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Note on the Kummer-Hilbert reciprocity law.

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1. Introduction.

Let p be an odd prime number, Q the field of rational p-adic numbers, ζ a fixed primitive p-th root of unity, and $k=Q(\zeta)$.

The classical Kummer-Hilbert reciprocity law was purely locally proved by K. Yamamoto [8] in the following form.[1)]

Let $\mathfrak{p}$ be the prime ideal, and π an arbitrary prime element in k. By making use of the polynomial

$$\mathrm{Log}\,(1+x)=\sum_{i=1}^{p-1}\frac{(-1)^{i-1}}{i}x^{i},$$

we define differential quotients $l_{\pi}^{(i)}(\nu)$, which are determined modulo p, for a principal unit ν in k as follows:

$$\mathrm{Log}\,\nu\equiv\sum_{i=1}^{p-1}\frac{1}{i!}l_{\pi}^{(i)}(\nu)\,\pi^{i}\qquad(\mathfrak{p}^{p}).$$

Then it is necessary and sufficient for ν to be a norm of an element of $K=k(\sqrt[p]{\mu})$, where μ is a principal unit in k, that we have

$$\sum_{n=1}^{p-1}(-1)^{n-1}l_{\pi}^{(n)}(\nu)\,l_{\pi}^{(p-n)}(\mu)\equiv 0\quad(p).$$

Since the Lemmas 5, 5′ in [8], which are of importance in the local proof, contain an error, we shall make an attempt to obtain explicit formulas of general forms correcting [8], and therefrom we shall show that we may derive the above reciprocity law naturally. In the last section of this note we also obtain a formal generalization of the classical differential quotients of Kummer.

We exclude the case that μ is primary, i.e., K/k does not ramify, in which case we have $l_{\pi}^{(i)}(\mu)\equiv 0$ (p) for all i and the above proposition is evidently true.

1) Cf. also Hilbert [4], Takagi [7], Artin and Hasse [1], Šafarevič [6], Kneser [5], Dwork [2].

Let v be the ramification constant of K/k and suppose $\mu = 1-\beta$, $\mathrm{Ord}_{\mathfrak{p}}(\beta) = s$, then we have $p = s+v$ ([**3**]). Now choose the least non negative integers c, d such that $cs-pd=1$ and fix a prime element $\Pi = (1-M)^c\tau^{-d}$ in K for an arbitrary prime element τ in k and a p-th root M of μ. The element Π is indeed a prime element since K/k is totally ramified and so the exponent of $1-M$ with respect to the prime ideal of K is s. Then our explicit formulas of norms read:

$$N_{K/k}(1-\Pi^m) \equiv (1-\gamma^m)(1-\gamma^v)^{e(m)} \qquad (\mathfrak{p}^{v+1}),$$

$$e(m) \equiv Am \sum_{m|n} \frac{1}{(p-n)!}\, l_\gamma^{(p-n)}(\mu) \quad (p), \quad m = 1, 2, \cdots, v.$$

Herein $N_{K/k}$ denotes of course the norm from K to k, A a certain constant depending only on τ, v and we put $\gamma = N_{K/k}\Pi$.

2. Several lemmas.

The following lemmas are as a matter of fact due to Yamamoto [8].

LEMMA 1. *If we define* $\sigma_t = S_{K/k}(\Pi^t)$, $\sigma = \sigma_1$ *and make use of the above notations, then*

$$\sigma_t \equiv p\tau^{-dt}\beta^{\left[\frac{ct}{p}\right]} \qquad (\mathfrak{p}^{v+1}).$$

Herein t *means an arbitrary positive integer,* $S_{K/k}$ *the trace from* K *to* k, *and* $[x]$ *Gauss' symbol indicating the greatest integer* $\leqq x$.

As a proof has been given in [8], we shall omit one. This lemma will be used only for $1 \leqq t \leqq v$.

LEMMA 2.

$$\frac{1}{p}\sigma_{tp} - \gamma^t \equiv \frac{1}{p}\sigma_t{}^p \equiv 0 \qquad (\mathfrak{p}^{v+1}),$$

$$\frac{1}{p}\sigma_p{}^p \equiv \frac{1}{p^2}\sigma^{p^2} \equiv 0 \qquad (\mathfrak{p}^{v+1}).$$

The upper congruence is valid for any positive integer t *if* $v \neq 1$ *but for* $t \geqq 2$ *if* $v=1$. *The lower one has no restriction.*

This lemma will be readily verified by the fact that $S_{K/k}(\mathfrak{P}^i) = \mathfrak{p}^j$, $j = \left[\frac{i+(p-1)(v+1)}{p}\right]$ for the prime ideal $\mathfrak{P}$ in K.

Now the polynomial $\mathrm{Log}(1+x)$ and the inverse polynomial $\mathrm{Exp}\, x = \sum_{i=0}^{p-1} \frac{1}{i!} x^i$ have the following properties.

If $\varepsilon_1, \varepsilon_2$ denote two principal units and α_1, α_2 non-unit integers in k respectively, then

$$\operatorname{Log}\varepsilon_1+\operatorname{Log}\varepsilon_2\equiv\operatorname{Log}\varepsilon_1\varepsilon_2,\quad \operatorname{Exp}\alpha_1\operatorname{Exp}\alpha_2\equiv\operatorname{Exp}(\alpha_1+\alpha_2)\quad(\mathfrak{p}^p),$$
$$\operatorname{Log}\operatorname{Exp}\alpha_1\equiv\alpha_1,\quad \operatorname{Exp}\operatorname{Log}\varepsilon_1\equiv\varepsilon_1\quad(\mathfrak{p}^p).$$

Furthermore we have

LEMMA 3. *If* $\mu\equiv 1+a\lambda$ $(\mathfrak{p}^2)$, $\lambda=1-\zeta$, *then*

$$\log\mu\equiv\operatorname{Log}\mu+a\operatorname{Log}\zeta\quad(\mathfrak{p}^p).$$

PROOF. Because $\operatorname{Ord}_p(\nu-1)>\frac{1}{p-1}$ implies that $\log\nu\equiv\operatorname{Log}\nu$ $(\mathfrak{p}^p)$, we have $\log\zeta^a\mu\equiv\operatorname{Log}\zeta^a\mu$ $(\mathfrak{p}^p)$, that is,

$$\log\mu\equiv\operatorname{Log}\mu+a\operatorname{Log}\zeta\quad(\mathfrak{p}^p).$$

The following lemma is of special interest and our proof is slightly different from [8].

LEMMA 4. $\operatorname{Log}\zeta\equiv\frac{1}{p}\lambda^p$ $(\mathfrak{p}^p)$. *In particular by using* $\tilde{\omega}=\sqrt[p-1]{-p}$ *such that* $\zeta\equiv 1+\tilde{\omega}$ $(\mathfrak{p}^p)$, *we have* $\tilde{\omega}\equiv\operatorname{Log}\zeta$, $\zeta\equiv\operatorname{Exp}\tilde{\omega}$ $(\mathfrak{p}^p)$.

PROOF. We make use of the polynomial $F(x)=\sum_{i=0}^{p-1}(1-x)^i=\sum_{j=0}^{p-1}(-1)^j\binom{p}{j}x^j$. The fact $F(\lambda)=0$ implies that $\frac{1}{p}\lambda^p=\sum_{i=0}^{p-1}(-1)^i\frac{1}{p}\binom{p}{i}\lambda^i$, from which we have $\frac{1}{p}\lambda^p\equiv\operatorname{Log}\zeta$ $(\mathfrak{p}^p)$. Here we have $\tilde{\omega}\equiv\frac{1}{p}\lambda^p$ $(\mathfrak{p}^p)$ for the element $\tilde{\omega}$ in the proposition. We have therefore $\tilde{\omega}\equiv\operatorname{Log}\zeta$ $(\mathfrak{p}^p)$ and $\zeta\equiv\operatorname{Exp}\tilde{\omega}$ $(\mathfrak{p}^p)$.

3. Computation of norms.

Our problem consists in calculating explicitly the norms in terms of the differential quotients defined in Introduction. We transform σ_t further as follows.

Determine a $(p-1)$-th root of unity ξ, by $\xi\beta\equiv\tau^s$ $({}^{\times}\mathfrak{p})$. Then we have for $t=1,2,\cdots,p-1$,

$$(*)\qquad \sigma_t\equiv-\xi^{-d}\frac{t}{s}\frac{1}{(p-t)!}l_\gamma^{(p-t)}(\mu)\gamma^v\qquad(\mathfrak{p}^{v+1}).$$

For, on one hand from Lemma 1 follows

$$\sigma_t\equiv 0\quad(\mathfrak{p}^{v+1})\quad\text{for } t\not\equiv p\quad(s),$$
$$\sigma_{p-si}\equiv-\xi^{(i-1)d}\gamma^v\quad(\mathfrak{p}^{v+1}).$$

On the other hand we have, by $\gamma^s\equiv\beta\xi^{-d}$ $(\mathfrak{p}^p)$,

$$\operatorname{Log}\mu\equiv-\sum_{i=1}^{p-1}\frac{1}{i}\xi^{di}\gamma^{si}\qquad(\mathfrak{p}^p),$$

so that

$$l_\gamma^{(t)}(\mu) \equiv \begin{cases} 0 & (p) \quad \text{for } s \nmid t\,, \\ -s(t-1)\,!\,\xi^{d\frac{t}{s}} & (p) \quad \text{for } s \mid t\,. \end{cases}$$

Therefore (*) follows immediately from these two congruences.

Now by solving an inequality $\left[\frac{mi+(p-1)(v+1)}{p}\right]-(p-1)\,\mathrm{Ord}_p(i) \leqq v$ for i, which is equivalent to $\frac{1}{i}\sigma_{mi} \not\equiv 0\ (\mathfrak{p}^{v+1})$, we obtain

$$\begin{aligned} \log N_{K/k}(1-\Pi^m) &= -\sum_{i=1}^{\infty}\frac{1}{i}\sigma_{mi} \\ &\equiv -\sum_{i=1}^{p-1}\frac{1}{i}\sigma_{mi} - \sum_{i=1}^{p-1}\frac{1}{pi}\sigma_{mpi} - \delta_{m,1}\frac{1}{p^2}\sigma_{p^2} \qquad (\mathfrak{p}^{v+1}) \\ &\equiv -\sum_{i=1}^{p-1}\frac{1}{i}\sigma_{mi} - \sum_{i=1}^{p-1}\frac{1}{i}\Big(\frac{1}{p}\sigma_{mi}^p + \gamma^{mi}\Big) - \delta_{m,1}\Big(\frac{1}{p^2}\sigma^{p^2} + \frac{1}{p}\gamma^p\Big) \quad (\mathfrak{p}^{v+1}) \\ &\equiv \begin{cases} -\sum_{i=1}^{p-1}\frac{1}{i}\sigma_{mi} - \sum_{i=1}^{p-1}\frac{1}{i}\gamma^{mi} - \delta_{m,1}\frac{1}{p}\gamma^p & (\mathfrak{p}^{v+1}),\quad v \neq 1\,, \\ -\sum_{i=1}^{p-1}\frac{1}{i}\sigma_{mi} - \sum_{i=1}^{p-1}\frac{1}{i}\gamma^{mi} - \delta_{m,1}\Big(\frac{1}{p}\gamma^p + \frac{1}{p}\sigma^p\Big) & (\mathfrak{p}^{v+1}),\quad v = 1\,. \end{cases} \end{aligned}$$

We have used Lemma 2 in the above transformation.

Regarding Lemma 3 we have

$$\log N_{K/k}(1-\Pi^m) \equiv \mathrm{Log}\, N_{K/k}(1-\Pi^m) + \delta_{m,1}\, a\, \mathrm{Log}\, \zeta \qquad (\mathfrak{p}^p)\,.$$

Of course a means a number satisfying $\zeta^a N_{K/k}(1-\Pi^m) \equiv 1\ (\mathfrak{p}^p)$,

i.e.,
$$a \equiv \begin{cases} -\frac{1}{\lambda}\gamma & (p), \quad v \neq 1\,, \\ -\frac{1}{\lambda}(\gamma+\sigma) & (p), \quad v = 1\,. \end{cases}$$

From these congruences and from Lemma 4 follows

$$a\,\mathrm{Log}\,\zeta \equiv \begin{cases} -\frac{1}{p}\gamma^p & (\mathfrak{p}^p), \quad v \neq 1\,, \\ -\frac{1}{p}(\gamma^p+\sigma^p) & (\mathfrak{p}^p), \quad v = 1\,. \end{cases}$$

Consequently in both cases where $v \neq 1$ and $v = 1$,

$$\operatorname{Log} N_{K/k}(1-\Pi^{m}) \equiv \sum_{i=1}^{p-1} \frac{1}{i}\sigma_{mi} - \sum_{i=1}^{p-1} \frac{1}{i}\gamma^{mi} \quad (\mathfrak{p}^{v+1}),$$

that is,

$$N_{K/k}(1-\Pi^{m}) \equiv (1-\gamma^{m}) \prod_{i=1}^{\left[\frac{v}{m}\right]} (1-\sigma_{mi})^{\frac{1}{i}} \quad (\mathfrak{p}^{v+1}).$$

Here by making use of the formula (*), we have

$$N_{K/k}(1-\Pi^{m}) \equiv (1-\gamma^{m})(1-\gamma^{v})^{e(m)} \quad (\mathfrak{p}^{v+1}),$$

$$e(m) \equiv \frac{m}{s}\xi^{-d} \sum_{m|n} \frac{1}{(p-n)!} l_{\gamma}^{(p-n)}(\mu) \quad (\mathfrak{p}^{v+1}) \quad \text{for } m=1,2,\cdots,v.$$

These are the desired explicit formulas of norms.

4. Reciprocity law.

Our explicit formulas mentioned above yield readily a reciprocity law with respect to a base γ.

If we put ν in a form of power products $\nu \equiv \prod_{m=1}^{v}(1-\gamma^{m})^{a_m}$ $(\mathfrak{p}^{v+1})$, then by Möbius' inversion formula we can verify

$$a_m \equiv -\frac{1}{m}\sum_{d|m} \mu\left(\frac{m}{d}\right)\frac{1}{(d-1)!} l_{\gamma}^{(d)}(\nu) \quad (p),$$

where $\mu(x)$ denotes Möbius' function.

The norm of element $E \equiv \prod_{m=1}^{v}(1-\Pi^{m})^{a_m}$ $(\mathfrak{p}^{v+1})$ can be expressed in terms of the differential quotients as follows.

$$N_{K/k}E \equiv \prod_{m=1}^{v}(1-\gamma^{m})^{a_m}(1-\gamma^{v})^{R(E)} \quad (\mathfrak{p}^{v+1}),$$

$$R(E) \equiv \sum_{m=1}^{v} a_m \frac{m}{s}\xi^{-d}\sum_{m|n}\frac{1}{(p-n)!} l_{\gamma}^{(p-n)}(\mu) \quad (p).$$

It follows from these formulas that we have the relation

$$\sum_{m=1}^{v}\sum_{d|m}\sum_{m|n} \mu\left(\frac{m}{d}\right)\frac{l_{\gamma}^{(d)}(\nu)}{(d-1)!}\frac{l_{\gamma}^{(p-n)}(\mu)}{(p-n)!} \equiv 0 \quad (p),$$

as a necessary and sufficient condition for ν to be a norm from K to k. This condition is equal to

$$\sum_{n=1}^{p-1}\frac{l_{\gamma}^{(n)}(\nu)}{(n-1)!}\frac{l_{\gamma}^{(p-n)}(\mu)}{(p-n)!} \equiv 0 \quad (p),$$

i.e.,

$$\sum_{n=1}^{p-1}(-1)^{n-1}l_{\Gamma}^{(n)}(\nu)\,l_{\Gamma}^{(p-n)}(\mu)\equiv 0 \qquad (p),$$

which gives us the reciprocity law desired.

In the next section we verify the so-called invariance to the effect that this conclusion is equivalent to

$$\sum_{n=1}^{p-1}(-1)^{n-1}l_{\pi}^{(n)}(\nu)\,l_{\pi}^{(p-n)}(\mu)\equiv 0 \qquad (p),$$

through a formal calculation by making use of Lagrange's inversion formula for power series. Herein π means an arbitrary prime element in k.

5. Proof of the invariance.

We verify the invariance following an idea of Yamamoto [8].

We define differentials of a non-unit integer α and an integer υ,

$$D_\pi^i\alpha\equiv a_i\ (p),\ i=1,2,\cdots,p-1,\quad D_\pi^0\alpha\equiv 0\quad (p),$$
$$D_\pi^i\upsilon\equiv c_i\ (p),\ i=0,1,\cdots,p-2,$$

by $\alpha\equiv\sum_{i=1}^{p-1}\frac{1}{i!}a_i\pi^i\ (\mathfrak{p}^p)$, $a_i\in\mathfrak{O}_Q$, $\upsilon\equiv\sum_{i=0}^{p-2}\frac{1}{i!}c_i\pi^i\ (\mathfrak{p}^{p-1})$, $c_i\in\mathfrak{O}_Q$. Also we define $\frac{d\alpha}{d\pi}=\sum_{i=0}^{p-2}\frac{1}{i!}a_{i+1}\pi^i$, $\frac{d\upsilon}{d\pi}=\sum_{i=0}^{p-3}\frac{1}{i!}c_{i+1}\pi^i$.

The following identities are valid for legitimately defined operators $D_\pi^i\ (D_\pi^1=D_\pi)$ and are readily proved.

$$(1)\quad D_\pi(\alpha\beta)\equiv\alpha D_\pi\beta+\beta D_\pi\alpha \qquad (p),$$

$$(2)\quad D_\pi^m(\alpha\beta)\equiv\sum_{i=0}^{m}\binom{m}{i}D_\pi^i\alpha\, D_\pi^{m-i}\beta \qquad (p),$$

$$(3)\quad D_\pi^{m+1}(\alpha\pi^i)\equiv(m+i)_i D_\pi^m\alpha \qquad (p),$$

$$(4)\quad D_\pi^m\frac{d\alpha}{d\pi}\equiv D_\pi^{m+1}\alpha \qquad (p).$$

Now Lagrange's inversion formula for power series is expressed as follows: If θ,π denote arbitrary two prime element in k, then

$$D_\theta^m\alpha\equiv D_\pi^{m-1}\Big(\frac{d\alpha}{d\pi}\Big(\frac{\pi}{\theta}\Big)^m\Big) \qquad (p).$$

For a proof of this formula one is referred to [8]. By making use of these notations we can simply express our norm condition as follows:

$$\sum_{n=1}^{p-1}(-1)^{n-1}\, l_\gamma^{(n)}(\nu)\, l_\gamma^{(p-n)}(\mu) \equiv D_\gamma^{p-1}\left(\operatorname{Log}\nu \frac{d}{d\gamma}\operatorname{Log}\mu\right) \qquad (p).$$

Consequently we have only to verify the invariance

$$D_\pi^{p-1}\left(\operatorname{Log}\nu \frac{d}{d\pi}\operatorname{Log}\mu\right) \equiv D_\gamma^{p-1}\left(\operatorname{Log}\nu \frac{d}{d\gamma}\operatorname{Log}\mu\right) \qquad (p).$$

Since $D_\theta^{p-1}\omega \equiv D_{\theta'}^{p-1}\omega\ (p)$ holds for two primes θ, θ' such that $\theta = a\theta'$ with $a \in \mathfrak{O}_Q$, and also $D_\pi^{p-1}\left(\omega \frac{d}{d\pi}\alpha\right)$ is linear with respect to an integer ω and a non-unit α, it suffices to show that

$$D_\pi^{p-1}\left(\pi^i \frac{d\pi^j}{d\pi}\right) \equiv D_\theta^{p-1}\left(\pi^i \frac{d\pi^j}{d\theta}\right) \qquad (p)$$

for $0 \leqq i \leqq p-1$, $1 \leqq j \leqq p-1$ under the assumption $D_\pi^0\left(\frac{\pi}{\theta}\right) \equiv 1\ (p)$.

Now by Lagrange's inversion formula indicated above we have

$$\pi \equiv \sum_{t=1}^{p-1} \frac{1}{t!} D_\pi^{t-1}\left(\left(\frac{\pi}{\theta}\right)^t\right)\theta^t \qquad (\mathfrak{p}^p).$$

Therefore

$$\pi^{i+j-1}$$

$$\equiv \sum_{t=i+j-1}\sum_S^{p-1} \frac{(i+j-1)!}{n_1!\cdots n_{p-1}!}\frac{1}{1!^{n_1}\cdots(p-1)!^{n_{p-1}}} D_\pi^0\left(\left(\frac{\pi}{\theta}\right)\right)^{n_1}\cdots D_\pi^{p-2}\left(\left(\frac{\pi}{\theta}\right)^{p-1}\right)^{n_{p-1}}\theta^t \ (\mathfrak{p}^p),$$

where the second summation extends over all partitions S: $n_1+\cdots+n_{p-1} = i+j-1$, $n_1+\cdots+(p-1)n_{p-1} = t$.

On one hand this equality and $\frac{d\pi}{d\theta} \equiv \sum_{t=0}^{p-2}\frac{1}{t!}D_\pi^t\left(\left(\frac{\pi}{\theta}\right)^{t+1}\right)\theta^t\ (\mathfrak{p}^{p+1})$ yield

$$D_\theta^{p-1}\left(\pi^i \frac{d\pi^j}{d\theta}\right) \equiv D_\theta^{p-1}\left(\pi^{i+j-1} j \frac{d\pi}{d\theta}\right) \qquad (p)$$

$$\equiv j\sum p \frac{1}{1!^{n_1}\cdots(p-1)!^{n_{p-1}}}\frac{(i+j-1)!}{n_1!\cdots n_{p-1}!} D_\pi^0\left(\left(\frac{\pi}{\theta}\right)\right)^{n_1}\cdots D_\pi^{p-2}\left(\left(\frac{\pi}{\theta}\right)^{p-1}\right)^{n_{p-1}} \ (p)$$

$$\equiv -j\,\delta_{p,i+j} D_\pi^0\left(\left(\frac{\pi}{\theta}\right)\right)^p \equiv -j\,\delta_{p,i+j} \qquad (p).$$

On the other hand

$$D_\pi^{p-1}\left(\pi^i \frac{d\pi^j}{d\pi}\right) \equiv D_\pi^{p-1}(j\,\pi^{i+j-1}) \equiv -j\,\delta_{p,i+j} \qquad (p).$$

This completes a proof of our assertion.

6. Kummer's differential quotients.

We can interpret classical Kummer's differential quotients as follows.

A principal unit ν has a unique expression with regard to a prime element π.

$$\nu=\sum_{i=0}^{p-2} c_i(1-\pi)^i\,,$$

where c_i are rational integers.

After Kummer and Hilbert we define for ν an adjoint polynomial of degree $p-1$,

$$\nu(x)=\sum_{i=0}^{p-2} c_i x^i-\frac{1}{p}\Big(\sum_{i=0}^{p-2} c_i-1\Big)\Big(\sum_{i=0}^{p-1} x^i\Big)\,,$$

so that $\nu(1)=1$, $\nu(1-\pi)\equiv\nu\ (p)$.

Suppose $L_\pi^{(i)}(\nu)\equiv\dfrac{d^i}{dv^i}\log\nu(e^v)\Big|_{v=0}\ (p)$ for $i=1,2,\cdots,p-1$, then we maintain the relations

$$l_{\mathrm{Log}(1-\pi)}^{(i)}(\nu)\equiv L_\pi^{(i)}(\nu)\quad(p)\qquad\text{for } i=1,2,\cdots,p-1\,.$$

For, by making use of Stirling numbers of the second kind $\mathfrak{S}(i,j)=\dfrac{1}{j\,!}\sum_\nu(-1)^{j-v}\binom{j}{v}v^i$, we have first

$$L_\pi^{(i)}(\nu)\equiv\sum_{j=1}^{i}\mathfrak{S}(i,j)\frac{d^j}{dx^j}\log\nu(x)\Big|_{x=1}\quad(p)\,.$$

Noticing that $\nu\equiv\sum_{i=0}^{p-1} b_i\pi^i\ (\mathfrak{p}^p)$, $b_i=(-1)^i\dfrac{\nu^{(i)}(x)}{i\,!}\Big|_{x=1}$, we obtain

$$\begin{aligned}\mathrm{Log}\,\nu&\equiv\sum_{n=1}^{p-1}(-1)^{n-1}\frac{1}{n}\Big(\sum_{i=1}^{p-1} b_i\pi^i\Big)^n\\&\equiv\sum_{i=1}^{p-1}L_\pi^{(i)}(\nu)\sum_{t=i}^{p-1}\frac{(-1)^t}{t\,!}S(t,i)\,\pi^t\quad(\mathfrak{p}^p)\,.\end{aligned}$$

This modification has been carried out by means of the orthogonality relation of Stirling numbers

$$\sum_t\mathfrak{S}(i,t)\,S(t,j)=\delta_{i,j}\,,\qquad\sum S(i,t)\,\mathfrak{S}(t,j)=\delta_{i,j}\,.$$

Herein $S(i,j)$ denotes the Stirling number of the first kind defined by $(x)_i=\sum S(i,j)x^j$.

Because we also have

$$\frac{(\mathrm{Log}(1-\pi))^i}{i\,!} \equiv \sum_{t=i}^{p-1} \frac{(-1)^t}{t\,!} S(t, i)\,\pi^t \qquad (\mathfrak{p}^p),$$

we finally conclude

$$\mathrm{Log}\,\nu \equiv \sum_{i=1}^{p-1} \frac{1}{i\,!} L_\pi^{(i)}(\nu)\,(\mathrm{Log}(1-\pi))^i \qquad (\mathfrak{p}^p),$$

that is,

$$l_{\mathrm{Log}(1-\pi)}^{(i)}(\nu) \equiv L_\pi^{(i)}(\nu) \quad (p) \qquad \text{for } i=1, 2, \cdots, p-1\,.$$

Especially if we take λ in place of π, $L_\pi^{(i)}(\nu)$ coincides just with Kummer's differential quotient, and then by Lemma 4 we have $\mathrm{Log}(1-\pi) \equiv \tilde{\omega}$ $(\mathfrak{p}^p)$, as was pointed out in [8].

Finally we should like to note such a reciprocity law that is expressed in terms of exponents of power products.

The relations

$$l_\pi^{(i)}(\mu) \equiv (i-1)\,! \sum_{d\mid i} a_d(\mu) \quad (p) \qquad \text{for } i=1, 2, \cdots, p-1\,,$$

immediately yield

$$\sum_{i=1}^{p-1} (-1)^{i-1}\, l_\pi^{(i)}(\nu)\, l_\pi^{(p-i)}(\mu) \equiv \sum_{i=1}^{p-1} \frac{1}{i} \sum_{d\mid i} \sum_{d'\mid p-i} a_d(\nu)\, a_{d'}(\mu) \qquad (p)\,.$$

It is necessary and sufficient for ν to be a norm from K to k that we have

$$\sum_{d=1}^{p-1} a_d(\nu) \sum_{d'=1}^{p-d} a_{d'}(\mu)\, f(d, d') \equiv 0 \qquad (p)\,.$$

Herein

$$f(d, d') = \sum_{\substack{t\equiv 0(d),\, t\equiv 0(d'),\\ 1\leqq t\leqq p-1}} \frac{1}{t} \text{ and } \nu \equiv \prod_{m=1}^{p-1} (1-\pi^m)^{a_m(\nu)}\ (\mathfrak{p}^p),\ \mu \equiv \prod_{m=1}^{p-1} (1-\pi^m)^{a_m(\mu)}\ (\mathfrak{p}^p)\,.$$

This formula as a matter of fact does not depend on the choice of prime element π.

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