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On the stable cohomology groups of certain Postnikov complexes

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Introduction.

It is an important but difficult problem of topology to compute the cohomology groups of Postnikov complexes $K(\pi, n; k; G, n+q)$. These cohomology groups become stable for large n; more precisely, $H^{n+i}(K(\pi, n; k; G, n+q); \Lambda)$ become independent of n for sufficiently large n. This "limit group" will be denoted by

$$A^{i}(\pi, k, G, q; \Lambda) = \lim H^{n+i}(K(\pi, n; k; G, n+q); \Lambda).$$

The purpose of this paper is to determine $A^{i}(\pi, k, G, 1; Z_2)$ (which we shall hereafter denote simply by $A^{i}(\pi, k, G; Z_2)$) for the case where each of π, G is generated by one element. Our result will be given as Theorem in § 3, after some preparations in §§ 1-2.

Our computation is based on some properties of secondary cohomology operations as given in § 2.

We shall indicate another geometrical method in the appendix.

In the case where $\pi = Z, G = Z_2, \Lambda = Z_2$ and $q = 1$, the preoblem was solved by H. Toda [**9**] by geometrical methods.

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§ 1. Preliminaries.

1. Let π, G be abelian groups and n, q positive integers. A Postnikov space $\mathcal{K}(\pi, n; k; G, n+q)$ with an invariant $k \in H^{n+q+1}(\pi, n; G)$ can be considered as a fibre space with the base space $\mathcal{K}(\pi, n)$ (Eilenberg-MacLane space) and the fibre $\mathcal{K}(G, n+q)$:

$$\mathcal{K}(\pi, n; k; G, n+q)/\mathcal{K}(G, n+q) = \mathcal{K}(\pi, n). \tag{1.1}$$

The projection and the inclusion of the fibering will be denoted by p, i respectively. Then we have the following exact sequence associated with (1.1) for

$i \leqq 2n+q-1$:

$$\text{(1.2)}\quad \cdots \longleftarrow H^{i+1}(\pi, n; Z_2) \xleftarrow{\tau} H^i(G, n+q; Z_2) \xleftarrow{i^*} H^i(\mathcal{K}(\pi, n; k; G, n+q); Z_2) \xleftarrow{p^*} H^i(\pi, n; Z_2) \longleftarrow \cdots,$$

where τ is the transgression.

It is known that the groups $H^{n+i}(\mathcal{K}(\pi, n; k; G, n+q); Z_2)$ become stable for sufficiently large n. We denote this group by $A^i(\pi, k, G, q; Z_2)$ and write

$$\text{(1.3)}\quad A^*(\pi, k, G, q; Z_2) = \sum_{i=0}^{\infty} A^i(\pi, k, G, q; Z_2).$$

If we denote as usual by $A^i(\pi; Z_2)$ the stable group $H^{n+i}(\mathcal{K}(\pi, n); Z_2)$ for large n, then we have (1.2)

$$\text{(1.4)}\quad \cdots \longleftarrow A^{i+1}(\pi; Z_2) \xleftarrow{\tau} A^{i-q}(G; Z_2) \xleftarrow{i^*} A^i(\pi, k, G, q; Z_2) \xleftarrow{p^*} A^i(\pi; Z_2) \longleftarrow \cdots.$$

We denote further by A^* the Steenrod algebra

$$A^*(Z_2; Z_2) = \lim H^*(Z_2, n; Z_2),$$

in which the multiplication is defined by the composition of the squaring operations Sq^r. The squaring operations in $A^*(\pi; Z_2)$, $A^*(G; Z_2)$ and $A^*(\pi, k, G, q; Z_2)$ define naturally the left A^*-module structure in these modules, and τ, i^*, p^* in exact sequence (1.4) are A^*-homomorphisms.

2. We need the following results on A^*.

Let $\alpha \in A^*$. The mapping $\beta \to \beta\alpha$ for every $\beta \in A^*$ will be denoted by α_*. Then we have the following exact sequences (cf. H. Toda [**9**] and A. Negishi [**4**]).

$$\text{(1.5)}\quad A^* \xrightarrow{\mathrm{Sq}^1_*} A^* \xrightarrow{\mathrm{Sq}^1_*} A^*,$$

$$\text{(1.6)}\quad A^* \xrightarrow{\mathrm{Sq}^2_*} A^* \xrightarrow{\mathrm{Sq}^2_*} A^*/A^*\mathrm{Sq}^1,$$

$$\text{(1.7)}\quad A^*/A^*\mathrm{Sq}^1 \xrightarrow{\mathrm{Sq}^3_*} A^* \xrightarrow{\mathrm{Sq}^2_*} A^*,$$

$$\text{(1.8)}\quad A^*/A^*\mathrm{Sq}^1 \xrightarrow{\mathrm{Sq}^5_*} A^*/A^*\mathrm{Sq}^1 \xrightarrow{\mathrm{Sq}^3_*} A^*,$$

$$\text{(1.9)}\quad A^*/A^*\mathrm{Sq}^1 \xrightarrow{\mathrm{Sq}^3_*} A^*/A^*\mathrm{Sq}^1 \xrightarrow{\mathrm{Sq}^3_*} A^*/A^*\mathrm{Sq}^1,$$

$$\text{(1.10)}\quad A^* \xrightarrow{\mathrm{Sq}^2_*} A^*/A^*\mathrm{Sq}^1 \xrightarrow{\mathrm{Sq}^5_*} A^*/A^*\mathrm{Sq}^1.$$

3. We shall use the following results on derived Bockstein cohomology operations.

Let

$$(1.11) \qquad 0 \longrightarrow Z_{2^q} \xrightarrow{f_q} Z_{2^{q+1}} \xrightarrow{g_q} Z_2 \longrightarrow 0\,,$$

$$(1.12) \qquad 0 \longrightarrow Z_2 \xrightarrow{f_q'} Z_{2^{q+1}} \xrightarrow{g_q'} Z_{2^q} \longrightarrow 0\,,$$

be exact sequences. The coboundary operators associated with (1.11), (1.12) are denoted by δ_q, δ_q' respectively. Then derived Bockstein cohomology operations $\Delta_2^q (q \geqq 1)$ were defined by T. Yamanoshita [**10**], such that for any pair of spaces (X,Y).

$$(1.13) \qquad \Delta_2^q : H^n(X, Y; Z_2) \frown \operatorname{Ker} \Delta_2^{q-1} \longrightarrow H^{n+1}(X, Y; Z_2)/\operatorname{Im} \delta'_{q-1}\,.$$

The following properties of Δ_2^q are known (cf. T. Yamanoshita [**10**]).

$$(1.14) \qquad \Delta_2^1 = \mathrm{Sq}^1 : H^n(X, Y; Z_2) \longrightarrow H^{n+1}(X, Y; Z_2)\,.$$

(1.15) The naturality $f^* \circ \Delta_2^q = \Delta_2^q \circ f^*$ holds for homomorphisms f^* of cohomology groups induced by a mapping $f : (X, Y) \to (X', Y')$.

(1.16) $\Delta_2^q \circ \Delta = \Delta \circ \Delta_2^q$ for the coboundary homomorphism Δ of cohomology sequence.

(1.17) $\Delta_2^q \circ \tau = \tau \circ \Delta_2^q$ for the transgression τ.

(1.18) $\Delta_2^r \circ \Delta_2^q = 0$.

Let $E/F = B$ be a fibering of a space E such that the local system formed by $H^i(F; Z_2)$ is trivial for each $i \geqq 0$, $H^i(B; Z_2) = 0$ for $0 < i < \lambda$, and $H^i(F; Z_2) = 0$ for $0 < i < \mu$. Let

$$(1.19) \qquad \cdots \longleftarrow H^i(F; Z_2) \xleftarrow{i^*} H^i(E; Z_2) \xleftarrow{p^*} H^i(B; Z_2) \xleftarrow{\tau} H^{i-1}(F; Z_2) \longleftarrow \cdots$$

be an exact sequence associated with the above fibering, where p is the projection, i is the inclusion, and τ is the transgression $(1 \leqq i < \lambda + \mu)$.

Then we have (cf. T. Yamanoshita [**10**] and H. Toda [**9**]):

(1.20) For $\alpha \in H^i(F; Z_2)$, $\beta \in H^i(B; Z_2)$, assume that $\Delta_2^r \beta = \{\tau\alpha\}$. Then there is an element $\tilde{\alpha} \in H^{i+1}(E; Z_2)$ such that $i^*\tilde{\alpha} = \mathrm{Sq}^1\alpha$ and $\Delta_2^{r+1} p^*\beta = \{\tilde{\alpha}\}$ $r \geqq 1$.

(1.21) For $\alpha \in H^i(E; Z_2)$, $\beta \in H^{i+1}(B; Z_2)$, assume that $\Delta_2^r \alpha = \{p^*\beta\}$. Then $\tau \circ \Delta_2^{r+1} \circ i^*(\alpha) = \{\mathrm{Sq}^1\beta\}$.

(1.22) For $\alpha \in H^i(F; Z_2)$, $\beta \in H^{i+1}(B; Z_2)$, assume that $\tau\alpha = \beta$, and $\beta \in \operatorname{Ker} \Delta_2^{r-1}$. Then there are elements $\tilde{\alpha} \in H^{i+1}(E; Z_2)$, $\gamma \in H^{i-2}(B; Z_2)$ such that $i^*\tilde{\alpha} = \mathrm{Sq}^1\alpha$, $\Delta_2^r \beta = \{\gamma\}$ and $\Delta_2^{r-1}\tilde{\alpha} = \{p^*\gamma\}$, $r \geqq 2$.

§ 2. Certain secondary cohomology operations.

Let $\sum_{i=1}^{k} \alpha_i \beta_i = 0$ be a relation with homogeneous degree $m+1$ in A^*, and C be a graded left free A^*-module generated by symbols $[\beta_i]$, where $\deg [\beta_i] = \deg \beta_i = \nu_i$:

$$C = \sum_{i=1}^{k} A^*[\beta_i]\,.$$

Let (d, z) be a pair, where d is a A^*-map of degree zero from C to A^* defined by $d[\beta_i]=\beta_i$, and $z=\sum_{i=1}^{k}\alpha_i[\beta_i]$.

For such a pair, J. F. Adams has defined axiomatically the stable secondary cohomology operation Φ_z such that

(2.1) $$\Phi_z : H^n(X; Z_2) \cap \operatorname{Ker}\beta_1 \cap \cdots \cap \operatorname{Ker}\beta_k \longrightarrow H^{n+m}(X; Z_2)/\sum_{i=1}^{k}\operatorname{Im}\alpha_i,$$

for any space X.

We use the following results in [**1**].

a) If Φ, Φ' are two operations associated with the same pair (d, z), then there is an element γ in $(A^*/dC)_m$ such that

(2.2) $$\Phi(u)-\Phi'(u)=\{\gamma(u)\},$$

for $u \in H^n(X; Z_2) \cap \operatorname{Ker}\beta_1 \cap \cdots \cap \operatorname{Ker}\beta_k$,

b) Suppose $z=\sum_t a_t z_t$, where $a_t \in A^*$, $z_t=\sum \alpha_{i,t}[\beta_i]$ and $dz_t=0$, and let Φ_t be an operation associated with the pair (d, z_t). Then there is an operation Φ associated with (d, z) such that

(2.3) $$\sum_t a_t\Phi_t(u)=\{\Phi(u)\} \mod \sum_{i,t}\operatorname{Im} a_t\alpha_{i,t},$$

for $u \in H^n(X; Z_2) \cap \operatorname{Ker}\beta_1 \cap \cdots \cap \operatorname{Ker}\beta_k$.

c) Let the following commutative diagram be given:

(2.4) $$\begin{array}{ccc} A^* & \overset{d}{\longleftarrow} & C \\ \mu\big\downarrow & & \big\downarrow\mu' \\ A^* & \overset{d'}{\longleftarrow} & C' \end{array}$$

in which d, d' are as above, $C'=\sum_{i=1}^{j}\alpha_i'[\beta_i']$, and μ, μ' are A^*-maps with the same degree. Let Φ be an operation associated with a pair (d, z). Then there is an operation Φ' associated with $(d', \mu' z)$ such that

(2.5) $$\Phi_z(\mu(u))=\{\Phi'_{\mu' z}(u)\},$$

for $u \in H^n(X; Z_2) \cap \operatorname{Ker}\beta_1' \cap \operatorname{Ker}\beta_2' \cap \cdots \cap \operatorname{Ker}\beta_j'$.

We put now:

$z(1,1)=Sq^1[Sq^1]$, $z(2,2)=Sq^2[Sq^2]+Sq^3[Sq^1]$, $z(3,3)=Sq^3[Sq^3]+Sq^5[Sq^1]$, $z(1,3)=Sq^1[Sq^3]$, $z(3,2)=Sq^3[Sq^2]$ and $z(5,3)=Sq^5[Sq^3]$. Operations associated with $z(1,1)$, $z(2,2)$, $z(3,3)$ are defined uniquely from c). We denote them with $\Phi(1,1)$, $\Phi(2,2)$, $\Phi(3,3)$ respectively. We have $\Phi(1,1)=\Delta_2^2$.

PROPOSITION 1.

1) $Sq^1\Delta_2^2 u=0$, *for* $u \in H^n(X; Z_2) \cap \operatorname{Ker} Sq^1$.

2) i) $Sq^2\Phi(2,2)u=\Delta_2^2 Sq^4 u+Sq^4\Delta_2^2 u \mod \operatorname{Im} Sq^1+\operatorname{Im} Sq^4Sq^1$, *for* $u \in H^n(X, Z_2) \cap \operatorname{Ker} Sq^1 \cap \operatorname{Ker} Sq^2$.

ii) $\Phi(2,2)Sq^2 u=\Phi(3,3)u+Sq^4\Delta_2^2 u \mod \operatorname{Im} Sq^2+\operatorname{Im} Sq^3+\operatorname{Im} Sq^4Sq^1$, *for* $u \in$

$H^n(X;Z_2)\cap \operatorname{Ker} Sq^1 \cap \operatorname{Ker} Sq^3$.

3) i) $\Phi(3,3)u = \Delta_2^2 Sq^4 u \mod \operatorname{Im} Sq^1$,

ii) $Sq^2\Phi(3,3)u = \Delta_2^2 Sq^4 Sq^2 u + Sq^4 \Delta_2^2 Sq^2 u + Sq^6 \Delta_2^2 u \mod \operatorname{Im} Sq^1 + \operatorname{Im} Sq^4 Sq^1 + \operatorname{Im} Sq^6 Sq^1$, *for* $u \in H^n(X;Z_2)\cap \operatorname{Ker} Sq^1 \cap \operatorname{Ker} Sq^3$.

PROOF. The proof of 1) is easily seen from (1.14) and (1.18).

The proof of 2), i)*).

Consider the following commutative diagram

$$\begin{array}{ccc} A^* & \xleftarrow{\ d\ } & A^*[Sq^1] \\ \big\downarrow Sq^4 & d' & \big\downarrow \mu' \\ A^* & \longleftarrow & A^*[Sq^1]+A^*[Sq^2]=C' \end{array}$$

where μ' is given by

$$\mu'[Sq^1] = Sq^4[Sq^1] + Sq^2Sq^1[Sq^2].$$

Then we have

$$\begin{aligned} \mu' z(1,1) &= Sq^5[Sq^1] + Sq^3Sq^1[Sq^2] \\ &= Sq^2(Sq^2[Sq^2] + Sq^3[Sq^1]) + Sq^4Sq^1[Sq^1] \\ &= Sq^2 z(2,2) + Sq^4 z(1,1). \end{aligned}$$

From the above c), there is an operation $\Phi_{\mu' z(1,1)}$ associated with $\mu' z(1,1) = Sq^2 z(2,2) + Sq^4 z(1,1)$ such that

$$\Delta_2^2 Sq^4 u = \Phi_{\mu' z(1,1)} u \mod \operatorname{Im} Sq^1,$$

for $u \in H^n(X;Z_2)\cap \operatorname{Ker} Sq^1 \cap \operatorname{Ker} Sq^2$.

On the other hand, from a) and b), there is an element γ in $(A^*/d'C')_5$ such that

$$(Sq^2\Phi(2,2) + Sq^4\Delta_2^2)u - \Phi_{\mu' z(1,1)} u = \gamma u \mod \operatorname{Im} Sq^1 + \operatorname{Im} Sq^4 Sq^1.$$

But, we have $(A^*/d'C')_5 = 0$, and so $\gamma = 0$. This yields the result 2), i).

In the same way, we can also prove the relations 2), ii) and 3), i). We omit the proofs of them.

Using the results 2), i) and ii), we have

$$\begin{aligned} Sq^2\Phi(3,3)u &= Sq^2\Phi(2,2)\cdot Sq^2 u + Sq^2 Sq^4 \Delta_2^2 u \\ &\qquad \mod \operatorname{Im} Sq^3Sq^1 + \operatorname{Im}(Sq^5 + Sq^4Sq^1) + \operatorname{Im} Sq^6Sq^1, \\ &= (\Delta_2^2 Sq^4 + Sq^4\Delta_2^2)\cdot Sq^2 u + (Sq^6 + Sq^5Sq^1)\Delta_2^2 u \\ &\qquad \mod \operatorname{Im} Sq^1 + \operatorname{Im} Sq^4Sq^1 + \operatorname{Im} Sq^6Sq^1, \\ &= \Delta_2^2 Sq^4 Sq^2 u + Sq^4\Delta_2^2 Sq^2 u + Sq^6 \Delta_2^2 u \\ &\qquad \mod \operatorname{Im} Sq^1 + \operatorname{Im} Sq^4Sq^1 + \operatorname{Im} Sq^6Sq^1, \end{aligned}$$

*) This was also proved by N. Shimada and T. Yamanoshita, not utilizing the result of Adams [**1**].

for $u \in H^n(X; Z_2) \cap \mathrm{Ker}\, Sq^1 \cap \mathrm{Ker}\, Sq^3$.
This completes the proof of (3), ii).

Operations associated with $z(1,3)$, $z(3,2)$, $z(5,3)$ are not uniquely determined.

If $\Phi(1,3)$, $\Phi'(1,3)$ are two operations associated with $z(1,3)$, we have

(2.6) $$\Phi(1,3)u - \Phi'(1,3)u = x Sq^2 Sq^1 u \mod \mathrm{Im}\, Sq^1\,,$$

for $u \in H^n(X; Z_2) \cap \mathrm{Ker}\, Sq^3$, x being zero or one.

For two operations $\Phi(3,2)$, $\Phi'(3,2)$ associated with $z(3,2)$, we have

(2.7) $$\Phi(3,2)u - \Phi'(3,2)u = x Sq^4 u \mod \mathrm{Im}\, Sq^3\,,$$

for $u \in H^n(X; Z_2) \cap \mathrm{Ker}\, Sq^2$, x being zero or one.

For two operations $\Phi(5,3)$, $\Phi'(5,3)$ associated with $z(5,3)$, we have

(2.8) $$\Phi(5,3)u - \Phi'(5,3)u = x Sq^7 u + y Sq^6 Sq^1 u + z Sq^4 Sq^2 Sq^1 u \mod \mathrm{Im}\, Sq^5\,,$$

for $u \in H^n(X; Z_2) \cap \mathrm{Ker}\, Sq^3$, x, y, z being zero or one.

Now we have

Proposition 2. *There exist secondary operations $\Phi(1,3)$, $\Phi(3,2)$ and $\Phi(5,3)$ associated respectively with $z(1,3)$, $z(3,2)$ and $z(5,3)$ such that*

1) i) $\Phi(3,2)u = \Delta_2^2 Sq^2 Sq^1 u \mod \mathrm{Im}\, Sq^1$,

ii) $Sq^2 \Phi(3,2)u = \Delta_2^2 Sq^4 Sq^2 Sq^1 u \mod \mathrm{Im}\, Sq^1$, *for* $u \in H^n(X; Z_2) \cap \mathrm{Ker}\, Sq^2$.

2) $\Phi(1,3)u = \Delta_2^2 Sq^2 u \mod \mathrm{Im}\, Sq^1$, *for* $u \in H^n(X; Z_2) \cap \mathrm{Ker}\, Sq^3$.

3) i) $\Phi(5,3)u = \Delta_2^2 Sq^4 Sq^2 u \mod \mathrm{Im}\, Sq^1$,

ii) $Sq^2 \Phi(5,3)u = Sq^6 \Delta_2^2 Sq^2 u \mod \mathrm{Im}\, Sq^6 Sq^1$, *for* $u \in H^n(X; Z_2) \cap \mathrm{Ker}\, Sq^3$.

Proof. 1) Applying c), we see easily that there is an operation $\Phi(3,2)$ associated with $z(3,2)$ such that

$$\Phi(3,2)u = \Delta_2^2 Sq^2 Sq^1 u \mod \mathrm{Im}\, Sq^1\,,$$

for $u \in H^n(X; Z_2) \cap \mathrm{Ker}\, Sq^2$.

Consider the following commutative diagram

$$\begin{array}{ccc} A^* & \xleftarrow{\ d\ } & A^*[Sq^1] \\ \Big\downarrow{\scriptstyle Sq^4Sq^2Sq^1} & & \Big\downarrow{\scriptstyle \mu'} \\ A^* & \xleftarrow{\ d'\ } & A^*[Sq^2] = C' \end{array}$$

where μ' is given by $\mu'[Sq^1] = Sq^4 Sq^2 [Sq^2]$. Then we have

$$\mu' z(1,1) = Sq^1 Sq^4 Sq^2 [Sq^2] = Sq^4 Sq^3 [Sq^2]\,.$$

Therefore, there is an operation $\Phi_{\mu' z(1,1)}$ associated with $\mu' z(1,1) = Sq^4 Sq^3 [Sq^2]$, such that

$$\Delta_2^2 Sq^4 Sq^2 Sq^1 u = \Phi_{\mu' z(1,1)} u \mod \mathrm{Im}\, Sq^1\,.$$

Since the operation $Sq^4 \Phi(3,2)$ is also associated with $\mu' z(1,1)$, there is an element γ in $(A^*/d'C')_8$ such that

$$\mathrm{Sq}^4\Phi(3,2)u-\Phi_{\mu' z(1,1)}u=\gamma u \mod \mathrm{Im}\,\mathrm{Sq}^5\mathrm{Sq}^2.$$

But we have $(A^*/d'C')_8=\{\mathrm{Sq}^8, \mathrm{Sq}^7\mathrm{Sq}^1\}$. Thus we may put

$$\mathrm{Sq}^4\Phi(3,2)u-\Phi_{\mu' z(1,1)}u=x\mathrm{Sq}^8u+y\mathrm{Sq}^7\mathrm{Sq}^1u \mod \mathrm{Im}\,\mathrm{Sq}^5\mathrm{Sq}^2,$$

where x, y are zero or one.

Operating Sq^1 from the left to the above, we have

$$\mathrm{Sq}^5\Phi(3,2)u=x\mathrm{Sq}^9u \mod 0.$$

Since $\mathrm{Sq}^3\mathrm{Sq}^5\Phi(3,2)=\mathrm{Sq}^7\mathrm{Sq}^1\Phi(3,2)=0$ and $\mathrm{Sq}^3(x\mathrm{Sq}^9)=x\mathrm{Sq}^{11}\mathrm{Sq}^1$, we have $x=0$. Thus we have

$$\mathrm{Sq}^4\Phi(3,2)u-\Phi_{\mu' z(1,1)}u=y\mathrm{Sq}^7\mathrm{Sq}^1u \mod \mathrm{Im}\,\mathrm{Sq}^5\mathrm{Sq}^2,$$

which shows

$$\mathrm{Sq}^4\Phi(3,2)u=\Phi_{\mu' z(1,1)}u=\Delta_2^2\mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1u \mod \mathrm{Im}\,\mathrm{Sq}^1.$$

Proof of 2) is easy, and so omitted.

3) Applying c), we see easily that there is an operation $\Phi'(5,3)$ associated with $z(5,3)$ such that

$$\Phi'(5,3)u=\Delta_2^2\mathrm{Sq}^4\mathrm{Sq}^2u \mod \mathrm{Im}\,\mathrm{Sq}^1,$$

for $u\in H^n(X;Z_2)\cap \mathrm{Ker}\,\mathrm{Sq}^3$.

Since the operation $\mathrm{Sq}^2\Phi'(5,3)+\mathrm{Sq}^6\Phi(1,3)$ is associated with the trivial relation $\mathrm{Sq}^2\mathrm{Sq}^5[\mathrm{Sq}^3]+\mathrm{Sq}^6\mathrm{Sq}^1[\mathrm{Sq}^3]$ in $A^*[\mathrm{Sq}^3]$, we may put

$$(\mathrm{Sq}^2\Phi'(5,3)+\mathrm{Sq}^6\Phi(1,3))u=x\mathrm{Sq}^9u+y\mathrm{Sq}^8\mathrm{Sq}^1u+z\mathrm{Sq}^7\mathrm{Sq}^2u \mod \mathrm{Im}\,\mathrm{Sq}^6\mathrm{Sq}^1,$$

where x, y, z are zero or one. Since

$$\mathrm{Sq}^3(\mathrm{Sq}^2\Phi'(5,3)+\mathrm{Sq}^6\Phi(1,3))u=0 \mod 0 \quad \text{and}$$

$$\mathrm{Sq}^3(x\mathrm{Sq}^9u+y\mathrm{Sq}^8\mathrm{Sq}^1u+z\mathrm{Sq}^7\mathrm{Sq}^2u)=x\mathrm{Sq}^{11}\mathrm{Sq}^1u+y\mathrm{Sq}^{11}\mathrm{Sq}^1u,$$

we have $x=y$, that is,

$$(\mathrm{Sq}^2\Phi'(5,3)+\mathrm{Sq}^6\Phi(1,3))u=x(\mathrm{Sq}^9+\mathrm{Sq}^8\mathrm{Sq}^1)+z\mathrm{Sq}^7\mathrm{Sq}^2u.$$

Next, operate Sq^2 to the above, then we have

$$\begin{aligned}\mathrm{Sq}^2(\mathrm{Sq}^2\Phi'(5,3)+\mathrm{Sq}^6\Phi(1,3))u&=\mathrm{Sq}^3\mathrm{Sq}^1\Phi'(5,3)u+\mathrm{Sq}^7\mathrm{Sq}^1\Phi(1,3)u\\&=0 \mod 0, \quad \text{and}\end{aligned}$$

$$\begin{aligned}&\mathrm{Sq}^2x(\mathrm{Sq}^9+\mathrm{Sq}^8\mathrm{Sq}^1)u+\mathrm{Sq}^2z\mathrm{Sq}^7\mathrm{Sq}^2u\\&\quad=x(\mathrm{Sq}^{10}\mathrm{Sq}^1+\mathrm{Sq}^{10}\mathrm{Sq}^1)u+z(\mathrm{Sq}^9\mathrm{Sq}^2+\mathrm{Sq}^8\mathrm{Sq}^3)u\\&\quad=z\mathrm{Sq}^9\mathrm{Sq}^2u,\end{aligned}$$

which show $z=0$. Thus we have

$$(\mathrm{Sq}^2\Phi'(5,3)+\mathrm{Sq}^6\Phi(1,3))u=x(\mathrm{Sq}^9+\mathrm{Sq}^8\mathrm{Sq}^1)u \mod \mathrm{Im}\,\mathrm{Sq}^6\mathrm{Sq}^1.$$

If we take $\Phi(5,3)=\Phi'(5,3)+x\mathrm{Sq}^7$ for the above $\Phi'(5,3)$, we have

$$\Phi(5,3)u=\Delta_2^2\mathrm{Sq}^4\mathrm{Sq}^2u \mod \mathrm{Im}\,\mathrm{Sq}^1, \quad \text{and}$$

$$\begin{aligned}&(\mathrm{Sq}^2\Phi(5,3)+\mathrm{Sq}^6\Phi(1,3))u\\ &\quad=\mathrm{Sq}^2\Phi'(5,3)u+x(\mathrm{Sq}^9+\mathrm{Sq}^8\mathrm{Sq}^1)u+\mathrm{Sq}^6\Phi(1,3)u \mod \mathrm{Im}\,\mathrm{Sq}^6\mathrm{Sq}^1\\ &\quad=0 \mod \mathrm{Im}\,\mathrm{Sq}^6\mathrm{Sq}^1.\end{aligned}$$

This completes the proof.

In the sequel, $\Phi(1,3)$, $\Phi(3,2)$, $\Phi(5,3)$ will always mean a fixed cohomology operation associated with $z(1,3)$, $z(3,2)$, $z(5,3)$, respectively, with the properties of the Proposition 2.

§ 3. Stable cohomology group $A^*(\pi, k^{(2)}, G\,; Z_2)$.

In this section, we shall consider the stable cohomology group $A^*(\pi, k^{(2)}, G\,; Z_2)$ determined by abelian groups π, G and an invariant $k^{(2)}\in A^2(\pi\,;G)$, where π and G have one generator and the invariant $k^{(2)}$ is non-trivial.

Let us denote by π one of the groups Z, Z_2 or $Z_{2^{q'+1}}$ $(q'\geqq 1)$, and by G one of Z_2 or $Z_{2^{q+1}}$ $(q\geqq 1)$.

Let

$$0\longrightarrow Z_2\overset{f_q'}{\longrightarrow}Z_{2^{q+1}}\longrightarrow Z_{2^q}\longrightarrow 0$$

be the exact sequence, then we see easily

(3.1) $\qquad A^2(\pi\,;G)\approx Z_2 \quad$ and $\quad f'_{q*}: A^2(\pi\,;Z_2)\approx A^2(\pi\,;Z_{2^{q+1}})$,

where f'_{q*} denotes the homomorphism of cohomology groups induced by the inclusion f_q' of coefficient groups.

Let u be the generator of degree zero of $A^*(\pi\,;Z_2)$, and a be the generator of degree zero of $A^*(G\,;Z_2)$. Then, from (3.1), our $A^*(\pi, k^{(2)}, G\,;Z_2)$ must be one of the following six types, as $k^{(2)}\in A^2(\pi\,;G)$ is non-trivial:

(3.2)
(1) $A^*(Z, \mathrm{Sq}^2u, Z_2\,;Z_2)$ (2) $A^*(Z_2, \mathrm{Sq}^2u, Z_2\,;Z_2)$
(3) $A^*(Z_{2^{q'+1}}, \mathrm{Sq}^2u, Z_2\,;Z_2)$ (4) $A^*(Z, f'_{q*}\mathrm{Sq}^2u, Z_{2^{q+1}}\,;Z_2)$
(5) $A^*(Z_2, f'_{q*}\mathrm{Sq}^2u, Z_{2^{q+1}}\,;Z_2)$ (6) $A^*(Z_{2^{q'+1}}, f'_{q*}\mathrm{Sq}^2u, Z_{2^{q+1}}\,;Z_2)$.

For convenience, we denote these types by $A^*(1)$, $A^*(2)$, $A^*(3)$, $A^*(4)$, $A^*(5)$ and $A^*(6)$, respectively, and write e. g. $A^i(1)$ for $A^i(Z, \mathrm{Sq}^2u, Z_2\,;Z_2)$. Then we have the following exact sequence:

$$(S_j):\quad \cdots\longleftarrow A^{i+1}(\pi\,;Z_2)\longleftarrow A^{i-1}(G\,;Z_2)\overset{i^*}{\longleftarrow}A^i(j)\overset{p^*}{\longleftarrow}A^i(\pi\,;Z_2)$$
$$\overset{\tau}{\longleftarrow}A^{i-2}(G\,;Z_2)\longleftarrow\cdots$$

where $j=1,2,3,4,5,6$.

We are now in a position to formulate our main theorem.

THEOREM.

(I) *$A^*(Z, \mathrm{Sq}^2u, Z_2 ; Z_2)$ is an A^*-module generated by elements $v=p^*u$ and $\Phi(2,2)v$ with basic relations*

$$\mathrm{Sq}^1v=\mathrm{Sq}^2v=\mathrm{Sq}^3\Phi(2,2)v=0\,.$$

In particular, we have

$$\Delta_2^2\mathrm{Sq}^4v=\mathrm{Sq}^2\Phi(2,2)v\,.$$

(II) *$A^*(Z_2, \mathrm{Sq}^2u, Z_2 ; Z_2)$ is an A^*-module generated by elements $v=p^*u$ and $\Phi(3,2)v$ with basic relations*

$$\mathrm{Sq}^2v=\mathrm{Sq}^1\Phi(3,2)v=\mathrm{Sq}^5\Phi(3,2)v=0\,.$$

In particular, we have

$$\Phi(3,2)v=\Delta_2^2\mathrm{Sq}^2\mathrm{Sq}^1v,\quad \mathrm{Sq}^4\Phi(3,2)v=\Delta_2^2\mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1v \mod \mathrm{Sq}^7\mathrm{Sq}^1v\,.$$

(III) *$A^*(Z_{2^{q'+1}}, \mathrm{Sq}^2u, Z_2 ; Z_2)$ is an A^*-module generated by elements $v=p^*u$, $\Delta_2^{q'+1}v$ and $\Phi(2,2)v$ with basic relations*

$$\mathrm{Sq}^1v=\mathrm{Sq}^2v=\mathrm{Sq}^1\Delta_2^{q'+1}v=0\,,\quad \textit{and}$$

$$\mathrm{Sq}^3\Phi(2,2)v=\begin{cases}0 & \textit{if}\ \ q'>1\\ \mathrm{Sq}^5\Delta_2^{q'+1}v & \textit{if}\ \ q'=1\,.\end{cases}$$

In particular, we have

$$\mathrm{Sq}^2\Phi(2,2)v=\Delta_2^2\mathrm{Sq}^4v+\mathrm{Sq}^4\Delta_2^2v\,.$$

(IV) *$A^*(Z, f'_{q*}\mathrm{Sq}^2u, Z_{2^{q+1}} ; Z_2)$ is an A^*-module generated by elements $v=p^*u$, b_1 such that $i^*b_1=a$, and $\Phi(3,3)v$ with basic relations*

$$\mathrm{Sq}^1v=\mathrm{Sq}^3v=\mathrm{Sq}^1\Phi(3,3)v=\mathrm{Sq}^3\Phi(3,3)v=0\,,\quad \textit{and}$$

$$\mathrm{Sq}^1b_1=\begin{cases}0 & \textit{if}\ \ q>1\\ \mathrm{Sq}^2v & \textit{if}\ \ q=1\,.\end{cases}$$

In particular, we have

$$\Delta_2^q b_1=\mathrm{Sq}^2v\,,\quad \Phi(3,3)v=\Delta_2^2\mathrm{Sq}^4v\quad \textit{and}\quad \mathrm{Sq}^2\Phi(3,3)v=\Delta_2^2\mathrm{Sq}^4\mathrm{Sq}^2v\,.$$

(V) *$A^*(Z_2, f'_{q*}\mathrm{Sq}^2u, Z_{2^{q+1}} ; Z_2)$ is an A^*-module generated by elements $v=p^*u$, b_1 such that $i^*b_1=a$ and $\Phi(5,3)v$ with basic relations*

$$\mathrm{Sq}^3v=\mathrm{Sq}^1\Phi(5,3)v=\mathrm{Sq}^2\Phi(5,3)v=0\,,\quad \textit{and}$$

$$\mathrm{Sq}^1b_1=\begin{cases}0 & \textit{if}\ \ q>1\\ \mathrm{Sq}^2v & \textit{if}\ \ q=1\,.\end{cases}$$

In particular, we have

$$\Delta_2^q b_1=\mathrm{Sq}^2v\quad \textit{and}\quad \Phi(5,3)v=\Delta_2^2\mathrm{Sq}^4\mathrm{Sq}^2v \mod \mathrm{Sq}^7v\,.$$

(VI) *$A^*(Z_{2^{q'+1}}, f'_{q*}\mathrm{Sq}^2u, Z_{2^{q+1}} ; Z_2)$ is an A^*-module generated by elements $v=p^*u$, $\Delta_2^{q'+1}v$, b^1 such that $i^*b^1=a$ and $\Phi(3,3)v$ with basic relations*

$$\mathrm{Sq}^1 v = \mathrm{Sq}^3 v = \mathrm{Sq}^1 \Delta_2^{q'+1} v = \mathrm{Sq}^1 \Phi(3,3) v = 0\,,$$

$$\mathrm{Sq}^1 b_1 = \begin{cases} 0 & \textit{if} \quad q > 1 \\ \mathrm{Sq}^2 v & \textit{if} \quad q = 1\,, \quad \textit{and} \end{cases}$$

$$\mathrm{Sq}^3 \Phi(3,3) v = \begin{cases} 0 & \textit{if} \quad q' > 1 \\ \mathrm{Sq}^7 \Delta_2^{q'+1} v & \textit{if} \quad q' = 1\,. \end{cases}$$

In particular, we have

$$\Delta_2^q b_1 = \mathrm{Sq}^2 v\,, \quad \Phi(3,3) v = \Delta_2^2 \mathrm{Sq}^4 v \quad \textit{and}$$

$$\Delta_2^2 \mathrm{Sq}^4 \mathrm{Sq}^2 v = \mathrm{Sq}^2 \Phi(3,3) v + \mathrm{Sq}^6 \Delta_2^2 v \mod \mathrm{Sq}^7 v\,.$$

To prove this theorem, we need some informations about the exact sequences (S_j).

First, it is well-known that

(3.3) $$A^i(Z, Z_2) \approx A^i / A^{i-1}\mathrm{Sq}^1\,, \quad A^i(Z_2\,; Z_2) \approx A^i \quad \text{and}$$

$$A^i(Z_{2^{q+1}}\,; Z_2) \approx A^i / A^{i-1}\mathrm{Sq}^1 \oplus A^{i-1} / A^{i-2}\mathrm{Sq}^1\,,$$

where $\oplus$ denote the direct sum.

Second, the transgression τ is determined by the following property.

PROPOSITION 3.

i) *In the cases* (1), (2) *and* (3), *we have*

$$\tau a = \mathrm{Sq}^2 u\,.$$

ii) *In the cases* (4), (5) *and* (6), *we have*

$$\tau a = 0 \quad \textit{and} \quad \tau \Delta_2^{q+1} a = \mathrm{Sq}^3 u\,.$$

The first part i) is a well-known result. To prove the second part ii), we require the following lemma.

Let

$$0 \longrightarrow Z_2 \xrightarrow{f_q'} Z_{2^{q+1}} \xrightarrow{g_q'} Z_{2^q} \longrightarrow 0\,,$$

$$0 \longrightarrow Z_{2^q} \xrightarrow{f_q} Z_{2^{q+1}} \xrightarrow{g_q} Z_2 \longrightarrow 0\,,$$

$$0 \longrightarrow Z_{2^{q+1}} \longrightarrow Z_{2^{2(q+1)}} \longrightarrow Z_{2^{q+1}} \longrightarrow 0$$

be exact sequences defined in usual ways. And we shall denote by δ_q', δ_q and δ the coboundary homomorphisms associated with the above sequences, respectively.

Consider the following diagram:

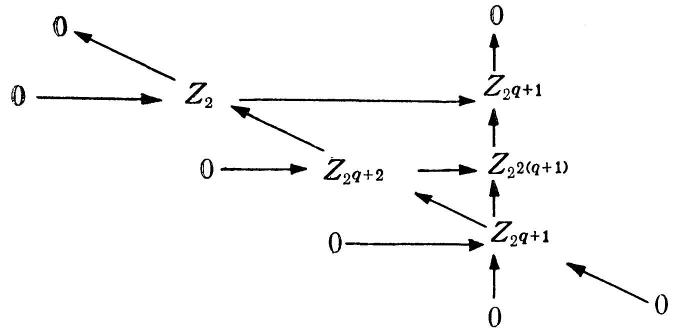
(3.4) $$\begin{array}{ccc} A^3(\pi\,; Z_{2^{q+1}}) & \overset{\delta}{\longleftarrow} & A^2(\pi\,; Z_{2^{q+1}}) \\ \downarrow g_{q*} & \mathrm{Sq}^1 = \Delta_2^1 & \uparrow f'_{q*} \quad \downarrow g_{q*} \\ A^3(\pi\,; Z_2) & \longleftarrow & A_2(\pi\,; Z_2) \end{array}$$

Then we have

LEMMA. *In the above diagram* (3.4), *we have*

$$g_{q*}f'_{q*}=0 \quad and \quad g_{q*}\delta f'_{q*}=\mathrm{Sq}^1 .$$

PROOF. The first part is clear, and so we shall prove only the second part. Let us consider the commutative diagram



From the above diagram, we see that $\delta f'_{q*}$ is equal to the coboundary homomorphism δ_{q+1}.

Similary the composition $g_{q*}\delta_{q+1}$ is equal to the coboundary homomorphism $\delta_1=\mathrm{Sq}^1$. Thus we have $g_{q*}\delta f'_{q*}=g_{q*}\delta_{q+1}=\mathrm{Sq}^1$.

PROOF OF PROPOSITION 3, ii).

Let c be a fundamental class of $A^0(Z_{2^{q+1}};Z_{2^{q+1}})$, and u be the non-zero element of $A^0(\pi;Z_2)$. Then we see easily that

$$g_{q*}c=a\,,\quad g_{q*}\delta c=\Delta_2^{q+1}a \quad \text{and} \quad \tau c=f'_{q*}\mathrm{Sq}^2u\,. \tag{3.5}$$

Since $\tau g_{q*}=g_{q*}\tau$ and $\tau\delta=\delta\tau$ hold, we see by using the lemma and (3.5) that

$$\begin{aligned}\tau a&=\tau g_{q*}c=g_{q*}\tau c\\&=g_{q*}f'_{q*}\mathrm{Sq}^2\\&=0\end{aligned}$$

$$\begin{aligned}\tau\Delta_2^{q+1}a&=\tau g_{q*}\delta c=g_{q*}\tau\delta c\\&=g_{q*}\delta f'_{q*}\mathrm{Sq}^2u\\&=\mathrm{Sq}^1\mathrm{Sq}^2u=\mathrm{Sq}^3u\,.\end{aligned}$$

This completes the proof.

PROOF OF THE THEOREM.

We begin with the proof of (II).

From the exactness of the sequence (S_2), we have an isomorphism $p^*:A^0(Z_2;Z_2)\approx A^0(2)$, therefore $v=p^*u$ is a generator of $A^*(2)$. The homomorphism τ: $A^*(Z_2;Z_2)\to A^*(Z_2;Z_2)$ is given by $\tau(a)=\mathrm{Sq}^2u$. Since $\tau\alpha a=\alpha\mathrm{Sq}^2u$ for each α in A^*, τ is equivalent to $\mathrm{Sq}^2_*:A^*\to A^*$. It follows from the exactness of the sequence (1.7) in § 1 that the kernel of Sq^2_* is

$$\mathrm{Sq}^3_*(A^*/A^*\mathrm{Sq}^1) = (A^*/A^*\mathrm{Sq}^1)\cdot \mathrm{Sq}^3\,,$$

that is, the kernel of τ is generated by $\mathrm{Sq}^3 a$.

From the exactness of (S_2), we see that $A^*(2)$ is generated by $v = p^*u$ and an element $b_4 \in A^4(2)$ such that

$$i^*b_4 = \mathrm{Sq}^3 a\,.$$

Since $\tau a = \mathrm{Sq}^2 u$, we have $\mathrm{Sq}^2 v = 0$, and so $\Phi(3,2)v$ is well-defined. As b_4 we may take $\Phi(3,2)v$ such that

$$\Phi(3,2)v = \Delta_2^2\mathrm{Sq}^2\mathrm{Sq}^1 v\,,$$

$$\mathrm{Sq}^4\Phi(3,2)v = \Delta_2^2\mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1 v \quad \mathrm{mod}\ \mathrm{Im}\,\mathrm{Sq}^1\,.$$

Then we have relations

$$\mathrm{Sq}^1 b_4 = \mathrm{Sq}^5 b_4 = 0\,.$$

Now let

$$\alpha v + \beta b_4 = 0\,, \qquad \alpha, \beta \in A^*$$

be a relation between generators v and b_4, then we have $i^*(\alpha v+\beta b_4) = i^*(\beta b_4) = \beta\mathrm{Sq}^3 a = 0$. From (1.8) in § 1, the kernel of $\mathrm{Sq}^3_* : A^* \to A^*$ is generated by Sq^1 and Sq^5. Therefore such a β is generated by Sq^1 and Sq^5, that is, there are some elements β_1 and β_2 in A^* such that

$$\beta = \beta_1\mathrm{Sq}^1 + \beta_2\mathrm{Sq}^5\,.$$

Since $\mathrm{Sq}^1 b_4 = \mathrm{Sq}^5 b_4 = 0$, we have $\beta b_4 = 0$, and so $\alpha v = 0$. Since the image of τ is generated by $\mathrm{Sq}^2 u$, such an α is generated by Sq^2, that is, there is an element α_1 in A^* such that

$$\alpha = \alpha_1\mathrm{Sq}^2\,.$$

Hence we have

$$\alpha v + \beta b_4 = \alpha_1\mathrm{Sq}^2 v + \beta_1\mathrm{Sq}^1 b_4 + \beta_2\mathrm{Sq}^5 b_4\,.$$

This shows that

$$\mathrm{Sq}^2 v = \mathrm{Sq}^1 b_4 = \mathrm{Sq}^5 b_4 = 0$$

are the basic relations of the generators.

The proof of (I) is similar to the above. We only use the exact sequences (S_1), (1.6), (1.7) and the Proposition 1 instead of (S_2), (1.7), (1.8) and the Proposition 2.

Proof of (V).

From the exactness of the sequence (S_5), we have an isomorphism $p^* : A^0(Z_2\,; Z_2) \approx A^0(5)$, therefore $v = p^*u$ is a generator of $A^*(5)$.

According to the Proposition 3, the homomorphism $\tau : A^*(Z_{2^{q+1}}\,; Z_2) \to A^*(Z_2\,; Z_2)$ is given by $\tau a = 0$ and $\tau\Delta_2^{q+1}a = \mathrm{Sq}^3 u$.

From the exactness of (1.8), the kernel of such a τ is generated by a and $\mathrm{Sq}^5\Delta_2^{q+1}a$. From the exactness of the sequence (S_5), we see that $A^*(5)$ is generated by $v = p^*u$, an element b_1 such that $i^*b_1 = a$ and an element b_7 in $A^7(5)$

such that $i^*b_7=\mathrm{Sq}^5\Delta_2^{q+1}a$. Since $\tau\Delta_2^{q+1}a=\mathrm{Sq}^3u$, we have $\mathrm{Sq}^3v=0$. Applying (1.21) to the exact sequence (S_5), we easily verify that

$$\Delta_2^q b_1=\mathrm{Sq}^2 v\,.$$

Then we have

$$\mathrm{Sq}^1 b_1=\begin{cases}0 & \text{if } q>1\\ \mathrm{Sq}^2 v & \text{if } q=1\,.\end{cases}$$

Since $\mathrm{Sq}^3v=0$, $\Phi(5,3)v$ is well-defined. As b_7 we may take $\Phi(5,3)v$ such that

$$\Phi(5,3)v=\Delta_2^2\mathrm{Sq}^4\mathrm{Sq}^2v \mod \mathrm{Sq}^7 v\,,\quad\text{and}$$
$$\mathrm{Sq}^2\Phi(5,3)v=\mathrm{Sq}^6\Delta_2^2\mathrm{Sq}^2v\,.$$

By applying (1.20) to the exact sequence (S_5), we see that in this case $\Delta_2^2\mathrm{Sq}^2v=0$.

From the above relations, we have

$$\mathrm{Sq}^1b_7=\mathrm{Sq}^2b_7=0\,.$$

Let

$$\alpha v+\beta b_1+\gamma b_7=0\,,\qquad \alpha,\beta,\gamma\in A^*$$

be a relation between the generators v, b_1 and b_7 taken as above. Then we have

$$i^*(\alpha v+\beta b_1+\gamma b_7)=i^*(\beta b_1+\gamma b_7)=\beta a+\gamma\mathrm{Sq}^5\Delta_2^{q+1}a=0\,.$$

Now we define a homomorphism

$$\varphi: A^*(Z_{2q+1}\,;Z_2)\to A^*(Z_{2q+1}\,;Z_2)$$

by $\varphi(a)=a$ and $\varphi(\Delta_2^{q+1}a)=\mathrm{Sq}_*^5(\Delta_2^{q+1}a)$.
From the exactness of (1.10), we see that the kernel of φ is generated by Sq^1a, $\mathrm{Sq}^1\Delta_2^{q+1}a$ and $\mathrm{Sq}^2\Delta_2^{q+1}a$. That is, β is generated by Sq^1, and γ is generated by Sq^1 and Sq^2:

$\beta=\beta_1\mathrm{Sq}^1$, $\gamma=\gamma_1\mathrm{Sq}^1+\gamma_2\mathrm{Sq}^2$ for some β_1, γ_1 and γ_2 in A^*.

Since

$$\mathrm{Sq}^1 b_1=\begin{cases}0 & \text{if } q>1\\ \mathrm{Sq}^2 v & \text{if } q=1\,,\end{cases}$$

and $\mathrm{Sq}^1b_7=\mathrm{Sq}^2b_7=0$, we have

$$\alpha v=\begin{cases}0 & \text{if } q>1\\ \beta_1\mathrm{Sq}^2 v & \text{if } q=1\,.\end{cases}$$

Since the image of τ is generated by Sq^3u, such an α (resp. $\alpha+\beta_1\mathrm{Sq}^2$) is generated by Sq^3. Therefore we may put

$$\alpha=\alpha_1\mathrm{Sq}^3\qquad \text{if } q>1\,,\ \text{and}$$
$$\alpha+\beta_1\mathrm{Sq}^2=\alpha_2\mathrm{Sq}^3\qquad \text{if } q=1\,.$$

Then we have

$$\alpha v+\beta b_1+\gamma b_7=\begin{cases}\alpha_1\mathrm{Sq}^3v+\beta_1\mathrm{Sq}^1b_1+\gamma_1\mathrm{Sq}^1b_7+\gamma_2\mathrm{Sq}^2b_7 & \text{if } q>1\\ \alpha_2\mathrm{Sq}^3v+\beta_1(\mathrm{Sq}^1b_1+\mathrm{Sq}^2v)+\gamma_1\mathrm{Sq}^1b_7+\gamma_2\mathrm{Sq}^2b_7 & \text{if } q=1,\end{cases}$$

which shows that our relations are basic.

PROOF OF (VI).

From the exactness of the sequence (S_6), we have an isomorphism $p^*: A^0(Z_{2^{q'+1}}; Z_2)\approx A^0(6)$, therefore $v=p^*u$ is a generator of $A^*(6)$. According to the Proposition 3, the homomorphism $\tau: A^*(Z_{2^{q+1}}; Z_2)\rightarrow A^*(Z_{2^{q'+1}}; Z_2)$ is given by $\tau a=0$ and $\tau\Delta_2^{q+1}a=\mathrm{Sq}^3u$. From the exact sequence (1.9), we see that the kernel of τ is generated by a and $\mathrm{Sq}^3\Delta_2^{q+1}a$.

From the exactness of (S_6), we see that there are elements b_1 in $A^1(6)$ and b_5 in $A^5(6)$ such that $i^*b_1=a$, $i^*b_5=\mathrm{Sq}^3\Delta_2^{q+1}a$, and $A^*(6)$ is generated by $v=p^*u$, $\Delta_2^{q'+1}v$, b_1 and b_5.

Since $\tau\Delta_2^{q+1}a=\mathrm{Sq}^3u$, we have $\mathrm{Sq}^3v=0$ and $\mathrm{Sq}^1v=0$, therefore $\Phi(3,3)v$ is well-defined. Then we may take $\Phi(3,3)v$ as b_5.

According to (1.21) and the Proposition 1, we have relations:

$$\Delta_2^q b_1=\mathrm{Sq}^2v,\quad \Phi(3,3)v=\Delta_2^2\mathrm{Sq}^4v$$

and $\mathrm{Sq}^2\Phi(3,3)v=\Delta_2^2\mathrm{Sq}^4\mathrm{Sq}^2v+\mathrm{Sq}^6\Delta_2^2v \bmod \mathrm{Sq}^7v$. This shows that

$$\mathrm{Sq}^1v=\mathrm{Sq}^3v=\mathrm{Sq}^1\Delta_2^{q'+1}v=\mathrm{Sq}^1b_5=0, \tag{3.6}$$

$$\mathrm{Sq}^1b_1=\begin{cases}0 & \text{if } q>1\\ \mathrm{Sq}^2v & \text{if } q=1, \text{ and}\end{cases}$$

$$\mathrm{Sq}^3b_5=\begin{cases}0 & \text{if } q'>1\\ \mathrm{Sq}^7\Delta_2^{q'+1}v & \text{if } q'=1.\end{cases}$$

Next we shall prove that the relations (3.6) are basic relations.

Let

$$\alpha v+\beta\Delta_2^{q'+1}v+\gamma b_1+\delta b_5=0,\qquad \alpha,\beta,\gamma,\delta\in A^*$$

be a relation between generators v, $\Delta_2^{q'+1}v$, b_1 and b_5. Then we have

$$\begin{aligned}&i^*(\alpha v+\beta\Delta_2^{q'+1}v+\gamma b_1+\delta b_5)\\ &=i^*(\gamma b_1+\delta b_5)\\ &=\gamma a+\delta\mathrm{Sq}^3\Delta_2^{q+1}a=0.\end{aligned}$$

From the exactness of (1.9), such a δ is generated by Sq^1 and Sq^3, that is, $\delta=\delta_1\mathrm{Sq}^1+\delta_2\mathrm{Sq}^3$ for some δ_1 and δ_2 of A^*. γ is generated by Sq^1, that is, $\gamma=\gamma_1\mathrm{Sq}^1$ for some γ_1 of A^*. From (3.6), we have

$$\alpha v+\beta\Delta_2^{q'+1}v=0 \qquad \text{if } q'>1 \text{ and } q>1,$$

$$(\alpha+\gamma_1\mathrm{Sq}^2)v+\beta\Delta_2^{q'+1}v=0 \qquad \text{if } q'>1 \text{ and } q=1,$$

$$\alpha v+(\beta+\delta_2\mathrm{Sq}^7)\Delta_2^{q'+1}v=0 \qquad \text{if } q'=1 \text{ and } q>1,$$
$$(\alpha+\beta_1\mathrm{Sq}^2)v+(\beta+\delta_2\mathrm{Sq}^7)\Delta_2^{q'+1}v=0 \qquad \text{if } q'=1 \text{ and } q=1.$$

On the other hand, since the image of τ is generated by Sq^3u, and $\mathrm{Sq}^1u=\mathrm{Sq}^1\Delta_2^{q'+1}u=0$, we may put for some α_1, α_2, α_1', α_2', β_1 and β_1' of A^*,

$$\alpha=\alpha_1\mathrm{Sq}^1+\alpha_2\mathrm{Sq}^3,\quad \beta=\beta_1\mathrm{Sq}^1 \qquad \text{if } q'>1 \text{ and } q>1,$$
$$\alpha+\gamma_1\mathrm{Sq}^2=\alpha_1'\mathrm{Sq}^1+\alpha_2'\mathrm{Sq}^3,\quad \beta=\beta_1\mathrm{Sq}^1 \qquad \text{if } q'>1 \text{ and } q=1,$$
$$\alpha=\alpha_1\mathrm{Sq}^1+\alpha_2\mathrm{Sq}^3,\quad \beta+\delta_2\mathrm{Sq}^7=\beta_1'\mathrm{Sq}^1 \qquad \text{if } q'=1 \text{ and } q>1,$$
$$\alpha+\gamma_1\mathrm{Sq}^2=\alpha_1'\mathrm{Sq}^1+\alpha_2'\mathrm{Sq}^3,\quad \beta+\delta_2\mathrm{Sq}^7=\beta_1'\mathrm{Sq}^1 \qquad \text{if } q'=1 \text{ and } q=1.$$

Then we have

$$\alpha v+\beta\Delta_2^{q'+1}v+\gamma b_1+\delta b_5$$
$$=\begin{cases} \alpha_1\mathrm{Sq}^1v+\alpha_2\mathrm{Sq}^3v+\beta_1\mathrm{Sq}^1\Delta_2^{q'+1}v+\gamma_1\mathrm{Sq}^1b_1+\delta_1\mathrm{Sq}^1b_5+\delta_2\mathrm{Sq}^3b_5 \\ \qquad\qquad \text{if } q'>1 \text{ and } q>1, \\ \alpha_1'\mathrm{Sq}^1v+\alpha_2'\mathrm{Sq}^3v+\beta_1\mathrm{Sq}^1\Delta_2^{q'+1}v+\gamma_1(\mathrm{Sq}^1b_1+\mathrm{Sq}^2v) \\ \qquad +\delta_1\mathrm{Sq}^1b_5+\delta_2\mathrm{Sq}^3b_5 \qquad \text{if } q'>1 \text{ and } q=1, \\ \alpha_1\mathrm{Sq}^1v+\alpha_2\mathrm{Sq}^3v+\beta_1'\mathrm{Sq}^1\Delta_2^{q'+1}v+\gamma_1\mathrm{Sq}^1b_1+\delta_1\mathrm{Sq}^1b_5 \\ \qquad +\delta_2(\mathrm{Sq}^3b_5+\mathrm{Sq}^7\Delta_2^{q'+1}v) \qquad \text{if } q'=1 \text{ and } q>1, \\ \alpha_1'\mathrm{Sq}^1v+\alpha_2'\mathrm{Sq}^3v+\beta_1'\mathrm{Sq}^1\Delta_2^{q'+1}v+\gamma_1(\mathrm{Sq}^1b_1+\mathrm{Sq}^2v) \\ \qquad +\delta_1\mathrm{Sq}^1b_5+\delta_2(\mathrm{Sq}^3b_5+\mathrm{Sq}^7\Delta_2^{q'+1}v) \qquad \text{if } q'=1 \text{ and } q=1. \end{cases}$$

This shows that (3.6) are basic relations between v, $\Delta_2^{q'+1}v$, b_1 and b_5.

The proofs of (III), (IV) are similar to the above, and so omitted.

Appendix

We shall show in these Appendix that we can obtain the above results in low dimensional cases also by geometrical considerations.

First we shall summarize the results of H. Toda [**7**], [**8**] and T. Yamanoshita [**11**] on stable homotopy groups of spheres: $G_i=\lim \pi_{n+i}(S^n)$ $(=\pi_{n+i}(S^n)$ for $i+1<n)$ for $i\leqq 10$.

$G_0=Z=\{\iota\}$,

$G_1=Z_2=\{\eta\}$, where η is a suspension of Hopf map $S^3\to S^2$,

$G_2=Z_2=\{\eta\circ\eta\}$,

$G_3=Z_8+Z_3$, where $Z_8=\{\nu\}$, and ν is a suspension of Hopf map $S^7\to S^4$,

$G_4=G_5=0$,

$G_6=Z_2=\{\nu\circ\nu\}$,

$G_7=Z_{16}+Z_3+Z_5$, where $Z_{16}=\{\sigma\}$, and σ is a suspension of Hopf map $S^{15}\to S^8$,

$G_8=Z_2+Z_2=\{\sigma\circ\eta, \varepsilon\}$, $\varepsilon=[\eta, 2\nu, \nu]$, where [, ,] denotes the toric construction [**7**].

$G_9 = Z_2 + Z_2 + Z_2 = \{\sigma\circ\eta\circ\eta, \varepsilon\circ\eta, \mu\}$, $\mu = [\eta, 16\iota, \sigma]$,

$G_{10} = Z_2 + Z_9$, where $Z_2 = \{\mu\circ\eta\}$.

We have relations

$$\eta\circ\nu = 0, \quad \sigma\circ\nu = 0, \quad \varepsilon\circ\eta\circ\eta = 0, \quad \eta\circ\eta\circ\eta = 4\nu.$$

(We shall use these results up to G_4 in the following. Further results on G_5, $G_6, \cdots$ would be needed, if we continue our computation to higher dimensional cases.)

Now let π and G be finitely generated abelian groups, and X_n be an $(n-1)$-connected CW-complex such that $\pi_n(X_n) = \pi$, $\pi_{n+1}(X_n) = G$, and with the Eilenberg-MacLane invariant $k^{n+2} \in H^{n+2}(\pi; G)$. n is supported to be sufficiently large.

The following are the CW-complexes X_n with the invariants corresponding to the cases (1)$\sim$(6), § 3. (X_n corresponding to the case (j) is denoted by $X_n(j)$) (cf. A_n^2-polyhedra [**3**]).

$X_n(1) = S^n$,

$X_n(2) = S_n \underset{2}{\cup} e^{n+1}$, where e^{n+1} is attached to S^n by a map of degree 2,

$X_n(3) = S_n \underset{2^{q'+1}}{\cup} e^{n+1}$, where e^{n+1} is attached to S^n by a map of degree $2^{q'+1}$, $q' \geqq 1$,

$X_n(4) = (S^n \vee S^{n+1}) \underset{\eta, 2^q}{\cup} e^{n+2}$, where $(S^n \vee S^{n+1})$ is a union of S^n and S^{n+1} with a single common point, and e^{n+2} is attached to $(S^n \vee S^{n+1})$ by a map η and of degree 2^q over S^{n+1},

$X_n(5) = (S^n \vee S^{n+1}) \underset{\eta, 2^q}{\cup} e^{n+2} \underset{2}{\cup} e^{n+1}$, where e^{n+1} is attached to S^n by a map of degree 2,

$X_n(6) = (S^n \vee S^{n+1}) \underset{\eta, 2^q}{\cup} e^{n+2} \underset{2^{q'+1}}{\cup} e^{n+1}$, where e^{n+1} is attached to S^n by a map of degree $2^{q'+1}$, $q' \geqq 1$.

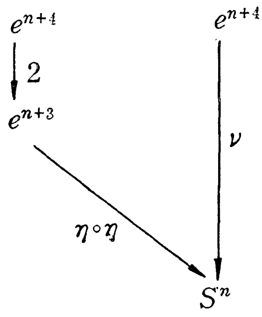
For such a complex X_n, we can construct by killing homotopy methods a CW-complex K_n, satisfy the conditions:

1) $\mathcal{K}(\pi, n) \supset K_n \supset X_n$,
2) $K_n^{n+2} = X_n$, and
3) $\pi_i(K_n) = 0$ for $n+2 < i$.

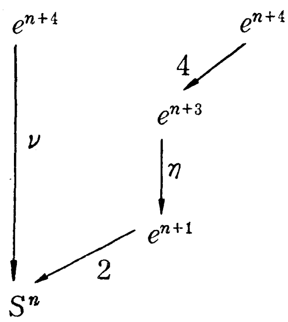
Then K_n is a complex of type $\mathcal{K}(\pi, n; k^{n+2}; G, n+1)$. From each $X_n(j)$ we obtain

$$K_n^{n+l}(j) \qquad (l = 1, 2, 3, \cdots)$$

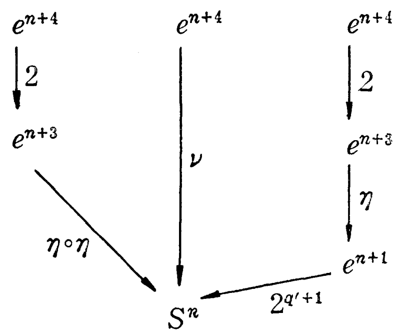
by step by step construction. For examples, $K_n^{n+4}(j)$, $j = 1, 2, \cdots, 6$ are given as follows.



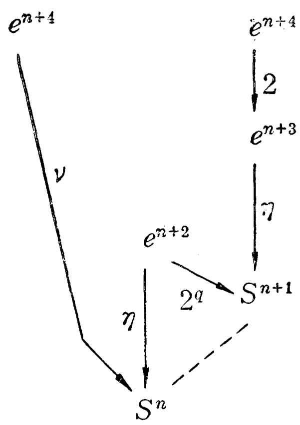
K_n^{n+4} (1)



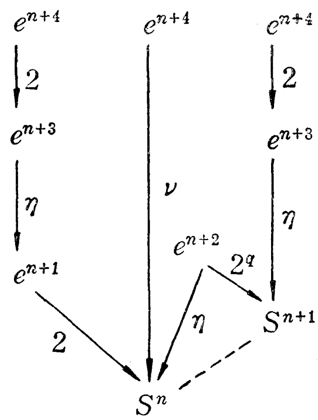
K_n^{n+4} (2)



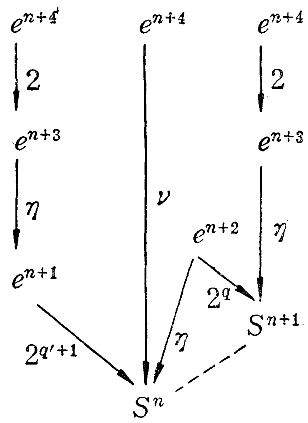
K_n^{n+4} (3)



K_n^{n+4} (4)



K_n^{n+4} (5)



K_n^{n+4} (6)

where $\cdots$ means union with a single common point.

By the construction, we have

$$A^i(j) = \lim H^{n+i}(K_n(j)\,;Z_2)\,.$$

From the cell structure of $K_n(j)$, we can obtain cohomological informations of $\mathcal{K}(\pi, n\,; k^{n+2}\,; G, n+1)$ in low dimensions. For examples, the Proposition 3 in § 3 is easily obtained from the aboves. We can also obtain the same relations of generators in $A^*(j)$.

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