only if*

$$\rho_{\mathfrak{h},\mathfrak{h}\cap\mathfrak{h}^{\tau}} F = \rho_{\mathfrak{h}^{\tau},\mathfrak{h}\cap\mathfrak{h}^{\tau}} I_{\tau} F \qquad \textit{for every } \tau \in \mathfrak{g}\,.$$

(Remark that the choice of the representatives τ_i in the decomposition $\mathfrak{g} = \bigcup \mathfrak{h}\tau_i\mathfrak{h}$ is arbitrary.) Thus, our definition of stability coincides with that of Cartan-Eilenberg [**2**, Chap. XII, § 9]. If $\mathfrak{h}$ is a normal subgroup of $\mathfrak{g}$, then the stable elements are precisely the $\mathfrak{g}$-invariant elements.

By successive applications of (11) we have

$$\mathrm{Ext}^p_{\mathfrak{h}}(A, B^{(q)}) \cong \prod_{\sigma_i} \mathrm{Ext}^p_{\mathfrak{h}\cap\mathfrak{h}^{\sigma_1}\cap\cdots\cap\mathfrak{h}^{\sigma_q}}(A, B)\,,$$

where $\{\sigma_1, \cdots, \sigma_q\}$ runs over the q-fold direct power of the set $\{\tau_i\}$. Noting that the choice of $\{\tau_i\}$ is arbitrary, Theorems 1 and 2 are stated as follows:

PROPOSITION 2. *Assume that* $\mathrm{Ext}^p_{\mathfrak{h}\cap\mathfrak{h}^{\sigma_1}\cap\cdots\cap\mathfrak{h}^{\sigma_{n-p}}}(A, B) = 0$ $(0 < p < n)$ *where* $\sigma_i \in \mathfrak{g}$ *are arbitrary, then the sequence*

$$0 \to \mathrm{Ext}^n_{\mathfrak{g},\mathfrak{h}}(A, B) \xrightarrow{\lambda} \mathrm{Ext}^n_{\mathfrak{g}}(A, B) \xrightarrow{\rho} \mathrm{Ext}^n_{\mathfrak{h}}(A, B)$$

is exact. If further $\mathrm{Ext}^p_{\mathfrak{h}\cap\mathfrak{h}^{\sigma_1}\cap\cdots\cap\mathfrak{h}^{\sigma_{n-p+1}}}(A, B) = 0$ $(0 < p < n)$ *for every* $\sigma_i \in \mathfrak{g}$, *the sequence*

$$0 \to \mathrm{Ext}^n_{\mathfrak{g},\mathfrak{h}}(A, B) \xrightarrow{\lambda} \mathrm{Ext}^n_{\mathfrak{g}}(A, B) \xrightarrow{\rho} [\mathrm{Ext}^n_{\mathfrak{h}}(A, B)]^s$$
$$\xrightarrow{\tau} \mathrm{Ext}^{n+1}_{\mathfrak{g},\mathfrak{h}}(A, B) \xrightarrow{\lambda} \mathrm{Ext}^{n+1}_{\mathfrak{g}}(A, B)$$

is exact.

Put $A = Z$. Then $\mathrm{Ext}^n_{\mathfrak{g}}(Z, B) = H^n(\mathfrak{g}, B)$ etc., and the exactness of above sequences is proved by Adamson [**1**] and Nakayama [**10**] respectively.

It $\mathfrak{h}$ is a normal subgroup, the assumptions of the Proposition reduce to $\mathrm{Ext}^p_{\mathfrak{h}}(A, B) = 0$ $(0 < p < n)$. For $A = Z$, we have $\mathrm{Ext}^n_{\mathfrak{g},\mathfrak{h}}(Z, B) \cong \mathrm{Ext}^n_{\mathfrak{g}/\mathfrak{h}}(Z, B^{\mathfrak{h}})$. The exact sequence in this case is obtained by Hochschild and Serre [**8**].

§ 5. Lie algebras.

Let $\mathfrak{g}$ be a Lie algebra over a commutative ring K with an identity, $\mathfrak{h}$ an ideal of $\mathfrak{g}$, and assume that $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{g}/\mathfrak{h}$ all have K-bases. Let $R=U(\mathfrak{g})$, $S=U(\mathfrak{h})$, the universal enveloping algebras of $\mathfrak{g}$ and $\mathfrak{h}$. If $\{v_\nu\}$ is an ordered basis of some complementary space of $\mathfrak{h}$ in $\mathfrak{g}$, then, denoting

$$v^E=\prod_\nu v_\nu{}^{e_\nu}$$

for an ordered sequence $E=\{e_\nu\}$ of non-negative integers for which $|E|=\sum e_\nu$ is finite, we have by Birkhoff-Witt theorem

$$U(\mathfrak{g})=\sum_E U(\mathfrak{h})v^E=\sum_E v^E U(\mathfrak{h})\,,$$

where the sum is direct and is taken over all possible E.

We write $F\leqq E$ for two sequences $E=\{e_\nu\}$ and $F=\{f_\nu\}$, if we have $f_\nu\leqq e_\nu$ for every ν; and in this case we can define $E-F=\{e_\nu-f_\nu\}$. Define also the generalized binomial coefficients

$$\binom{E}{F}=\sum_\nu\binom{e_\nu}{f_\nu}=\binom{E}{E-F}\,.$$

With these notations, we shall prove as a preliminary the following relations

$$\text{(12)}\quad \sum_{G\leqq F\leqq E}\binom{E}{F}\binom{F}{G}v^{F-G}(v^{E-F})_*=\sum_{G\leqq F\leqq E}\binom{E}{F}\binom{F}{G}(v^{F-G})_*v^{E-F}=\begin{cases}1 & (G=E),\\ 0 & (G\neq E),\end{cases}$$

where $*$ denotes the anti-automorphism of $U(\mathfrak{g})$ which coincides with $x\to -x$ on $U^1(\mathfrak{g})=\mathfrak{g}$. This is clear for $G=E$. Let $G\neq E$, and say $E=\{e_\nu\}$, $F=\{f_\nu\}$, $G=\{g_\nu\}$. If s is the last index for which $e_s>g_s$, the expression

$$\cdots v_{s-1}^{\,f_{s-1}-g_{s-1}}\,v_s^{\,e_s-g_s}\,v_{s-1}^{\,e_{s-1}-f_{s-1}}\cdots$$

appears in the left hand side of (12) with the multiplicity

$$\pm\sum_{\nu\neq s}\binom{e_\nu}{f_\nu}\binom{f_\nu}{g_\nu}\left\{\binom{e_s}{e_s}\binom{e_s}{g_s}-\binom{e_s}{e_s-1}\binom{e_s-1}{g_s}+\cdots\pm\binom{e_s}{g_s}\binom{g_s}{g_s}\right\}$$

$$=\pm\sum_{\nu\neq s}\binom{e_\nu}{f_\nu}\binom{f_\nu}{g_\nu}\cdot\binom{e}{g_s}\left\{\binom{e_s-g_s}{0}-\binom{e_s-g_s}{1}+\cdots\pm\binom{e_s-g_s}{e_s-g_s}\right\}=0\,.$$

We also remark that, since $\mathfrak{h}$ is an ideal, $\mathfrak{h}$ is viewed as a (left) $U(\mathfrak{g})$-module. If we denote the operation of $u\in U(\mathfrak{g})$ to $x\in\mathfrak{h}$ by $u(x)$, we verify easily by induction on $|E|$:

$$\text{(13)}\qquad v^E x=\sum_{F\leqq E}\binom{E}{F}v^{E-F}(x)\cdot v^F,\qquad (x\in\mathfrak{h})\,.$$

For $\mathfrak{g}$-modules A and B, $\mathrm{Hom}_K(A, B)$ is viewed as a $\mathfrak{g}$-module with the operation

$$x(f) = x\cdot f - f\cdot x, \qquad x\in\mathfrak{g},\ \ f\in \mathrm{Hom}_K(A, B).$$

The subspace $\mathrm{Hom}_{\mathfrak{h}}(A, B)$ is then $\mathfrak{g}$-invariant, and can be regarded as a $U(\mathfrak{g})$-module. One verifies easily the following explicit formula

$$v^E(f) = \sum_{F\leqq E} \binom{E}{F} v^{E-F}\cdot f\cdot (v^F)_*. \tag{14}$$

Replacing A by its projective resolution, $\mathrm{Ext}_{\mathfrak{h}}(A, B)$ becomes a $U(\mathfrak{g})$-module, and in fact a $U(\mathfrak{g}/\mathfrak{h})$-module since $x(f)=0$ for $x\in\mathfrak{h}$. An element $F\in\mathrm{Ext}_{\mathfrak{h}}(A, B)$ is called $\mathfrak{g}$-invariant if $xF=0$ for every $x\in\mathfrak{g}$, or equivalently if $v^E F=0$ for every $E\neq 0$. Now we shall show that F *is invariant if and only if it is stable in the sense of* § 2.

Thus, let A be a $\mathfrak{g}$-module. Then there exists an isomorphism of $U(\mathfrak{h})$-modules

$$U(\mathfrak{g})\otimes_{\mathfrak{h}} A \cong \sum_E A_E, \quad A_E\cong A. \tag{15}$$

Indeed, define a $U(\mathfrak{h})$-homomorphism $\varphi: U(\mathfrak{g})\otimes_K A\to \sum A_E$ by

$$\varphi(v^E\otimes a) = \sum_{F\leqq E}\binom{E}{F} v^{E-F} a_F.$$

If $x\in\mathfrak{h}$, we have by (13)

$$\begin{aligned}\varphi(v^E x\otimes a) &= \sum_{F\leqq E}\binom{E}{F} v^{E-F}(x)\varphi(v^F\otimes a)\\ &= \sum_{F\leqq E}\sum_{G\leqq F}\binom{E}{F}\binom{F}{G} v^{E-F}(x)v^{F-G}a_G = \sum_{G\leqq E}\sum_{G\leqq F\leqq E}\binom{E}{F}\binom{F}{G} v^{E-F}(x)v^{F-G}a_G,\end{aligned}$$

while

$$\begin{aligned}\varphi(v^E\otimes xa) &= \sum_{G\leqq E}\binom{E}{G} v^{E-G}xa_G = \sum_{G\leqq E}\sum_{H\leqq E-G}\binom{E}{G}\binom{E-G}{H} v^{E-G-H}(x)v^H a_G\\ &= \sum_{G\leqq E}\sum_{G\leqq F\leqq E}\binom{E}{G}\binom{E-G}{F-G} v^{E-F}(x)v^{F-G}a_G.\end{aligned}$$

Since $\binom{E}{F}\binom{F}{G} = \binom{E}{G}\binom{E-G}{F-G}$, we have $\varphi(v^E x\otimes a) = \varphi(v^E\otimes xa)$. Hence φ induces a $U(\mathfrak{h})$-homomorphism $U(\mathfrak{g})\otimes_{\mathfrak{h}} A\to \sum A_E$ which we shall denote also by φ.

Next define a linear mapping $\psi: \sum A_E\to U(\mathfrak{g})\otimes_{\mathfrak{h}} A$ by

$$\psi(a_E) = \sum_{F\leqq E}\binom{E}{F} v^F\otimes (v^{E-F})_* a.$$

Then we have by (12)

$$\varphi\psi(a_E)=\sum_{F\leqq E}\sum_{G\leqq F}\binom{E}{F}\binom{F}{G}v^{F-G}v_*^{E-F}a_G=a_E\,,$$

and similarly $\psi\varphi=$ the identity. Hence the isomorphism (15) is established.

Let B be an $\mathfrak{h}$-module. If we put $\mathrm{Hom}_{\mathfrak{h}}(A_E, B)=\mathrm{Hom}_{\mathfrak{h}}(A, B)_E$, we have at once

(16) $$\mathrm{Hom}_{\mathfrak{h}}(U(\mathfrak{g})\otimes_{\mathfrak{h}} A, B)\cong\prod_E \mathrm{Hom}_{\mathfrak{h}}(A, B)_E\,,$$

$\prod$ denoting the Cartesian product. The E-component f'_E of $f'\in\mathrm{Hom}_{\mathfrak{h}}(U(\mathfrak{g})\otimes_{\mathfrak{h}} A, B)$ is given by

$$f'_E(a)=f'(\psi a_E)=\sum_{F\leqq E}\binom{E}{F}f'(v^F\otimes v_*^{E-F}a)\,.$$

Replacing A by a $\mathfrak{g}$-projective resolution Y of A, *we get the following isomorphism*

(17) $$\mathrm{Ext}_{\mathfrak{h}}(U(\mathfrak{g})\otimes_{\mathfrak{h}} A, B)\cong\prod_E \mathrm{Ext}_{\mathfrak{h}}(A, B)_E\,,$$

for a $\mathfrak{g}$-module A and an $\mathfrak{h}$-module B.

Now assume that A is an $\mathfrak{h}$-module and B a $\mathfrak{g}$-module. Then $U(\mathfrak{g})\otimes_{\mathfrak{h}} A$ is a $\mathfrak{g}$-module. Define a mapping $\varphi'\colon \mathrm{Hom}_{\mathfrak{h}}(U(\mathfrak{g})\otimes_{\mathfrak{h}} A, B)\to\prod \mathrm{Hom}_{\mathfrak{h}}(A, B)_E$ by $\varphi'(f')_E=f'_{(E)}$, where

$$f'_{(E)}(a)=v^E(f')\ (1\otimes a)\,.$$

This mapping φ' is in fact an isomorphism, the inverse mapping being given by

$$\psi'\{f_E\}(v_*^E\otimes a)=\sum_{F\leqq E}\binom{E}{F}v_*^{E-F}f_F(a),\quad \{f_E\}\in\prod \mathrm{Hom}_{\mathfrak{h}}(A, B)_E\,.$$

Replacing A by an $\mathfrak{h}$-projective resolution, we get also an isomorphism (17), for an $\mathfrak{h}$-module A and a $\mathfrak{g}$-module B.

Now assume that both A and B are $\mathfrak{g}$-modules. Then we have an isomorphism (17) by either one of the above considerations. Let us follow the first. Then for $f\in\mathrm{Hom}_{\mathfrak{h}}(Y_n, B)$ where Y is a $\mathfrak{g}$-projective resolution of A, the E-component of $f\pi$ is given by

$$(f\pi)_E(y)=\sum_{F\leqq E}\binom{E}{F}f(v^Fv_*^{E-F}y)=\begin{cases} f(y) & (E=0)\\ 0 & (E\neq 0)\end{cases}$$

while the E-component of $s\alpha f$ is given by

$$(s\alpha f)_E(y)=\sum_{F\leqq E}\binom{E}{F}v^Ff(v_*^{E-F}y)=v^E(f)(y)\,.$$

Hence $F \in \mathrm{Ext}^n_{\mathfrak{h}}(A, B)$ is stable if and only if $v^E(F)=0$ $(E \neq 0)$, as desired. (If we follow the second, the E-component of $f\pi$ is $v^E(f)$, while that of $s\alpha f$ is f or 0 according to $E=0$ or $E \neq 0$.)

Since the condition $\mathrm{Ext}^p_{\mathfrak{h}}(A, B^{(q)})=0$ reduces by (17) to $\mathrm{Ext}^p_{\mathfrak{h}}(A, B)=0$, Theorems 1 and 2 are stated in our case as follows:

PROPOSITION 3. *Let* $\mathfrak{h}$ *be an ideal of* $\mathfrak{g}$, *and assume that* $\mathfrak{g}, \mathfrak{h}$ *and* $\mathfrak{g}/\mathfrak{h}$ *all have K-bases. Assume furthermore that* $\mathrm{Ext}^p_{\mathfrak{h}}(A, B)=0$ $(0<p<n)$. *Then the following sequence is exact:*

$$0 \to \mathrm{Ext}^n_{\mathfrak{g},\mathfrak{h}}(A, B) \xrightarrow{\lambda} \mathrm{Ext}^n_{\mathfrak{g}}(A, B) \xrightarrow{\rho} [\mathrm{Ext}^n_{\mathfrak{h}}(A, B)]^{\mathfrak{g}}$$
$$\xrightarrow{\tau} \mathrm{Ext}^{n+1}_{\mathfrak{g},\mathfrak{h}}(A, B) \xrightarrow{\lambda} \mathrm{Ext}^{n+1}_{\mathfrak{g}}(A, B).$$

In the case $A=K$ this proposition is proved by Hochschild and Serre [**9**].

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