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On the Goldbach problem in an algebraic number field II.

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§ 4. Treatment of $I_s(\mu;\lambda)$ (I).

Let λ be a totally positive integer with sufficiently large norm $N(\lambda)$ and $\Omega(\lambda)$ be the set of all prime numbers ω such that

$$(4.1)\qquad \begin{aligned} &0<\omega^{(q)}\leqq\lambda^{(q)} && (q=1,2,\cdots,r_1),\\ &|\omega^{(p)}|\leqq|\lambda^{(p)}| && (p=r_1+1,\cdots,r_1+r_2).\end{aligned}$$

We shall define a trigonometrical sum

$$(4.2)\qquad S(z;\lambda)=\sum_{\omega\in\Omega(\lambda)}e^{2\pi iS(\omega z)},$$

where $z=(z_1,z_2,\cdots,z_n)$ is a point of E.

We know by (2.1) that $z_1,z_2,\cdots,z_n$ are written in the form

$$z_j=\sum_{k=1}^{n}x_k\delta_k^{(j)}\qquad (j=1,2,\cdots,n)$$

with real numbers $x_1,x_2,\cdots,x_n$. Taking $x_1,x_2,\cdots,x_n$ as variables, we consider an integral

$$(4.3)\qquad I_s(\mu:\lambda)=\int_{-1/2}^{1/2}\!\!\cdots\!\int S(z;\lambda)^s e^{-2\pi iS(\mu z)}dx_1dx_2\cdots dx_n,$$

where s is a positive rational integer, μ is a totally positive integer and the domain of integration is given by the conditions

$$|x_j|\leqq\frac{1}{2}\qquad (j=1,2,\cdots,n).$$

We see that $I_s(\mu;\lambda)$ is equal to the number of the s-tuples $(\omega_1,\omega_2,\cdots,\omega_s)$ of prime numbers which satisfy the following conditions:

$$\mu=\omega_1+\omega_2+\cdots+\omega_s,$$
$$\omega_j\in\Omega(\lambda)\qquad (j=1,2,\cdots,s).$$

Therefore, for any totally positive unit η we have

$$I_s(\eta\mu;\eta\lambda)=I_s(\mu;\lambda).$$

On the other hand, by suitable choice of a totally positive unit η_0 we have

$$c_1N(\lambda)^{1/n}<|\lambda^{(j)}\eta_0^{(j)}|<c_2N(\lambda)^{1/n}\qquad (j=1,2,\cdots,n).$$

Taking $\lambda\eta_0$ instead of λ, we shall assume that λ in (4.2) satisfies the inequalities

$$c_1 N(\lambda)^{1/n} < |\lambda^{(j)}| < c_2 N(\lambda)^{1/n} \qquad (j=1,2,\cdots,n).$$

If we put

$$N=\max(\lambda^{(1)},\cdots,\lambda^{(r_1)},|\lambda^{(r_1+1)}|,\cdots,|\lambda^{(n)}|),$$

then N is sufficiently large and the inequalities

$$cN<|\lambda^{(j)}|\leqq N \qquad (j=1,2,\cdots,n) \tag{4.4}$$

are satisfied.

Now we take positive constants $\sigma, \sigma_1, \sigma_2, \sigma_3\ \sigma_4$ and σ_5 such that

$$\sigma\geqq 3, \qquad \sigma_2>\sigma_4, \qquad \sigma_1+\sigma\geqq\sigma_4, \tag{4.5}$$

$$\min\left(\frac{\sigma_3}{n}-1, \frac{\sigma_4}{n}, \sigma_1-1\right)\geqq(4+n)\sigma+3r+8, \tag{4.6}$$

$$\min\left(\sigma_2, \frac{\sigma_5}{n}, \sigma_1-1\right)\geqq(4+n)\sigma+3r+12+\sigma_3 \qquad (r=r_1+r_2-1). \tag{4.7}$$

(We can easily find such constants.) Putting

$$H=\frac{N}{(\log N)^{\sigma_1}}, \qquad T=(\log N)^{\sigma_2}, \tag{4.8}$$

we consider the Farey division of E with respect to (H, T). In this and following paragraphs we shall always use the notations H and T in the meaning of (4.8).

We shall now define a division of E, which is slightly different from the Farey division with respect to (H, T).

Let Γ be the set of numbers γ of K such that $(\gamma^{(1)},\gamma^{(2)},\cdots,\gamma^{(n)})\in E$ and $\gamma\to\mathfrak{a}$ with $N(\mathfrak{a})\leqq T^n$. For every $\gamma\in\Gamma$ we define a domain $B_\gamma\subset E$ as follows:

$$B_\gamma=\Big\{z\,;z\in E,\ |z_j-\gamma_1^{(j)}|\leqq\frac{T^{n-1}}{H}\ (j=1,2,\cdots,n) \quad \text{for any } \gamma_1\equiv\gamma \pmod{\mathfrak{d}^{-1}}\Big\} \tag{4.9}$$

and put $B^0=E-\bigcup_{\gamma\in\Gamma}B_\gamma$.

Let E_γ be a domain defined by (2.4). If $z\in E_\gamma$, then we have

$$|z_j-\gamma_0^{(j)}|\leqq\frac{T^{n-1}}{HN(\mathfrak{a})}\leqq\frac{T^{n-1}}{H} \qquad (j=1,2,\cdots,n)$$

for a certain γ_0 such that $\gamma_0\equiv\gamma \pmod{\mathfrak{d}^{-1}}$. Hence we have

$$B^0\subset E^0, \qquad B_\gamma\supset E_\gamma \qquad (\gamma\in\Gamma).$$

Moreover, we shall prove that

$$B_{\gamma_1}\cap B_{\gamma_2}=\phi \qquad (\gamma_1,\gamma_2\in\Gamma,\gamma_1\neq\gamma_2). \tag{4.10}$$

If $B_{\gamma_1}\cap B_{\gamma_2}\neq\phi$, then there would exist a point $z\in B_{\gamma_1}\cap B_{\gamma_2}$ and, choosing suitably γ_1^0 and γ_2^0 such that $\gamma_1^0 \equiv \gamma_1 \pmod{\mathfrak{d}^{-1}}$ and $\gamma_2^0 \equiv \gamma_2 \pmod{\mathfrak{d}^{-1}}$, we have

$$|\gamma_1^{0(j)} - \gamma_2^{0(j)}| \leqq |z_j - \gamma_1^{0(j)}| + |z_j - \gamma_2^{0(j)}| \leqq \frac{2T^{n-1}}{H} \qquad (j = 1, 2, \cdots, n). \tag{4.11}$$

Let $\mathfrak{a}_1$ and $\mathfrak{a}_2$ be the denominators of γ_1 and γ_2 respectively, then $\gamma_1^0 - \gamma_2^0 \equiv (\mathfrak{d}\mathfrak{a}_1\mathfrak{a}_2)^{-1}$. Since $\gamma_1^0 \neq \gamma_2^0$, (4.11) would give

$$\frac{2^n T^{n(n-1)}}{H^n} \geqq |N(\gamma_1^0 - \gamma_2^0)| \geqq \frac{1}{N(\mathfrak{d}\mathfrak{a}_1\mathfrak{a}_2)} \geqq \frac{1}{DT^{2n}}.$$

But this inequality is not true for sufficiently large N and so (4.10) is proved.

In the integral of (4.3) we shall change the variables of integration $x_1, x_2, \cdots, x_n$ into $X_1(z), X_2(z), \cdots, X_n(z)$. Then we have

$$I_s(\mu; \lambda) = 2^{r_2}\sqrt{D} \int \underset{\mathfrak{B}}{\cdots} \int S(z; \lambda)^s e^{-2\pi i S(\mu z)} dx(z), \tag{4.12}$$

where

$$\mathfrak{B} = \{x(z); (z_1, z_2, \cdots, z_n) \in E\}$$

and we write

$$dx(z) = dX_1(z) dX_2(z) \cdots dX_n(z).$$

Now we define subdomains of $\mathfrak{B}$ as follows:

$$\mathfrak{B}_\gamma = \{x(z); (z_1, z_2, \cdots, z_n) \in B_\gamma\} \qquad (\gamma \in \Gamma)$$

$$\mathfrak{B}^0 = \mathfrak{B} - \bigcup_{\gamma \in \Gamma} \mathfrak{B}_\gamma.$$

Then we see that

$$\mathfrak{B}_{\gamma_1} \cap \mathfrak{B}_{\gamma_2} = \phi \qquad (\gamma_1, \gamma_2, \in \Gamma, \gamma_1 \neq \gamma_2)$$

and we write

$$I_s(\mu; \lambda) = 2^{r_2}\sqrt{D} \Big\{ \int \underset{\mathfrak{B}^0}{\cdots} \int + \sum_{\gamma \in \Gamma} \int \underset{\mathfrak{B}_\gamma}{\cdots} \int \Big\} S(z; \lambda)^s e^{-2\pi i S(\mu z)} dx(z). \tag{4.13}$$

In the following paragraphs § 5 and § 6, we shall estimate the trigonometrical sum $S(z; \lambda)$ on $\mathfrak{B}^0$ and $\mathfrak{B}_\gamma$ $(\gamma \in \Gamma)$ respectively.

§ 5. Estimation of $S(z; \lambda)$ (I).

In this paragraph we assume that $z = (z_1, z_2, \cdots, z_n)$ belongs to E^0 which is defined by the Farey division with respect to (H, T).

Let $\mathfrak{M}_0$ be the set of all integers ν of K which satisfy the following conditions:

$$\frac{N}{(\log N)^\sigma} < \nu^{(q)} \leqq \lambda^{(q)} \qquad (q = 1, 2, \cdots, r_1),$$

$$\frac{N}{(\log N)^\sigma} < |\nu^{(p)}| \leqq |\lambda^{(p)}| \qquad (p = r_1 + 1, \cdots, r_1 + r_2).$$

Since N is sufficiently large, we see from (4.4) that $\mathfrak{M}_0$ is not empty.

Let $\mathfrak{N}$ be the product of all prime ideals $\mathfrak{p}$ with $N(\mathfrak{p}) \leqq N^{n/2}$, then an integer $\nu \in \mathfrak{M}_0$ which is prime to $\mathfrak{N}$ must be a prime number. Therefore, we have

(5.1)
$$\sum_{\omega \in \mathfrak{M}_0} e^{2\pi i S(\omega z)} = \sum_{\substack{\nu \in \mathfrak{M}_0 \\ (\nu, \mathfrak{N}) = 1}} e^{2\pi i S(\nu z)},$$

where the left-hand side is a sum taken over all prime numbers ω in $\mathfrak{M}_0$ and the right-hand side is a sum taken over integers ν of $\mathfrak{M}_0$ such that $(\nu, \mathfrak{N}) = 1$.

Using Möbius function $\mu(\mathfrak{a})$ for ideals, we can write the right-hand side of (5.1) as follows:

$$\sum_{\substack{\nu \in \mathfrak{M}_0 \\ (\nu, \mathfrak{N}) = 1}} e^{2\pi i S(\nu z)} = \sum_{\nu \in \mathfrak{M}_0} e^{2\pi i S(\nu z)} \sum_{\mathfrak{a} | (\nu, \mathfrak{N})} \mu(\mathfrak{a})$$
$$= \sum_{\mathfrak{a} | \mathfrak{N}} \mu(\mathfrak{a}) \sum_{\substack{\mathfrak{a} | (\nu) \\ \nu \in \mathfrak{M}_0}} e^{2\pi i S(\nu z)} = \sum_{\mathfrak{a} | \mathfrak{N}} \mu(\mathfrak{a}) \sum_{\substack{\nu \in \mathfrak{M}_0 \\ \nu \in \mathfrak{a}}} e^{2\pi i S(\nu z)}.$$

Therefore, putting for any ideal $\mathfrak{a}$

$$I(\mathfrak{a}) = \sum_{\substack{\nu \in \mathfrak{M}_0 \\ \nu \in \mathfrak{a}}} e^{2\pi i S(\nu z)},$$

we have

(5.2)
$$S(z\,;\lambda) = \sum_{\substack{\mathfrak{a} | \mathfrak{N} \\ N\mathfrak{a} \leqq N^n}} \mu(\mathfrak{a}) I(\mathfrak{a}) + O\left(\frac{N^n}{(\log N)^\sigma}\right)$$
$$= \sum_{\substack{\mathfrak{a} | \mathfrak{N} \\ N\mathfrak{a} \leqq N^n,\ \mu(\mathfrak{a}) = 1}} I(\mathfrak{a}) - \sum_{\substack{\mathfrak{a} | \mathfrak{N} \\ N\mathfrak{a} \leqq N^n,\ \mu(\mathfrak{a}) = -1}} I(\mathfrak{a}) + O\left(\frac{N^n}{(\log N)^\sigma}\right).$$

Now we shall consider the first sum in the right-hand side of (5.2). We put

$$S_0 = \sum_{\substack{\mathfrak{a} | \mathfrak{N} \\ N\mathfrak{a} \leqq N^n,\ \mu(\mathfrak{a}) = 1}} I(\mathfrak{a})$$

and

$$\tau = \frac{N^n}{(\log N)^{\sigma_3}}$$

and divide S_0 into three parts:

$$S_0 = \sum_{N\mathfrak{a} \leqq (\log N)^{\sigma_4}} + \sum_{(\log N)^{\sigma_4} < N\mathfrak{a} \leqq \tau} + \sum_{\tau < N\mathfrak{a} \leqq N^n} = S_1 + S_2 + S_3,$$

where σ_3 and σ_4 are the constants defined in the previous paragraph.

We shall estimate these sums S_j $(j = 1, 2, 3)$ one after another.

(i) Estimation of S_1.

Let $\mathfrak{C}$ be an ideal class of K. We define a sum

$$S(\mathfrak{C}) = \sum_{\substack{N\mathfrak{a} \leqq (\log N)^{\sigma_4} \\ \mathfrak{a} \in \mathfrak{C}}} | I(\mathfrak{a}) |,$$

where $\mathfrak{a}$ runs through all ideals belonging to $\mathfrak{C}$ with $N(\mathfrak{a}) \leqq (\log N)^{\sigma}$. Then we have

$$|S_1| \leqq \sum_{\mathfrak{C}} S(\mathfrak{C}), \tag{5.3}$$

where $\mathfrak{C}$ runs through all ideal classes of K. Therefore it suffices to estimate $S(\mathfrak{C})$.

Let $\mathfrak{a}_0$ be an ideal belonging to $\mathfrak{C}$, then each ideal $\mathfrak{a}$ in $\mathfrak{C}$ is the product of $\mathfrak{a}_0$ and a certain number $\alpha \in \mathfrak{a}_0^{-1}$, that is,

$$\mathfrak{a} = \alpha \mathfrak{a}_0 \qquad (\alpha \in \mathfrak{a}_0^{-1}). \tag{5.4}$$

Moreover, we may assume that α in (5.4) satisfies the inequalities

$$c_0 \leqq |\alpha^{(j)}| \leqq cN(\mathfrak{a})^{1/n} \qquad (j = 1, 2, \cdots, n).$$

Let $\rho_1, \rho_2, \cdots, \rho_n$ be a basis of $\mathfrak{a}_0$ such that

$$|\rho_j^{(k)}| \leqq c \qquad (j, k = 1, 2, \cdots, n),$$

then $\alpha\rho_1, \alpha\rho_2, \cdots, \alpha\rho_n$ is a basis of $\mathfrak{a} = \alpha\mathfrak{a}_0$ satisfying the inequalities

$$|\alpha^{(k)}\rho_j^{(k)}| \leqq cN(\mathfrak{a})^{1/n} \qquad (j, k = 1, 2, \cdots, n).$$

Therefore, by Lemma 3.6, we have

$$I(\mathfrak{a}) \ll N^{n-1} \min_{1 \leqq j \leqq n} (N, \|S(\alpha\rho_j z)\|^{-1})$$

so that

$$S(\mathfrak{C}) \ll N^{n-1} \sum_{\substack{\alpha \in \mathfrak{a}_0^{-1} \\ c_0 \leqq |\alpha| \leqq c(\log N)^{\sigma_4/n}}} \min_{1 \leqq j \leqq n} (N, \|S(\alpha\rho_j z)\|^{-1}), \tag{5.5}$$

where α runs through all elements of $\mathfrak{a}_0^{-1}$ such that

$$c_0 \leqq |\alpha^{(j)}| \leqq c(\log N)^{\sigma_4/n} \qquad (j = 1, 2, \cdots, n).$$

If we put $V = c_1(\log N)^{\sigma_4/n}$ for a suitable positive number c_1, then the inequality (3.61) holds, on account of the inequality $\sigma_2 > \sigma_4$ in (4.5). Therefore, we can apply Theorem 3.2 to the estimation of the sum in the right-hand side of (5.5) and, putting

$$V = c_1(\log N)^{\sigma_4/n}, \qquad U = N, \qquad \mathfrak{c} = 1$$

in (3.63), we obtain

$$S(\mathfrak{C}) \ll N^n(\log N)^{\sigma_4}\left(\frac{1}{(\log N)^{\sigma_1}} + \frac{1}{(\log N)^{\sigma_1 - 1 + \sigma_4/n}} + \frac{\log N}{N}\right),$$

whence follows

$$S(\mathfrak{C}) \ll \frac{N^n}{(\log N)^{\sigma}}$$

on account of (4.5) and (4.6).

Thus we have

$$S_1 \ll \frac{N^n}{(\log N)^{\sigma}}.$$

(ii) Estimation of S_2.

We define two sets of ideals as follows:

$$M_1=\left\{\mathfrak{a}\,;\,(\log N)^{\sigma_3-n\sigma_5}\leqq N(\mathfrak{a})\leqq\frac{N^n}{(\log N)^{\sigma_4}}\right\},$$

$$M_2=\left\{\mathfrak{a}\,;\,\mathfrak{a}\mid\mathfrak{N},\quad \mu(\mathfrak{a})=1,\quad (\log N)^{\sigma_4}<N(\mathfrak{a})\leqq\frac{N^n}{(\log N)^{\sigma_3}}\right\}.$$

Then we have

$$S_2=\sum_{\mathfrak{b}\in M_2}\sum_{\substack{\nu\in\mathfrak{M}_0\\ \nu\in\mathfrak{b}}}e^{2\pi iS(\nu z)}=\sum_{\mathfrak{b}\in M_2}\sum_{\substack{\nu\in\mathfrak{M}_0\\ \frac{(\nu)}{\mathfrak{b}}\in M_1}}e^{2\pi iS(\nu z)}.$$

This is a sum of the type treated in Theorem 3.5. Therefore, putting

$$N_0=\frac{N}{(\log N)^{\sigma}},\quad U_2^*=\frac{N^n}{(\log N)^{\sigma_4}},\quad V_1^*=(\log N)^{\sigma_4},\quad V_2^*=\frac{N^n}{(\log N)^{\sigma_3}},\quad \mathfrak{c}=1$$

in (3.97) and noting that the condition (3.66) is satisfied, we obtain

$$S_2\ll N^n(\log N)^{\frac{n\sigma}{4}+\frac{3r}{4}+2}$$
$$\times\left(\frac{1}{(\log N)^{\sigma_2}}+\frac{1}{(\log N)^{\sigma_4/n}}+\frac{(\log N)^{\sigma_1}}{N}+\frac{1}{(\log N)^{\sigma_1-1}}+\frac{1}{(\log N)^{\frac{\sigma_3}{n}-1}}\right)^{1/4},$$

whence follows

$$S_2\ll\frac{N^n}{(\log N)^{\sigma}}$$

on account of (4.6) and (4.7).

(iii) Estimation of S_3.

We shall put

$$A=\left\{\mathfrak{a}\,;\,\mathfrak{a}\mid\mathfrak{N},\quad \mu(\mathfrak{a})=1,\quad \frac{N^n}{(\log N)^{\sigma_3}}<N(\mathfrak{a})\leqq N^n\right\},$$

then we have

$$S_3=\sum_{\mathfrak{a}\in A}\sum_{\substack{\nu\in\mathfrak{M}_0\\ \nu\in\mathfrak{a}}}e^{2\pi iS(\nu z)}$$

(5.6)

$$=\sum_{1\leqq N\mathfrak{a}\leqq(\log N)^{\sigma_3}}\sum_{\substack{\nu\in\mathfrak{M}_0\\ \frac{(\nu)}{\mathfrak{a}}\in A}}e^{2\pi iS(\nu z)}.$$

We denote by A_j the set of the ideals of A which are divisible by exact j prime divisors whose norms exceed $(\log N)^{\sigma_5}$ with σ_5 in (4.7) and divide the inner sum in the last term of (5.6) as follows:

$$\sum_{\substack{\nu\in\mathfrak{M}_0\\ \frac{(\nu)}{\mathfrak{a}}\in A}}e^{2\pi iS(\nu z)}=\sum_{j}\sum_{\substack{\nu\in\mathfrak{M}_0\\ \frac{(\nu)}{\mathfrak{a}}\in A_j}}e^{2\pi i(\nu z)}.$$

The range of the indices j of A_j is given as follows:

$$0 \leqq j \leqq \log N.$$

Let ν be an integer of $\mathfrak{M}_0$ such that $(\nu)/\mathfrak{a} \in A_0$ with $1 \leqq N(\mathfrak{a}) \leqq (\log N)^{\sigma_5}$. If this $(\nu)/\mathfrak{a}$ has k prime divisors, then

$$(\log N)^{k\sigma_5} \geqq \tau$$

so that

$$k \geqq \frac{\log N}{2\sigma_5 \log\log N}. \tag{5.7}$$

If we denote by $\tau(\mathfrak{b})$ the number of the divisors of ideal $\mathfrak{b}$, then (5.7) gives

$$\tau\left(\frac{(\nu)}{\mathfrak{a}}\right) = 2^k > N^{\frac{1}{4\sigma_5 \log\log N}}.$$

Therefore, we have

$$\Big|\sum_{\substack{\nu \in \mathfrak{M}_0 \\ \frac{(\nu)}{\mathfrak{a}} \in A_0}} e^{2\pi i S(\nu z)}\Big| \cdot N^{\frac{1}{4\sigma_5 \log\log N}} \leqq \sum_{\substack{\nu \in \mathfrak{M}_0 \\ \frac{(\nu)}{\mathfrak{a}} \in A_0}} \tau\left(\frac{(\nu)}{\mathfrak{a}}\right)$$

$$\leqq \sum_{\substack{\nu \in \mathfrak{M}_0 \\ \nu \in \mathfrak{a}}} \tau\left(\frac{(\nu)}{\mathfrak{a}}\right) = \sum_{\substack{\nu \in \mathfrak{M}_0 \\ \nu \in \mathfrak{a}}} \sum_{\mathfrak{b} \mid \frac{(\nu)}{\mathfrak{a}}} 1 = \sum_{1 \leqq N\mathfrak{b} \leqq \frac{N^n}{N\mathfrak{a}}} \sum_{\substack{\nu \in \mathfrak{M}_0 \\ \nu \in \mathfrak{b}\mathfrak{a}}} 1$$

$$\ll \frac{N^n}{N(\mathfrak{a})} \sum_{N\mathfrak{b} \leqq \frac{N^n}{N\mathfrak{a}}} \frac{1}{N(\mathfrak{b})} \ll \frac{N^n}{N(\mathfrak{a})} \log N.$$

Hence

$$\sum_{\substack{\nu \in \mathfrak{M}_0 \\ \frac{(\nu)}{\mathfrak{a}} \in A_0}} e^{2\pi i S(\nu z)} \ll \frac{N^n}{N(\mathfrak{a})} (\log N)^{-\sigma_5}. \tag{5.8}$$

Now we put

$$T_k(\mathfrak{a}) = \sum_{\substack{\nu \in \mathfrak{M}_0 \\ \frac{(\nu)}{\mathfrak{a}} \in A_k}} e^{2\pi i S(\nu z)} \qquad (k \geqq 1).$$

Moreover we define a set

$A_l^* = \{\mathfrak{a}\,;\, \mathfrak{a} \mid \mathfrak{N},\ \ \mu(\mathfrak{a}) = -1,\ \ N(\mathfrak{a}) \leqq N^n,\ \ \mathfrak{a}$ is divisible by exact l prime divisors whose norms exceed $(\log N)^{\sigma_5}\}$

and a sum

$$T_k^*(\mathfrak{a}) = \sum_{(\log N)^{\sigma_5} < N\mathfrak{p} \leqq N^{n/2}} \sum_{\substack{\nu \in \mathfrak{M}_0 \\ \frac{(\nu)}{\mathfrak{a}\mathfrak{p}} \in A^*_{k-1}}} e^{2\pi i S(\nu z)} \qquad (k \geqq 1). \tag{5.9}$$

where outer sum is taken over all prime ideals $\mathfrak{p}$ such that $(\log N)^{\sigma_5} < N(\mathfrak{p}) \leqq N^{n/2}$ and the inner sum is taken over all integers $\nu \in \mathfrak{M}_0$ such that $(\nu)/\mathfrak{a}\mathfrak{p} \in A^*_{k-1}$.

We shall divide the inner sum in (5.9) into two parts, by the condition $\nu \notin \mathfrak{a}\mathfrak{p}^2$ or $\nu \in \mathfrak{a}\mathfrak{p}^2$. Then we have the following estimations;

$$\sum_{(\log N)^{\sigma_s}<N\mathfrak{p}\leqq N^{n/2}} \sum_{\substack{\nu\in\mathfrak{M}_0 \\ \frac{(\nu)}{\mathfrak{a}\mathfrak{p}}\in A^*_{k-1},\ \nu\notin\mathfrak{a}\mathfrak{p}^2}} e^{2\pi i S(\nu z)}$$

$$= \sum_{\substack{\nu\in\mathfrak{M}_0 \\ \frac{(\nu)}{\mathfrak{a}}\in A_k}} e^{2\pi i S(\nu z)} \sum_{\substack{\mathfrak{p}\mid(\nu)/\mathfrak{a} \\ (\log N)^{\sigma_s}<N\mathfrak{p}\leqq N^{n/2}}} 1 = kT_k(\mathfrak{a})$$

and

$$\Big| \sum_{(\log N)^{\sigma_s}<N\mathfrak{p}\leqq N^{n/2}} \sum_{\substack{\nu\in\mathfrak{M}_0 \\ \frac{(\nu)}{\mathfrak{a}\mathfrak{p}}\in A^*_{k-1},\ \nu\in\mathfrak{a}\mathfrak{p}^2}} e^{2\pi i S(\nu z)} \Big|$$

$$\leqq \sum_{(\log N)^{\sigma_s}<N\mathfrak{p}\leqq N^{n/2}} \sum_{\substack{\nu\in\mathfrak{M}_0 \\ \nu\in\mathfrak{a}\mathfrak{p}^2}} 1 \ll \sum_{(\log N)^{\sigma_s}<N\mathfrak{p}\leqq N^{n/2}} \Big(1+\frac{N^n}{N(\mathfrak{a}\mathfrak{p}^2)}\Big)$$

$$\ll N^{n/2}+\frac{N^n}{N(\mathfrak{a})} \sum_{(\log N)^{\sigma_s}<N\mathfrak{p}} \frac{1}{N(\mathfrak{p})^2} \ll \frac{N^n}{N(\mathfrak{a})}(\log N)^{-\sigma_s}.$$

Therefore

$$T_k^*(\mathfrak{a}) = kT_k(\mathfrak{a})+O\Big(\frac{N^n}{N(\mathfrak{a})}(\log N)^{-\sigma_s}\Big).$$

From this result and (5.8) follows

$$\sum_{\substack{\nu\in\mathfrak{M}_0 \\ \frac{(\nu)}{\mathfrak{a}}\in A}} e^{2\pi i S(\nu z)} = \sum_{1\leqq k}\frac{1}{k}T_k^*(\mathfrak{a})+O\Big(\frac{N^n}{N(\mathfrak{a})}(\log N)^{-\sigma_s}\Big)$$

$$+O\Big(\frac{N^n}{N(\mathfrak{a})}(\log N)^{-\sigma_s}\sum_{1\leqq k\leqq \log N}\frac{1}{k}\Big)$$

$$= \sum_{1\leqq k}\frac{1}{k}T_k^*(\mathfrak{a})+O\Big(\frac{N^n}{N(\mathfrak{a})}(\log N)^{1-\sigma_s}\Big).$$

Putting this result in (5.6), we have

(5.10) $$S_3 = \sum_{1\leqq N\mathfrak{a}\leqq(\log N)^{\sigma_s}} \sum_{1\leqq k}\frac{1}{k}T_k^*(\mathfrak{a})+O(N^n(\log N)^{2-\sigma_s}).$$

Now we put

$$M = \{\mathfrak{p}\,;\,(\log N)^{\sigma_s} < N(\mathfrak{p}) \leqq N^{n/2}\},$$

then

$$T_k^*(\mathfrak{a}) = \sum_{\mathfrak{p}\in M} \sum_{\substack{\nu\in\mathfrak{M}_0 \\ \frac{(\nu)}{\mathfrak{a}\mathfrak{p}}\in A^*_{k-1}}} e^{2\pi i S(\nu z)}.$$

which is of the same type as was treated in Theorem 3.5. Therefore, putting

$$N_0=\frac{N}{(\log N)^{\sigma}},\quad U_2^*=N^n,\quad V_1^*=(\log N)^{\sigma_s},\quad V_2^*=N^{n/2},\quad \mathfrak{c}=\mathfrak{a}$$

in (3.97), we have

$$T_k^*(\mathfrak{a}) \ll \frac{N^n}{N(\mathfrak{a})^{3/4}} (\log N)^{\frac{n\sigma}{4}+\frac{3r}{4}+2}$$

$$\times\left(\frac{1}{(\log N)^{\sigma_3}}+\frac{1}{(\log N)^{\sigma_3/n}}+\frac{(\log N)^{\sigma_3/n}}{N}+\frac{1}{(\log N)^{\sigma_1-1}}+\frac{(\log N)^{1+\sigma_3/n}}{N^{1/2}}\right)^{1/4}$$

$$\ll \frac{N^n}{N(\mathfrak{a})^{3/4}} (\log N)^{\frac{n\sigma}{4}+\frac{3r}{4}+2}\left(\frac{1}{(\log N)^{\sigma_3}}+\frac{1}{(\log N)^{\sigma_3/n}}+\frac{1}{(\log N)^{\sigma_1-1}}\right)^{1/4}$$

and, putting then this result in (5.10), we obtain

$$S_3 \ll N^n (\log N)^{\frac{\sigma_3}{4}+\frac{n\sigma}{4}+\frac{3r}{4}+3}\left(\frac{1}{(\log N)^{\sigma_3}}+\frac{1}{(\log N)^{\sigma_3/n}}+\frac{1}{(\log N)^{\sigma_1-1}}\right)^{1/4}$$

$$+\frac{N^n}{(\log N)^{\sigma_3-2}},$$

whence follows

$$S_3 \ll \frac{N^n}{(\log N)^{\sigma}}$$

on account of (4.6) and (4.7).

Thus we finally obtain

$$S_0 \ll \frac{N^n}{(\log N)^{\sigma}}.$$

In the similar way, we can estimate the second sum in the right-hand side of (5.2).

Thus we have

THEOREM 5.1. *Let $S(z;\lambda)$ be the trigonometrical sum defined in* §4. *If z belongs to E^0, then we have*

$$S(z;\lambda) \ll \frac{N^n}{(\log N)^{\sigma}}$$

with $\sigma \geqq 3$.

§6. Estimation of $S(z;\lambda)$ (II).

We quote from [3] the prime number theorem in the slightly simple form:

LEMMA 6.1. *Let $\mathfrak{a}$ be an ideal and ρ be a totally positive integer prime to $\mathfrak{a}$. Let $N_1, N_2, \cdots, N_n$ be positive numbers such that*

$$N_{p'} = N_p \qquad (p = r_1+1, \cdots, r_1+r_2),$$

$$N_j \leqq N_k^{a} \qquad (j, k = 1, 2, \cdots, n)$$

with a large constant a. Moreover we take r_2 positive numbers ϑ_p $(r_1+1 \leqq p \leqq r_1+r_2)$ such that $0 < \vartheta_p \leqq 1$ $(p = r_1+1, \cdots, r_1+r_2)$.

We denote by $\pi(\mathfrak{a}, \rho; N, \vartheta) = \pi(\mathfrak{a}, \rho; N_1, \cdots, N_n; \vartheta_{r_1+1}, \cdots, \vartheta_{r_1+r_2})$ the number of prime numbers ω which satisfy the following conditions

$$\omega \equiv \rho \pmod{\mathfrak{a}},$$

$$0 < \omega^{(q)} \leqq N_q \qquad (q=1, 2, \cdots, r_1),$$
$$|\omega^{(p)}| \leqq N_p \qquad (p=r_1+1, \cdots, r_1+r_2),$$
$$0 \leqq \arg \omega^{(p)} < 2\pi\vartheta_p \qquad (p=r_1+1, \cdots, r_1+r_2).$$

Then we have

$$\pi(\mathfrak{a}, \rho; N, \vartheta) = \frac{w \prod_p \vartheta_p}{2^{r_1} hR\varphi(\mathfrak{a})} \int_2^{N_j^{e_j}} \!\!\cdots\! \int \frac{dt_1 dt_2 \cdots dt_{r+1}}{\log(t_1 t_2 \cdots t_{r+1})} \tag{6.1}$$
$$+O(N_1 N_2 \cdots N_n e^{-c\sqrt{\log(N_1 N_2 \cdots N_n)}}),$$

where h is the class number of K, R is the regulator of K, w is the number of the roots of unity in K, $\varphi(\mathfrak{a})$ is Euler's function for ideals and the domain of integration is defined as follows:

$$2 \leqq t_j \leqq N_j^{e_j} \qquad (j=1, 2, \cdots, r_1+r_2)$$

with $e_j=1$ $(j \leqq r_1)$, $=2$ $(j \geqq r_1+1)$, $r=r_1+r_2-1$ and the notation $\prod_p$ means a product over $p=r_1+1, \cdots, r_1+r_2$.

If $N(\mathfrak{a}) \leqq (\log(N_1 \cdots N_n))^A$ for a positive constant A, then the constants in the error term are independent of $\mathfrak{a}$.

From now on, we shall use the notations h, R, w and $\prod_p$ in the meaning of Lemma 6.1.

THEOREM 6.1. *Let $z=(z_1, z_2, \cdots, z_n)$ be a point belonging to B_Γ which is defined by (4.9) with $\gamma \to \mathfrak{a}$. We can choose a suitable number γ_0 such that $\gamma_0 \equiv \gamma \pmod{\mathfrak{d}^{-1}}$ and*

$$|z_j - \gamma_0^{(j)}| \leqq \frac{T^{n-1}}{H} \qquad (j=1, 2, \cdots, n).$$

We shall put $y_j = z_j - \gamma_0^{(j)}$ $(j=1, 2, \cdots, n)$.

Then we have

$$S(z; \lambda) = \frac{w\mu(\mathfrak{a})}{2^{r_1} hR\varphi(\mathfrak{a})} \int_0^1 \!\!\cdots\! \int \prod_p d\theta_p \int_{N^{e_j/2}}^{|\lambda^{(j)}|^{e_j}} \!\!\cdots\! \int \frac{e^{2\pi i S(\tilde{t}y)}}{\log(t_1 \cdots t_{r+1})} dt_1 \cdots dt_{r+1} \tag{6.2}$$
$$+O\left(\frac{N^n}{(\log N)^{a-b+1}}\right),$$

where the domain of the integration is given by the conditions

$$0 \leqq \theta_p \leqq 1 \qquad (p=r_1+1, \cdots, r_1+r_2),$$
$$N^{e_j/2} \leqq t_j \leqq |\lambda^{(j)}|^{e_j} \qquad (j=1, 2, \cdots, r_1+r_2)$$

with $e_j=1$ $(j \leqq r_1)$, $=2$ $(j > r_1)$. In the integrand, we put

$$\tilde{t}_q = t_q \qquad (q=1, 2, \cdots, r_1),$$
$$\left.\begin{aligned}\tilde{t}_p &= \sqrt{t_p}\, e^{2\pi i\theta_p} \\ \tilde{t}_{p'} &= \sqrt{t_p}\, e^{-2\pi i\theta_p}\end{aligned}\right\} \qquad (p=r_1+1, \cdots, r_1+r_2).$$

Moreover, $b=(n-1)\sigma_2+\sigma_1$ *and we can take* $\mathfrak{a}$ *sufficiently large.*

PROOF. We shall divide the sum $S(z;\lambda)$ into two parts;

$$S(z;\lambda)=\sum_{\substack{(\omega,\mathfrak{a})=1\\ \omega\in\Omega(\lambda)}}+\sum_{\substack{(\omega,\mathfrak{a})\neq 1\\ \omega\in\Omega(\lambda)}}=S_1+S_2.$$

First we have

$$|S_2|\leqq\sum_{\substack{|\omega|\leqq N\\ (\omega)|\mathfrak{a}}}1,$$

where the sum is taken over all prime numbers ω such that $(\omega)|\mathfrak{a}$ and $|\omega^{(j)}|\leqq N$ $(j=1,2,\cdots,n)$. Denoting by $\sum_{(\omega)}$ a sum taken over all prime principal ideals dividing $\mathfrak{a}$, we have

$$|S_2|\leqq\sum_{(\omega)}\sum_{|\varepsilon\omega|\leqq N}1,$$

where the inner sum gives the number of units ε such that $|\varepsilon^{(j)}\omega^{(j)}|\leqq N$ $(j=1,2,\cdots,n)$. Therefore, applying Lemma 3.4, we have

$$\begin{aligned} S_2 &\ll(\log N)^r\sum_{\mathfrak{p}|\mathfrak{a}}1 \ll(\log N)^r\log(N(\mathfrak{a})+1)\\ &\ll(\log N)^{r+1}. \end{aligned}\tag{6.3}$$

Now we shall consider S_1.

Let $\Omega_1(\lambda)$ be the set of prime numbers ω which satisfy the following conditions:

$$\sqrt{N}<\omega^{(q)}\leqq\lambda^{(q)} \qquad (q=1,2,\cdots,r_1),$$

$$\sqrt{N}<|\omega^{(p)}|\leqq|\lambda^{(p)}| \qquad (p=r_1+1,\cdots,r_1+r_2).$$

Then we have

$$\begin{aligned} S_1 &=\sum_{\substack{(\omega,\mathfrak{a})=1\\ \omega\in\Omega_1(\lambda)}}e^{2\pi i S(\omega z)}+O(N^{n-1/2})\\ &=S_1'+O(N^{n-1/2}). \end{aligned}\tag{6.4}$$

We denote by ρ the elements of the complete system of residues mod $\mathfrak{a}$ which are totally positive and prime to $\mathfrak{a}$. If the summation $\sum_\rho$ is used for the sum over these ρ, then

$$\begin{aligned} S_1' &=\sum_\rho e^{2\pi i S(\rho\gamma)}\sum_{\substack{\omega\equiv\rho(\mathfrak{a})\\ \omega\in\Omega_1(\lambda)}}e^{2\pi i S(\omega y)}\\ &=\sum_\rho e^{2\pi i S(\rho\gamma)}S_\rho(y). \end{aligned}\tag{6.5}$$

Now, we shall divide two intervals $[\sqrt{N},N]$ and $[0,1]$ as follows:

$$M_0=\sqrt{N}<M_1<M_2<\cdots<M_{l-1}<M_l=N,$$

$$\Theta_0=0<\Theta_1<\Theta_2<\cdots<\Theta_{m-1}<\Theta_m=1,$$

where

$$M_{j+1}-M_j \ll \frac{N}{(\log N)^a} \qquad (j=0,1,\cdots,l-1),$$

(6.6)
$$\Theta_{j+1}-\Theta_j \ll \frac{1}{(\log N)^a} \qquad (j=0,1,\cdots,m-1),$$

$$l \ll (\log N)^a, \qquad m \ll (\log N)^a$$

with $a > b = (n-1)\,\sigma_2+\sigma_1$. Moreover we assume that each of the $\lambda^{(1)},\cdots,\lambda^{(r_1)}$, $|\lambda^{(r_1+1)}|,\cdots,|\lambda^{(r_1+r_2)}|$ is equal to one of $M_0,\cdots,M_l$.

By these divisions of $[\sqrt{N}, N]$ and $[0,1]$ the set $\Omega_1(\lambda)$ is divided into $O((\log N)^{an})$ subsets each of which consists of the prime numbers ω such that

$$M_{i_{q-1}} < \omega^{(q)} \leqq M_{i_q} \qquad (q=1,2,\cdots,r_1),$$

$$\left.\begin{array}{l} M_{i_{p-1}} < |\,\omega^{(p)}\,| \leqq M_{i_p} \\ 2\pi\Theta_{j_{p-1}} < \arg\,\omega^{(p)} \leqq 2\pi\Theta_{j_p} \end{array}\right\} \qquad (p=r_1+1,\cdots,r_1+r_2).$$

We take one of these subsets and denote it by $\Omega(M;\Theta)$. We shall write, for brevity, the conditions for $\omega\in\Omega(M;\Theta)$ as follows;

$$M_q' < \omega^{(q)} \leqq M_q \qquad (q=1,2,\cdots,r_1),$$

$$\left.\begin{array}{l} M_p' < |\,\omega^{(p)}\,| \leqq M_p \\ 2\pi\Theta_p' < \arg\,\omega^{(p)} \leqq 2\pi\Theta_p \end{array}\right\} \qquad (p=r_1+1,\cdots,r_1+r_2).$$

Now we write

(6.7)
$$S_\rho(y) = \sum_{M,\,\Theta} S_\rho(y\,;M,\Theta)$$

with

$$S_\rho(y\,;M,\Theta) = \sum_{\substack{\omega\equiv\rho\,(\mathfrak{a}) \\ \omega\in\Omega(M;\,\Theta)}} e^{2\pi i S(\omega y)}\,.$$

The sum in (6.7) is taken over all possible M_{i_j} $(j=1,2,\cdots,r_1+r_2)$ and Θ_{j_p} $(p=r_1+1,\cdots,r_1+r_2)$.

We put

$$\tilde{M}_q = M_q \qquad (q=1,2,\cdots,r_1),$$

$$\left.\begin{array}{l} \tilde{M}_p = M_p e^{2\pi i\Theta_p} \\ \tilde{M}_{p'} = M_p e^{-2\pi i\Theta_p} \end{array}\right\} \qquad (p=r_1+1,\cdots,r_1+r_2),$$

then, noting that

$$|\,y_j\,| \leqq \frac{(\log N)^b}{N} \qquad (j=1,2,\cdots,n)$$

and for $\omega\in\Omega(M;\Theta)$

$$\omega^{(j)}-\tilde{M}_j \ll \frac{N}{(\log N)^a} \qquad (j=1,2,\cdots,n),$$

we have

$$e^{2\pi i S(\omega y)} = e^{2\pi i S(\tilde{M} y)} + O((\log N)^{b-a})\,.$$

Therefore we have

$$S_\rho(y\,;M,\Theta)=(e^{2\pi\,S(\tilde{M}y)}+O((\log N)^{b-a})\sum_{\substack{\omega\equiv\rho(\mathfrak{a})\\ \omega\in\Omega(M;\,\Theta)}}1\,.$$

We now apply Lemma 6.1 to this last sum. Then we obtain

$$(6.8)\qquad \sum_{\substack{\omega\equiv\rho(\mathfrak{a})\\ \omega\in\Omega(M;\,\Theta)}}1=\frac{w}{2^{r_1}hR\varphi(\mathfrak{a})}\prod_p(\Theta_p-\Theta_p')\int_{M_j'^{e_j}}^{M_j^{e_j}}\!\!\cdots\!\int\frac{dt_1\cdots dt_{r+1}}{\log(t_1\cdots t_{r+1})}+O(N^ne^{-c\sqrt{\log N}})\,,$$

where the domain of integration is given by the conditions

$$M_j'^{e_j}\leqq t_j\leqq M_j^{e_j}\qquad (j=1,2,\cdots,r+1)\,.$$

Since $N(\mathfrak{a})\leqq T^n=(\log N)^{n\sigma_1}$, the constants in the error term in (6.8) are independent of $\mathfrak{a}$. Therefore, putting

$$J(M)=\int_{M_j'^{e_j}}^{M_j^{e_j}}\!\!\cdots\!\int\frac{dt_1dt_2\cdots dt_{r+1}}{\log(t_1t_2\cdots t_{r+1})}\,,$$

we have

$$(6.9)\qquad \begin{aligned}S_\rho(y\,;M,\Theta)&=\frac{w}{2^{r_1}hR\varphi(\mathfrak{a})}\prod_p(\Theta_p-\Theta_p')J(M)e^{2\pi i\,S(\tilde{M}y)}\\&\quad+O(N^ne^{-c\sqrt{\log N}})+O\Big(\frac{\prod_p(\Theta_p-\Theta_p')J(M)}{\varphi(\mathfrak{a})(\log N)^{a-b}}\Big)\\&=\frac{w}{2^{r_1}hR\varphi(\mathfrak{a})}e^{2\pi i\,S(\tilde{M}y)}\int_{\Theta_p'}^{\Theta_p}\!\!\cdots\!\int J(M)\prod_p d\theta_p\\&\quad+O(N^ne^{-c\sqrt{\log N}})+O\Big(\frac{\prod_p(\Theta_p-\Theta_p')J(M)}{\varphi(\mathfrak{a})(\log N)^{a-b}}\Big)\,.\end{aligned}$$

Now we define $\tilde{t}_1,\tilde{t}_2,\cdots,\tilde{t}_n$ as in our Theorem and assume

$$M_j'^{e_j}\leqq t_j\leqq M_j^{e_j}\qquad (j=1,2,\cdots,r_1+r_2)\,,$$
$$\Theta_p'\leqq\theta_p\leqq\Theta_p\qquad (p=r_1+1,\cdots,r_1+r_2)\,,$$

then

$$S(\tilde{M}y)=S(\tilde{t}y)+O((\log N)^{b-a})\,,$$

which gives

$$S_\rho(y\,;M,\Theta)=\frac{w}{2^{r_1}hR\varphi(\mathfrak{a})}\int_{\Theta_p'}^{\Theta_p}\!\!\cdots\!\int\int_{M_j'^{e_j}}^{M_j^{e_j}}\!\!\cdots\!\int\frac{e^{2\pi i\,S(\tilde{t}y)}}{\log(t_1t_2\cdots t_{r+1})}dt_1\cdots dt_{r+1}\prod_p d\theta_p$$
$$+O(N^ne^{-c\sqrt{\log N}})+O\Big(\frac{\prod_p(\Theta_p-\Theta_p')J(M)}{\varphi(\mathfrak{a})(\log N)^{a-b}}\Big)\,.$$

Therefore we have

$$S_\rho(y)=\frac{w}{2^{r_1}hR\varphi(\mathfrak{a})}\int_0^1\!\!\cdots\!\!\int\prod_p d\theta_p\int_{N^{e_j/2}}^{|\lambda^{(j)}|^{e_j}}\!\!\cdots\!\!\int\frac{e^{2\pi iS(t\tilde{y})}}{\log(t_1t_2\cdots t_{r+1})}dt_1dt_2\cdots dt_{r+1}$$

$$+O(N^ne^{-c\sqrt{\log N}})+O\Big(\frac{N^n}{\varphi(\mathfrak{a})(\log N)^{a-b+1}}\Big),$$

whence follows

$$(6.10)\qquad \begin{aligned} S_1=&\frac{w\mu(\mathfrak{a})}{2^{r_1}hR\varphi(\mathfrak{a})}\int_0^1\!\!\cdots\!\!\int\prod_p d\theta_p\int_{N^{e_j/2}}^{|\lambda^{(j)}|^{e_j}}\!\!\cdots\!\!\int\frac{e^{2\pi iS(t\tilde{y})}}{\log(t_1t_2\cdots t_{r+1})}dt_1dt_2\cdots dt_{r+1}\\ &+O\Big(\frac{N^n}{(\log N)^{a-b+1}}\Big),\end{aligned}$$

since

$$\sum_{\substack{\rho \bmod \mathfrak{a}\\ (\rho,\,\mathfrak{a})=1}} e^{2\pi iS(\gamma\rho)}=\mu(\mathfrak{a}).$$

Theorem 6.1 follows from (6.3) and (6.10).

§ 7. Treatment of $I_s(\mu;\lambda)$ (II).

Now we shall return to $I_s(\mu;\lambda)$ defined in (4.3). From now on we assume that $s\geqq 3$.

First we have, by Theorem 5.1,

$$(7.1)\qquad \begin{aligned} &\int_{\mathfrak{B}^\circ}\!\!\cdots\!\!\int S(z;\lambda)^se^{-2\pi iS(\mu z)}dx(z)\\ &\ll\frac{N^{n(s-2)}}{(\log N)^{\sigma(s-2)}}\int_{-1/2}^{1/2}\!\!\cdots\!\!\int|S(z;\lambda)|^2dx_1dx_2\cdots dx_n\\ &=\frac{N^{n(s-2)}}{(\log N)^{\sigma(s-2)}}\sum_{\omega\in\Omega(\lambda)}1\ll\frac{N^{n(s-1)}}{(\log N)^{\sigma(s-2)+1}}\ll\frac{N^{n(s-1)}}{(\log N)^{s+1}}.\end{aligned}$$

Now we put

$$(7.2)\qquad W=\frac{2^{r_1}hR}{w}$$

and

$$J(y;\lambda)=\int_0^1\!\!\cdots\!\!\int\prod_p d\theta_p\int_{N^{e_j/2}}^{|\lambda^{(j)}|^{e_j}}\!\!\cdots\!\!\int\frac{e^{2\pi iS(t\tilde{y})}}{\log(t_1t_2\cdots t_{r+1})}dt_1dt_2\cdots dt_{r+1}$$

which is the integral in (6.2). It is obvious that

$$J(y;\lambda)\ll\frac{N^n}{\log N}.$$

If z is a point of B_γ with $\gamma\to\mathfrak{a}$, then we have by Theorem 6.1.

$$S(z;\lambda)^s=\frac{\mu(\mathfrak{a})^s}{W^s\varphi(\mathfrak{a})^s}J(y;\lambda)^s+O\Big(\frac{N^{ns}}{(\log N)^{a-b+s}}\Big)$$

and

$$\int_{\mathfrak{B}_\gamma}\cdots\int S(z\,;\lambda)^s e^{-2\pi i S(\mu z)} dx(z)$$

$$(7.3)\qquad = \frac{\mu(\mathfrak{a})^s}{W^s\varphi(\mathfrak{a})^s} e^{-2\pi i S(\mu\gamma)} \int_{\mathfrak{B}_0}\cdots\int J(y\,;\lambda)^s e^{-2\pi i S(\mu y)} dx(y) + O\Big(\frac{N^{n(s-1)}}{(\log N)^{a-b(n+1)+s}}\Big),$$

where

$$\mathfrak{B}_0 = \Big\{x(y)\,;\, |y_j| \leqq \frac{(\log N)^b}{N} \qquad (j=1, 2, \cdots, n)\Big\}.$$

The error term of (7.3) follows from

$$\int_{\mathfrak{B}_\gamma}\cdots\int dx(z) \ll \frac{(\log N)^{bn}}{N^n}.$$

Summing up the both sides of (7.3) over all $\gamma \in \Gamma$, we have

$$\sum_{\gamma\in\Gamma}\int_{\mathfrak{B}_\gamma}\cdots\int S(z\,;\lambda)^s e^{-2\pi i S(\mu z)} dx(z)$$

$$(7.4)\qquad = \frac{1}{W^s}\sum_{\gamma\in\Gamma} \frac{\mu(\mathfrak{a})^s}{\varphi(\mathfrak{a})^s} e^{-2\pi i S(\mu\gamma)} \int_{\mathfrak{B}_0}\cdots\int J(y\,;\lambda)^s e^{-2\pi i S(\mu y)} dx(y) + O(N^{n(s-1)}(\log N)^{-a+b(n+1)+2n\sigma_2-s}),$$

since

$$\sum_{\gamma\in\Gamma} 1 \ll \sum_{N\mathfrak{a}\leqq T^n} N(\mathfrak{a}) \ll T^{2n} = (\log N)^{2n\sigma_2}.$$

Therefore, putting

$$a = b(n+1)+2n\sigma_2+1,$$

$$R(\mu\,;\lambda) = \int_{\mathfrak{B}_0}\cdots\int J(y\,;\lambda)^s e^{-2\pi i S(\mu y)} dx(y)$$

and

$$G(\mathfrak{a}, \mu) = \sum_{\substack{\gamma\to\mathfrak{a} \\ \gamma \bmod \mathfrak{d}^{-1}}} e^{-2\pi i S(\mu\gamma)},$$

where γ runs through a complete system of residues mod $\mathfrak{d}^{-1}$ such that $\gamma\to\mathfrak{a}$, we have by (7.1) and (7.4)

$$(7.5)\qquad I_s(\mu\,;\lambda) = \frac{2^{r_2}\sqrt{D}}{W^s} R(\mu\,;\lambda) \sum_{N\mathfrak{a}\leqq T^n} \frac{\mu(\mathfrak{a})^s}{\varphi(\mathfrak{a})^s} G(\mathfrak{a}, \mu) + O\Big(\frac{N^{n(s-1)}}{(\log N)^{s+1}}\Big).$$

Now we shall prove

LEMMA 7.1. *We have*

$$(7.6)\qquad \frac{1}{\varphi(\mathfrak{a})} \ll \frac{\log(N(\mathfrak{a})+1)}{N(\mathfrak{a})}.$$

PROOF. We have

$$\log\frac{N(\mathfrak{a})}{\varphi(\mathfrak{a})}=-\sum_{\mathfrak{p}\mid\mathfrak{a}}\log\Big(1-\frac{1}{N(\mathfrak{p})}\Big)=\sum_{\mathfrak{p}\mid\mathfrak{a}}\frac{1}{N(\mathfrak{p})}+O(1),$$

where $\mathfrak{p}$ runs through all prime divisors of $\mathfrak{a}$. We know, by the prime ideal theorem,

$$\pi_K(x)=\sum_{N\mathfrak{p}\leqq x}1=\int_2^x\frac{dt}{\log t}+O(xe^{-c\sqrt{\log x}}).$$

(See Landau [2]). Therefore

$$\sum_{N\mathfrak{p}\leqq x}\frac{1}{N(\mathfrak{p})}=\sum_{m=1}^{[x]}\frac{\pi_K(m)-\pi_K(m-1)}{m}=\sum_{m=2}^{[x]-1}\pi_K(m)\Big(\frac{1}{m}-\frac{1}{m+1}\Big)+\frac{\pi_K([x])}{[x]}$$

$$=\int_2^{[x]}\frac{\pi_K(u)}{u^2}du+O(1)=\int_2^x\Big(\int_2^u\frac{dt}{\log t}\Big)\frac{du}{u^2}+O(1)$$

$$=\int_2^x\frac{dt}{t\log t}+O(1)=\log\log x+O(1).$$

Since

$$\sum_{\mathfrak{p}\mid\mathfrak{a}}\frac{1}{N(\mathfrak{p})}\leqq\sum_{N\mathfrak{p}\leqq N\mathfrak{a}}\frac{1}{N(\mathfrak{p})}\leqq\log\log(N(\mathfrak{a})+1)+c,$$

we have

$$\log\frac{N(\mathfrak{a})}{\varphi(\mathfrak{a})}\leqq\log\log(N(\mathfrak{a})+1)+c,$$

and obtain (7.6).

Now we put

$$\kappa=b(n+1)+1 \tag{7.7}$$

and define a set $\mathfrak{D}(\lambda)$ of integers ν of K which satisfy the following conditions:

$$\lambda^{(q)}-\frac{N}{(\log N)^\kappa}<\nu^{(q)}\leqq\lambda^{(q)}\qquad(q=1,2,\cdots,r_1),$$

(7.8)

$$|\lambda^{(p)}-\nu^{(p)}|\leqq\frac{N}{(\log N)^\kappa}\qquad(p=r_1+1,\cdots,r_1+r_2).$$

Assume that $\mu\in\mathfrak{D}(\lambda)$. Then we have

$$e^{-2\pi iS(\mu y)}=e^{-2\pi iS(\lambda y)}+O\Big(\frac{N}{(\log N)^\kappa}\max_{1\leqq j\leqq n}(|X_j(y)|)\Big),$$

therefore

$$R(\lambda,\lambda)-R(\mu,\lambda)\ll\frac{N^{ns+1}}{(\log N)^{s+\kappa}}\int\cdots\int\max_{1\leqq j\leqq n}(|X_j(y)|)dx(y)$$

$$\ll\frac{N^{ns+1}}{(\log N)^{s+\kappa}}\cdot\frac{(\log N)^{b(n+1)}}{N^{n+1}}=\frac{N^{n(s-1)}}{(\log N)^{s+1}}$$

so that, by Lemma 7.1, we have

$$I_s(\mu\,;\lambda)=\frac{2^{r_2}\sqrt{D}}{W^s}R(\lambda,\lambda)\sum_{N\mathfrak{a}\leqq T^n}\frac{\mu(\mathfrak{a})^s}{\varphi(\mathfrak{a})^s}G(\mathfrak{a},\mu)+O\Big(\frac{N^{n(s-1)}}{(\log N)^{s+1}}\Big).$$

We shall sum up the both sides over all $\mu\in\mathfrak{D}(\lambda)$. Then we have

$$(7.9)\qquad \begin{aligned} T(\lambda)&=\sum_{\mu\in\mathfrak{D}(\lambda)}I_s(\mu\,;\lambda)\\ &=\frac{2^{r_2}\sqrt{D}}{W^s}R(\lambda,\lambda)\sum_{N\mathfrak{a}\leqq T^n}\frac{\mu(\mathfrak{a})^s}{\varphi(\mathfrak{a})^s}\sum_{\mu\in\mathfrak{D}(\lambda)}G(\mathfrak{a},\mu)+O\Big(\frac{N^{ns}}{(\log N)^{s+n\kappa+1}}\Big),\end{aligned}$$

since

$$\sum_{\mu\in\mathfrak{D}(\lambda)}1=\frac{2^{r_2}\pi^{r_2}N^n}{\sqrt{D}\,(\log N)^{n\kappa}}+O\Big(\frac{N^{n-1}}{(\log N)^{\kappa(n-1)}}\Big)$$

on account of Lemma 3.2.

We shall consider a sum

$$S_{\mathfrak{a}}=\sum_{\mu\in\mathfrak{D}(\lambda)}G(\mathfrak{a},\mu)\qquad (N(\mathfrak{a})\leqq T^n).$$

If $\mathfrak{a}=\mathfrak{o}$, then $G(\mathfrak{a},\mu)=1$ for all $\mu\in\mathfrak{D}(\lambda)$.

Assume that $\mathfrak{a}\neq\mathfrak{o}$ and take a number $\gamma\in\Gamma$ with $\gamma\rightarrow\mathfrak{a}$. If we put

$$I_\gamma=\sum_{\mu\in\mathfrak{D}(\lambda)}e^{-2\pi iS(\mu\gamma)},$$

then, by Lemma 3.5, we have

$$I_\gamma\ll\frac{N^{n-1}}{(\log N)^{\kappa(n-1)}}\min_{1\leqq j\leqq n}\Big(\frac{N}{(\log N)^{\kappa}},\ \|S(\rho_j\gamma)\|^{-1}\Big),$$

where $\rho_1,\rho_2,\cdots,\rho_n$ is a basis of $\mathfrak{o}$ such that

$$S(\rho_j\delta_k)=\begin{cases}1 & \text{if } j=k\\ 0 & \text{if } j\neq k\end{cases}\qquad (j,k=1,2,\cdots,n).$$

Since the assumption $\mathfrak{a}\neq\mathfrak{o}$ means that not all $\|S(\rho_j\gamma)\|$ vanish, we have

$$S_{\mathfrak{a}}=\sum_{\substack{\gamma\rightarrow\mathfrak{a}\\ \gamma \bmod \mathfrak{d}^{-1}}}I_\gamma\ll\frac{N^{n-1}}{(\log N)^{\kappa(n-1)}}\sum_{\substack{\gamma \bmod \mathfrak{d}^{-1}\\ \gamma\rightarrow\mathfrak{a}}}\min_{1\leqq j\leqq n}(\|S(\rho_j\gamma)\|^{-1}).$$

We shall denote by $L_{\mathfrak{a}}$ the sum in this right-hand side.

Now we write

$$S(\rho_j\gamma)=a_j+d_j$$

with rational integer a_j and $-1/2<d_j\leqq 1/2$ $(j=1,2,\cdots,n)$ and put

$$\vartheta=\sum_{j=1}^n a_j\delta_j,\qquad \zeta=\sum_{j=1}^n d_j\delta_j.$$

Then $\gamma=\vartheta+\zeta$ and

$$L_{\mathfrak{a}} \ll \sum_{\substack{\gamma \to \mathfrak{a} \\ \gamma \bmod \mathfrak{b}^{-1}}} \min_{1 \leqq j \leqq n} \left(\frac{1}{|X_j(\zeta)|} \right).$$

We take n rational integers $g_1, g_2, \cdots, g_n$ and define a parallelotope $B(g)$ in n-dimensional euclidean space as follows:

$$B(g) = \left\{ (x_1, \cdots, x_n) ; \frac{1}{3(DN(\mathfrak{a}))^{1/n}} \left(g_j - \frac{1}{2} \right) < x_j \leqq \frac{1}{3(DN(\mathfrak{a}))^{1/n}} \left(g_j + \frac{1}{2} \right) \quad (j = 1, 2, \cdots, n) \right\}.$$

Since $\zeta \in (\mathfrak{a}\mathfrak{d})^{-1}$, the number of γ in $L_{\mathfrak{a}}$ such that $x(\zeta) \in B(g)$ is at most one. Therefore we have

$$L_{\mathfrak{a}} \ll N(\mathfrak{a})^{1/n} \sum_{\{g\} \neq \{0\}} \min_{1 \leqq j \leqq n} \left(\frac{1}{|g_j|} \right),$$

where $g_1, g_2, \cdots, g_n$ in the sum run through all n rational integers for which $B(g)$ contain the points $x(\zeta)$ defined by γ.

The range of $\{g_1, g_2, \cdots, g_n\}$ is roughly given by the conditions

$$g_j \ll N(\mathfrak{a})^{1/n} \qquad (j = 1, 2, \cdots, n).$$

Therefore, applying Lemma 3.3, we obtain

$$L_{\mathfrak{a}} \ll N(\mathfrak{a}) \log N,$$

which gives

$$S_{\mathfrak{a}} \ll N(\mathfrak{a}) \frac{N^{n-1}}{(\log N)^{\kappa(n-1)-1}}$$

and

$$\sum_{N\mathfrak{a} \leqq T^n} \frac{\mu(\mathfrak{a})^s}{\varphi(\mathfrak{a})^s} \sum_{\mu \in \mathfrak{D}(\lambda)} G(\mathfrak{a}, \mu) = \sum_{\mu \in \mathfrak{D}(\lambda)} 1 + O\left(\sum_{N\mathfrak{a} \leqq T^n} \frac{N(\mathfrak{a})}{\varphi(\mathfrak{a})^s} \cdot \frac{N^{n-1}}{(\log N)^{\kappa(n-1)-1}} \right)$$

$$= \frac{2^{r_2} \pi^{r_2} N^n}{\sqrt{D} (\log N)^{\kappa n}} + O\left(\frac{N^{n-1}}{(\log N)^{\kappa(n-1)-1}} \right).$$

Putting this result in (7.9), we have

$$T(\lambda) = \frac{2^{2r_2} \pi^{r_2} N^n R(\lambda, \lambda)}{W^s (\log N)^{n\kappa}} \left(1 + O\left(\frac{(\log N)^{\kappa+1}}{N} \right) \right) + O\left(\frac{N^{ns}}{(\log N)^{n\kappa+s+1}} \right). \tag{7.10}$$

On the other hand, by the definitions of $I_s(\mu ; \lambda)$ and $\mathfrak{D}(\lambda)$, (4.3) and (7.8) respectively, we see that $T(\lambda)$ is equal to the number of the s-tuples $(\omega_1, \omega_2, \cdots, \omega_s)$ of prime numbers which satisfy the following conditions:

$$\lambda^{(q)}-\frac{N}{(\log N)^{\kappa}}<\omega_1^{(q)}+\omega_2^{(q)}+\cdots+\omega_s^{(q)}\leqq\lambda^{(q)} \qquad (q=1,2,\cdots,r_1),$$

$$(\mathrm{C}_\omega) \qquad |\omega_1^{(p)}+\omega_2^{(p)}+\cdots+\omega_s^{(p)}-\lambda^{(p)}|\leqq\frac{N}{(\log N)^{\kappa}} \qquad (p=r_1+1,\cdots,r_1+r_2),$$

$$\omega_j\in\Omega(\lambda) \qquad (j=1,2,\cdots,s).$$

In the following paragraph § 8, we shall reduce these conditions to that connected with integers.

§ 8. Some relations between prime numbers and integers.

We put

$$C_0=\left(\frac{2^{r_2}\pi^{r_2}nW}{\sqrt{D}}\right)^{1/n}, \qquad \kappa_0=\kappa+1+\frac{1}{n}$$

with W and κ in (7.2) and (7.7) respectively, and

$$N_0=\frac{N}{(\log N)^{\kappa_0}},$$

$$Y=C_0N_0(\log N_0)^{1/n},$$

$$G=[(\log N)^{\kappa_0}],$$

where $[x]$ means the integral part of real number x.

We take $r_1+r_2=r+1$ positive rational integers $g_1, g_2, \cdots, g_{r+1}$ such that

$$(8.1) \qquad 1\leqq g_j\leqq c(\log N)^{\kappa+1} \qquad (j=1,2,\cdots,r_1+r_2)$$

and r_2 positive rational integers h_p such that $1\leqq h_p\leqq G$ $(p=r_1+1,\cdots,r_1+r_2)$.

Let $n(N_0; g, h)=n(N_0; g_1,\cdots,g_{r+1},h_{r_1+1},\cdots,h_{r+1})$ be the number of integers ν which satisfy the following conditions:

$$(g_q-1)N_0<\nu^{(q)}\leqq g_qN_0 \qquad (q=1,2,\cdots,r_1),$$

$$(g_p-1)N_0<|\nu^{(p)}|\leqq g_pN_0$$

$$\frac{2\pi}{G}(h_p-1)<\arg\nu^{(p)}\leqq\frac{2\pi}{G}h_p. \qquad (p=r_1+1,\cdots,r_1+r_2),$$

Then we have, by Lemma 3.2,

$$(8.2) \qquad n(N_0; g, h)=\left(\frac{2\pi}{G}\right)^{r_2}\frac{N_0^n}{\sqrt{D}}\prod_{p=r_1+1}^{r_1+r_2}(2g_p-1)\left(1+O\left(\frac{1}{\sqrt{N_0}}\right)\right),$$

where the constants in the error term are independent of g_j $(1\leqq j\leqq r+1)$ and h_p $(r_1+1\leqq p\leqq r_1+r_2)$.

In the following lines of this paragraph, we shall use symbols O and $\ll$ when the constants in them are independent of g_j $(1\leqq j\leqq r+1)$ and h_p $(r_1+1\leqq p\leqq r_1+r_2)$.

Let $\pi(Y;g,h)=\pi(Y;g_1,\cdots,g_{r+1},h_{r_1+1},\cdots,h_{r+1})$ be the number of prime numbers ω which satisfy the following conditions

$$0<\omega^{(q)}\leqq g_q Y \qquad (q=1,2,\cdots,r_1),$$

$$\left.\begin{array}{l}|\omega^{(p)}|\leqq g_p Y\\ \frac{2\pi}{G}(h_p-1)<\arg\omega^{(p)}\leqq\frac{2\pi}{G}h_p\end{array}\right\} \qquad (p=r_1+1,\cdots,r_1+r_2),$$

then we have, by Lemma 6.1,

$$\pi(Y;g,h)=\frac{1}{WG^{r_2}}\int_2^{(g_jY)^{e_j}}\cdots\int\frac{dt_1\cdots dt_{r+1}}{\log(t_1\cdots t_{r+1})}+O(N^ne^{-c\sqrt{\log N}}),$$

where the domain of integration is given by the conditions

$$2\leqq t_j\leqq(g_jY)^{e_j} \qquad (j=1,2,\cdots,r+1).$$

This integral can be easily estimated. Putting

$$\prod_p=\prod_{p=r_1+1}^{r_1+r_2},\qquad \sum_j=\sum_{j=1}^{r+1},\qquad \prod_j=\prod_{j=1}^{r+1},$$

we have

$$\pi(Y;g,h)=\frac{1}{WG^{r_2}}\cdot\frac{Y^n\prod_j g_j^{e_j}}{\log(Y^n\prod_j g_j^{e_j})}\left(1+O\left(\frac{1}{\log N}\right)\right)+O(N^ne^{-c\sqrt{\log N}})$$

(8.3)

$$=\frac{1}{WG^{r_2}}\cdot\frac{Y^n\prod_j g_j^{e_j}}{\log(Y^n\prod_j g_j^{e_j})}\left(1+O\left(\frac{1}{\log N}\right)\right)$$

$$=\frac{1}{WG^{r_2}}\cdot\frac{Y^n\prod_j g_j^{e_j}}{n\log Y}\left(1-\frac{\sum_j e_j\log g_j}{n\log Y}+O\left(\frac{1}{\log N}\right)\right).$$

Let $g_1',g_2',\cdots,g_{r+1}'$ be positive rational integers derived from $g_1,g_2,\cdots,g_{r+1}$ by substracting 1 from some of $g_1,g_2,\cdots,g_{r+1}$. Then we have

$$\pi(Y;g',h)=\frac{1}{WG^{r_2}}\cdot\frac{Y^n\prod_j g_j'^{e_j}}{n\log Y}\left(1-\frac{\sum_j e_j\log g_j'}{n\log Y}+O\left(\frac{1}{\log N}\right)\right)$$

$$=\frac{1}{WG^{r_2}}\cdot\frac{Y^n\prod_j g_j'^{e_j}}{n\log Y}\left(1-\frac{\sum_j e_j\log g_j}{n\log Y}+O\left(\frac{1}{\log N}\right)\right),$$

since

$$\sum_j e_j\log g_j-\sum_j e_j\log g_j'\ll 1.$$

Therefore, if we denote by $\pi^*(Y;g,h)=\pi^*(Y;g_1,\cdots,g_{r+1},h_{r_1+1},\cdots,h_{r+1})$ the number of the prime numbers ω which satisfy the following conditions

$$(g_q-1)Y<\omega^{(q)}\leqq g_qY \qquad (q=1,2,\cdots,r_1),$$

$$(g_p-1)Y<|\omega^{(p)}|\leqq g_pY \qquad (p=r_1+1,\cdots,r_1+r_2),$$

$$\frac{2\pi}{G}(h_p-1)<\arg\omega^{(p)}\leqq\frac{2\pi}{G}h_p,$$

then we have

$$\pi^*(Y;g,h)=\frac{1}{G^{r_2}}\prod_p(2g_p-1)\frac{Y^n}{nW\log Y}$$

$$\times\Big(1-\frac{\sum_j e_j\log g_j}{n\log Y}+O\Big(\frac{1}{\log N}\Big)\Big) \tag{8.4}$$

$$=\Big(\frac{2\pi}{G}\Big)^{r_2}\frac{N_0^n}{\sqrt{D}}\prod_p(2g_p-1)\Big(1-\frac{\log\log N}{n\log N}-\frac{\sum_j e_j\log g_j}{n\log N}+O\Big(\frac{1}{\log N}\Big)\Big).$$

Comparing the results (8.2) and (8.4), we see that the inequality

$$\pi^*(Y;g,h)\leqq n(N_0;g,h) \tag{8.5}$$

is true for sufficiently large N and for any $g_1,g_2,\cdots,g_{r+1}$ satisfying (8.1).

We shall denote by $\bar{\pi}^*(Y;g,h)$ the set of the prime numbers belonging to $\pi^*(Y;g,h)$. Similarly we define a set $\bar{n}(N_0;g,h)$. Above inequality (8.5) shows that we can construct a mapping $\phi=\phi(g,h)$

$$\phi:\ \bar{\pi}^*(Y;g,h)\rightarrow\bar{n}(N_0;g,h), \tag{8.6}$$

which always maps the different elements of $\bar{\pi}^*(Y;g,h)$ into the different elements of $\bar{n}(N_0;g,h)$, that is,

$$\phi(\omega)\neq\phi(\omega_1) \qquad (\text{if}\quad \omega\neq\omega_1). \tag{8.7}$$

Moreover, we can easily prove that

$$\omega^{(j)}-\frac{Y}{N_0}\phi(\omega)^{(j)}\ll\frac{N}{(\log N)^{\kappa+1}} \qquad (j=1,2,\cdots,n) \tag{8.8}$$

for $\omega\in\bar{\pi}^*(Y;g,h)$.

Now we put

$$Z=C_0N_0(\log N_0)^{1/n}\Big(1+a\frac{\log\log N}{\log N}\Big)$$

with $a=(\kappa_0+1)/n$ and define a set of prime numbers $\pi^*(Z;g,h)$ similarly to $\pi^*(Y;g,h)$. Then we have

$$\pi^*(Z;g,h)=\prod_p(2g_p-1)\frac{Z^n}{G^{r_2}nW\log Z}\Big(1-\frac{\sum_j e_j\log g_j}{n\log Z}+O\Big(\frac{1}{\log N}\Big)\Big)$$

$$=\Big(\frac{2\pi}{G}\Big)^{r_2}\frac{N_0^n}{\sqrt{D}}\prod_p(2g_p-1)\Big(1+a\frac{\log\log N}{\log N}\Big)^n$$

$$\times\Big(1-\frac{\log\log N}{n\log N}-\frac{\sum_j e_j\log g_j}{n\log N}+O\Big(\frac{1}{\log N}\Big)\Big)$$

$$=\left(\frac{2\pi}{G}\right)^{r_2}\frac{N_0^n}{\sqrt{D}}\prod_p(2g_p-1)\left(1+\left(na-\frac{1}{n}\right)\frac{\log\log N}{\log N}\right.$$

$$\left.-\frac{\sum_j e_j\log g_j}{n\log N}+O\left(\frac{1}{\log N}\right)\right).$$

Since

$$\sum_j e_j\log g_j\leqq n\left(\kappa_0-\frac{1}{n}\right)\log\log N+O(1),$$

we have, for sufficiently large N,

$$\pi^*(Z;g,h)\geqq\left(\frac{2\pi}{G}\right)^{r_2}\frac{N_0^n}{\sqrt{D}}\prod_p(2g_p-1)\left(1+\frac{\log\log N}{2\log N}\right) \tag{8.9}$$

so that

$$\pi^*(Z;g,h)\geqq n(N_0;g,h). \tag{8.10}$$

Therefore, we can also construct a mapping $\psi=\psi(g,h)$

$$\psi:\quad \bar{n}(N_0;g,h)\to\bar{\pi}^*(Z;g,h) \tag{8.11}$$

such that

$$\psi(\nu)\neq\psi(\nu_1)\qquad(\text{if}\quad\nu\neq\nu_1) \tag{8.12}$$

Moreover, we can prove that

$$\psi(\nu)^{(j)}-\frac{Z}{N_0}\nu^{(j)}\ll\frac{N}{(\log N)^{\kappa+1}}\qquad(j=1,2,\cdots,n) \tag{8.13}$$

for $\nu\in\bar{n}(N_0;g,h)$.

Now we define rational integers $G_1,G_2,\cdots,G_{r+1}$ and $G_1',G_2',\cdots,G_{r+1}'$ by the following conditions:

$$\begin{aligned}&G_qY<\lambda^{(q)}\leqq(G_q+1)Y\\&G_q'Z<\lambda^{(q)}\leqq(G_q'+1)Z\end{aligned}\qquad(q=1,2,\cdots,r_1),$$

$$\begin{aligned}&G_pY<|\lambda^{(p)}|\leqq(G_p+1)Y\\&G_p'Z<|\lambda^{(p)}|\leqq(G_p'+1)Z\end{aligned}\qquad(p=r_1+1,\cdots,r_1+r_2).$$

It is obvious that $G_j'\leqq G_j\ll(\log N)^{\kappa+1}$ $(j=1,2,\cdots,n)$.

We shall denote by $\mathfrak{L}_1$ the set of all integers ν such that

$$0<\nu^{(q)}\leqq(G_q+1)N_0\qquad(q=1,2,\cdots,r_1),$$

$$|\nu^{(p)}|\leqq(G_p+1)N_0\qquad(p=r_1+1,\cdots,r_1+r_2)$$

and by $\mathfrak{L}_2$ the set of all integers ν such that

$$0<\nu^{(q)}\leqq G_q'N_0\qquad(q=1,2,\cdots,r_1),$$

$$|\nu^{(p)}|\leqq G_p'N_0\qquad(p=r_1+1,\cdots,r_1+r_2).$$

$\mathfrak{L}_1$ and $\mathfrak{L}_2$ are divided into subsets as follows:

$$\mathfrak{L}_1 = \sum_h{}' \sum_{g_1=1}^{G_1+1} \cdots \sum_{g_{r+1}=1}^{G_{r+1}+1} \bar{n}(N_0\,; g, h)\,,$$

$$\mathfrak{L}_2 = \sum_h{}' \sum_{g_1=1}^{G_1'} \cdots \sum_{g_{r+1}=1}^{G_{r+1}'} \bar{n}(N_0\,; g, h)\,,$$

where we use the abbreviation:

$$\sum_h{}' = \sum_{h_{r_1+1}=1}^{G} \cdots \sum_{h_{r+1}=1}^{G}\,.$$

It is obvious that

$$\Omega(\lambda) \subset \sum_h{}' \sum_{g_1=1}^{G_1+1} \cdots \sum_{g_{r+1}=1}^{G_{r+1}+1} \bar{\pi}^*(Y\,; g, h)$$

and

$$\Omega(\lambda) \supset \sum_h{}' \sum_{g_1=1}^{G_1'} \cdots \sum_{g_{r+1}=1}^{G_{r+1}'} \bar{\pi}^*(Z\,; g, h)\,.$$

Now we shall define a mapping $\tilde{\phi} : \Omega(\lambda) \rightarrow \mathfrak{L}_1$ as follows:

$$\tilde{\phi}(\omega) = \phi(g, h)(\omega) \qquad (\text{if} \quad \omega \in \bar{\pi}^*(Y\,; g, h))\,,$$

then, by (8.7) and (8.8), we see that

$$\tilde{\phi}(\omega) \neq \tilde{\phi}(\omega_1) \qquad (\omega, \omega_1 \in \Omega(\lambda), \omega \neq \omega_1)\,, \tag{8.14}$$

$$\omega^{(j)} - \frac{Y}{N_0}\tilde{\phi}(\omega)^{(j)} \ll \frac{N}{(\log N)^{\kappa+1}} \qquad (\omega \in \Omega(\lambda),\ j = 1, 2, \cdots, n)\,. \tag{8.15}$$

Similarly we can define a mapping $\tilde{\psi} : \mathfrak{L}_2 \rightarrow \Omega(\lambda)$ such that

$$\tilde{\psi}(\nu) \neq \tilde{\psi}(\nu_1) \qquad (\nu, \nu_1 \in \mathfrak{L}_2, \nu \neq \nu_1)\,, \tag{8.16}$$

$$\tilde{\psi}(\nu)^{(j)} - \frac{Z}{N_0}\nu^{(j)} \ll \frac{N}{(\log N)^{\kappa+1}} \qquad (\nu \in \mathfrak{L}_2\,; j = 1, 2, \cdots, n)\,. \tag{8.17}$$

Choosing a suitable positive constant B, we can write the conditions (8.15) and (8.17) as follows:

$$\left| \omega^{(j)} - \frac{Y}{N_0}\tilde{\phi}(\omega)^{(j)} \right| \leqq \frac{BN}{(\log N)^{\kappa+1}} \qquad (\omega \in \Omega(\lambda)\,; j = 1, 2, \cdots, n)\,, \tag{8.15$'$}$$

$$\left| \tilde{\psi}(\nu)^{(j)} - \frac{Z}{N_0}\nu^{(j)} \right| \leqq \frac{BN}{(\log N)^{\kappa+1}} \qquad (\nu \in \mathfrak{L}_2\,; j = 1, 2, \cdots, n)\,. \tag{8.17$'$}$$

Now we shall return to $T(\lambda)$ in § 7, which is the number of the s-tuples $(\omega_1, \omega_2, \cdots, \omega_s)$ of prime numbers satisfying the conditions (C_ω). We denote by $\bar{T}(\lambda)$ the set of these s-tuples.

Let T_1 be the number of the s-tuples $(\nu_1, \nu_2, \cdots, \nu_s)$ of integers which satisfy the following conditions:

$$\lambda^{(q)}-\frac{N}{(\log N)^{\kappa}}-\frac{sBN}{(\log N)^{\kappa+1}} \qquad (q=1,2,\cdots,r_1)$$

$$<\frac{Y}{N_0}(\nu_1^{(q)}+\nu_2^{(q)}+\cdots+\nu_s^{(q)})\leqq\lambda^{(q)}+\frac{sBN}{(\log N)^{\kappa+1}},$$

$$(\mathrm{C}_1) \qquad \left|\frac{Y}{N_0}(\nu_1^{(p)}+\nu_2^{(p)}+\cdots+\nu_s^{(p)})-\lambda^{(p)}\right|\leqq\frac{N}{(\log N)^{\kappa}}+\frac{2sBN}{(\log N)^{\kappa+1}}$$

$$(p=r_1+1,\cdots,r_1+r_2),$$

$$0<\nu_j^{(q)}\leqq\frac{N_0}{Y}\Big(\lambda^{(q)}+(sB+C_0)\frac{N}{(\log N)^{\kappa+1}}\Big) \qquad (q=1,2,\cdots,r_1),$$

$$|\nu_j^{(p)}|\leqq\frac{N_0}{Y}\Big(|\lambda^{(p)}|+\frac{C_0N}{(\log N)^{\kappa+1}}\Big)$$

$$(p=r_1+1,\cdots,r_1+r_2), \quad (j=1,2,\cdots,s).$$

We shall denote by $\bar{T}_1$ the set of these s-tuples.

If we take $(\omega_1,\omega_2,\cdots,\omega_s)\in\bar{T}(\lambda)$, then we can prove without difficulty that $(\tilde{\phi}(\omega_1),\tilde{\phi}(\omega_2),\cdots,\tilde{\phi}(\omega_s))\in\bar{T}_1$. Moreover, if $(\omega_1,\omega_2,\cdots,\omega_s)$ and $(\omega_1^0,\omega_2^0,\cdots,\omega_s^0)$ are different elements of $\bar{T}(\lambda)$, then it follows from (8.14) that $(\tilde{\phi}(\omega_1),\tilde{\phi}(\omega_2),\cdots,\tilde{\phi}(\omega_s))\neq(\tilde{\phi}(\omega_1^0),\tilde{\phi}(\omega_2^0),\cdots,\tilde{\phi}(\omega_s^0))$. Therefore we have

$$T_1\geqq T(\lambda).$$

Similarly, if we denote by T_2 the number of the s-tuples $(\nu_1,\nu_2,\cdots,\nu_s)$ of integers which satisfy the following conditions:

$$\lambda^{(q)}-\frac{N}{(\log N)^{\kappa}}+(sB+C_0)\frac{N}{(\log N)^{\kappa+1}} \qquad (q=1,2,\cdots,r_1)$$

$$<\frac{Z}{N_0}(\nu_1^{(q)}+\nu_2^{(q)}+\cdots+\nu_s^{(q)})\leqq\lambda^{(q)}-(sB+C_0)\frac{N}{(\log N)^{\kappa+1}},$$

$$(\mathrm{C}_2) \qquad \left|\lambda^{(p)}-\frac{Z}{N_0}(\nu_1^{(p)}+\nu_2^{(p)}+\cdots+\nu_s^{(p)})\right|\leqq\frac{N}{(\log N)^{\kappa}}-2(sB+C_0)\frac{N}{(\log N)^{\kappa+1}}$$

$$(p=r_1+1,\cdots,r_1+r_2),$$

$$0<\nu_j^{(q)}\leqq\frac{N_0}{Z}\Big(\lambda^{(q)}-\frac{C_0N}{(\log N)^{\kappa+1}}\Big) \qquad (q=1,2,\cdots,r_1),$$

$$|\nu_j^{(p)}|\leqq\frac{N_0}{Z}\Big(|\lambda^{(p)}|-\frac{C_0N}{(\log N)^{\kappa+1}}\Big)$$

$$(p=r_1+1,\cdots,r_1+r_2), \quad (j=1,2,\cdots,s),$$

then we have an inequality

$$T(\lambda)\geqq T_2.$$

Thus the estimation of $T(\lambda)$ is reduced to that of T_1 and T_2.

§ 9. The number of the representations of a totally positive integer as the sums of s totally positive integers.

Let M be a sufficiently large number and $M_1, M_2, \cdots, M_n$ be positive numbers such that

$$c_1 M < M_j < c_2 M \qquad (j=1,2,\cdots,n),$$
$$M_{p'} = M_p \qquad (p=r_1+1,\cdots,r_1+r_2).$$

Let $z=(z_1, z_2, \cdots, z_n)$ be a point of E and define a sum

$$T(z\,;M)=\sum_{\nu} e^{2\pi i S(\nu z)}, \tag{9.1}$$

where ν runs through all integers such that

$$\begin{aligned} 0<\nu^{(q)} &\leqq M_q \qquad (q=1,2,\cdots,r_1), \\ |\nu^{(p)}| &\leqq M_p \qquad (p=r_1+1,\cdots,r_1+r_2). \end{aligned} \tag{9.2}$$

We put

$$a=\frac{n+1}{n+2}$$

and divide E into two parts D_0 and D_1:

$$D_1=\{z\,; z\in E,\ \ |z_j|\leqq M^{-a} \qquad (j=1,2,\cdots,n)\},$$
$$D_0=E-D_1.$$

By Lemma 3.5, we have

$$T(z\,;M)\ll M^{n-1}\min_{1\leqq j\leqq n}(M,\ \|S(\rho_j z)\|^{-1}),$$

where $\rho_1, \rho_2, \cdots, \rho_n$ is a basis of $\mathfrak{o}$ such that

$$S(\rho_j\delta_k)=\begin{cases} 1 & \text{if } \ j=k \\ 0 & \text{if } \ j\neq k \end{cases} \qquad (j\ k,=1,2,\cdots n).$$

Now we write

$$z_j=\sum_{k=1}^{n} x_k \delta_k^{(j)} \qquad (j=1,2,\cdots,n)$$

with real numbers $x_1, x_2, \cdots, x_n$. If $z\in D_0$, there exists at least one x_l $(1\leqq l\leqq n)$ such that $|x_l|\geqq cM^{-a}$. Therefore we have

$$T(z\,;M)\ll M^{n-1}\min_{1\leqq j\leqq n}(M,\ |x_j|^{-1})\ll M^{n-1+a} \tag{9.3}$$

for $z\in D_0$.

Now we assume that $z\in D_1$. We take an integer ν satisfying the condition (9.2) and write

$$\nu=\sum_{i=1}^{n} m_i\rho_i.$$

with rational integers $m_1, m_2, \cdots, m_n$. We put

$$\xi_j=\sum_{i=1}^{n} t_i \rho_i^{(j)} \qquad (j=1,2,\cdots,n)$$

with $m_i < t_i \leqq m_i+1$ $(i=1,2,\cdots,n)$. Then we have

$$S(\nu z)-S(\xi z) \ll \max_{1\leqq j\leqq n}(|z_j|) \ll M^{-a}$$

so that

$$\begin{aligned} e^{2\pi i S(\nu z)} &= e^{2\pi i S(\xi z)}+O(M^{-a}) \\ &= \int_{m_i}^{m_i+1}\!\!\cdots\!\int e^{2\pi i S(\xi z)} dt_1 \cdots dt_n + O(M^{-a}) , \end{aligned}$$

where the domain of integration is given as follows:

$$m_i \leqq t_i \leqq m_i+1 \qquad (i=1,2,\cdots,n).$$

Therefore we have

$$T(z\,;M)=\sum_{\{m\}}\int_{m_i}^{m_i+1}\!\!\cdots\!\int e^{2\pi i S(\xi z)} dt_1 \cdots dt_n + O(M^{n-a}) , \tag{9.4}$$

where the sum is taken over all $(m_1, m_2, \cdots, m_n)$ which are derived from integers ν satisfying the condition (9.2). The number of these integers is $O(M^n)$, which gives the error term in the right-hand side of (9.4).

From now on we shall use abbreviations for the notations of products;

$$\prod_q=\prod_{q=1}^{r_1}, \qquad \prod_p=\prod_{p=r_1+1}^{r_1+r_2}, \qquad \prod_j=\prod_{j=1}^{r+1}.$$

If we put

$$\begin{aligned} U_q &= \xi_q && (q=1,2,\cdots,r_1), \\ U_p &= |\xi_p| && \\ \theta_p &= \arg \xi_p && (p=r_1+1,\cdots,r_1+r_2), \end{aligned}$$

then we have

$$\frac{\partial(t_1, t_2, \cdots, t_n)}{\partial(U_1, \cdots, U_{r+1}, \theta_{r_1+1}, \cdots, \theta_{r+1})}=\frac{2^{r_2}}{\sqrt{D}}\prod_p U_p$$

and

$$\begin{aligned} T(z\,;M) &= \frac{2^{r_2}}{\sqrt{D}}\int\cdots\int e^{2\pi i S(\xi z)}\prod_p U_p \prod_j dU_j \prod_p d\theta_p + O(M^{n-a}) \\ &= \frac{2^{r_2}}{\sqrt{D}}\prod_q\int_0^{M_q} e^{2\pi i z_q U} dU \prod_p \int_0^{2\pi} d\theta_p \int_0^{M_p} e^{2\pi i (z_p \xi_p + z_{p'} \xi_{p'})} U_p dU_p \\ &\quad + O(M^{n-a}). \end{aligned}$$

We shall put

$$\phi(z\,;M)=\prod_q\int_0^{M_q} e^{2\pi i z_q U} dU \prod_p \int_0^{2\pi} d\theta_p \int_0^{M_p} e^{2\pi i (z_p \xi_p + z_{p'} \xi_{p'})} U_p dU_p \tag{9.5}$$

and estimate it. It is obvious that

$$\int_0^{M_q} e^{2\pi i z_q U} dU \ll \min\left(M, \frac{1}{|z_q|}\right) \qquad (q=1, 2, \cdots, r_1). \tag{9.6}$$

As for the integral in the second factor of the right-hand side of (9.5), putting

$$\varphi_p = \arg z_p \qquad (p = r_1+1, \cdots, r_1+r_2),$$

we have

$$\int_0^{2\pi}\int_0^{M_p} e^{2\pi i (z_p \xi_p + z_{p'} \xi_{p'})} U_p dU_p d\theta_p = \int_0^{2\pi}\int_0^{M_p} e^{4\pi i U |z_p| \cos(\theta+\varphi_p)} U dU d\theta .$$

By partial-integration we have

$$\int_0^M e^{4\pi i U|z| \cos(\theta+\varphi)} U dU$$

$$= M\int_0^M e^{4\pi i U|z| \cos(\theta+\varphi)} dU - \int_0^M \left\{\int_0^U e^{4\pi i t|z| \cos(\theta+\varphi)} dt\right\} dU .$$

Since

$$\int_0^{2\pi}\int_0^U e^{4\pi i t|z| \cos(\theta+\varphi)} dt d\theta \ll \min\left(U, \frac{1}{|z|}\right)$$

(Siegel [**6**, (83)]), we obtain

$$\int_0^{2\pi}\int_0^{M_p} e^{4\pi i U|z_p| \cos(\theta+\varphi_p)} U dU d\theta \ll M \min\left(M, \frac{1}{|z_p|}\right) \quad (p = r_1+1, \cdots, r_1+r_2). \tag{9.7}$$

If we put

$$F(z) = M^{r_2} \prod_{j=1}^{r+1} \min\left(M, \frac{1}{|z_j|}\right),$$

then it follows from (9.6) and (9.7) that

$$\phi(z\,; M) \ll F(z) .$$

Now we take a rational integer $s \geqq 3$ and a totally positive integer μ and define an integral

$$J_s(\mu\,; M) = \int_{-1/2}^{1/2} \cdots \int T(z\,; M)^s e^{-2\pi i S(\mu z)} dx_1 dx_2 \cdots dx_n . \tag{9.8}$$

We define two sets $\mathfrak{D}_0$ and $\mathfrak{D}_1$ in n-dimensional euclidean space as follows:

$$\mathfrak{D}_0 = \{x(z)\,; z = (z_1, z_2, \cdots, z_n) \in D_0\},$$

$$\mathfrak{D}_1 = \{x(z)\,; z = (z_1, z_2, \cdots, z_n) \in D_1\}$$

and we divide the integral (9.8) into two parts:

$$J_s(\mu\,; M) = 2^{r_2}\sqrt{D} \left\{\int_{\mathfrak{D}_0} \cdots \int + \int_{\mathfrak{D}_1} \cdots \int\right\} T(z\,; M)^s e^{-2\pi i S(\mu z)} dx(z) . \tag{9.9}$$

As for the first integral in this right-hand side, we have by (9.3)

$$\int_{\mathfrak{D}_0}\cdots\int T(z\,;M)^s e^{-2\pi i S(\mu z)}dx(z)$$

$$\ll M^{(n-1+a)(s-2)}\int_{-1/2}^{1/2}\cdots\int |T(z\,;M)|^2 dx_1\cdots dx_n \ll M^{(n-1+a)(s-2)+n}.$$

If $z\in D_1$, then, using the estimation for $T(z\,;M)^s$;

$$T(z\,;M)^s=\frac{2^{r_2 s}}{D^{s/2}}\phi(z\,;M)^s+O(M^{ns-a}),$$

we have

$$\begin{aligned} J_s(\mu\,;M)&=\frac{2^{r_2(s+1)}}{D^{(s-1)/2}}\int_{\mathfrak{D}_1}\cdots\int \phi(z\,;M)^s e^{-2\pi i S(\mu z)}dx(z)\\ &\quad+O(M^{ns-a})\int_{\mathfrak{D}_1}\cdots\int dx(z)+O(M^{n(s-1)-(1-a)(s-2)})\\ &=\frac{2^{r_2(s+1)}}{D^{(s-1)/2}}\int_{\mathfrak{D}_1}\cdots\int \phi(z\,;M)^s e^{-2\pi i S(\mu z)}dx(z)\\ &\quad+O(M^{ns-a(1+n)})+O(M^{n(s-1)-(1-a)(s-2)}), \end{aligned}\tag{9.10}$$

since

$$\int_{\mathfrak{D}_1}\cdots\int dx(z)\ll M^{-na}.$$

Now we put

$$\begin{aligned} u_q&=z_q && (q=1,2,\cdots,r_1),\\ u_p&=|z_p| && \\ \varphi_p&=\arg z_p && (p=r_1+1,\cdots,r_1+r_2) \end{aligned}$$

and

$$a_0=\min((n+1)a-n,\ (a-1)(s-2)),$$

then we have from (9.10)

$$J_s(\mu\,;M)=\frac{2^{r_2(s+1)}}{D^{(s-1)/2}}\int_{X_1}\cdots\int \phi(z\,;M)^s e^{-2\pi i S(\mu z)}\prod_p u_p\prod_j du_j\prod_p d\varphi_p$$
$$+O(M^{n(s-1)-a_0}),$$

where the domain of integration is given as follows:

$$X_1:\quad \begin{aligned} &|u_q|\leqq M^{-a} && (q=1,2,\cdots,r_1),\\ &0\leqq u_p\leqq M^{-a} && \\ &0\leqq \varphi_p\leqq 2\pi && (p=r_1+1,\cdots r_1+r_2). \end{aligned}$$

We consider a domain X containing X_1 which is defined by the conditions:

$$X:\quad \begin{aligned} &|u_q|<\infty && (q=1,2,\cdots,r_1),\\ &0\leqq u_p<\infty && \\ &0\leqq \varphi_p\leqq 2\pi && (p=r_1+1,\cdots,r_1+r_2), \end{aligned}$$

and estimate

$$I=\int\underset{X-X_1}{\cdots}\int F(z)^s\prod_p u_p\prod_j du_j\prod_p d\varphi_p .$$

Easily we have

$$\int_0^\infty \min\left(M, \frac{1}{u}\right)^s du \ll M^{s-1},$$

$$\int_0^\infty \min\left(M, \frac{1}{u}\right)^s udu \ll M^{s-2},$$

$$\int_{M^{-a}}^\infty \min\left(M, \frac{1}{u}\right)^s du \ll M^{a(s-1)},$$

$$\int_{M^{-a}}^\infty \min\left(M, \frac{1}{u}\right)^s udu \ll M^{a(s-2)},$$

which gives

$$I \ll M^{n(s-1)-(1-a)(s-2)} .$$

Therefore we have

$$J_s(\mu\,;M)=\frac{2^{r_2(s+1)}}{D^{(s-1)/2}}\int\underset{X}{\cdots}\int \phi(z\,;M)^s e^{-2\pi i S(\mu z)}\prod_p u_p d\varphi_p\prod_j du_j \tag{9.11}$$
$$+O(M^{n(s-1)-a_0}) .$$

Since

$$\phi(z\,;M)=M_1M_2\cdots M_n\prod_q\int_0^1 e^{2\pi i M_q z_q U_q}dU_q$$
$$\times\prod_p\int_0^{2\pi}d\theta_p\int_0^1 e^{2\pi i M_p(z_p\xi_p+z_{p'}\xi_{p'})}U_p dU_p ,$$

we have

$$J_s(\mu\,;M)=\frac{2^{r_2(s+1)}}{D^{(s-1)/2}}(M_1M_2\cdots M_n)^{s-1}\int\underset{X}{\cdots}\int \phi(z)^s e^{-2\pi i S(\tilde{\mu} z)}\prod_p u_p d\varphi_p\prod_j du_j \tag{9.12}$$
$$+O(M^{n(s-1)-a_0}) ,$$

where

$$\tilde{\mu}^{(j)}=\frac{\mu^{(j)}}{M_j} \qquad (j=1,2,\cdots,n)$$

and

$$\phi(z)=\prod_q\int_0^1 e^{2\pi i U_q z_q}dU_q\prod_p\int_0^{2\pi}d\theta_p\int_0^1 e^{2\pi i(z_p\xi_p+z_{p'}\xi_{p'})}U_p dU_p .$$

The last function $\phi(z)$ is also written as follows:

$$\phi(z)=\prod_q\int_0^1 e^{2\pi i t z_q}dt\prod_p\iint_{u^2+v^2\leqq 1} e^{2\pi i(2uX_p(z)-2vX_{p'}(z))}dudv .$$

Therefore we have

$$\int\underset{X}{\cdots}\int \phi(z)^s e^{-2\pi i S(\tilde{\mu} z)}\prod_p u_p d\varphi_p\prod_j du_j$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty}\!\!\cdots\!\int \phi(z)^s e^{-2\pi i S(\tilde{\mu} z)} dx(z) \\
(9.13)\qquad &= \prod_q \int_{-\infty}^{\infty} \Big(\int_0^1 e^{2\pi i wt}\, dt\Big)^s e^{-2\pi i \tilde{\mu}^{(q)} w} dw \\
&\times \frac{1}{2^{2r_2}} \prod_p \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Big(\iint_{u^2+v^2\leqq 1} e^{2\pi i(ux+vy)} du dv\Big)^s e^{-2\pi i(X_p(\tilde{\mu})x + X_{p'}(\tilde{\mu})y)} dx dy\,.
\end{aligned}
$$

We shall denote this integral by J_0.

Now we put

$$K_q(\mu) = \int_{-\infty}^{\infty} \phi_1(w)^s e^{-2\pi i \tilde{\mu}^{(q)} w} dw \qquad (q=1, 2, \cdots, r_1),$$

$$\phi_1(w) = \int_0^1 e^{2\pi i wt} dt\,,$$

$$K_p(\mu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_2(x, y)^s e^{-2\pi i(X_p(\tilde{\mu})x + X_{p'}(\tilde{\mu})y)} dx dy \quad (p = r_1+1, \cdots, r_1+r_2),$$

$$\phi_2(x, y) = \iint_{u^2+v^2\leqq 1} e^{2\pi i(ux+vy)} du dv\,.$$

Then we have

$$J_0 = \frac{1}{2^{2r_2}} \prod_q K_q(\mu) \prod_p K_p(\mu)\,. \tag{9.14}$$

Now we write

$$\phi_1(w)^s = \int_0^1\!\!\cdots\!\int e^{2\pi i w(t_1+t_2+\cdots+t_s)} dt_1 dt_2 \cdots dt_s\,,$$

then, putting $\xi = t_1+t_2+\cdots+t_s$, we have

$$\phi_1(w)^s = \int_{-\infty}^{\infty} F(\xi)\, e^{2\pi i w\xi} d\xi\,,$$

where

$$F(\xi) = \int\!\!\cdots\!\int_{B_1} dt_1 dt_2 \cdots dt_{s-1}$$

with the domain of integration

$$B_1: \quad \begin{aligned} &0 \leqq t_j \leqq 1 \qquad (j=1, 2, \cdots, s-1),\\ &0 \leqq \xi - (t_1+t_2+\cdots+t_{s-1}) \leqq 1 \end{aligned}$$

so that

$$K_q(\mu) = \int_{-\infty}^{\infty} \Big\{\int_{-\infty}^{\infty} F(\xi)\, e^{2\pi i w\xi} d\xi\Big\}\, e^{-2\pi i \tilde{\mu}^{(q)} w} dw \qquad (q=1, 2, \cdots, r_1).$$

It is obvious that $F(\xi)$ is a continuous function of ξ, therefore we have, applying the theory of Fourier integrals,

$$K_q(\mu) = F(\tilde{\mu}^{(q)}) \qquad (q=1, 2, \cdots, r_1).$$

Now we assume that

$$M_q \geqq \mu^{(q)} \qquad (q=1, 2, \cdots, r_1),$$

then we can easily calculate $K_q(\mu)$:

(9.15) $$K_q(\mu)=\frac{1}{\Gamma(s)}\left(\frac{\mu^{(q)}}{M_q}\right)^{s-1} \qquad (q=1, 2, \cdots, r_1).$$

In the similar way, we have

$$K_p(\mu)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(U, V)\, e^{2\pi i(Ux+Vy)} dU dV\right\} e^{-2\pi i(X_p(\tilde{\mu})x+X_{p'}(\tilde{\mu})y)} dx dy$$
$$(p=r_1+1, \cdots, r_1+r_2),$$

where

$$G(U, V)=\int\cdots\int_{B_2} du_1 du_2 \cdots du_{s-1} dv_1 dv_2 \cdots dv_{s-1}$$

is a $2(s-1)$-fold integral with the domain of integration

$$B_2: \quad \begin{aligned} &u_j^2+v_j^2 \leqq 1 \qquad (j=1, 2, \cdots, s-1), \\ &(u_1+\cdots+u_{s-1}-U)^2+(v_1+v_2+\cdots+v_{s-1}-V)^2 \leqq 1. \end{aligned}$$

Therefore we have, by the theory of Fourier integrals,

(9.16) $$K_p(\mu)=G(X_p(\tilde{\mu}), X_{p'}(\tilde{\mu}))=\int\cdots\int_{B_p(\mu)} u_1 \cdots u_{s-1} du_1 \cdots du_{s-1} d\varphi_1 \cdots d\varphi_{s-1},$$

where $B_p(\mu)$ is defined as follows:

$$B_p(\mu): \quad \begin{aligned} &0 \leqq u_j \leqq 1, \quad 0 \leqq \varphi_j \leqq 2\pi \qquad (j=1, 2, \cdots, s-1), \\ &\left| u_1 e^{i\varphi_1}+\cdots+u_{s-1}e^{i\varphi_{s-1}}-\frac{|\mu^{(p)}|}{M_p}\right| \leqq 1. \end{aligned}$$

By the above results (9.12)-(9.16), we have

(9.17) $$J_s(\mu\,; M)=\frac{2^{r_2(s-1)}(\mu^{(1)}\cdots\mu^{(r_1)})^{s-1}}{D^{(s-1)/2}((s-1)!\,)^{r_1}}\prod_p M_p^{2(s-1)}K_p(\mu)+O(M^{n(s-1)-a_0}).$$

Obviously, $J_s(\mu\,; M)$ is equal to the number of the s-tuples $(\nu_1, \nu_2, \cdots, \nu_s)$ of integers which satisfy the following conditions:

$$\begin{aligned} &\mu=\nu_1+\nu_2+\cdots+\nu_s, \\ &0<\nu_j^{(q)} \leqq M_q \qquad (q=1, 2, \cdots, r_1), \quad (j=1, 2, \cdots, s), \\ &|\nu_j^{(p)}| \leqq M_p \qquad (p=r_1+1, \cdots r_1+r_2) \end{aligned}$$

with

$$M_q \geqq \mu^{(q)} \qquad (q=1, 2, \cdots, r_1).$$

Now we take a sufficiently small number $\delta>0$ such that $\delta M \geqq 1$ and n real numbers $\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n$ which satisfy the following conditions:

$$\begin{aligned} &0<\tilde{x}_q \leqq M_q && (q=1,2,\cdots,r_1), \\ (9.18)\qquad &\tilde{x}_q = M_q+O(M\delta) && \\ &|\tilde{x}_p+i\tilde{x}_{p'}| = M_p+O(M\delta) && (p=r_1+1,\cdots,r_1+r_2). \end{aligned}$$

We shall define a set $\mathfrak{M}(\tilde{x})$ of integers μ such that

$$\begin{aligned} (9.19)\qquad &\tilde{x}_q - M\delta < \mu^{(q)} \leqq \tilde{x}_q && (q=1,2,\cdots,r_1), \\ &|\tilde{x}_p+i\tilde{x}_{p'}-\mu^{(p)}| \leqq M\delta && (p=r_1+1,\cdots,r_1+r_2). \end{aligned}$$

Since $K_p(\mu)=O(1)$ and $\mu^{(q)} \leqq M_q$ $(q=1,\cdots,r_1)$ for $\mu \in \mathfrak{M}(\tilde{x})$, we have from (9.17)

$$J_s(\mu\,;M) = \frac{2^{r_2(s-1)}(M_1M_2\cdots M_n)^{s-1}}{D^{(s-1)/2}((s-1)\,!\,)^{r_1}} \prod_p K_p(\mu) + O(\delta M^{n(s-1)}) + O(M^{n(s-1)-a_0})$$

for $\mu \in \mathfrak{M}(\tilde{x})$.

Now we consider an integral

$$I = \int\underset{B_0}{\cdots}\int u_1 \cdots u_{s-1} du_1 \cdots du_{s-1} d\varphi_1 \cdots d\varphi_{s-1}$$

with the domain of integration

$$B_0: \quad \begin{aligned} &0 \leqq u_j \leqq 1, \quad 0 \leqq \varphi_j \leqq 2\pi \qquad (j=1,2,\cdots,s-1), \\ &|u_1e^{i\varphi_1}+\cdots+u_{s-1}e^{i\varphi_{s-1}}-1| \leqq 1, \end{aligned}$$

and we shall prove

$$(9.20)\qquad K_p(\mu) = I + O(\delta) \qquad (\mu \in \mathfrak{M}(\tilde{x})).$$

We change the variables in I and $K_p(\mu)$; $x_j = u_j \cos\varphi_j$, $y_j = u_j \sin\varphi_j$ $(j=1,2,\cdots, s-1)$ and write

$$K_p(\mu) = \int_{B_p'(\mu)}\int dx_1dx_2\cdots dx_{s-1}dy_1dy_2\cdots dy_{s-1}$$

with

$$B_p'(\mu): \quad \begin{aligned} &x_j^2+y_j^2 \leqq 1 \qquad (j=1,2,\cdots,s-1), \\ &(|\tilde{\mu}^{(p)}| - \sum_{j=1}^{s-1} x_j)^2 + (\sum_{j=1}^{s-1} y_j)^2 \leqq 1 \end{aligned}$$

and

$$I = \int\underset{B_0'}{\cdots}\int dx_1dx_2\cdots dx_{s-1}dy_1dy_2\cdots dy_{s-1}$$

with

$$B_0': \quad \begin{aligned} &x_j^2+y_j^2 \leqq 1 \qquad (j=1,2,\cdots,s-1), \\ &(1-\sum_{j=1}^{s-1} x_j)^2 + (\sum_{j=1}^{s-1} y_j)^2 \leqq 1. \end{aligned}$$

Clearly, $|I-K_p(\mu)|$ does not exceed the volume of $V=(B_p'(\mu)-B_0')\cup(B_0'-B_p'(\mu))$ in $2(s-1)$-dimensional euclidean space, $(x_1,\cdots,y_1,\cdots)$ being the points of this space. If $(x_1, \cdots, x_{s-1}, y_1, \cdots, y_{s-1})$ is a point of V for given $y_1, y_2, \cdots, y_{s-1}$, then $x_1, x_2, \cdots, x_{s-1}$ satisfy the conditions

$$|x_j| \leqq 1 \qquad (j=1, 2, \cdots, s-1),$$

$$f(y_1, \cdots, y_{s-1}) \leqq x_1+x_2+\cdots+x_{s-1} \leqq f(y_1, \cdots, y_{s-1})+c\delta$$

with a certain function $f(y_1, \cdots, y_{s-1})$ of $y_1, y_2, \cdots, y_{s-1}$. Therefore we have

$$\int \underset{V}{\cdots} \int dx_1 dx_2 \cdots dx_{s-1} dy_1 dy_2 \cdots dy_{s-1} = \int \cdots \int dy_1 \cdots dy_{s-1} \int \cdots \int dx_1 \cdots dx_{s-1}$$

$$\ll \delta \int_{-1}^{1} \cdots \int dy_1 \cdots dy_{s-1} \ll \delta$$

and the assertion (9.20) is proved.

If we put

$$\sigma(s) = \int \underset{B^0}{\cdots} \int du_1 \cdots du_{s-1} d\varphi_1 \cdots d\varphi_{s-1}$$

with

$$B^0: \quad \begin{array}{l} 0 \leqq u_j \leqq 1, \quad 0 \leqq \varphi_j \leqq 2\pi \qquad (j=1, 2, \cdots, s-1), \\ |\sqrt{u_1}\, e^{i\varphi_1} + \cdots + \sqrt{u_{s-1}}\, e^{i\varphi_{s-1}} - 1| \leqq 1, \end{array}$$

then $\sigma(s) = 2^{s-1} I$. Therefore we have for $\mu \in \mathfrak{M}(\tilde{x})$

$$J_s(\mu\,; M) = \frac{\sigma(s)^{r_2}}{D^{\frac{s-1}{2}}((s-1)\,!\,)^{r_1}} (M_1 M_2 \cdots M_n)^{s-1} + O(\delta M^{n(s-1)}) + O(M^{n(s-1)-a_0}). \tag{9.21}$$

Now we sum up the both sides of (9.21) over all $\mu \in \mathfrak{M}(\tilde{x})$. It is obvious that

$$\sum_{\mu \in \mathfrak{M}(\tilde{x})} 1 = \frac{2^{r_2} \pi^{r_2}}{\sqrt{D}} M^n \delta^n + O(M^{n-1} \delta^{n-1}).$$

Therefore we have

$$\begin{aligned} \sum_{\mu \in \mathfrak{M}(\tilde{x})} J_s(\mu\,; M) &= \frac{(2\pi\sigma(s))^{r_2}}{D^{s/2}((s-1)\,!\,)^{r_1}} M^n \delta^n (M_1 M_2 \cdots M_n)^{s-1} \\ &\quad + O(M^{ns} \delta^{n+1}) + O(M^{ns-a_0} \delta^n) + O(M^{ns-1} \delta^{n-1}) \\ &= \frac{(2\pi\sigma(s))^{r_2}}{D^{s/2}((s-1)\,!\,)^{r_1}} M^n \delta^n (M_1 M_2 \cdots M_n)^{s-1} (1 + O(\delta) + O(M^{-a_0}) \\ &\qquad + O(\delta^{-1} M^{-1})). \end{aligned} \tag{9.22}$$

This left-hand side is equal to the number of the s-tuples $(\nu_1, \nu_2, \cdots, \nu_s)$ of integers which satisfy the condition

$$\begin{aligned} &\tilde{x}_q - M\delta < \nu_1^{(q)} + \nu_2^{(q)} + \cdots + \nu_s^{(q)} \leqq \tilde{x}_q \qquad (q=1, 2, \cdots, r_1), \\ &|\tilde{x}_p + i\tilde{x}_{p'} - (\nu_1^{(p)} + \nu_2^{(p)} + \cdots + \nu_s^{(p)})| \leqq M\delta \qquad (p=r_1+1, \cdots, r_1+r_2), \\ &0 < \nu_j^{(q)} \leqq M_q \quad (q=1, 2, \cdots, r_1), \\ &|\nu_j^{(p)}| \leqq M_p \quad (p=r_1+1, \cdots, r_1+r_2), \qquad (j=1, 2, \cdots, s) \end{aligned} \tag{9.23}$$

provided that $M\delta \geqq 1$ and

$$
\begin{aligned}
&c_1M \leqq M_j \leqq c_2M && (j=1,2,\cdots,n), \\
(9.24)\qquad &M_q - c_3M\delta \leqq \tilde{x}_q \leqq M_q && (q=1,2,\cdots,r_1), \\
&M_p - c_4M\delta \leqq |\tilde{x}_p + i\tilde{x}_{p'}| \leqq M_p + c_5M\delta && (p=r_1+1,\cdots,r_1+r_2)
\end{aligned}
$$

for suitable positive constants c_1, c_2, c_3, c_4 and c_5.

§ 10. Generalization of Goldbach-Vinogradov's theorem.

Using the notations in § 8, we shall put, in (9.23),

$$\tilde{x}_q = \frac{N_0}{Y}\Big(\lambda^{(q)} + \frac{sBN}{(\log N)^{\kappa+1}}\Big) \qquad (q=1,2,\cdots,r_1),$$

$$\tilde{x}_p + i\tilde{x}_{p'} = \frac{N_0}{Y}\lambda^{(p)} \qquad (p=r_1+1,\cdots,r_1+r_2),$$

$$M_q = \frac{N_0}{Y}\Big(\lambda^{(q)} + (sB+C_0)\frac{N}{(\log N)^{\kappa+1}}\Big) \qquad (q=1,2,\cdots,r_1),$$

$$M_p = \frac{N_0}{Y}\Big(|\lambda^{(p)}| + \frac{C_0N}{(\log N)^{\kappa+1}}\Big) \qquad (p=r_1+1,\cdots,r_1+r_2),$$

$$M = \frac{NN_0}{Y}, \qquad \delta = \frac{1}{(\log N)^{\kappa}}\Big(1+\frac{2sB}{\log N}\Big).$$

We see that, after these substitutions, (9.23) coincides to the conditions (C_1) for T_1 in § 8. Moreover, the conditions (9.24) are satisfied. Therefore, we can now estimate T_1 by making use of the results in § 9.

Since

$$M\delta = \frac{N}{C_0(\log N)^{\kappa+1/n}}\Big(1+O\Big(\frac{\log\log N}{\log N}\Big)\Big),$$

$$M_1M_2\cdots M_n = \frac{N(\lambda)}{C_0^{\,n}\log N}\Big(1+O\Big(\frac{\log\log N}{\log N}\Big)\Big),$$

we have by (9.22)

$$
\begin{aligned}
T_1 &= \frac{(2\pi\sigma(s))^{r_2}}{D^{s/2}((s-1)!)^{r_1}}\cdot\frac{N^n\,N(\lambda)^{s-1}}{C_0^{\,ns}(\log N)^{n\kappa+s}}\Big(1+O\Big(\frac{\log\log N}{\log N}\Big)\Big) \\
(10.1)\qquad &= \frac{(2^{1-s}\pi^{1-s}\sigma(s))^{r_2}}{n^sW^s((s-1)!)^{r_1}}\cdot\frac{N^n\,N(\lambda)^{s-1}}{(\log N)^{n\kappa+s}}\Big(1+O\Big(\frac{\log\log N}{\log N}\Big)\Big).
\end{aligned}
$$

In the similar way, putting

$$\tilde{x}_q = \frac{N_0}{Z}\Big(\lambda^{(q)} - (sB+C_0)\frac{N}{(\log N)^{\kappa+1}}\Big)\Big) \qquad (q=1,2,\cdots,r_1),$$

$$\tilde{x}_p + i\tilde{x}_{p'} = \frac{N_0}{Z}\lambda^{(p)} \qquad (p=r_1+1,\cdots,r_1+r_2),$$

$$M_q=\frac{N_0}{Z}\Big(\lambda^{(q)}-\frac{C_0N}{(\log N)^{\kappa+1}}\Big) \qquad (q=1,2,\cdots,r_1),$$

$$M_p=\frac{N_0}{Z}\Big(|\lambda^{(p)}|-\frac{C_0N}{(\log N)^{\kappa+1}}\Big) \qquad (p=r_1+1,\cdots,r_1+r_2),$$

$$M=\frac{NN_0}{Z}, \qquad \delta=\frac{1}{(\log N)^{\tau}}\Big(1-\frac{2(sB+C_0)}{\log N}\Big),$$

we also have

$$T_2=\frac{(2^{1-s}\pi^{1-s}\sigma(s))^{r_2}}{n^sW^s((s-1)!)^{r_1}}\cdot\frac{N^n N(\lambda)^{s-1}}{(\log N)^{n\kappa+s}}\Big(1+O\Big(\frac{\log\log N}{\log N}\Big)\Big). \tag{10.2}$$

These two results (10.1) and (10.2) give the asymptotic formula for $T(\lambda)$:

$$T(\lambda)=\frac{(2^{1-s}\pi^{1-s}\sigma(s))^{r_2}}{n^sW^s((s-1)!)^{r_1}}\cdot\frac{N^n N(\lambda)^{s-1}}{(\log N)^{n\kappa+s}}\Big(1+O\Big(\frac{\log\log N}{\log N}\Big)\Big),$$

Comparing this result with another asymptotic formula (7.10) for $T(\lambda)$, we have

$$\frac{(2^{1-s}\pi^{1-s}\sigma(s))^{r_2}}{n^sW^s((s-1)!)^{r_1}}\cdot\frac{N^n N(\lambda)^{s-1}}{(\log N)^{n\kappa+s}}\Big(1+O\Big(\frac{\log\log N}{\log N}\Big)\Big)$$
$$=\frac{2^{2r_2}\pi^{r_2}N^nR(\lambda,\lambda)}{W^s(\log N)^{n\kappa}}\Big(1+O\Big(\frac{(\log N)^{\kappa+1}}{N}\Big)\Big),$$

which gives an asymptotic formula for $R(\lambda,\lambda)$:

$$R(\lambda,\lambda)=\frac{\sigma(s)^{r_2}}{n^s((s-1)!)^{r_1}(\pi^s2^{1+s})^{r_2}}\cdot\frac{N(\lambda)^{s-1}}{(\log N)^s}\Big(1+O\Big(\frac{\log\log N}{\log N}\Big)\Big).$$

Putting this result in (7.5), we finally obtain

$$I_s(\lambda\,;\lambda)=\frac{w^s\sigma(s)^{r_2}D^{1/2}}{((s-1)!)^{r_1}(2^{r_1+r_2}\pi^{r_2}hR)^s}\cdot\frac{N(\lambda)^{s-1}}{(\log N(\lambda))^s}\sum_{N\mathfrak{a}\leqq T^n}\frac{\mu(\mathfrak{a})^s}{\varphi(\mathfrak{a})^s}G(\mathfrak{a},\lambda)$$
$$+O\Big(\frac{N^{n(s-1)}\log\log N}{(\log N)^{s+1}}\Big).$$

Now we define the singular series:

$$\mathfrak{S}_s(\lambda)=\sum_{\mathfrak{a}}\frac{\mu(\mathfrak{a})^s}{\varphi(\mathfrak{a})^s}G(\mathfrak{a},\lambda),$$

where $\mathfrak{a}$ runs through all integral ideals. This series is convergent and

$$\mathfrak{S}_s(\lambda)-\sum_{N\mathfrak{a}\leqq T^n}\frac{\mu(\mathfrak{a})^s}{\varphi(\mathfrak{a})^s}G(\mathfrak{a},\lambda)\ll\sum_{N\mathfrak{a}>T^n}\frac{N(\mathfrak{a})}{\varphi(\mathfrak{a})^s}$$
$$\ll T^{-n/2}\ll(\log N)^{-1}$$

on account of Lemma 7.1.

Therefore we have

$$I_s(\lambda\,;\lambda)=\frac{w^s\sigma(s)^{r_2}D^{1/2}}{((s-1)\,!\,)^{r_1}(2^{r_1+r_2}\pi^{r_2}hR)^s}\mathfrak{S}_s(\lambda)\frac{N(\lambda)^{s-1}}{(\log N(\lambda))^s}+O\Big(\frac{N(\lambda)^{s-1}\log\log N(\lambda)}{(\log N(\lambda))^{s+1}}\Big).$$

The following properties of the singular series $\mathfrak{S}_s(\lambda)$ has already been studied. (See Rademacher [4]).

$\mathfrak{S}_s(\lambda)$ is written in the form of an infinite product as follows:

$$\mathfrak{S}_s(\lambda)=\prod_{\mathfrak{p}\mid\lambda}\Big(1+\frac{(-1)^s}{(N(\mathfrak{p})-1)^{s-1}}\Big)\prod_{\mathfrak{p}\nmid\lambda}\Big(1+\frac{(-1)^{s+1}}{(N(\mathfrak{p})-1)^s}\Big),$$

where first product is taken over all prime divisors of λ and second product is taken over all other prime ideals.

Let $\mathfrak{L}$ be the product of all prime ideals $\mathfrak{p}$ with $N(\mathfrak{p})=2$. (If no such ideal exists, we put $\mathfrak{L}=\mathfrak{o}$). We shall call an integer μ of K *even*, if $\mu\in\mathfrak{L}$, and *odd*, if $(\mu,\mathfrak{L})=1$. Then we see that, if both s and λ are even or odd, then

$$\mathfrak{S}_s(\lambda)\geqq c>0$$

and in other case, $\mathfrak{S}_s(\lambda)=0$.

Now collecting all our results, we have

THEOREM 10.1. *Let λ be a totally positive integer of K and s be a rational integer $\geqq 3$. We denote by $I_s(\lambda)$ the number of the s-tuples $(\omega_1,\omega_2,\cdots,\omega_s)$ of prime numbers of K which satisfy the following conditions*

$$\lambda=\omega_1+\omega_2+\cdots+\omega_s\,,$$
$$\left.\begin{array}{ll}0<\omega_j^{(q)}\leqq\lambda^{(q)} & (q=1,2,\cdots,r_1)\,,\\ |\,\omega_j^{(p)}\,|\leqq|\,\lambda^{(p)}\,| & (p=r_1+1,\cdots,r_1+r_2)\end{array}\right.\quad (j=1,2,\cdots,s).$$

Then we have

$$I_s(\lambda)=\frac{w^s\sigma(s)^{r_2}D^{1/2}}{((s-1)\,!\,)^{r_1}(2^{r_1+r_2}\pi^{r_2}hR)^s}\mathfrak{S}_s(\lambda)\frac{N(\lambda)^{s-1}}{(\log N(\lambda))^s}+O\Big(\frac{N(\lambda)^{s-1}\log\log N(\lambda)}{(\log N(\lambda))^{s+1}}\Big),$$

where D is the absolute value of the discriminant of K, w is the number of the roots of unity in K, h is the class number and R is the regulator of K, $\sigma(s)$ is a $2(s-1)$-fold integral:

$$\sigma(s)=\int\!\underset{B}{\cdots}\!\int du_1\cdots du_{s-1}d\varphi_1\cdots d\varphi_{s-1}$$

with the domain of integration

$$B:\quad\begin{array}{l}0\leqq u_j\leqq 1\,,\qquad 0\leqq\varphi_j\leqq 2\pi\qquad\qquad (j=1,2,\cdots,s-1)\,,\\ |\,\sqrt{u_1}\,e^{i\varphi_1}+\cdots+\sqrt{u_{s-1}}\,e^{i\varphi_{s-1}}-1\,|\leqq 1\,.\end{array}$$

$\mathfrak{S}_s(\lambda)$ is the singular series which is written in the form of an infinite product:

$$\mathfrak{S}_s(\lambda)=\prod_{\mathfrak{p}\mid\lambda}\Big(1+\frac{(-1)^s}{(N(\mathfrak{p})-1)^{s-1}}\Big)\prod_{\mathfrak{p}\nmid\lambda}\Big(1+\frac{(-1)^{s+1}}{(N(\mathfrak{p})-1)^s}\Big).$$

If both s and λ are even or odd, then

$$\mathfrak{S}_s(\lambda) \geqq c > 0$$

and otherwise $\mathfrak{S}_s(\lambda) = 0$.

§ 11. Generalization of Estermann's theorem.

In this paragraph, we assume that N is a sufficiently large rational integer and we take positive constants σ, σ_1 and σ_2 as in (4.5), (4.6) and (4.7). We put

$$H = \frac{N}{(\log N)^{\sigma_1}}, \qquad T = (\log N)^{\sigma_1}$$

and consider the division of E into B^0 and B_γ ($\gamma \in \Gamma$) which are defined by (4.9).

LEMMA 11.1. *Let $z = (z_1, z_2, \cdots, z_n)$ be a point of B_γ with $\gamma \to \mathfrak{a}$. Then we have*

$$S(z\,; N) = \frac{w\sqrt{D}}{2^{r_1}hR} \cdot \frac{\mu(\mathfrak{a})}{\varphi(\mathfrak{a})} \sum_{\mu \in \Lambda_0(N)} \frac{e^{2\pi i S(\mu y)}}{\log N(\mu)} + O\left(\frac{N^n}{(\log N)^{a-b+1}}\right), \tag{11.1}$$

where $S(z\,; N)$ is the trigonometrical sum defined by (4.2), *a is a positive constant which can be taken sufficiently large,* $b = (n-1)\sigma_2 + \sigma_1$,

$$y_j = z_j - \gamma^{(j)} \qquad (j = 1, 2, \cdots, n)$$

and $\Lambda_0(N)$ is the set of integers μ such that

$$0 < \mu^{(q)} \leqq N \qquad (q = 1, 2, \cdots, r_1),$$
$$|\mu^{(p)}| \leqq N \qquad (p = r_1+1, \cdots, r_1+r_2),$$
$$1 < N(\mu).$$

PROOF. Let $\Lambda_1(N)$ be the set of integers μ such that

$$\sqrt{N} < \mu^{(q)} \leqq N \qquad (q = 1, 2, \cdots, r_1),$$
$$\sqrt{N} < |\mu^{(p)}| \leqq N \qquad (p = r_1+1, \cdots, r_1+r_2).$$

We divide the intervals $[\sqrt{N}, N]$ and $[0, 1]$ as we did in (6.6), that is,

$$M_0 = \sqrt{N} < M_1 < M_2 < \cdots < M_{l-1} < M_l = N,$$
$$\Theta_0 = 0 < \Theta_1 < \Theta_2 < \cdots < \Theta_{m-1} < \Theta_m = 1,$$

where

$$M_{j+1} - M_j \ll \frac{N}{(\log N)^a} \qquad (j = 0, 1, \cdots, l-1),$$
$$\Theta_{j+1} - \Theta_j \ll \frac{1}{(\log N)^a} \qquad (j = 0, 1, \cdots, m-1),$$
$$l \ll (\log N)^a, \qquad m \ll (\log N)^a.$$

In the similar way as we defined the set $\Omega(M, \Theta)$ in § 6, we now define the set $\Lambda(M, \Theta)$ of integers ν of K such that

$$M_q' < \nu^{(q)} \leqq M_q \qquad (q=1,2,\cdots,r_1),$$
$$\left.\begin{aligned} M_p' &< |\nu^{(p)}| \leqq M_p \\ 2\pi\Theta_p' &< \arg \nu^{(p)} \leqq 2\pi\Theta_p \end{aligned}\right\} \qquad (p=r_1+1,\cdots,r_1+r_2).$$

We shall define a sum as follows:

$$I(M,\Theta)=\sum_{\mu\in A(M,\Theta)}\frac{1}{\log N(\mu)}, \tag{11.2}$$

where μ runs through all elements of $A(M,\Theta)$.

Let μ be an element of $A(M,\Theta)$ and $\rho_1,\rho_2,\cdots,\rho_n$ be a basis of $\mathfrak{o}$. Then μ is written in the following form

$$\mu=\sum_{i=1}^{n} m_i\rho_i .$$

If we put

$$\xi_j=\sum_{i=1}^{n}u_i\rho_i^{(j)} \qquad (j=1,2,\cdots,n)$$

with $m_i \leqq u_i \leqq m_i+1$ $(i=1,2,\cdots,n)$, then

$$c\sqrt{N}\leqq|\xi_j| \qquad (j=1,2,\cdots,n),$$
$$|\mu^{(j)}-\xi_j|\leqq c \qquad (j=1,2,\cdots,n),$$
$$\frac{1}{\log N(\mu)}-\frac{1}{\log N(\xi)}=\frac{\log N(\mu/\xi)}{\log N(\mu)\log N(\xi)}=\frac{\log N\left(1+\frac{\mu-\xi}{\xi}\right)}{\log N(\mu)\log N(\xi)}$$
$$\ll\frac{1}{\sqrt{N}(\log N)^2},$$

and

$$\frac{1}{\log N(\mu)}=\int_{m_i}^{m_i+1}\!\!\cdots\!\int\frac{1}{\log N(\xi)}\,du_1\cdots du_n+O\left(\frac{1}{\sqrt{N}(\log N)^2}\right). \tag{11.3}$$

Summing up both sides of (11.3) over all $\mu\in A(M,\Theta)$, we have

$$I(M,\Theta)=\frac{2^{r_2}}{\sqrt{D}}\int\!\cdots\!\int_B\frac{1}{\log N(\xi)}\,dx(\xi)+O(N^{n-1/2}), \tag{11.4}$$

where the domain of integration is defined as follows:

$$B:\quad \begin{aligned} &M_q'\leqq X_q(\xi)\leqq M_q && (q=1,2,\cdots,r_1),\\ &\left.\begin{aligned} &M_p'^2\leqq X_p^2(\xi)+X_{p'}^2(\xi)\leqq M_p^2\\ &2\pi\Theta_p'\leqq\arg(X_p(\xi)+iX_{p'}(\xi))\leqq 2\pi\Theta_p\end{aligned}\right\} && (p=r_1+1,\cdots,r_1+r_2).\end{aligned}$$

If we put

$$t_q=\xi_q \qquad (q=1,2,\cdots,r_1),$$
$$\left.\begin{aligned}\sqrt{t_p}\,e^{2\pi i\theta_p}&=\xi_p\\ \sqrt{t_p}\,e^{-2\pi i\theta_p}&=\xi_{p'}\end{aligned}\right\} \qquad (p=r_1+1,\cdots,r_1+r_2),$$

then we have from (11.4)

$$(11.5)\qquad I(M,\Theta)=\frac{1}{\sqrt{D}}\int_{\Theta_p'}^{\Theta_p}\overset{\cdots}{\cdots}\int\int_{M_j'^{e_j}}^{M_j^{e_j}}\overset{\cdots}{\cdots}\int\frac{dt_1\cdots dt_{r+1}}{\log(t_1\cdots t_{r+1})}\,d\theta_{r_1+1}\cdots d\theta_{r+1}+O(N^{n-1/2}),$$

where the domain of integration is defined as follows:

$$M_j'^{e_j}\leqq t_j\leqq M_j^{e_j}\qquad (j=1,2,\cdots,r+1),$$

$$\Theta_p'\leqq\theta_p\leqq\Theta_p\qquad (p=r_1+1,\cdots,r_1+r_2)$$

with $e_j=1$ $(j\leqq r_1)$, $=2$ $(j\geqq r_1+1)$.

Comparing this result (11.5) with (6.9), a formula for $S_\rho(y;M,\Theta)$, and using the same notations as in § 6, we have

$$S_\rho(y;M,\Theta)=\frac{w\sqrt{D}}{2^{r_1}hR\varphi(\mathfrak{a})}\,e^{2\pi i S(\tilde{M}y)}\sum_{\mu\in A(M,\Theta)}\frac{1}{\log N(\mu)}$$

$$+O(N^n e^{-c\sqrt{\log N}})+O\Big(\frac{\prod_p(\Theta_p-\Theta_p')J(M)}{\varphi(\mathfrak{a})(\log N)^{a-b}}\Big)+O\Big(\frac{N^{n-1/2}}{\varphi(\mathfrak{a})}\Big).$$

Moreover, we have for $\mu\in A(M,\Theta)$

$$S(\mu y)=S(\tilde{M}y)+O((\log N)^{b-a}).$$

Therefore

$$e^{2\pi i S(\tilde{M}y)}\sum_{\mu\in A(M,\Theta)}\frac{1}{\log N(\mu)}=\sum_{\mu\in A(M,\Theta)}\frac{e^{2\pi i S(\mu y)}}{\log N(\mu)}+O\Big(\frac{N^n}{(\log N)^{a(n+1)-b+1}}\Big),$$

since Lemma 3.2 shows that

$$\sum_{\mu\in A(M,\Theta)}1\ll\frac{N^n}{(\log N)^{an}},$$

so we have

$$S_\rho(y;M,\Theta)=\frac{w\sqrt{D}}{2^{r_1}hR\varphi(\mathfrak{a})}\sum_{\mu\in A(M,\Theta)}\frac{e^{2\pi i S(\mu y)}}{\log N(\mu)}+O(N^n e^{-c\sqrt{\log N}})$$

$$+O\Big(\frac{\prod_p(\Theta_p-\Theta_p')J(M)}{\varphi(\mathfrak{a})(\log N)^{a-b}}\Big)+O\Big(\frac{N^n}{\varphi(\mathfrak{a})(\log N)^{a(n+1)-b+1}}\Big),$$

$$S_\rho(y)=\frac{w\sqrt{D}}{2^{r_1}hR\varphi(\mathfrak{a})}\sum_{\mu\in A_1(N)}\frac{e^{2\pi i S(\mu y)}}{\log N(\mu)}+O\Big(\frac{N^n}{\varphi(\mathfrak{a})(\log N)^{a-b+1}}\Big)$$

and finally

$$S(z;N)=\frac{w\sqrt{D}}{2^{r_1}hR}\cdot\frac{\mu(\mathfrak{a})}{\varphi(\mathfrak{a})}\sum_{\mu\in A_1(N)}\frac{e^{2\pi i S(\mu y)}}{\log N(\mu)}+O\Big(\frac{N^n}{(\log N)^{a-b+1}}\Big).$$

Since

$$\sum_{\mu\in A_0(N)-A_1(N)}1\ll N^{n(1/2)},$$

we complete the proof.

Now we define a function $g_1(z)=g_1(z_1, z_2, \cdots, z_n)$ as follows:

$$g_1(z)=\sum_{\mu\in A_0(N)} \frac{e^{2\pi i S(\mu z)}}{\log N(\mu)}. \tag{11.6}$$

We denote by $A(t)$, for positive real number t, the set of integers ν such that

$$0<\nu^{(q)}\leqq t \qquad (q=1, 2, \cdots, r_1),$$

$$|\nu^{(p)}|\leqq t \qquad (p=r_1+1, \cdots, r_1+r_2)$$

and consider the square of $g_1(z)$:

$$g_1{}^2(z)=\sum_{\mu\in A(2N)} B(\mu)\, e^{2\pi i (S\mu z)}$$

with

$$B(\mu)=\sum_{\substack{\mu=\nu_1+\nu_2,\\ \nu_i\in A_0(N)}} \frac{1}{\log N(\nu_1)\log N(\nu_2)}.$$

If we put for any ideal $\mathfrak{a}$

$$g_2(z;\mathfrak{a})=\sum_{\substack{\gamma\to\mathfrak{a}\\ \gamma \bmod \mathfrak{d}^{-1}}} g_1{}^2(z-\gamma), \tag{11.7}$$

where γ runs through a complete system of residues mod $\mathfrak{d}^{-1}$ with $\gamma\to\mathfrak{a}$, then we have

$$g_2(z;\mathfrak{a})=\sum_{\mu\in A(2N)} B(\mu)G(\mathfrak{a}, \mu)e^{2\pi i S(\mu z)} \tag{11.8}$$

with

$$G(\mathfrak{a}, \mu)=\sum_{\substack{\gamma\to\mathfrak{a}\\ \gamma \bmod \mathfrak{d}^{-1}}} e^{-2\pi i S(\mu\gamma)}.$$

As for this sum $G(\mathfrak{a}, \mu)$, we have, by Rademacher [4],

$$G(\mathfrak{a}, \mu)=\sum_{\mathfrak{c}\mid(\mathfrak{a}, \mu)} N(\mathfrak{c})\mu(\mathfrak{a}/\mathfrak{c}).$$

Hence

$$|G(\mathfrak{a}, \mu)|\leqq\sum_{\mathfrak{c}\mid(\mathfrak{a}, \mu)} N(\mathfrak{c})=N((\mathfrak{a}, \mu))\sum_{\mathfrak{c}\mid(\mathfrak{a}, \mu)}\frac{1}{N(\mathfrak{c})}$$

$$\leqq N((\mathfrak{a}, \mu))\sum_{N\mathfrak{c}\leqq N\mathfrak{a}}\frac{1}{N(\mathfrak{c})}\ll N((\mathfrak{a}, \mu))(1+\log N(\mathfrak{a})).$$

Therefore we have by (11.8)

$$g_2(z;\mathfrak{a})\ll N^n(1+\log N(\mathfrak{a}))\sum_{\mu\in A(2N)} N((\mathfrak{a}, \mu)),$$

since $B(\mu)\ll N^n$. In this right-hand side, the sum over μ is estimated as follows:

$$\sum_{\mu\in A(2N)} N((\mathfrak{a},\mu)) = \sum_{\mathfrak{c}\mid\mathfrak{a}} \sum_{\substack{\mu\in A(2N)\\ (\mathfrak{a},\mu)=\mathfrak{c}}} N(\mathfrak{c}) \leqq \sum_{\mathfrak{c}\mid\mathfrak{a}} \sum_{\substack{\mu\in A(2N)\\ \mu\in\mathfrak{c}}} N(\mathfrak{c})$$

$$\ll N^n \sum_{\mathfrak{c}\mid\mathfrak{a}} 1 = N^n \tau(\mathfrak{a}).$$

Now we shall prove

$$\tau(\mathfrak{a}) \ll N(\mathfrak{a})^{\delta} \tag{11.9}$$

for any given positive constant δ.

Consider the set of pairs $(m, \mathfrak{p})$ of rational integers m and prime ideals $\mathfrak{p}$ such that

$$1+m > N(\mathfrak{p})^{m\delta}.$$

Then it is obvious that this set is finite. Therefore, decomposing $\mathfrak{a}$ into the product of prime divisors as follows; $\mathfrak{a} = \mathfrak{p}_1^{\alpha_1}\mathfrak{p}_2^{\alpha_2}\cdots\mathfrak{p}_t^{\alpha_t}$, we have

$$\tau(\mathfrak{a}) = (1+\alpha_1)(1+\alpha_2)\cdots(1+\alpha_t) \ll N(\mathfrak{p}_1)^{\alpha_1\delta} N(\mathfrak{p}_2)^{\alpha_2\delta}\cdots N(\mathfrak{p}_t)^{\alpha_t\delta}$$

and the assertion (11.9) is proved.

Thus we have

$$g_2(z;\mathfrak{a}) \ll N^{2n}(1+\log N(\mathfrak{a}))\tau(\mathfrak{a}) \ll N^{2n}N(\mathfrak{a})^{\varepsilon}, \tag{11.10}$$

with a sufficiently small positive constant ε.

If we define a function of $z=(z_1, z_2, \cdots, z_n)$,

$$g_3(z) = \sum_{\mathfrak{a}} \frac{\mu(\mathfrak{a})^2}{\varphi(\mathfrak{a})^2} g_2(z;\mathfrak{a}), \tag{11.11}$$

where $\mathfrak{a}$ runs through all ideals, then $g_3(z)$ converges on account of the estimation (11.10) for $g_2(z;\mathfrak{a})$.

Moreover we put

$$\begin{aligned} F(z) &= \sum_{N\mathfrak{a}\leqq T^n} \frac{\mu(\mathfrak{a})^2}{\varphi(\mathfrak{a})^2} g_2(z;\mathfrak{a}) \\ &= \sum_{N\mathfrak{a}\leqq T^n} \frac{\mu(\mathfrak{a})^2}{\varphi(\mathfrak{a})^2} \sum_{\substack{\gamma-\mathfrak{a}\\ \gamma \bmod \mathfrak{d}^{-1}}} g_1^2(z-\gamma). \end{aligned} \tag{11.12}$$

LEMMA 11.2. *If $z=(z_1, z_2, \cdots, z_n)$ is a point of E, then we have*

$$g_1(z) \ll N^{n-1} \min_{1\leqq j\leqq n} (N, |z_j|^{-1}). \tag{11.13}$$

PROOF. First we have

$$g_1(z) \ll N^n. \tag{11.14}$$

Let $A_2(N)$ be the set of integers ν such that

$$1 < \nu^{(q)} \leqq N \qquad (q=1, 2, \cdots, r_1),$$

$$1 < |\nu^{(p)}| \leqq N \qquad (p=r_1+1, \cdots, r_1+r_2)$$

and put

$$h(z)=\sum_{\mu\in A_2(N)}\frac{e^{2\pi iS(\mu z)}}{\log N(\mu)},$$

then we have

$$g_1(z)=h(z)+O(N^{n-1}).$$

Let ρ be one of $\rho_1, \rho_2, \cdots, \rho_n$, a basis of $\mathfrak{o}$, then

$$g_1(z)e^{2\pi iS(\rho z)}=h(z)e^{2\pi iS(\rho z)}+O(N^{n-1})$$

and

$$g_1(z)(e^{2\pi iS(\rho z)}-1)=\sum_{\substack{\mu\in A_2(N)\\ \mu-\rho\in A_2(N)}} e^{2\pi iS(\mu z)}\Big(\frac{1}{\log N(\mu-\rho)}-\frac{1}{\log N(\mu)}\Big)$$

$$+O\Big(\sum_{\substack{\mu\in A_2(N)\\ \mu-\rho\notin A_2(N)}}1+\sum_{\substack{\mu\notin A_2(N)\\ \mu-\rho\in A_2(N)}}1\Big)+O(N^{n-1})$$

$$=\sum_{\substack{\mu\in A_2(N)\\ \mu-\rho\in A_2(N)}} e^{2\pi iS(\mu z)}\Big(\frac{1}{\log N(\mu-\rho)}-\frac{1}{\log N(\mu)}\Big)+O(N^{n-1}).$$

In the last sum,

$$\frac{1}{\log N(\mu-\rho)}-\frac{1}{\log N(\mu)}\ll\frac{|\log N(1-\rho/\mu)|}{(\log N(\mu))^2}\ll\frac{1}{(\log N(\mu))^2}\sum_{j=1}^{n}\frac{1}{|\mu^{(j)}|}.$$

Therefore we have

$$g_1(z)(e^{2\pi iS(\rho z)}-1)\ll\sum_{\mu\in A_2(N)}\sum_{j=1}^{n}\frac{1}{|\mu^{(j)}|(\log N(\mu))^2}+O(N^{n-1}). \tag{11.15}$$

Moreover we have

$$\sum_{\mu\in A_2(N)}\frac{1}{|\mu^{(j)}|(\log N(\mu))^2}\leqq\sum_{m=1}^{N-1}\frac{1}{m}\sum_{\substack{\mu\in A_2(N)\\ m<|\mu^{(j)}|\leqq m+1}}\frac{1}{(\log N(\mu))^2}$$

$$\leqq\sum_{m=1}^{N-1}\frac{1}{m(\log(m+1))^2}\sum_{\substack{\mu\in A_2(N)\\ m<|\mu^{(j)}|\leqq m+1}}1\ll N^{n-1}\sum_{m=1}^{N-1}\frac{1}{m(\log(m+1))^2}\ll N^{n-1}.$$

Hence

$$g_1(z)(e^{2\pi iS(\rho z)}-1)\ll N^{n-1}. \tag{11.16}$$

Therefore, by (11.14) and (11.16),

$$g_1(z)\ll N^{n-1}\min_{1\leqq j\leqq n}(N,\ \|S(\rho_j z)\|^{-1}). \tag{11.17}$$

Since $(z_1, z_2, \cdots, z_n)\in E$, writing

$$z_j=\sum_{i=1}^{n}x_i\delta_i^{(j)} \qquad (j=1,2,\cdots,n)$$

with a basis $\delta_1, \delta_2, \cdots, \delta_n$ of $\mathfrak{d}^{-1}$ such that

$$S(\delta_i\rho_j)=\begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases} \qquad (i,j=1,2,\cdots,n),$$

we see that

(11.18) $$\|S(\rho_j z)\|=|S(\rho_j z)|=|x_j| \qquad (j=1,2,\cdots,n).$$

It is obvious that

(11.19) $$|z_j|\ll\max(|x_1|,|x_2|,\cdots,|x_n|) \qquad (j=1,2,\cdots,n).$$

Hence we have, from (11.17), (11.18) and (11.19),

$$g_1(z)\ll N^{n-1}\min_{1\leqq j\leqq n}(N,\ |z_j|^{-1}).$$

LEMMA 11.3. *If $z=(z_1 z_2,\cdots,z_n)$ is a point of B_{γ_1} and $\gamma\neq\gamma_1, \gamma\in\Gamma$, then we have*

(11.20) $$g_1(z-\gamma)\ll N^{n-1}(\log N)^{2\sigma_1}.$$

PROOF. Let $z^0=(z_1^0,\cdots,z_n^0)$ be a point of E such that $z^0\equiv z-\gamma \pmod{\mathfrak{d}^{-1}}$, then by Lemma 11.2 we have

(11.21) $$g_1(z-\gamma)=g_1(z^0)\ll N^{n-1}\min_{1\leqq j\leqq n}(N,\ |z_j^0|^{-1}).$$

On the other hand, there exists a certain number γ_2 such that $\gamma_1\equiv\gamma_2 \pmod{\mathfrak{d}^{-1}}$ and

(11.22) $$|z_j-\gamma_2^{(j)}|\leqq\frac{(\log N)^b}{N} \qquad (j=1,2,\cdots,n).$$

We put $z^0=z-\gamma+\beta$. Since $\gamma\not\equiv\gamma_1 \pmod{\mathfrak{d}^{-1}}$, we see that $\gamma_2-\gamma+\beta$ is a non-vanishing element of $(\mathfrak{d}\mathfrak{a}_1\mathfrak{a})^{-1}$, where $\gamma\to\mathfrak{a}$ and $\gamma_1\to\mathfrak{a}_1$. Therefore

$$|N(\gamma_2-\gamma+\beta)|\geqq\frac{1}{N(\mathfrak{d}\mathfrak{a}\mathfrak{a}_1)}\geqq\frac{1}{DT^{2n}}$$

and there exists an index l $(1\leqq l\leqq n)$ such that

(11.23) $$|\gamma_2^{(l)}-\gamma^{(l)}+\beta^{(l)}|\geqq\frac{D^{-1/n}}{T^2}.$$

From (11.22) and (11.23) follows that

$$|z_l^0|\geqq\big||\gamma_2^{(l)}-\gamma^{(l)}+\beta^{(l)}|-|z_l-\gamma_2^{(l)}|\big|\geqq\frac{c}{T^2}=\frac{c}{(\log N)^{2\sigma_1}}.$$

Putting this result in (11.21), we complete the proof.

LEMMA 11.4. *We put*

$$D_0=\left(\frac{w\sqrt{D}}{2^{r_1}hR}\right)^2$$

and assume that $a\geqq 2\sigma+b+1$ in Lemma 11.1.

Then we have

(11.24) $$S(z;N)^2-D_0F(z)\ll\frac{N^{2n}}{(\log N)^{2\sigma}}.$$

for every $z=(z_1, z_2, \cdots, z_n)\in E$.

PROOF. First we assume that $z\in B_{\gamma_1}$ with $\gamma_1\to\mathfrak{a}$. Then Lemma 11.1 shows that

$$S(z\,;N)=\frac{w\sqrt{D}}{2^{r_1}hR}\cdot\frac{\mu(\mathfrak{a})}{\varphi(\mathfrak{a})}g_1(z-\gamma_1)+O\Big(\frac{N^n}{(\log N)^{2\sigma}}\Big) \tag{11.25}$$

so that

$$S(z\,;N)^2=D_0\frac{\mu(\mathfrak{a})^2}{\varphi(\mathfrak{a})^2}g_1^2(z-\gamma_1)+O\Big(\frac{N^{2n}}{(\log N)^{2\sigma}}\Big). \tag{11.26}$$

On the other hand, using Lemma 11.3 and Lemma 7.1, we have

$$F(z)-\frac{\mu(\mathfrak{a})^2}{\varphi(\mathfrak{a})^2}g_1^2(z-\gamma_1)=\sum_{N\mathfrak{a}\leqq T^n}\frac{\mu(\mathfrak{a})^2}{\varphi(\mathfrak{a})^2}\sum_{\substack{\gamma\to\mathfrak{a},\ \gamma\neq\gamma_1\\ \gamma \bmod \mathfrak{d}^{-1}}}g_1^2(z-\gamma)$$

$$\ll\sum_{N\mathfrak{a}\leqq T^n}\frac{1}{\varphi(\mathfrak{a})}N^{2n-2}(\log N)^{4\sigma_2}\ll N^{2n-2}(\log N)^{4\sigma_2+2}.$$

Therefore we have

$$S(z\,;N)^2-D_0F(z)\ll\frac{N^{2n}}{(\log N)^{2\sigma}}. \tag{11.27}$$

Now assume that $z\in B^0$, then Theorem 5.1 shows that

$$S(z\,;N)\ll\frac{N^n}{(\log N)^{\sigma}}. \tag{11.28}$$

We shall take $\gamma\in\Gamma$ and define a point $z^0=(z_1{}^0, z_2{}^0, \cdots, z_n{}^0)$ of E such that $z^0=z-\gamma+\beta$ with a certain $\beta\in\mathfrak{d}^{-1}$. Since $z=(z_1, z_2, \cdots, z_n)\in B^0$, there exists an index j $(1\leqq j\leqq n)$ such that

$$|z_j-\gamma^{(j)}+\beta^{(j)}|\geqq\frac{(\log N)^b}{N}.$$

Therefore, by Lemma 11.2, we have

$$g_1(z-\gamma)=g_1(z^0)\ll N^n(\log N)^{-b}$$

and consequently

$$F(z)\ll\sum_{N\mathfrak{a}\leqq T^n}\frac{1}{\varphi(\mathfrak{a})}\cdot\frac{N^{2n}}{(\log N)^{2b}}\ll\frac{N^n}{(\log N)^{2b-2}}. \tag{11.29}$$

By (11.28), (11.29) and (4.6), which shows that $2b-2\geqq 2\sigma$, we have

$$S(z\,;N)^2-D_0F(z)\ll\frac{N^{2n}}{(\log N)^{2\sigma}}$$

for $z\in B^0$. Thus the proof is completed.

LEMMR 11.5. *For* $z=(z_1, \cdots, z_n)\in E$, *we have*

$$S(z\,;N)^2-D_0g_3(z)\ll\frac{N^{2n}}{(\log N)^{2\sigma}}. \tag{11.30}$$

PROOF. Using the estimation (11.10) for $g_2(z\,;\mathfrak{a})$ and Lemma 7.1, we have

$$g_3(z)-F(z)=\sum_{N\mathfrak{a}>T^n}\frac{\mu(\mathfrak{a})^2}{\varphi(\mathfrak{a})^2}g_2(z\,;\mathfrak{a})\ll N^{2n}\sum_{N\mathfrak{a}>T^n}\frac{1}{N(\mathfrak{a})^{2-\varepsilon}}$$

$$\ll\frac{N^{2n}}{(\log N)^{(1-\varepsilon)n\sigma_2}}\ll\frac{N^{2n}}{(\log N)^{n\sigma_2/2}}.$$

Since $n\sigma_2\geqq 4\sigma$, our Lemma follows directly from Lemma 11.4.

LEMMA 11.6. *If we put*

$$J(N)=\int_{-1/2}^{1/2}\!\!\cdots\!\!\int|S(z\,;N)^2-D_0\,g_3(z)|^2\,dx_1dx_2\cdots dx_n\,, \tag{11.31}$$

then we have

$$J(N)\ll\frac{N^{3n}}{(\log N)^{2\sigma}}. \tag{11.32}$$

PROOF. We have, by Lemma 11.5,

$$J(N)\ll\frac{N^{2n}}{(\log N)^{2\sigma}}\int_{-1/2}^{1/2}\!\!\cdots\!\!\int|S(z\,;N)^2-D_0\,g_3(z)|\,dx_1dx_2\cdots dx_n\,.$$

Therefore, it suffices to prove

$$\int_{-1/2}^{1/2}\!\!\cdots\!\!\int|S(z\,;N)^2-D_0\,g_3(z)|\,dx_1\cdots dx_n\ll N^n\,. \tag{11.33}$$

First we obtain

$$\int_{-1/2}^{1/2}\!\!\cdots\!\!\int|S(z\,;N)|^2\,dx_1\cdots dx_n=\sum_{\omega\in\Omega(N)}1\ll\frac{N^n}{\log N}. \tag{11.34}$$

Now we have

$$\int_{-1/2}^{1/2}\!\!\cdots\!\!\int|g_1(z-\gamma)|^2\,dx_1\cdots dx_n=\sum_{\mu\in\Lambda_0(N)}\frac{1}{(\log N(\mu))^2}\ll\frac{N^n}{(\log N)^2}\,,$$

$$\int_{-1/2}^{1/2}\!\!\cdots\!\!\int|g_2(z\,;\mathfrak{a})|\,dx_1\cdots dx_n\ll\varphi(\mathfrak{a})\frac{N^n}{(\log N)^2}$$

and

$$\int_{-1/2}^{1/2}\!\!\cdots\!\!\int\left|\sum_{N\mathfrak{a}\leqq N^{2n}}\frac{\mu(\mathfrak{a})^2}{\varphi(\mathfrak{a})^2}g_2(z\,;\mathfrak{a})\right|dx_1\cdots dx_n$$

$$\ll\frac{N^n}{(\log N)^2}\sum_{N\mathfrak{a}\leqq N^{2n}}\frac{1}{\varphi(\mathfrak{a})}\ll\frac{N^n}{(\log N)^2}\sum_{N\mathfrak{a}\leqq N^{2n}}\frac{\log N(\mathfrak{a})}{N(\mathfrak{a})}\ll N^n.$$

Finally we have, by (11.10),

$$\int_{-1/2}^{1/2}\!\!\cdots\!\!\int\left|\sum_{N\mathfrak{a}>M^{2n}}\frac{\mu(\mathfrak{a})^2}{\varphi(\mathfrak{a})^2}g_2(z\,;\mathfrak{a})\right|dx_1\cdots dx_n$$

$$\ll N^{2n}\sum_{N\mathfrak{a}>N^{2n}}\frac{1}{N(\mathfrak{a})^{2-\varepsilon}}\ll N^{2\varepsilon n}\ll N^n.$$

Hence

$$\int_{-1/2}^{1/2}\cdots\int |g_3(z)|\,dx_1\cdots dx_n \ll N^n. \tag{11.35}$$

This results (11.35) and (11.34) give (11.33) and then we complete the proof.

Now we consider the square of $S(z;N)$:

$$S(z;N)^2=\sum_{\mu\in A(2N)} A(\mu)e^{2\pi i S(\mu z)},$$

where

$$A(\mu)=\sum_{\substack{\mu=\omega_1+\omega_2\\ \omega_i\in\Omega(N)}} 1$$

is the number of the representations of μ as the sums of two prime numbers belonging to $\Omega(N)$.

We write

$$\begin{aligned} g_3(z)&=\sum_{\mathfrak{a}}\frac{\mu(\mathfrak{a})^2}{\varphi(\mathfrak{a})^2}\sum_{\substack{\gamma\to\mathfrak{a}\\ \gamma \bmod \mathfrak{b}^{-1}}}\sum_{\mu\in A(2N)} B(\mu)e^{-2\pi i S(\gamma\mu)}e^{2\pi i S(\mu z)}\\ &=\sum_{\mu\in A(2N)} B(\mu)\mathfrak{S}(\mu)e^{2\pi i S(\mu z)},\end{aligned}$$

where

$$\mathfrak{S}(\mu)=\sum_{\mathfrak{a}}\frac{\mu(\mathfrak{a})^2}{\varphi(\mathfrak{a})^2}G(\mathfrak{a},\mu).$$

$\mathfrak{S}(\mu)$ is a convergent series and we can write $\mathfrak{S}(\mu)$ in the form of an infinite product:

$$\mathfrak{S}(\mu)=\prod_{\mathfrak{p}|\mu}\Big(1+\frac{1}{N(\mathfrak{p})-1}\Big)\prod_{\mathfrak{p}\nmid\mu}\Big(1-\frac{1}{(N(\mathfrak{p})-1)^2}\Big).$$

In the first product, $\mathfrak{p}$ runs through all prime divisors of μ and in the second product, $\mathfrak{p}$ runs through other prime ideals. We shall define even or odd integer of K as in § 10. Then we see that

$$\mathfrak{S}(\mu)\geqq c>0 \qquad (\text{if } \mu \text{ is even}),$$

$$\mathfrak{S}(\mu)=0 \qquad (\text{if } \mu \text{ is odd}).$$

Now we see that

$$J(N)=\sum_{\mu\in A(2N)}\{A(\mu)-D_0B(\mu)\mathfrak{S}(\mu)\}^2. \tag{11.36}$$

From the definition of $B(\mu)$ follows

$$B(\mu)\geqq\frac{1}{n^2(\log N)^2}\sum_{\substack{\mu=\nu_1+\nu_2\\ \nu_i\in A_0(N)}} 1$$

and, since the number of the units ε such that $|\varepsilon^{(j)}|\leqq N$ $(j=1,2,\cdots,n)$ is $O((\log N)^r)$, we have

$$B(\mu) \geqq \frac{1}{(n \log N)^2} \Big(\sum_{\substack{\mu=\nu_1+\nu_2 \\ \nu_i \in A(N)}} 1 - c(\log N)^r \Big).$$

Now we put $\xi = (2\sigma-4)/(2n+1)$ and take an integer μ such that

$$\frac{N}{(\log N)^\xi} \leqq \mu^{(q)} \leqq N \qquad (q=1,2,\cdots,r_1),$$

(11.37)

$$\frac{N}{(\log N)^\xi} \leqq |\mu^{(p)}| \leqq N \qquad (p=r_1+1,\cdots,r_1+r_2).$$

Then we see that for such μ

(11.38) $$B(\mu) \geqq c\frac{N(\mu)}{(\log N)^2} \geqq \frac{cN^n}{(\log N)^{2+n\xi}}.$$

Let Q_1 be the number of even integers μ which satisfy the condition (11.37) and for which $A(\mu)=0$. Then we have from (11.36) and (11.38)

$$J(N) \geqq \frac{cQ_1 N^{2n}}{(\log N)^{4+2n\xi}}.$$

On the other hand, Lemma 11.6 shows that

$$J(N) \ll \frac{N^{3n}}{(\log N)^{2\sigma}}.$$

Therefore, we have

(11.39) $$Q_1 \ll \frac{N^n}{(\log N)^\xi}.$$

Now let $Q(N)$ be the number of even integers μ such that

$$\mu \in A(N), \qquad A(\mu)=0,$$

then we have from (11.36) and (11.39)

(11.40) $$Q(N) \ll \frac{N^n}{(\log N)^\xi} + Q_1 \ll \frac{N^n}{(\log N)^\xi}.$$

Thus we can prove

THEOREM 11.1. *Almost all totally positive even integers of K are represented as the sums of two totally positive odd prime numbers of K.*

PROOF. Let $P(N)$ be the number of even integers μ such that $\mu \in A(N)$. Then we have

$$P(N) \geqq cN^n.$$

Hence (11.40) shows that almost all even integers in $A(N)$ are represented as the sums of two totally positive prime numbers.

Now assume that $\mu=\omega_1+\omega_2$, where μ is an even integer in $A(N)$ and ω_1 and ω_2 are prime numbers in $\Omega(N)$ at least one of which is not odd. Suppose ω_1 is not odd, then $(\omega_1)=\mathfrak{p}$ is a prime ideal with $N(\mathfrak{p})=2$. Since μ is even, $\mu \in \mathfrak{p}$, which implies $(\omega_2)=\mathfrak{p}$. Therefore we see that the number of even integers $\mu \in \Lambda(N)$ which are represented as the sums of two prime numbers in $\Omega(N)$, but not of two odd prime numbers, does not exceed the numbers of the pairs (ω_1, ω_2) of prime numbers such that $\omega_1, \omega_2 \in \Omega(N)$ and $N(\omega_1) = N(\omega_2) = 2$. Applying Lemma 3.4, we see that the latter is $O((\log N)^{2r})$.

Hence we obtain the proof.

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