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On a certain system with infinite induction.

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Introduction

K. Schütte gave the system $\boldsymbol{Z}$ of the theory of natural numbers which contains the infinite induction, and he proved that G. Gentzen's elimination theorem holds in $\boldsymbol{Z}$ [1].

To every proof-figure in $\boldsymbol{Z}$ corresponds an ordinal number of the second class which is called the order of the proof-figure. In this paper we prove some metatheorems on $\boldsymbol{Z}$ by applying G. Gentzen's elimination theorem for proof-figures of finite order due to K. Schütte [1]. Our proofs are not necessarily based on the finite stand-point.

In §1 we formulate the system $\boldsymbol{Z}$ into G. Gentzen's style.

In §2 we give another proof of the consistency of $\boldsymbol{Z}$ and G. Gentzen's elimination theorem for proof-figures of any order in $\boldsymbol{Z}$, which is given in the following stronger form: for every proof-figure in $\boldsymbol{Z}$ we have a proof-figure of finite order to the same end-sequent which contains no cut. Moreover we prove that any arithmetical formula is decidable in $\boldsymbol{Z}$, i. e. if A is an arbitrary arithmetical formula, then either A or non-A is provable in $\boldsymbol{Z}$.

In §3 as an application of results in §1 we prove the consistency of G. Gentzen's LK with number-theoretic axioms containing the complete induction without use of the transfinite induction to Cantor's first ε-number ε_0.

§1. System.

In this section we formulate an ω-complete system $\boldsymbol{Z}$ of arithmetic into G. Gentzen style.

1. Symbols

We use the following fundamental symbols; symbol 0, bound variables x, y, z etc., function symbols $'$, $+$, $\cdot$, predicate symbol $=$, logical symbols $\wedge$, $\neg$, $\forall$ and symbol $\rightarrow$.

If necessary we use several letters for abbreviation.

2. *Terms* are constructed as follows:

(1) the symbol 0 is a term; (2) if t is a term, so is t', and if t_1 and t_2 are terms, so are t_1+t_2 and $t_1\cdot t_2$.

In particular terms of the form $0, 0', 0'', 0''', \cdots$ are called *numerals*.

3. *Formulas* are constructed as follows:

(1) if t_1 and t_2 are terms, then $t_1 = t_2$ is a prime formula and a prime formula is a formula; (2) if A is a formula, so is $\neg A$; (3) if A and B are formulas, so is $A \wedge B$, and (4) if $F(t)$ is a formula, so is $\forall x F(x)$.

The number of logical symbols in a formula is called the *degree* of the formula.

4. We call a figure of the following form a *sequent*, $A_1, \cdots, A_\mu \rightarrow B_1, \cdots, B_\nu$ where $A_1, \cdots, A_\mu, B_1, \cdots, B_\nu$ are arbitrary formulas. And it may happen that $\mu = 0$ or $\nu = 0$. We say that $A_1, \cdots, A_\mu$ are in the antecedent and $B_1, \cdots, B_\nu$ are in the succedent in the sequent.

5. Sequents of the following forms are called *beginning sequents*:

(1) a sequent of the form $\rightarrow P$, where P is a true prime formula, and (2) a sequent of the form $P \rightarrow$, where P is a false prime formula.

6. Rules of inference

If $S_1, \cdots, S_m$ and S are sequents, then a figure of the form

$$\frac{S_1, \cdots, S_m}{S}$$

is called a *rule of inference*. $S_1, \cdots, S_m$ are called the *upper sequents* and S is called the *lower sequent* of the rule of inference. In our case $\boldsymbol{Z}$ contains the following rules of inference.

In what follows, capital Greek letters Γ, Π etc. express finite sequences of formulas.

(1) Structural rules of inference

Thinning in antecedent

$$\frac{\Gamma \rightarrow \Delta}{D, \Gamma \rightarrow \Delta}$$

Thinning in succedent

$$\frac{\Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta, D}$$

Interchange in antecedent

$$\frac{\Gamma, D, C, \Pi \rightarrow \Delta}{\Gamma, C, D, \Pi \rightarrow \Delta}$$

Interchange in succedent

$$\frac{\Gamma \rightarrow \Delta, D, C, \Lambda}{\Gamma \rightarrow \Delta, C, D, \Lambda}$$

Contraction in antecedent

$$\frac{D, D, \Gamma \rightarrow \Delta}{D, \Gamma \rightarrow \Delta}$$

Contraction in succedent

$$\frac{\Gamma \rightarrow \Delta, D, D}{\Gamma \rightarrow \Delta, D}$$

where C and D are arbitrary formulas called *principal formulas* of each rule of inference.

(2) Logical rules of inference

$\wedge$-in antecedent

$$\frac{A, \Gamma \rightarrow \Delta}{A \wedge B, \Gamma \rightarrow \Delta} \qquad \frac{B, \Gamma \rightarrow \Delta}{A \wedge B, \Gamma \rightarrow \Delta}$$

$\wedge$-in succedent

$$\frac{\Gamma \rightarrow \Delta, A \qquad \Gamma \rightarrow \Delta, B}{\Gamma \rightarrow \Delta, A \wedge B}$$

$\neg$-in antecedent

$$\frac{\Gamma \rightarrow \Delta, A}{\neg A, \Gamma \rightarrow \Delta}$$

$\neg$-in succedent

$$\frac{A, \Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta, \neg A}$$

where A and B are arbitrary formulas called *side formulas* of each rule of inference. $A \wedge B$ or $\neg A$ is called *principal formula* of each rule of inference.

$\forall$-in antecedent

$$\frac{F(\boldsymbol{n}), \Gamma \rightarrow \Delta}{\forall x F(x), \Gamma \rightarrow \Delta}$$, where $\boldsymbol{n}$ is an arbitrary numeral.

$\forall$-in succedent

$$\frac{\Gamma \rightarrow \Delta, F(\boldsymbol{n}) \text{ for every numeral } \boldsymbol{n}}{\Gamma \rightarrow \Delta, \forall x F(x)}.$$

This is called the infinite induction. $F(\boldsymbol{n})$ is called the *side formula* and $\forall x F(x)$ is called the *principal formula* of each rule of inference.

(3) Cut

$$\frac{\Gamma \rightarrow \Delta, D \qquad D, \Pi \rightarrow \Lambda}{\Gamma, \Pi \rightarrow \Delta, \Lambda}$$

where D is an arbitrary formula called the *cut-formula*. We define the *degree* of a cut as the degree of the cut-formula of this cut.

7. We introduce a concept "*proof-figure*" into the system. Under a proof-figure we understand a figure of finite or infinite sequents, built up in the following manner: uppermost sequents are always beginning sequents; every sequent is a lower sequent of at most one rule of inference; and every sequent, except just one, the end-sequent, is upper sequent of just one rule of inference. To every sequent of proof-figure corresponds an ordinal number of the second class as follows. (1) The ordinal number of a beginning sequent is zero. (2) The ordinal number of the lower sequent of a structural rule of inference is equal to that of the upper sequent. (3) The ordinal number of the lower sequent of a cut or a logical rule of inference is greater than those of upper sequents.

Only proof-figures which have the maximum of degrees of cut are under our consideration.

8. When a formula A contains no predicate symbol except $=$ and no function symbol except $'$, $+$, $\cdot$, A is said to be *arithmetical*. To simplify the treatment, we assume that the system contains only arithmetical formulas.

§ **2.** In this section we give a consistency proof of the system $\boldsymbol{Z}$ and give a proof of the elimination theorem in general form, starting with K. Schütte's elimination theorem for proof-figures of finite order.

1. Elimination theorem for proof-figures of finite order (due to K. Schütte [**1**]).

If a sequent $\Gamma \rightarrow \Delta$ is provable by a finite order, then we have a proof-figure to $\Gamma \rightarrow \Delta$ without cut, having also a finite order.

For simplicity we say that $\Gamma \rightarrow \Delta$ is *finitely provable* or *α-provable* in case that $\Gamma \rightarrow \Delta$ has a proof-figure of finite order or order α.

2. LEMMA. *If t and s are terms with the same numerical value, then the sequent $F(t) \rightarrow F(s)$ is $2n$-provable, where n is the degree of $F(t)$.*

It is easily proved by induction on the degree of the formula $F(t)$.

3. THEOREM. *Let $A_1, \cdots, A_\mu \rightarrow B_1, \cdots, B_\nu$ be a sequent where $\mu+\nu \neq 0$ and $m_1, \cdots, m_\mu$, $n_1, \cdots, n_\nu$ be the degrees of $A_1, \cdots, A_\mu$, $B_1, \cdots, B_\nu$ respectively. If the sequent $A_1, \cdots, A_\mu \rightarrow B_1, \cdots, B_\nu$ is provable without cut, then some sequent $A_i \rightarrow$ is m_i-provable without cut $(1 \leqq i \leqq \mu)$ or some sequent $\rightarrow B_j$ is n_j-provable without cut $(1 \leqq j \leqq \nu)$.*

PROOF. We prove by transfinite induction on the order of the proof-figure to $A_1, \cdots, A_\mu \rightarrow B_1, \cdots, B_\nu$.

If the order is zero, then it is clear. If the last rule of inference (denoted by $\mathfrak{L}$) is a logical rule of inference, then we have six cases, $\mathfrak{L}$ is $\wedge$-in succedent, $\wedge$-in antecedent, $\neg$-in succedent, $\neg$-in antecedent, $\forall$-in succedent and $\forall$-in antecedent.

In the case where $\mathfrak{L}$ is $\wedge$-in succedent let it be

$$\frac{A_1, \cdots, A_\mu \rightarrow B_1, \cdots, B_{\nu-1}, C \qquad A_1, \cdots, A_\mu \rightarrow B_1, \cdots, B_{\nu-1}, D}{A_1, \cdots, A_\mu \rightarrow B_1, \cdots, B_{\nu-1}, C \wedge D}$$

where $C \wedge D$ is B_ν.

If $A_i \rightarrow$ is not m_i-provable without cut for every i $(1 \leqq i \leqq \mu)$ and $\rightarrow B_j$ is not n_j-provable without cut for every j $(1 \leqq j \leqq \nu-1)$, then both $\rightarrow C$ and $\rightarrow D$ are $(n_\nu-1)$-provable without cut by the assumption of transfinite induction. Therefore $\rightarrow C \wedge D$ is n_ν-provable without cut.

In the case where $\mathfrak{L}$ is $\wedge$-in antecedent, $\neg$-in succedent, $\neg$-in antecedent or $\forall$-in antecedent we prove in the same way as in $\wedge$-in succedent.

In the case where $\mathfrak{L}$ is $\forall$-in succedent let it be

$$\frac{A_1, \cdots, A_\mu \rightarrow B_1, \cdots, B_{\nu-1}, F(\boldsymbol{n}) \text{ for every numeral } \boldsymbol{n}}{A_1, \cdots, A_\mu \rightarrow B_1, \cdots, B_{\nu-1}, \forall x F(x)}.$$

If $A_i \rightarrow$ is not m_i-provable without cut for every i $(1 \leqq i \leqq \mu)$ and $\rightarrow B_j$ is not n_j-provable without cut for every j $(1 \leqq j \leqq \nu-1)$, then $\rightarrow F(\boldsymbol{n})$ is $(n_\nu-1)$-provable without cut for every numeral $\boldsymbol{n}$. Therefore the sequent $\rightarrow \forall x F(x)$

is n_ν-provable without cut.

4. THEOREM. *Let F be an arbitrary formula. Then it is impossible that both the sequents* $\rightarrow F$ *and* $F\rightarrow$ *are provable without cut.*

PROOF. If $\rightarrow F$ is provable without cut, then $\rightarrow F$ is finitely provable without cut from Theorem 3. Similary $F\rightarrow$ is finitely provable without cut. Therefore if both $\rightarrow F$ and $F\rightarrow$ were provable without cut, then the sequent '$\rightarrow$' should be finitely provable. This should contradict to Theorem 1 in this section.

5. THEOREM (*Generalization of Theorem* 3).

Let $A_1, \cdots, A_\mu \rightarrow B_1, \cdots, B_\nu$ *be a sequent where* $\mu+\nu \neq 0$ *and* $m_1, \cdots, m_\mu, n_1, \cdots, n_\nu$ *be the degrees of* $A_1, \cdots, A_\mu, B_1, \cdots, B_\nu$ *respectively. If the sequent* $A_1, \cdots, A_\mu \rightarrow B_1, \cdots, B_\nu$ *is provable, then some sequent* $A_i \rightarrow$ *is* m_i*-provable without cut or some sequent* $\rightarrow B_j$ *is* n_j*-provable without cut.*

PROOF. We prove by transfinite induction on the order of the proof-figure to $A_1, \cdots, A_\mu \rightarrow B_1, \cdots, B_\nu$. In case that the last rule of inference is not a cut we can prove in the same way as in Theorem 3. In case that the last rule of inference is a cut, let it be

$$\frac{A_1, \cdots, A_{\mu_1} \rightarrow B_1, \cdots, B_{\nu_1}, C \qquad C, A_{\mu_1+1}, \cdots, A_\mu \rightarrow B_{\nu_1+1}, \cdots, B_\nu}{A_1, \cdots, A_\mu \rightarrow B_1, \cdots, B_\nu}.$$

If $A_i \rightarrow$ is not m_i-provable without cut for every i $(1 \leqq i \leqq \mu)$ and $\rightarrow B_j$ is not n_j-provable without cut for every j $(1 \leqq j \leqq \nu)$, then both $\rightarrow C$ and $C\rightarrow$ are m-provable without cut by the assumption of transfinite indudtion, where m is the degree of C. This contradicts to Theorem 4.

6. THEOREM (*Consistency theorem in general form*).

The sequent '$\rightarrow$' *is not provable in* $\boldsymbol{Z}$.

PROOF. If the sequent '$\rightarrow$' is provable in $\boldsymbol{Z}$, then the last rule of inference to '$\rightarrow$' is of the form

$$\frac{\rightarrow F \qquad F\rightarrow}{\rightarrow}\ \text{cut}.$$

Therefore both sequents $\rightarrow F$ and $F\rightarrow$ are provable in $\boldsymbol{Z}$. From Theorem 5 then both $\rightarrow F$ and $F\rightarrow$ are finitely provable without cut. This contradicts to Theorem 4.

7. THEOREM (*Elimination theorem in general form*).

If a sequent $\Gamma \rightarrow \Delta$ *is provable in* $\boldsymbol{Z}$, *then* $\Gamma \rightarrow \Delta$ *is finitely provable without cut.*

PROOF. Let $\Gamma \rightarrow \Delta$ be $A_1, \cdots, A_\mu \rightarrow B_1, \cdots, B_\nu$. From Theorem 6 it follows that $\mu+\nu \neq 0$. Therefore some sequent $A_i \rightarrow$ is finitely provable without cut or some sequent $\rightarrow B_j$ is finitely provable without cut. In each case $\Gamma \rightarrow \Delta$ is finitely provable without cut.

8. THEOREM (*Decidability Theorem*).

Let F be an arithmetical formula of the degree n. Then either $\rightarrow F$ or $F \rightarrow$ is n-provable without cut.

PROOF. By Lemma 2 the sequent $F \rightarrow F$ is $2n$-provable without cut. Therefore we obtain Theorem 8 from Theorem 3 and Theorem 1.

9. DEFINITION. We say that a system $\boldsymbol{S}$ contains the system $\boldsymbol{Z}$, when the system $\boldsymbol{S}$ satisfies the following conditions.

(1) Terms and formulas in $\boldsymbol{Z}$ are also terms and formulas in $\boldsymbol{S}$ respectively.

(2) Provable sequents in $\boldsymbol{Z}$ are also provable in $\boldsymbol{S}$.

10. THEOREM. *Let F be an arithmetical formula and $\boldsymbol{S}$ contains the system $\boldsymbol{Z}$. Then $\rightarrow F$ or $F \rightarrow$ is provable in $\boldsymbol{S}$.*

§ **3.** A consistency-proof of G. Gentzen's LK with number-theoretic axioms containing the complete induction.

1. We obtain G. Gentzen's LK with number-theoretic axioms containing the complete induction by modifying the system $\boldsymbol{Z}$ as follows.

(1) To symbols we add free variables a, b, c etc.

(2) In the construction rule of terms we add '*free variables are terms*'.

(3) Beginning sequents are the following, excluding those of $\boldsymbol{Z}$.

(3.1) *Arithmetical beginning sequents* are the following:

$$\rightarrow t=t; \quad s=t \rightarrow t=s; \quad s=t, t=u \rightarrow s=u;$$

$$s'=t' \rightarrow s=t; \quad s=t \rightarrow s'=t';$$

$$\rightarrow t+0=t; \quad \rightarrow s+t'=(s+t)';$$

$$\rightarrow t \cdot 0=0; \quad \rightarrow s \cdot t'=s \cdot t+s;$$

where s, t and u are arbitrary terms.

(3.2) *Logical beginning sequents* are sequents of the form

$$D \rightarrow D$$

where D is an arbitrary formula.

(4) Rules of inference $\forall$-in antecedent and $\forall$-in succedent in $\boldsymbol{Z}$ are omitted. And we introduce new rules of inference:

$\forall$-in antecedent

$$\frac{F(t), \Gamma \rightarrow \Delta}{\forall x F(x), \Gamma \rightarrow \Delta}$$

where t is an arbitrary term.

$\forall$-in succedent

$$\frac{\Gamma \rightarrow \Delta, F(a)}{\Gamma \rightarrow \Delta, \forall x F(x)}$$

where a is a free variable not contained in the lower sequent.

CI (complete induction)

$$\frac{F(a), \Gamma \rightarrow \Delta, F(a')}{F(0), \Gamma \rightarrow \Delta, F(t)},$$

where t is an arbitrary term and a is a free variable not contained in the lower sequent.

In what follows we denote the system given here by LK. To distinguish between proof-figures in LK and proof-figures in $\boldsymbol{Z}$ we use the terminologies LK-proof-figures and $\boldsymbol{Z}$-proof-figures.

2. In the following manner an ordinal number smaller than ω^2 corresponds to every sequent in an LK-proof-figure. This is called the *order of the sequent.* The order of the end-sequent is called the *order of the proof-figure.*

(1) The order of a beginning sequent is ω.

(2) The order of the lower sequent of a structural rule of inference is equal to that of the upper sequent.

(3) In a logical rule of inference with one upper sequent or a rule of inference CI, the order of the lower sequent is $\alpha+\omega$, where α is the order of the upper sequent.

(4) In a logical rule of inference with two upper sequents or a cut the order of the lower sequent is $\max(\alpha_1, \alpha_2)+\omega$, where α_1 and α_2 are orders of two upper sequents.

It is clear that order of every LK-proof-figure is smaller than ω^2.

3. We transform an LK-proof-figure to a $\boldsymbol{Z}$-proof-figure.

When Γ is $A_1(a_1, \cdots, a_m), \cdots, A_\mu(a_1, \cdots, a_m)$, so we express $A_1(t_1, \cdots, t_m), \cdots, A_\mu(t_1, \cdots, t_m)$ by $\Gamma(t_1, \cdots, t_m)$.

THEOREM. *Let $\Gamma(a_1, \cdots, a_m) \rightarrow \Delta(a_1, \cdots, a_m)$ be a sequent not containing free variables except $a_1, \cdots, a_m$ and be α-provable in* LK. *If $\boldsymbol{n}_1, \cdots, \boldsymbol{n}_m$ are arbitrary numerals, then we have a $\boldsymbol{Z}$-proof-figure to the sequent $\Gamma(\boldsymbol{n}_1, \cdots, \boldsymbol{n}_m) \rightarrow \Delta(\boldsymbol{n}_1, \cdots, \boldsymbol{n}_m)$ with an order β $(<\alpha)$.*

PROOF. We prove by induction on the number of rules of inference in the LK-proof-figure to the sequent $\Gamma(a_1, \cdots, a_m) \rightarrow \Delta(a_1, \cdots, a_m)$.

In case that $\Gamma(a_1, \cdots, a_m) \rightarrow \Delta(a_1, \cdots, a_m)$ is a beginning sequent, the sequent $\Gamma(\boldsymbol{n}_1, \cdots, \boldsymbol{n}_m) \rightarrow \Delta(\boldsymbol{n}_1, \cdots, \boldsymbol{n}_m)$ is finitely provable in $\boldsymbol{Z}$. In case that $\Gamma(a_1, \cdots, a_m) \rightarrow \Delta(a_1, \cdots, a_m)$ is not a beiginning sequent, we denote the last rule of inference by $\mathfrak{L}$. If $\mathfrak{L}$ is a structural rule of inference, then it is clear.

We have only to prove in cases where $\mathfrak{L}$ is $\forall$-in antecedent, $\forall$-in succedent, or CI. In other cases we can prove similarly.

In case that $\mathfrak{L}$ is $\forall$-in antecedent, it is of the form

$$\frac{F(t(a_1, \cdots, a_m), a_1, \cdots, a_m), \Pi(a_1, \cdots, a_m) \rightarrow \Delta(a_1, \cdots, a_m)}{\forall x F(x, a_1, \cdots, a_m), \Pi(a_1, \cdots, a_m) \rightarrow \Delta(a_1, \cdots, a_m)}.$$

If the order of the upper sequent is α_1, then $\alpha=\alpha_1+\omega$. By the assumption of the induction the sequent

$$F(t(\boldsymbol{n}_1,\cdots,\boldsymbol{n}_m),\boldsymbol{n}_1,\cdots,\boldsymbol{n}_m),\Pi(\boldsymbol{n}_1,\cdots,\boldsymbol{n}_m)\rightarrow\Delta(\boldsymbol{n}_1,\cdots,\boldsymbol{n}_m)$$

is γ-provable in $\boldsymbol{Z}$ ($\gamma<\alpha_1<\alpha$). Let $\boldsymbol{n}$ be the numerical value of $t(\boldsymbol{n}_1,\cdots,\boldsymbol{n}_m)$. From Lemma 2 in § 2 the sequent

$$F(\boldsymbol{n},\boldsymbol{n}_1,\cdots,\boldsymbol{n}_m)\rightarrow F(t(\boldsymbol{n}_1,\cdots,\boldsymbol{n}_m),\boldsymbol{n}_1,\cdots,\boldsymbol{n}_m)$$

is $2d$-provable in $\boldsymbol{Z}$, where d is the degree of $F(\boldsymbol{n},\boldsymbol{n}_1,\cdots,\boldsymbol{n}_m)$. Hence the sequent

$$F(\boldsymbol{n},\boldsymbol{n}_1,\cdots,\boldsymbol{n}_m),\Pi(\boldsymbol{n}_1,\cdots,\boldsymbol{n}_m)\rightarrow\Delta(\boldsymbol{n}_1,\cdots,\boldsymbol{n}_m)$$

is $(\max(2d,\gamma)+1)$-provable in $\boldsymbol{Z}$ ($\max(2d,\gamma)+1<\alpha$). Therefore the sequent

$$\forall xF(x,\boldsymbol{n}_1,\cdots,\boldsymbol{n}_m),\Pi(\boldsymbol{n}_1,\cdots,\boldsymbol{n}_m)\rightarrow\Delta(\boldsymbol{n}_1,\cdots\boldsymbol{n}_m)$$

is $(\max(2d,\gamma)+1+1)$-provable in $\boldsymbol{Z}$ ($\max(2d,\gamma)+1+1<\alpha$).

In case that $\mathfrak{L}$ is $\forall$-in succedent, it is of the form

$$\frac{\Gamma(a_1,\cdots,a_m)\rightarrow\Lambda(a_1,\cdots,a_m),F(a,a_1,\cdots,a_m)}{\Gamma(a_1,\cdots,a_m)\rightarrow\Lambda(a_1,\cdots,a_m),\forall xF(x,a_1,\cdots,a_m)}\,.$$

If the order of the upper sequent is α_1, then $\alpha=\alpha_1+\omega$. By the assumption of the induction the sequent

$$\Gamma(\boldsymbol{n}_1,\cdots,\boldsymbol{n}_m)\rightarrow\Lambda(\boldsymbol{n}_1,\cdots,\boldsymbol{n}_m),F(\boldsymbol{n},\boldsymbol{n}_1,\cdots,\boldsymbol{n}_m)$$

is γ_n-provable in $\boldsymbol{Z}$ ($\gamma_n<\alpha_1<\alpha$) for every numeral $\boldsymbol{n}$. Therefore we have a $\boldsymbol{Z}$-proof-figure of the order $\lim\limits_{n}\gamma_n+1(\leqq\alpha_1+1<\alpha)$ to the sequent

$$\Gamma(\boldsymbol{n}_1,\cdots,\boldsymbol{n}_m)\rightarrow\Lambda(\boldsymbol{n}_1,\cdots,\boldsymbol{n}_m),\forall xF(x,\boldsymbol{n}_1,\cdots,\boldsymbol{n}_m)\,.$$

In case that $\mathfrak{L}$ is CI, it is of the form

$$\frac{F(a,a_1,\cdots,a_m),\Pi(a_1,\cdots,a_m)\rightarrow\Lambda(a_1,\cdots a_m),F(a',a_1,\cdots,a_m)}{F(0,a_1,\cdots,a_m),\Pi(a_1,\cdots,a_m)\rightarrow\Lambda(a_1,\cdots a_m),F(t(a_1,\cdots,a_m),a_1,\cdots,a_m)}\,.$$

If the order of the upper sequent is α_1, then $\alpha=\alpha_1+\omega$. By the assumption of the induction for every numeral $\boldsymbol{n}$ the sequent

$$F(\boldsymbol{n},\boldsymbol{n}_1,\cdots,\boldsymbol{n}_m),\Pi(\boldsymbol{n}_1,\cdots,\boldsymbol{n}_m)\rightarrow\Lambda(\boldsymbol{n}_1,\cdots,\boldsymbol{n}_m),F(\boldsymbol{n}',\boldsymbol{n}_1,\cdots,\boldsymbol{n}_m)$$

is γ_n-provable in $\boldsymbol{Z}$ ($\gamma_n<\alpha_1<\alpha$). Now the sequents

$$F(0,\boldsymbol{n}_1,\cdots,\boldsymbol{n}_m),\Pi(\boldsymbol{n}_1,\cdots,\boldsymbol{n}_m)\rightarrow\Lambda(\boldsymbol{n}_1,\cdots,\boldsymbol{n}_m),F(0',\boldsymbol{n}_1,\cdots,\boldsymbol{n}_m)$$

and

$$F(0',\boldsymbol{n}_1,\cdots,\boldsymbol{n}_m),\Pi(\boldsymbol{n}_1,\cdots,\boldsymbol{n}_m)\rightarrow\Lambda(\boldsymbol{n}_1,\cdots,\boldsymbol{n}_m),F(0'',\boldsymbol{n}_1,\cdots,\boldsymbol{n}_m)$$

are γ_0-and γ_1-provable in $\boldsymbol{Z}$. Therefore the sequent

$$F(0,\boldsymbol{n}_1,\cdots,\boldsymbol{n}_m),\Pi(\boldsymbol{n}_1,\cdots,\boldsymbol{n}_m)\rightarrow\Lambda(\boldsymbol{n}_1,\cdots,\boldsymbol{n}_m),F(0'',\boldsymbol{n}_1,\cdots,\boldsymbol{n}_m)$$

is $(\max(\gamma_0,\gamma_1)+1)$-provable in $\boldsymbol{Z}$ ($\max(\gamma_0,\gamma_1)+1<\alpha_1+1<\alpha$). Similarly for every numeral $\boldsymbol{z}$ the sequent

$$F(0, \boldsymbol{n}_1, \cdots, \boldsymbol{n}_m), \Pi(\boldsymbol{n}_1, \cdots, \boldsymbol{n}_m) \to \Lambda(\boldsymbol{n}_1, \cdots, \boldsymbol{n}_m), F(\boldsymbol{z}, \boldsymbol{n}_1, \cdots, \boldsymbol{n}_m)$$

is $(\max(\gamma_0, \gamma_1, \cdots, \gamma_z)+z)$-provable in $\boldsymbol{Z}$ $(\max(\gamma_0, \gamma_1, \cdots, \gamma_z)+z < \alpha_1+z < \alpha)$. Therefore when the numerical value of $t(\boldsymbol{n}_1, \cdots, \boldsymbol{n}_m)$ is $\boldsymbol{l}$, so the sequent

$$F(0, \boldsymbol{n}_1, \cdots, \boldsymbol{n}_m), \Pi(\boldsymbol{n}_1, \cdots, \boldsymbol{n}_m) \to \Lambda(\boldsymbol{n}_1, \cdots, \boldsymbol{n}_m), F(\boldsymbol{l}, \boldsymbol{n}_1, \cdots, \boldsymbol{n}_m)$$

is $(\max(\gamma_0, \gamma_1, \cdots, \gamma_l)+l)$-provable in $\boldsymbol{Z}$. Hence the sequent

$$F(0, \boldsymbol{n}_1, \cdots, \boldsymbol{n}_m), \Pi(\boldsymbol{n}_1, \cdots, \boldsymbol{n}_m) \to \Lambda(\boldsymbol{n}_1, \cdots, \boldsymbol{n}_m), F(t(\boldsymbol{n}_1, \cdots, \boldsymbol{n}_m), \boldsymbol{n}_1, \cdots, \boldsymbol{n}_m)$$

is $(\max(\gamma_0, \gamma_1, \cdots, \gamma_l)+l+2d+1)$-provable in $\boldsymbol{Z}$ $(\max(\gamma_0, \gamma_1, \cdots, \gamma_l)+l+2d+1 < \alpha)$ where d is the degree of $F(l, n_1, \cdots, \boldsymbol{n}_m)$.

4. We assume that the sequent '$\to$' is provable in LK.

Then we have a $\boldsymbol{Z}$-proof-figure to '$\to$' of an order smaller than ω^2. For $\boldsymbol{Z}$-proof-figures with order smaller than ω^2 we can prove Theorems 3, 4 and 5 in § 2 only by using transfinite induction to ω^2. For such proof-figures, therefore, we have Theorem 6 in § 2 only by using transfinite induction to ω^2. This contradicts to the assumption. Hence we proved without use of transfinite induction on ordinal numbers greater than ω^2 that the sequent '$\to$' is not provable in LK.

Reference

[1] K. Schütte, Beweistheoretische Erfassung der unendlichen Induktion in der Zahlentheorie, Math. Ann., 122 (1951), 369-389.