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Remarks on the truth definition

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Let S be a mathematical theory and suppose that there is given a sequence $A_0(a), A_1(a), A_2(a), \cdots$ of all the formulae with a variable a. We define $g(A_i(a))$ by i. (g is a symbol outside of S.)

Tarski's truth theory ([**4**], [**5**]) shows that S is inconsistent, if S contains two formulae $E(a)$ and $Tr(a)$ satisfying the following conditions:

$$E(i), Tr(g(A_i(i))) \rightarrow \rightarrow E(i), Tr(g(A_i(i))) \qquad i=0,1,2,\cdots \tag{1}$$

$$A_i(a) \rightarrow Tr(i) \qquad i=0,1,2,\cdots \tag{2}$$

$$Tr(i) \rightarrow A_i(a) \qquad i=0,1,2,\cdots. \tag{3}$$

(In this paper, we use Gentzen's sequence developed in [**1**].)
Contradiction follows even in the case that (2) and (3) are ascertained to be satisfied only when they do not actually contain the variable a. In fact, such sequences mean

$$A \rightarrow Tr(g(A)) \tag{2*}$$

$$Tr(g(A)) \rightarrow A \tag{3*}$$

for all formulae A without free variable. Let $E(a)$ be the m-th formula $A_m(a)$, and consider the special case

$$E(m), Tr(g(A_m(m)) \rightarrow$$

$$\rightarrow E(m), Tr(g(A_m(m)))$$

of (1). Applying (2)* and (3)* to the formula $A_m(m)$, we obtain

$$E(m), A_m(m) \rightarrow$$

and

$$\rightarrow E(m), A_m(m).$$

These two sequences imply a contradiction, since $E(m)$ is $A_m(m)$.

Though (2) may be read "if $A_i(a)$ holds, then $Tr(i)$ also holds" and (3) may be read in an analogous manner, no contradiction may be derived if these

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colloquial expressions are formulated not in the form of axioms but in the schemata of inference:

$$(2)' \qquad \frac{\to A_i(a)}{\to Tr(i)} \qquad\qquad (3)' \qquad \frac{\to Tr(i)}{\to A_i(a)}.$$

In this paper we shall show that there are a great many consistent systems each of which contains two formulae $E(a)$ and $Tr(a)$, satisfies (1) and admitts (2)′ and (3)′.

Let S_0 be an arbitrary consistent system, which contains the theory of natural numbers ([**1**]) and does not contain predicate $E(a)$ or $Tr(a)$.

The system S_1 is called *E-Tr*-extension of S_0, if S_1 is obtained from S_0 by adding new predicates $E(a)$ and $Tr(a)$, axioms (1) and inferences (2)′, (3)′ under the presupposition that a sequence $A_0(a), A_1(a), A_2(a), \cdots$ of all the formulae with a variable a (which may contain new predicates E and Tr) is fixed and $g(A_i(a))$ is defined by i.

THEOREM. *If S_0 is consistent, then E-Tr-extension of S_0 is also consistent.*

PROOF. First we shall prove the consistency of the system S_2, which is obtained from S_0 by adding (1) and the following inferences (4);

$$(4) \qquad \frac{Tr(i_1), \cdots, Tr(i_n) \to Tr(i)}{Tr(i_1), \cdots, Tr(i_n) \to A_i(a)} \quad \text{and} \quad \frac{Tr(i_1), \cdots, Tr(i_n), Tr(i) \to}{Tr(i_1), \cdots, Tr(i_n), A_i(a) \to}$$

where $i, i_1, \cdots, i_n$ are integers and i is different from $i_1, \cdots, i_n$.

To prove this, we have only to prove that any sequence of the form

$$Tr(i_1), \cdots, Tr(i_n) \to Tr(i) \quad (i \neq i_1 \text{ and } \cdots \text{ and } i \neq i_n)$$

cannot be *provable from* (1) in S_0, which is easily proved by substitution of $a \neq i$ for $Tr(a)$.

Now we shall prove the following lemma.

LEMMA. *Let S_3 be a consistent extension of S_2 and $A_{i_0}(a)$ be provable in S_3. Then $Tr(i_0)$ is consistent with S_3.*

PROOF. We have only to prove that $Tr(i_0), \Gamma \to \Delta$ is provable in S_3, if $\Gamma \to \Delta$ is provable from $\to Tr(i_0)$ and S_3. If $\Gamma \to \Delta$ is a beginning sequence of S_2 or $\to Tr(i_0)$, then the lemma is clear. Here we have only to prove the lemma under the hypothesis that $Tr(i_0), \Pi \to \Lambda$ is provable in S_3 for any upper sequence $\Pi \to \Lambda$ of $\Gamma \to \Delta$. We must treat many cases but every case is trivial and the lemma is proved.

In virtue of this lemma, we see that (2)′ holds in the maximal consistent extension of S_2 and that the theorem holds.

Using the results in [**2**] or [**3**], we see easily the following corollary.

COROLLARY. *If S is consistent, then there exists a consistent extension $\tilde{S}$ of S, in which* (1), (2)′, (3)′ *hold.*

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