

Axioms of infinity of set theory

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Mahlo, in his penetrating papers [5], [6], proposed an axiom of set theory, assuring, roughly speaking, the existence of sets with very large cardinals. Recently Lévy [3], [4] gave an elegant equivalent axiom, which he called the axiom of strong infinity. Mahlo's axiom can be also stated as follows in making use of the concept of the super-complete model introduced by Shepherdson [8]. (We remind that a set a is called super-complete if and only if

$$\forall x \forall y (x \in a \wedge (y \subseteq x \vee y \in x) \rightarrow y \in a);$$

and that a set a is said to be a model of a set theory T , if and only if the theory holds, when all quantifiers of T are restricted on a .)

Let f be a function from sets to sets. A set a will be called a *fixed point* of f , if and only if

$$\forall x (x \in a \rightarrow f(x) \in a).$$

A class A will be called *dense*, if and only if for every function there exists a fixed point of this function which is an element of A . Then Mahlo's axiom means: The class of all the super-complete models of Bernays-Gödel's set theory is dense.

To go further, let us denote with M^0 the Bernays-Gödel's set theory. In adding Mahlo's axiom to M^0 , we obtain a new set theory M^1 , with 'much more set' than in M^0 . More generally, a stronger set theory M^{i+1} is obtained in adding the axiom: The class of all the super-complete models of M^i is dense, to M^i , $i=0, 1, 2, \dots$. These M^i will be called *Mahlo's set theories*.

Let $\tilde{\mathfrak{R}}(B)$ mean that B is the class of all the super-complete models of a Mahlo's set theory. $\mathfrak{R}(A)$ (for a class A) will mean

$$\exists B \exists x (\tilde{\mathfrak{R}}(B) \wedge \forall a (a \in B \wedge x \subseteq a \wedge x \in a \rightarrow a \in A)).$$

Then we have easily

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1. $\mathfrak{R}(0)$.
2. $\mathfrak{R}(G_0)$,
where G_0 is the class of all the super-complete models of Bernays-Gödel's set theory.
3. $\forall A \forall B (\mathfrak{R}(A) \wedge A \subseteq B \vdash \mathfrak{R}(B))$.
4. $\forall A \forall x (\mathfrak{R}(A) \vdash \mathfrak{R}((Kx)(a \in A \wedge x \subseteq a \wedge x \in a)))$,
where $(Kx)\mathfrak{A}(a)$ means the class of sets satisfying $\mathfrak{A}(a)$, i. e. $\{a \mid \mathfrak{A}(a)\}$ in the usual notation.
5. $\forall A (\mathfrak{R}(A) \vdash 'A \text{ is dense}')$.
6. If a closed formula \mathfrak{A} without \mathfrak{R} or $\tilde{\mathfrak{R}}$ is provable in this system, then the class of all the super-complete models of \mathfrak{A} satisfies \mathfrak{R} .

Now we can make correspond to each function f from sets to sets, a function F from sets to classes defined by

$$F(x) = (Kb)(f(x) \in b).$$

A set a is then a fixed point of f , if and only if

$$(*) \quad \forall x (x \in a \vdash a \in F(x)).$$

In the following a set a will be called a fixed point of a function F from sets to classes, if and only if $(*)$ holds. Using this notion, 5 can be strengthened into the following

- 5'. Let F be a function from sets to classes such that $\forall x (\mathfrak{R}(F(x)))$, and B the class of all the fixed points of F . Then we have $\mathfrak{R}(B)$. Symbolically:

$$\forall A (\forall x \mathfrak{R}(A''\{x\}) \vdash \mathfrak{R}((Kx)\forall x (x \in a \vdash a \in A''\{x\}))),$$

where $A''\{x\}$ means $F(x)$ (cf. [1]).

In adding a new predicate \mathfrak{R} satisfying 1, 2, 3, 4, 5', 6 to Bernays-Gödel's set theory, we obtain a theory $T^0(\mathfrak{R})$, which is stronger than any Mahlo's theory. We can now strengthen this in doing as follows.

Let a be a super-complete set and k a set of subsets of a . The pair $\langle ka \rangle$ is called a super-complete model of $T^0(\mathfrak{R})$, if and only if the theory holds when every $\mathfrak{R}(\ast)$ in $T^0(\mathfrak{R})$ is replaced by $\ast \in k$ and all quantifiers in $T^0(\mathfrak{R})$ are restricted on a . The "Mahlo's axiom for $T^0(\mathfrak{R})$ " will mean:

$$(Kx)\exists k (\langle ka \rangle \text{ is a super-complete model of } T(\mathfrak{R})) \text{ is dense.}$$

In adding this axiom to $T^0(\mathfrak{R})$, we obtain $T^1(\mathfrak{R})$ and in repeating this, $T^i(\mathfrak{R})$, $i = 1, 2, \dots$. Thus a new predicate \mathfrak{G}_1 , and a set theory $T^0(\mathfrak{R}, \mathfrak{G}_1)$ will be obtained from $T^0(\mathfrak{R})$, just as we have obtained \mathfrak{R} and $T^0(\mathfrak{R})$ from M^0 . In $T^0(\mathfrak{R}, \mathfrak{G}_1)$ we have axioms corresponding to axioms 1-6, 5' in $T^0(\mathfrak{R})$. In particular 5' becomes

$$\forall A (\forall x \mathfrak{G}_1(A''\{x\}) \vdash \mathfrak{G}_1((K\langle ba \rangle)\forall x (x \in a \vdash a \in \mathfrak{D}(A''\{x\}))),$$

where \mathfrak{D} means "domain" (cf. [1]).

Beginning again from $T^0(\mathfrak{R}, \mathfrak{G}_1)$ we can define $T^i(\mathfrak{R}, \mathfrak{G}_1)$ $i=1, 2, \dots$, a new predicate \mathfrak{G}_2 , a new set theory $T^i(\mathfrak{R}, \mathfrak{G}_1, \mathfrak{G}_2)$ and so on.

In the next §1, we shall give a system of axioms of a set theory, which is stronger than all these theories $T^i(\mathfrak{R}, \mathfrak{G}_1, \mathfrak{G}_2, \dots)$. In §2 we give some properties of this system. It will be shown that our system contains 'infinite induction' as an inference scheme. It has a role of an axiom of infinity, because it assures the existence of the super-complete model of the system without the infinite induction. In §3 we discuss the questions about relative consistency of this system with a certain weaker system. We have not treated the problem of relative consistency of Gödel's $V=L$ with our system, because this seems to be very difficult. Finally we give in §4 still another system of axioms of the same nature.

It seems to the author that our system cannot be strengthened any more without introducing metamathematical concepts like the Gödel number into the system. He is not quite sure of the consistency of the system.

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§1. Axioms and inferences.

We use the notations and notions in [1]. Other than basic symbols in [1] we use basic symbols K and \mathfrak{G} .

The axioms fall into six groups. Groups A, D and E are the same as in [1]. Instead of Groups B and C we use the following Groups B' and C'.

Group B'. $\forall X_1 \dots \forall X_m \forall x_1 \dots \forall x_n \forall x (x \in K(u) \mathfrak{A}(u) \vdash \mathfrak{A}(x))$.

We shall consider the notations $E, \cap, -, \mathfrak{D}, \mathfrak{B}, \mathfrak{C}nv, \mathfrak{C}nv_2, \mathfrak{C}nv_3$ as the adequate abbreviations expressed by basic symbols. (We use \cap instead of \cdot in [1], because we have to use \cdot as the multiplication of integer. $\mathfrak{B}(A)$ expresses $V \times A$ in [1].)

Group C' consists of C1, C2, C3 and C'4.

C'4. $\forall A \forall x \exists y \forall z (z \in y \vdash z \in A \wedge z \in x)$.

C'4 is Zermelo's Aussonderungssaxiom and is weaker than C4 (axiom of replacement). C4 follows, however, from C'4 and other axioms.

Group F.

1. $\neg \mathfrak{G}(0)$.
2. $\mathfrak{G}(\mathfrak{B}(H_0))$, where H_0 is defined to be an abbreviation of $(Ka)(\forall x \forall y (x \in a \wedge (y \subseteq x \vee y \in x) \vdash y \in a) \wedge a \neq 0)$.
3. $\forall A \forall B (\mathfrak{G}(A) \wedge A \subseteq B \vdash \mathfrak{G}(B))$.
4. $\forall A \forall x (\mathfrak{G}((A) \vdash \mathfrak{G}(K \langle ba \rangle) (\langle ba \rangle \in A \wedge x \subseteq a \wedge x \in a)))$, where $(K \langle ba \rangle) \mathfrak{A}(b, a)$ is an abbreviation of $(Ku) \exists b \exists a (u = \langle ba \rangle \wedge \mathfrak{A}(b, a))$.

5. $\forall A(\mathfrak{G}(A) \vdash \mathfrak{G}((K \langle ba \rangle)(a \times (\mathfrak{D}(A) \cap a) \in b)))$.
6. $\forall A(\forall x \mathfrak{G}(A''\{x\}) \vdash \mathfrak{G}((K \langle ba \rangle) \forall x(x \in a \vdash \langle ba \rangle \in A''\{x\})))$.

Other than inferences in [1] we use infinite induction and \mathfrak{G} -inference. In order to state \mathfrak{G} -inference, we shall first define some notations. Let U be a formula or a term and a be a set variable not contained in U . Then we define U^a from U by transforming all the quantifiers and $\mathfrak{M}(\)$ in U as follows: $\forall A, \exists B, (Kx), \mathfrak{M}(T)$ to $\forall A(A \subseteq a \vdash \)$, $\exists B(B \subseteq a \wedge \)$, $(Kx)(x \in a \wedge \)$, $T^a \in a$ respectively.

Though deviating from [1], we define $a < b$ to be $a \subseteq b \wedge a \in b$. Then it is easily seen that $(\forall x \mathfrak{F}(x))^a$ is $\forall x(x < a \vdash \mathfrak{F}^a(x))$.

(b/\mathfrak{G}) (substitution of b for \mathfrak{G}) is defined as follows. Let U be a formula or a term. $U(b/\mathfrak{G})$ is obtained from U by transforming all the forms $\mathfrak{G}(T)$ in U to $T \in b$.

\mathfrak{G} -inference. If \mathfrak{A} is a closed formula which holds, then

$$\mathfrak{G}((K \langle ba \rangle)(\mathfrak{A}^a(b/\mathfrak{G}))) \text{ also holds.}$$

Infinite induction. If $\mathfrak{A}(i)$ holds for every actually given integer

$$i(i=0, \{0\}, \{0, \{0\}\}, \dots), \text{ then } \forall i(i \in \omega \vdash \mathfrak{A}(i)) \text{ holds.}$$

REMARK. The following axiom F5' has a nicer form than F5:

$$F5'. \forall A(\mathfrak{G}(A) \vdash \mathfrak{G}((K \langle ba \rangle)(A \cap a \ni b))).$$

F5 follows from F5' (and other axioms) taking $V \times \mathfrak{D}(A)$ in place of A .

But the author is afraid if it is "too strong". It is to be noted that the axiom of the form

$$A \vdash \mathfrak{G}((K \langle ba \rangle)(\mathfrak{A}^a(b/\mathfrak{G})))$$

will lead to a contradiction by similar arguments as in [11] with the aid of 'Gödel number'. (See below.)

§ 2. Some properties of the system.

PROPOSITION 1. $\mathfrak{G}(V)$.

PROOF. This follows from F2 and F3.

PROPOSITION 2.

$$\forall x(x \in c \vdash \mathfrak{G}(A''\{x\})) \rightarrow \mathfrak{G}((K \langle ba \rangle) \forall x(x \in c \vdash \langle ba \rangle \in A''\{x\})).$$

PROOF. We define B by the following:

$$x \in c \rightarrow B''\{x\} = A''\{x\},$$

$$x \notin c \rightarrow B''\{x\} = V.$$

We see clearly $\forall x \mathfrak{G}(B''\{x\})$. By F6 and F4 we have

$$\mathfrak{G}((K \langle ba \rangle)(\forall x(x \in a \vdash \langle ba \rangle \in B''\{x\}) \wedge c \subseteq a \wedge c \in a)),$$

therefore, $\mathfrak{G}((K\langle ba \rangle)(\forall x(x \in c \vdash \langle ba \rangle \in B''\{x\}) \wedge c \subseteq a \wedge c \in a))$. In virtue of this and F3 we have the proposition.

As a special case of Proposition 2 we have

PROPOSITION 3. $\mathfrak{G}(A), \mathfrak{G}(B) \rightarrow \mathfrak{G}(A \cap B)$.

DEFINITION. Let $\mathfrak{I}(A)$ be a term. $\mathfrak{E}(A, \mathfrak{I}, \alpha)$ is defined to be

$$(Ka) \forall B(B''\{0\} = A \wedge \forall \beta(\beta < \alpha \vdash B''\{\beta+1\} = \mathfrak{I}(B\{\beta\}))$$

$$\wedge \forall \beta(\beta \leq \alpha \wedge \beta \in K_{II} \vdash B''\{\beta\} = (Kx) \forall \gamma(\gamma < \beta \vdash x \in B''\{\gamma\})) \vdash a \in B''\{\alpha\}.$$

The following proposition is familiar.

PROPOSITION 4. $\mathfrak{E}(A, \mathfrak{I}, 0) = A$

$$\mathfrak{E}(A, \mathfrak{I}, \alpha+1) = \mathfrak{I}(\mathfrak{E}(A, \mathfrak{I}, \alpha))$$

$$\alpha \in K_{II} \rightarrow \mathfrak{E}(A, \mathfrak{I}, \alpha) = (Ka) \forall \beta(\beta < \alpha \vdash a \in \mathfrak{E}(A, \mathfrak{I}, \beta)).$$

We have easily

PROPOSITION 5. $\mathfrak{G}(A), \forall X(\mathfrak{G}(X) \vdash \mathfrak{G}(\mathfrak{I}(X))) \rightarrow \forall \alpha(\mathfrak{G}(\mathfrak{E}(A, \mathfrak{I}, \alpha)))$.

PROPOSITION 6. $\mathfrak{G}(A), \forall X(\mathfrak{G}(X) \vdash \mathfrak{G}(\mathfrak{I}(X)))$

$$\rightarrow \mathfrak{G}((K\langle ba \rangle) \forall \alpha(\alpha \in a \vdash \langle ba \rangle \in \mathfrak{E}(A, \mathfrak{I}, \alpha))).$$

PROOF. We define B as follows:

$$B''\{x\} = \mathfrak{E}(A, \mathfrak{I}, x) \quad \text{if } x \in On,$$

$$B''\{x\} = V \quad \text{otherwise.}$$

Then we have the proposition by F6.

DEFINITION. $\mathfrak{G}^0(A)$ is an abbreviation of $\mathfrak{G}(\mathfrak{B}(A))$.

The following proposition is clear.

PROPOSITION 7.

1. $\neg \mathfrak{G}^0(0)$.
 2. $\mathfrak{G}^0(H_0)$.
 3. $\forall A \forall B(\mathfrak{G}^0(A) \wedge A \subseteq B \vdash \mathfrak{G}^0(B))$.
 4. $\forall A \forall x(\mathfrak{G}^0(A) \vdash \mathfrak{G}^0((Ka)(a \in A \wedge x \subseteq a \wedge x \in a)))$.
 5. $\forall A(\forall x \mathfrak{G}^0(A\{x\}) \rightarrow \mathfrak{G}^0((Ka) \forall x(x \in a \vdash a \in A''\{x\})))$.
 6. $\forall x(x \in c \vdash \mathfrak{G}^0(A''\{x\})) \rightarrow \mathfrak{G}^0((Ka) \forall x(x \in c \vdash a \in A''\{x\}))$.
 7. $\mathfrak{G}^0(A), \forall X(\mathfrak{G}^0(X) \vdash \mathfrak{G}^0(\mathfrak{I}(X))) \rightarrow \mathfrak{G}^0((Ka) \forall \alpha(\alpha \in a \vdash a \in \mathfrak{E}(A, \mathfrak{I}, \alpha)))$.
 8. $\mathfrak{G}^0(A), \forall X(\mathfrak{G}^0(X) \vdash \mathfrak{G}^0(\mathfrak{I}(X))) \rightarrow \forall \alpha(\mathfrak{G}^0(\mathfrak{E}(A, \mathfrak{I}, \alpha)))$.
 9. $\mathfrak{G}^0(A), \forall X(\mathfrak{G}^0(X) \vdash \mathfrak{G}^0((Ka)\mathfrak{A}(a, X))) \rightarrow \mathfrak{G}^0(Ka) \forall \alpha(\alpha \in a \vdash \mathfrak{A}(a, \mathfrak{E}(A, \mathfrak{I}, \alpha)))$,
- where $\mathfrak{I}(X)$ is an abbreviation of $(Ka)\mathfrak{A}(a, X)$.

PROPOSITION 8. $\mathfrak{G}^0(A), \mathfrak{Fnc}(F) \rightarrow \mathfrak{G}^0((Ka)(a \in A \wedge \forall x(x \in a \vdash F'x \in a)))$.

PROOF. We define B by the following:

$$\forall x(B''\{x\} = (Ka)(a \in A \wedge F'x \in a)).$$

By Proposition 7 we have $\forall x \mathfrak{G}^0(B''\{x\})$. Hence follows the proposition from Proposition 7.

PROPOSITION 9. $\mathfrak{G}^0(A), \forall x(x \in a \vdash \mathfrak{Fnc}(B''\{x\}))$

$$\rightarrow \mathfrak{G}^0((Kb)(b \in A \wedge \forall x \forall y(x \in a \wedge y \in b \vdash (B''\{x\})'y \in b))).$$

PROOF. The proposition follows from Proposition 7 and Proposition 8.

We have axiom of replacement, that is,

PROPOSITION 10. $\forall x \forall A (\text{Un}(A) \vdash \exists y \forall u (u \in y \vdash \exists v (v \in x \wedge \langle uv \rangle \in A)))$.

PROOF. In virtue of C'4, we have only to prove

$$\forall x \forall A (\text{Un}(A) \vdash \exists y \forall v (v \in x \vdash A'v \in y)),$$

which is clear from Proposition 8.

DEFINITION. G_0 is an abbreviation of $H_0 \cap (Ka)\mathfrak{A}^a$, where \mathfrak{A} is the conjunction of the axiom A4 and axioms of Group C. Clearly we have $\mathfrak{G}^0(G_0)$.

The following proposition is clear.

PROPOSITION 11.

1. $b \in a, c \in a \rightarrow (\{bc\})^a = \{bc\}$.
2. $a \in G_0, b \in a, c \in a \rightarrow (\langle bc \rangle)^a = \langle bc \rangle$.
3. $A \subseteq a, a \in G_0 \rightarrow (\mathfrak{B}(A))^a = a \times A$,

where a, b, c , and A are considered as variables.

PROPOSITION 12. If $\mathfrak{B}(A)$ contains no \mathfrak{G} -symbol, $\forall X \mathfrak{B}(X)$ is closed and $\forall A (\mathfrak{G}^0(A) \vdash \mathfrak{B}(A))$ holds, then $\forall A (\mathfrak{G}^0(A) \vdash \mathfrak{G}^0((Ka)\mathfrak{B}^a(a \cap A)))$ holds.

PROOF. From the hypothesis of the proposition and \mathfrak{G} -inference follows $\mathfrak{G}(C)$, where C is an abbreviation of

$$(K\langle ba \rangle)(a \in G_0 \wedge \forall x (x \subseteq a \wedge a \times x \in b \vdash \mathfrak{B}^a(x))).$$

B is defined to be

$$(K\langle ba \rangle)(a \times (A \cap a) \in b).$$

We have only to prove

$$\langle ba \rangle \in C, a \in A, \langle ba \rangle \in B \rightarrow \mathfrak{B}^a(a \cap A),$$

which is clear.

DEFINITION. $Mo(A)$ is an abbreviation of

$$\forall F (\mathfrak{Fnc}(F) \vdash \exists x (x \in A \wedge x \in H_0 \wedge \forall y (y \in x \vdash F'y \in x))),$$

which is called *Mahlo's formula* for A and is essentially due to Mahlo ([5] and [6]).

$M(a, A)$ is an abbreviation of

$$\forall x (x \subseteq a \wedge \mathfrak{Fnc}(x) \vdash \exists y (y \in a \cap A \wedge y \in H_0 \wedge \forall z (z \in y \vdash x'z \in y))).$$

The following proposition is clear.

PROPOSITION 13.

1. $a \in G_0 \rightarrow (Mo(A)) \vdash M(a, A)$. (A is a variable.)
2. $\neg Mo(0)$.
3. $\forall A \forall x (Mo(A) \vdash Mo((Ka)(a \in A \wedge x < a))$.
4. $\forall A \forall B (Mo(A) \wedge A \subseteq B \vdash Mo(B))$.

$Mo(A)$ has similar properties with $\mathfrak{G}^0(A)$.

PROPOSITION 14. $Mo(A), \mathfrak{Fnc}(F) \rightarrow Mo((Ka)(a \in A \wedge \forall x(x \in a \vdash F'x \in a)))$.

PROOF. $Mo((Ka)(a \in A \wedge \forall x(x \in a \vdash F'x \in a)))$ is equivalent to

$$\forall H(\mathfrak{Fnc}(H) \vdash \exists y(y \in (Ka)(a \in A \wedge \forall x(x \in a \vdash F'x \in a) \\ \wedge y \in H_0 \wedge \forall z(z \in y \vdash H'z \in y))).$$

We have, therefore, only to prove

$$Mo(A), \mathfrak{Fnc}(F), \mathfrak{Fnc}(H) \rightarrow \exists y(y \in A \wedge y \in H_0 \\ \wedge \forall x(x \in y \vdash F'x \in y) \wedge \forall z(z \in y \vdash H'z \in y)),$$

which is clear by introducing new function F_1 as follows:

$$F_1'x = \{F'x, H'x\}.$$

Moreover, we have the following proposition.

PROPOSITION 15. $\mathfrak{G}^0(A) \rightarrow Mo(A)$.

PROOF. The proposition is clear by Proposition 8.

However, the following example shows the difference between $\mathfrak{G}^0(A)$ and $Mo(A)$. We set

$$A = G_0$$

and

$$B = (Ka)\exists x(\mathfrak{Fnc}(x) \wedge \forall i\forall j(i \in \omega \wedge j \in \omega \wedge i < j \vdash \overline{x'i} < \overline{x'j}) \wedge a = \bigcup_{i \in \omega} x'i).$$

Then we see $Mo(A)$, $Mo(B)$ and $A \cap B = 0$.

PROPOSITION 16. $\mathfrak{G}^0(A) \rightarrow \mathfrak{G}^0((Ka)M(a, A))$.

PROOF. This follows from Proposition 12 and 13.

PROPOSITION 17. $\mathfrak{G}^0(A) \rightarrow \mathfrak{G}^0((Ka)\forall\alpha(\alpha \in a \vdash M(a, \mathfrak{E}(A, \mathfrak{I}, \alpha))))$, where $\mathfrak{I}(X)$ is an abbreviation of $(Ka)M(a, X)$.

PROOF. This follows from Propositions 15 and 7.

A term $T(A_1, \dots, A_n)$ is called elementary, if it is constructed just by E , \cap , $-$, \mathfrak{D} , \mathfrak{B} , \mathfrak{Cnb} , \mathfrak{Cnb}_2 , \mathfrak{Cnb}_3 , A_1, \dots, A_n .

If a term T is elementary and closed, then we define Gödel number $g(T)$ of T as follows:

1. $g(E) = 3$.
2. $g(T_1 \cap T_2) = 2 \cdot 3^{g(T_1)} \cdot 5^{g(T_2)}$.
3. $g(-T) = 2^2 \cdot 3^{g(T)}$.
4. $g(\mathfrak{D}(T)) = 2^3 \cdot 3^{g(T)}$.
5. $g(\mathfrak{B}(T)) = 2^4 \cdot 3^{g(T)}$.
6. $g(\mathfrak{Cnb}(T)) = 2^5 \cdot 3^{g(T)}$.
7. $g(\mathfrak{Cnb}_2(T)) = 2^6 \cdot 3^{g(T)}$.
8. $g(\mathfrak{Cnb}_3(T)) = 2^7 \cdot 3^{g(T)}$.

By the familiar method we can define a term $\mathfrak{R}(A)_{\mathfrak{I}}$ satisfying the following conditions:

1. $\mathfrak{N}(3) = E$.
2. $a \in \omega, b \in \omega \rightarrow \mathfrak{N}(2 \cdot 3^a \cdot 5^b) = \mathfrak{N}(a) \cap \mathfrak{N}(b)$.
3. $a \in \omega \rightarrow \mathfrak{N}(2^2 \cdot 3^a) = \neg \mathfrak{N}(a)$.
4. $a \in \omega \rightarrow \mathfrak{N}(2^3 \cdot 3^a) = \mathfrak{D}(\mathfrak{N}(a))$.
5. $a \in \omega \rightarrow \mathfrak{N}(2^4 \cdot 3^a) = \mathfrak{B}(\mathfrak{N}(a))$.
6. $a \in \omega \rightarrow \mathfrak{N}(2^5 \cdot 3^a) = \mathfrak{Cnb}(\mathfrak{N}(a))$.
7. $a \in \omega \rightarrow \mathfrak{N}(2^6 \cdot 3^a) = \mathfrak{Cnb}_2(\mathfrak{N}(a))$.
8. $a \in \omega \rightarrow \mathfrak{N}(2^7 \cdot 3^a) = \mathfrak{Cnb}_3(\mathfrak{N}(a))$.

According to [1], a sequence of natural numbers n_0, n_1, n_2, \dots are called “definable (in this system)” if and only if there exists a term $T(i)$ satisfying the following condition :

1. $T(i) = n_i$ for every $i (i = 0, 1, 2, \dots)$.
2. $T(0)$ has no free variable.

A sequence $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2, \dots$ of closed formulas is called “definable” if and only if the sequence of usual Gödel numbers of $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2, \dots$ (not in the sense $g(T)$) is definable.

We see clearly that if the sequence $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2, \dots$ is definable, then the sequence

$$g((K \langle ba \rangle)(\mathfrak{A}_i^a(b/\mathfrak{G}) \wedge a \in G_0)) \quad (i = 0, 1, 2, \dots)$$

is also definable.

Let $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2, \dots$ be a definable sequence of provable formulas. Then there exists a term $T(i)$ such that

$$T(i) = g((K \langle ba \rangle)(\mathfrak{A}_i^a(b/\mathfrak{G}) \wedge a \in G_0)) \quad (i = 0, 1, 2, \dots).$$

Then $\forall i (i \in \omega \vdash \mathfrak{G}(\mathfrak{N}(T(i))))$. Therefore we have

$$\mathfrak{G}((K \langle ba \rangle) \forall i (i \in \omega \vdash \langle ba \rangle \in \mathfrak{N}(T(i)))) ,$$

which we set as $\mathfrak{G}(A)$. Here A means the class of all the super-complete inner models, in which $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2, \dots$ and Bernays-Gödel’s set theory hold.

Let $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2, \dots$ be a definable sequence of closed provable formulas and $\mathfrak{B}_0, \mathfrak{B}_1, \mathfrak{B}_2, \dots$ be a definable sequence of all the axioms (which contains axioms of equality other than A, B’, C’, D, E and F). Then we define $\mathfrak{C}_0, \mathfrak{C}_1, \mathfrak{C}_2, \dots$ as follows :

$$\mathfrak{C}_0 \text{ is } \mathfrak{A}_0 \wedge \mathfrak{B}_0 .$$

$$\mathfrak{C}_{2n+1} \text{ is } \mathfrak{G}((K \langle ba \rangle)(\mathfrak{C}_{2n}^a(b/\mathfrak{G}) \wedge a \in G_0)) \quad (n = 0, 1, 2, \dots) .$$

$$\mathfrak{C}_{2n+2} \text{ is } \mathfrak{C}_0 \wedge \dots \wedge \mathfrak{C}_{2n+1} \wedge \mathfrak{A}_{n+1} \wedge \mathfrak{B}_{n+1} \quad (n = 0, 1, 2, \dots) .$$

The sequence $\mathfrak{C}_0, \mathfrak{C}_1, \mathfrak{C}_2, \dots$ is clearly definable and is called a derived sequence of $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2, \dots$.

The following proposition is proved by the familiar method.

PROPOSITION 18. *If $\mathfrak{A} \rightarrow \mathfrak{B}$ is closed and is probable without infinite induc-*

tion or \mathfrak{G} -inference or axioms A, B', C', D, E, F, then also

$$\exists x(x < a), \mathfrak{A}^a(b/\mathfrak{G}) \rightarrow \mathfrak{B}^a(b/\mathfrak{G})$$

is provable without infinite induction or \mathfrak{G} -inference or axioms A, B', C', D, E, F.

PROPOSITION 19. Let $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2, \dots$ be a definable sequence of closed probable formulas and $\mathfrak{C}_0, \mathfrak{C}_1, \mathfrak{C}_2, \dots$ be a derived sequence of $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2, \dots$. If \mathfrak{B} is provable from $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2, \dots$, axioms of equality, A, B', C', D, E, F and \mathfrak{G} -inference (but without infinite induction), then there exists \mathfrak{C}_n such that $\mathfrak{C}_n \rightarrow \mathfrak{B}$ is provable without infinite induction or \mathfrak{G} -inference, or A, B', C', D, E, F.

PROOF. We prove this by induction on the number of inferences to prove \mathfrak{B} . Clearly we have only to prove this in the case where \mathfrak{B} is of the form

$$\mathfrak{G}((K \langle ba \rangle) \mathfrak{D}^a(b/\mathfrak{G}))$$

and there exists \mathfrak{C}_m such that $\mathfrak{C}_m \rightarrow \mathfrak{D}$ is provable without infinite induction or \mathfrak{G} -inference, or A, B', C', D, E, F. By Proposition 18 we see easily that

$$(K \langle ba \rangle) \mathfrak{D}^a(b/\mathfrak{G}) \cong (K \langle ba \rangle) (\mathfrak{C}_m^a(b/\mathfrak{G}) \wedge a \in G_0)$$

is provable without infinite induction or \mathfrak{G} -inference or A, B', C', D, E, F. Hence follows the proposition.

§ 3. Inversion of \mathfrak{G} -inference.

In this section we shall consider the following inference as a basic inference:

Inversion of \mathfrak{G} -inference. If \mathfrak{A} is a closed formula and $\mathfrak{G}((K \langle ba \rangle) (\mathfrak{A}^a(b/\mathfrak{G})))$ holds, then \mathfrak{A} also holds.

First we shall reduce axioms C'4, D and E to the following weaker axioms C4', D' and E'.

$$C4'. \quad \forall x \forall y \exists z \forall u (u \in z \rightarrow u \in x \wedge u \in y).$$

$$D'. \quad \forall x (\neg \mathfrak{G}_m(x) \rightarrow \exists y (y \in x \wedge \mathfrak{G}_x(y, x))).$$

$$E'. \quad \forall x (\forall u (u \in x \rightarrow u \neq 0) \wedge \forall u \forall v (u \in x \wedge v \in x \wedge u \neq v \rightarrow \mathfrak{G}_x(u, v)) \\ \rightarrow \exists y \forall z (z \in x \rightarrow \exists ! u (u \in z \cap y))).$$

PROPOSITION 20. C'4 to can be reduced C4' under inversion of \mathfrak{G} -inference.

PROOF. We have only to prove $\mathfrak{G}^0(B)$, where B is

$$(Ka) \forall z \forall x (z \subseteq a \wedge x < a \rightarrow \exists y (y < a \wedge \forall u (u < a \rightarrow (u \in y \rightarrow u \in x \wedge u \in z)))).$$

Clearly $H_0 \subseteq B$ under C4' therefore we have the proposition.

PROPOSITION 21. D can be reduced to D' under inversion of \mathfrak{G} -inference.

PROOF. We have only to prove $\mathfrak{G}^0(B)$, where B is

$$(Ka) \forall z \forall x (z \subseteq a \wedge x < a \wedge x \in z \rightarrow \exists y (y < a \wedge y \in z \wedge \forall u (u < a \rightarrow \neg (u \in z \wedge u \in y)))).$$

Clearly $H_0 \subseteq B$ follows from D', therefore we have the proposition.

PROPOSITION 22. Axioms E can be reduced to E' under inversion of \mathfrak{G} -inference.

Now we shall reduce Group B' to Group B . To the end, we use the method of [10]. As we use Gentzen's sequence in [10], we shall explain notations. We say 'a closed formula \mathfrak{A} is an axiom of a system S ', if and only if ' $\rightarrow\mathfrak{A}$ is a beginning sequence of S '.

The system S_0 is obtained from our original system by adding the inversion of \mathfrak{G} -inference.

Group B'' is defined by

$$\forall X_1 \dots \forall X_m \forall x_1 \dots \forall x_n \exists Y \forall x (x \in Y \rightarrow \mathfrak{A}(x)).$$

S_1 is obtained from S by replacing Group B' by Group B'' and all the axioms and inferences containing K other than Group B' by an equivalent axiom without K , e. g.

$$\mathfrak{G}((K \langle ba \rangle) \mathfrak{A}^a(b/\mathfrak{G}))$$

is replaced by

$$\forall X (\forall u (u \in X \rightarrow \exists b \exists a (u = \langle ba \rangle \wedge \mathfrak{A}^a(b/\mathfrak{G}))) \rightarrow \mathfrak{G}(X)),$$

which we shall denote by $\rho(\mathfrak{A})$.

Though our system contains infinite induction, the method of [10] is still valid for our system. $((Ku)\mathfrak{A}(u))^*$ is defined as $\{X\} \forall x (x \in X \rightarrow \mathfrak{A}^*(x))$, while other *-operations are defined similarly as in [10]. Then we have the following

LEMMA. Let \mathfrak{A} be a formula and T a term with only free variable A_1, \dots, A_n . Then the following sequences are provable simultaneously in S_1 :

$$a \in H_0, A_1 \subseteq a, \dots, A_n \subseteq a \rightarrow \mathfrak{A}^{*a}(b/\mathfrak{G}) \rightarrow (\mathfrak{A}^a(b/\mathfrak{G}))^*;$$

$$a \in H_0, A_1 \subseteq a, \dots, A_n \subseteq a \rightarrow A \subseteq a \wedge T^{*a}(b/\mathfrak{G})(A) \rightarrow (T^a(b/\mathfrak{G}))^*(A).$$

Propositions 1-5 of [10] can be proved for our system by the literally same proof and Proposition 6 of [10] can be proved for our system as follows:

PROPOSITION 6 OF [10], If $\Gamma \rightarrow \Delta$ is provable in S_0 , then $\Gamma^* \rightarrow \Delta^*$ is provable in S_1 .

PROOF. Suppose $\Gamma \rightarrow \Delta$ be a beginning sequence. Then the proposition is clear. Therefore we have only to prove the proposition under the hypothesis that $\Pi^* \rightarrow A^*$ is provable in S_1 for every upper sequence $\Pi \rightarrow A$ of $\Gamma \rightarrow \Delta$. We have only to consider the following cases:

1)
$$\frac{\rightarrow \mathfrak{A}}{\rightarrow \mathfrak{G}((K \langle ba \rangle) \mathfrak{A}^a(b/\mathfrak{G}))}$$

where $\Gamma \rightarrow \Delta$ is $\rightarrow \mathfrak{G}((K \langle ba \rangle) \mathfrak{A}^a(b/\mathfrak{G}))$. By the hypothesis we have $\rightarrow \mathfrak{A}^*$ therefore we have

$$\rightarrow \forall X (\forall u (u \in X \rightarrow \exists b \exists a (u = \langle ba \rangle \wedge \mathfrak{A}^{*a}(b/\mathfrak{G}))) \rightarrow \mathfrak{G}(X)),$$

whence follows the proposition by the preceding Lemma.

$$2) \quad \frac{\rightarrow \mathfrak{G}(\langle K \langle ba \rangle \rangle \mathfrak{A}^a(b/\mathfrak{G}))}{\rightarrow \mathfrak{A}}$$

where $\Gamma \rightarrow \Delta$ is $\rightarrow \mathfrak{A}$. This case is proved by the same way as in 1).

$$3) \quad \frac{\rightarrow \mathfrak{A}(0) \quad \rightarrow \mathfrak{A}(1) \quad \rightarrow \mathfrak{A}(2) \quad \dots}{\rightarrow \forall x(x \in \omega \vdash \mathfrak{A}(x))}$$

where $\Gamma \rightarrow \Delta$ is $\rightarrow \forall x(x \in \omega \vdash \mathfrak{A}(x))$. By the hypothesis the following sequences are provable in S_1 :

$$\rightarrow \mathfrak{A}^*(0), \quad \rightarrow \mathfrak{A}^*(1), \quad \rightarrow \mathfrak{A}^*(2), \quad \dots,$$

therefore we have $\rightarrow \forall x(x \in \omega \vdash \mathfrak{A}^*(x))$, whence follows $\rightarrow (\forall x(x \in \omega \vdash \mathfrak{A}(x)))^*$.

Therefore we can reduce S_0 to S_1 . Now we shall prove that Group B'' can be reduced to Group B. To the end, we have only to prove $\forall u(u \in A \vdash \mathfrak{A}(u)) \vdash \mathfrak{G}(A)$, where $\mathfrak{A}(u)$ is

$$\begin{aligned} \exists b \exists a(u = \langle ba \rangle \wedge \forall z_1 \dots \forall z_m \forall x_1 \dots \forall x_n (z_1 \subseteq a \wedge \dots \wedge z_n \subseteq a \wedge x_1 < a \wedge \dots \wedge x_n < a \\ \vdash \exists y(y \subseteq a \wedge \forall x(x < a \vdash (x \in y \vdash \mathfrak{B}(x)))))) \end{aligned}$$

and \mathfrak{B} does not contain \mathfrak{G} . Clearly we have only to prove $\mathfrak{B}(H_0) \subseteq A$, which follows from Group B and the other axioms (mainly C'4).

Now, we shall reduce the inversion of \mathfrak{G} -inference. S_2 is obtained from S_1 by excluding inversion of \mathfrak{G} -inference.

First we shall prove the following lemma.

LEMMA. *Let Δ_0 be a system of axioms and consistent with S_2 . If $\rho(\mathfrak{A})$ is provable from Δ_0 in S_2 , then \mathfrak{A} and Δ_0 are consistent from \mathfrak{A} and Δ_0 are consistent with S_2 .*

PROOF. We have only to prove that if $\Gamma \rightarrow \Delta$ is provable from \mathfrak{A} and Δ_0 in S_2 , then $\mathfrak{A}, \Gamma \rightarrow \Delta$ is provable from Δ_0 in S_2 .

If $\Gamma \rightarrow \Delta$ is $\rightarrow \mathfrak{A}$ or $\rightarrow \mathfrak{B}$, where \mathfrak{B} is an axiom of Δ_0 , or a beginning sequence of S_2 , then $\mathfrak{A}, \Gamma \rightarrow \Delta$ is clearly provable from Δ_0 in S_2 .

Therefore we have only to prove that $\mathfrak{A}, \Gamma \rightarrow \Delta$ is provable from Δ_0 in S_2 under the hypothesis that $\mathfrak{A}, \Pi \rightarrow A$ is provable from Δ_0 in S_2 for every upper sequence $\Pi \rightarrow A$ of $\Gamma \rightarrow \Delta$.

We have only to prove the case where $\Pi \rightarrow A$ is $\rightarrow \mathfrak{B}$ and $\Gamma \rightarrow \Delta$ is $\rightarrow \rho(\mathfrak{B})$.

From the hypothesis $\rightarrow \rho(\mathfrak{A} \vdash \mathfrak{B})$ and $\rightarrow \rho(\mathfrak{A})$ are provable from Δ_0 in S_2 , therefore $\rightarrow \rho(\mathfrak{B})$ is also provable from Δ_0 in S_2 .

In virtue of this lemma we can easily see that the inversion of \mathfrak{G} -inference holds in the maximal consistent system containing S_2 , which we can obtain by the usual method.

PROPOSITION 23. *If A, B, C', D', F, \mathfrak{G} -inference and infinite induction are consistent, then the inversion of \mathfrak{G} -inference is consistent with our original system.*

§ 4. Another system of axioms.

In this section we shall give another system of set theory with strong axioms of infinity, which is closely related to the former system.

The new system is obtained from the former system by the following modifications:

- 1) Introduce new basic symbols $\mathfrak{Z}(\)$ and $\mathfrak{Z}(\ , \)$ and eliminate \mathfrak{G} .
- 2) Replace Group F and \mathfrak{G} -inference by the following Group H and \mathfrak{Z} -inference.
 - Group H.
 1. $\neg \mathfrak{Z}(0)$.
 2. $\mathfrak{Z}(H_0)$, where H_0 is defined to be an abbreviation of

$$(Ka)(\forall x \forall y (x \in a \wedge (y \in x \vee y \subseteq x) \rightarrow y \in a) \wedge a \neq 0)$$
 3. $\forall A \forall B (\mathfrak{Z}(A) \wedge A \subseteq B \rightarrow \mathfrak{Z}(B))$.
 4. $\forall A \forall x (\mathfrak{Z}(A) \rightarrow \mathfrak{Z}((Ka)(a \in A \wedge x < a)))$, where $b < a$ is an abbreviation of

$$b \subseteq a \wedge b \in a$$
 5. $\forall A (\forall x \mathfrak{Z}(A''\{x\}) \rightarrow \mathfrak{Z}((Ka) \forall x (x \in a \rightarrow a \in A''\{x\})))$.
 6. $\forall x \forall A (\mathfrak{Z}(x, A) \rightarrow \mathfrak{Z}(x, x \cap A))$.
 7. $\forall A (\mathfrak{Z}(A) \rightarrow \mathfrak{Z}((Ka) \mathfrak{Z}(a, A)))$.

\mathfrak{Z} -inference.

If \mathfrak{A} is a closed formula and holds, then $\mathfrak{Z}(K(a)\mathfrak{A}^a)$ also holds.

The following properties are also proved in the new system in the same way as in the former system.

PROPOSITION 1. $\mathfrak{Z}(V)$.

PROPOSITION 2. $\forall x (x \in c \rightarrow \mathfrak{Z}(A''\{x\})) \rightarrow \mathfrak{Z}((Ka) \forall x (x \in c \rightarrow a \in A''\{x\}))$.

PROPOSITION 3. $\mathfrak{Z}(A), \mathfrak{Z}(B) \rightarrow \mathfrak{Z}(A \cap B)$.

DEFINITION. Let $\mathfrak{I}(A)$ be a term. $\mathfrak{E}(A, \mathfrak{I}, \alpha)$ is defined to be $(Ka) \forall B (B''\{0\} = A \wedge \forall \beta (\beta < \alpha \rightarrow B''\{\beta+1\} = \mathfrak{I}(B''\{\beta\}))$

$\wedge \forall \beta (\beta \leq \alpha \wedge \beta \in K_{II} \rightarrow B''\{\beta\} = (Kx) \forall r (r < \beta \rightarrow x \in B''\{r\})) \rightarrow a \in B''\{\alpha\}$.

PROPOSITION 4. $\mathfrak{Z}(A), \forall X (\mathfrak{Z}(X) \rightarrow \mathfrak{Z}(\mathfrak{I}(X))) \rightarrow \mathfrak{Z}(Ka) \forall \alpha (\alpha \in a \rightarrow a \in \mathfrak{E}(A, \mathfrak{I}, \alpha))$.

PROPOSITION 5. $\mathfrak{Z}(A), \mathfrak{Fnc}(F) \rightarrow \mathfrak{Z}((Ka)(a \in A \wedge \forall x (x \in a \rightarrow F'x \in a)))$.

PROPOSITION 6. $\mathfrak{Z}(A), \forall x (x \in a \rightarrow \mathfrak{Fnc}(B''\{x\})) \rightarrow \mathfrak{Z}((Kb)(b \in A \wedge \forall x \forall y (x \in a \wedge y \in b \rightarrow (B''\{x\})'y \in b)))$.

PROPOSITION 7. $\forall x \forall A (\text{Un}(A) \rightarrow \exists y \forall u (u \in y \rightarrow \exists v (v \in x \wedge \langle uv \rangle \in A)))$.

PROPOSITION 8. If $\mathfrak{B}(A)$ contains no free variable other than A and $\forall A (\mathfrak{Z}(A) \rightarrow \mathfrak{B}(A))$ holds, then $\forall A (\mathfrak{Z}(A) \rightarrow \mathfrak{Z}((Ka) \mathfrak{B}^a(a \cap A)))$ also holds.

PROOF. In virtue of \mathfrak{Z} -inference follows $\mathfrak{Z}(C)$, where C is an abbreviation of

$$(Ka) \forall x (x \subseteq a \wedge \mathfrak{Z}(a, x) \rightarrow \mathfrak{B}(x)).$$

B is defined to be $(Ka) \mathfrak{Z}(a, A)$. We have only to prove

$$a \in C, a \in B \rightarrow \mathfrak{B}^a(a \cap A),$$

which is clear.

PROPOSITION 9. $\mathfrak{Z}(A) \rightarrow Mo(A)$.

PROPOSITION 10. $\mathfrak{Z}(A) \rightarrow \mathfrak{Z}((Ka)M(a, A))$.

PROPOSITION 11. $\mathfrak{Z}(A) \rightarrow \mathfrak{Z}((Ka)\forall\alpha(\alpha \in a \rightarrow M(a, \mathfrak{E}(A, \mathfrak{I}, \alpha))))$, where $\mathfrak{I}(X)$ is an abbreviation of $(Ka)M(a, X)$.

I_n is defined to be

$$(Ka)\exists x\exists y(a = \langle xy \rangle \wedge \mathfrak{Z}(x, y)).$$

A term $T(A_1, \dots, A_n)$ is called elementary, if it is constructed just by $E, I_n, \cap, -, \mathfrak{D}, \mathfrak{B}, \mathfrak{C}nv, \mathfrak{C}nv_2, \mathfrak{C}nv_3, A_1, \dots, A_n$.

If a term T is elementary and closed, then we define Gödel number $g(T)$ of T in the same way as in § 2, adding the condition

$$g(I_n) = 7.$$

We define a term $\mathfrak{R}(A)$ in the same way as in § 2, adding the condition

$$\mathfrak{R}(7) = I_n.$$

The concepts that a sequence of natural numbers and a sequence of closed formulas are 'definable' are defined in the same way as in § 2.

Then for any definable sequence $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2, \dots$ of provable formulas, there exists a term $T(i)$ such that

$$T(i) = g((Ka)(\mathfrak{A}_i^a \wedge a \in G_0)).$$

By an analogous argument as in § 2, we have

$$\mathfrak{Z}((Ka)\forall i(i \in \omega \rightarrow a \in \mathfrak{R}(T(i)))) ,$$

which we set as $\mathfrak{Z}(A)$. Here A means the class of all the complete inner models, in which $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2, \dots$ and Bernays-Gödel's set theory hold.

A derived sequence of a definable sequence of closed provable formulas is defined in the same way as in § 2, by replacing the axiom F there by the axiom H.

PROPOSITION 12. *Let $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2, \dots$ be a definable sequence of closed provable formulas and $\mathfrak{C}_0, \mathfrak{C}_1, \mathfrak{C}_2, \dots$ be a derived sequence of $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2, \dots$. If \mathfrak{B} is provable from $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2, \dots$, axiom of equality, A, B', C', D, E, H, and \mathfrak{Z} -inference (but without infinite induction), then there exists \mathfrak{C}_n such that $\mathfrak{C}_n \rightarrow \mathfrak{B}$ is provable without infinite induction or \mathfrak{Z} -infinite or A, B', C', D, E, H.*

We shall further consider the following inference as a basic inference.

Inversion of \mathfrak{Z} -inference.

If \mathfrak{A} is a closed formula and $\mathfrak{Z}((Ka)\mathfrak{A}^a)$ holds, then \mathfrak{A} also holds.

PROPOSITION 13. *Under inversion of \mathfrak{Z} -inference, C'4, D and E can be reduced to C'4, D' and E', respectively.*

Group B''' is defined to be all the axioms of Group B and the following B9.

B9. $\exists A \forall x \forall y (\langle xy \rangle \in A \leftrightarrow \mathfrak{I}(x, y))$.

PROPOSITION 14. *If $A, B'', C', D', H, \mathfrak{I}$ -inference and infinite induction are consistent, then the inversion of \mathfrak{I} -inference are consistent with our original system.*

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