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On harmonic and Killing tensor fields in a Riemannian manifold with boundary

By Tsunero TAKAHASHI

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Introduction. The study of harmonic and Killing vector fields and Ricci curvature in Riemannian manifolds without boundary has been started by Bochner [1][1]. Lichnerowicz [11], Mogi [12], Tomonaga [16] and Yano [17], [18] have extended Bochner's results to harmonic and Killing tensor fields of order $p(>1)$. They also have studied similar problems in manifolds with certain inequalities satisfied by the Riemann-Christoffel, projective, conformal and concircular curvature tensor fields [2], [3], [4], [12], [16], [18], [22].

On the other hand, harmonic tensor fields in Riemannian manifolds with boundary have been studied by Conner [6] and by Duff and Spencer [7]. Also Nakae [13] has treated curvatures and relative Betti numbers. Hsing [10] and Yano [19], [20], [21] have studied harmonic and Killing tensor fields in Riemannian manifolds with boundary.

The purpose of the present paper is to study, using integral formulas, harmonic and Killing tensor fields and also conformal vector fields in Riemannian manifolds with boundary, and extend the results for manifolds without boundary shown in [22] to the case of manifolds with boundary.

In § 1, we give general notations for skew-symmetric tensor fields and introduce a quadratic form $F^{(p)}$ which will play an important role in this paper. In this section we try to extend the notion of the Ricci curvature, and obtain a geometrical meaning of the quadratic form $F^{(p)}$.

In § 2, we give a definition of the compact Riemannian manifold with boundary. We introduce the quadratic form $H^{(p)}$ and $\hat{H}^{(p)}$ which will play important roles togther with $F^{(p)}$. Stokes' theorem is proved in this section.

§ 1 and § 2 are the preparations for § 3, § 4 and § 5.

In § 3, we obtain a necessary and sufficient condition for a skew-symmetric tensor field to be a harmonic or Killing tensor field, and for a vector field to be a conformal vector field.

In § 4, we study non-existence of harmonic and Killing tensor fields and also of conformal vector fields under certain conditions for the quadratic forms $F^{(p)}$, $H^{(p)}$ and $\hat{H}^{(p)}$.

1) Numbers in brackets refer to the bibliography at the end of the paper.

Some of the results in § 3 and § 4 have been obtained essentially by Yano [**20**] and Yano and the present author [**23**]. But we state the theorems in slightly different forms.

In § 5, we study the manifold in which certain inequalities are satisfied by Riemann-Christoffel projective, conformal or concircular curvature tensor fields.

As to tensor calculus, we refer to Eisenhart [**9**] or Schouten [**15**] and in notations of skew-symmetric tensors we refer to de Rham [**14**].

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§ 1. Skew-symmetric tensor fields.

Let V_m be an $m(\geqq 2)$ dimensional orientable Riemannian manifold of class C^∞ with positive definite metric and denote by g the fundamental tensor field of V_m.

We shall fix an orientation of V_m and unless otherwise stated, every coordinate system of V_m considered is assumed to be positively ordered.

Consider a p-tensor (or a tensor of order p) v and a q-tensor w at a point of V_m.

If $p \geqq q$, we denote by $v \llcorner w$ and $w \lrcorner v$ the $(p-q)$-tensors at the point with covariant components in each coordinate system given respectively by

$$(v \llcorner w)_{\lambda_1 \cdots \lambda_{p-q}} = \frac{1}{q\,!}\, v_{\lambda_1 \cdots \lambda_{p-q} \alpha_1 \cdots \alpha_q} w^{\alpha_1 \cdots \alpha_q}$$ [2]

and

$$(w \lrcorner v)_{\lambda_1 \cdots \lambda_{p-q}} = \frac{1}{q\,!}\, w^{\alpha_1 \cdots \alpha_q} v_{\alpha_1 \cdots \alpha_q \lambda_1 \cdots \lambda_{p-q}}\,,$$

where $v_{\lambda_1 \cdots \lambda_p}$ are the covariant components of v and $w^{\lambda_1 \cdots \lambda_q}$ the contravariant components of w.

When, in particular, $p=q$, four scalars $v \llcorner w$, $w \lrcorner v$, $v \lrcorner w$ and $w \llcorner v$ can be defined and are equal to each other. We call this scalar inner product of v and w, and we denote it by $\langle v, w \rangle$ or $\langle w, v \rangle$. Since the metric of V_m is positive definite, $\langle v, v \rangle$ is always positive for non zero tensor v. We write the square root of $\langle v, v \rangle$ as $\| v \|$.

If v, w and u are the tensors of order p, q and $p+q$ respectively, we have the formula

$$\langle v \lrcorner u, w \rangle = \langle v, u \llcorner w \rangle\,. \tag{1.1}$$

If v is a skew-symmetric p-tensor and w is an arbitrary q-tensor $(q \leqq p)$, it is easily found that

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2) Throughout the paper, the Greek indices take the values $1, 2, \cdots, m$ and the Latin indices take the values $1, 2, \cdots, m-1$.

$$v \llcorner w = (-1)^{(p-q)q} w \lrcorner v\,.$$

If v is skew-symmetric and w is symmetric or v is symmetric and w is skew-symmetric, we have

$$v \llcorner w = w \lrcorner v = 0\,.$$

Take a coordinate system (ξ^κ) and denote by $\mathfrak{g}$ the determinant formed by the covariant components of g.

Consider the quantities $e_{\lambda_1\cdots\lambda_m}$ in the coordinate system (ξ^κ) given by

$$e_{\lambda_1\cdots\lambda_m} = \begin{cases} \sqrt{\mathfrak{g}}, & \text{when } (\lambda_1, \cdots, \lambda_m) \text{ is an even permutation} \\ & \text{of } (1, 2, \cdots, m) \\ -\sqrt{\mathfrak{g}}, & \text{when } (\lambda_1, \cdots, \lambda_m) \text{ is an odd permutation} \\ & \text{of } (1, 2, \cdots, m) \\ 0, & \text{otherwise.} \end{cases}$$

Then $e_{\lambda_1\cdots\lambda_m}$ define a skew-symmetric tensor field of order m. We denote this tensor field by e.

For a skew-symmetric tensor v, $*v$ is defined by

$$*v = v \lrcorner e\,.$$

It is well known that for a skew-symmetric p-tensor v, we have

(1.2) $$**v = (-1)^{(m-p)p} v\,.$$

If v and w are skew-symmetric p-tensors, using (1.1) and (1.2) we get

(1.3) $$\langle *v, *w \rangle = \langle v, w \rangle\,.$$

The exterior product of the skew-symmetric tensor v and w is denoted by $v \wedge w$ and is given by

$$(v \wedge w)_{\lambda_1\cdots\lambda_p\lambda_{p+1}\cdots\lambda_{p+q}} = \frac{(p+q)}{p!\, q!} v_{[\lambda_1\cdots\lambda_p} w_{\lambda_{p+1}\cdots\lambda_{p+q}]}\,.$$

From this definition we get

$$w \wedge v = (-1)^{pq} v \wedge w\,.$$

Moreover, when $q \leqq p$, we have

$$v \llcorner w = (-1)^{(m-p+q)(p-q)} *(w \wedge *v)\,.$$

Now, we introduce an operator F and a quadratic form $F^{(p)}$ at each point of V_m.

At a point of V_m, we take a coordinate system (ξ^κ) and denote by $K_{\nu\mu\lambda\kappa}$ and $K_{\mu\lambda}$ the covariant components of the curvature tensor field and the Ricci tensor field respectively.

For a skew-symmetric p-tensor v at the point, Fv is a skew-symmetric p-tensor at the point and given by

$$(Fv)_{\lambda_1\cdots\lambda_p}=pK_{\alpha[\lambda_1}v^{\alpha}{}_{\lambda_2\cdots\lambda_p]}+\frac{p(p-1)}{2}K_{\alpha\beta[\lambda_1\lambda_2}v^{\alpha\beta}{}_{\lambda_3\cdots\lambda_p]}\,.$$

$F^{(p)}$ is the quadratic form on the vector space consisting of all skew-symmetric p-tensors at the point and for a skew-symmetric p-tensor v, $F^{(p)}(v)$ is defined by

$$F^{(p)}(v)=\langle Fv,v\rangle=\langle v,Fv\rangle\,.$$

When v is a unit vector, $F^{(1)}(v)$ has the form

$$F^{(1)}(v)=K_{\mu\lambda}v^{\mu}v^{\lambda}\,.$$

The right hand member of the above identity is the so-called Ricci curvature with respect to the direction v.

To obtain a similar relation for the quadratic form $F^{(p)}$ $(p>1)$ and the curvature, we first extend the notion of the Ricci curvature.

Consider a p-dimensional sub-space S of the tangent space of V_m at the point and choose an orthonormal base $(\underset{(1)}{u},\underset{(2)}{u},\cdots,\underset{(m)}{u})$ such that $(\underset{(1)}{u},\cdots,\underset{(p)}{u})$ spans S, then we have $p(m-p)$ two-dimensional subspaces S_{ij} spanned by $\underset{(i)}{u}$ and $\underset{(j)}{u}$ $(i=1,\cdots,p,j=p+1,\cdots,m)$. Denoting by $K(S_{ij})$ the sectional curvatures with respect to S_{ij}, that is,

$$K(S_{ij})=-K_{\nu\mu\lambda\kappa}\underset{(i)}{u}^{\nu}\underset{(j)}{u}^{\mu}\underset{(i)}{u}^{\lambda}\underset{(j)}{u}^{\kappa}\,,$$

and summing up them, we have a quantity

$$R(S)=\sum_{i=1}^{p}\sum_{j=p+1}^{m}K(S_{ij})\,.$$

By a simple calculation we can find that $R(S)$ is written in the form

$$R(S)=\sum_{i=1}^{p}K_{\mu\lambda}\underset{(i)}{u}^{\mu}\underset{(i)}{u}^{\lambda}+\sum_{i=1}^{p}\sum_{j=1}^{p}K_{\nu\mu\lambda\kappa}\underset{(i)}{u}^{\nu}\underset{(j)}{u}^{\mu}\underset{(i)}{u}^{\lambda}\underset{(j)}{u}^{\kappa}\,. \tag{1.4}$$

This shows that $R(S)$ is independent of the choice of the last $m-p$ vectors $\underset{(p+1)}{u},\cdots,\underset{(m)}{u}$ of the orthonormal base.

The first p vectors $\underset{(1)}{u},\cdots,\underset{(p)}{u}$ form an orthonormal base of S. If we can show that the right hand member of (1.4) does not depend on the choice of the orthonormal base of S, we can conclude that the quantity $R(S)$ defined above is independent of the choice of the orthonormal base such as $(\underset{(1)}{u},\cdots,\underset{(m)}{u})$.

For this purpose we make the exterior product

$$u=\underset{(1)}{u}\wedge\cdots\wedge\underset{(p)}{u}$$

and calculate the value of $F^{(p)}(u)$. We then get

$$F^{(p)}(u)=\sum_{i=1}^{p}K_{\mu\lambda}\underset{(i)}{u}^{\mu}\underset{(i)}{u}^{\lambda}+\sum_{i=1}^{p}\sum_{j=1}^{p}K_{\nu\mu\lambda\kappa}\underset{(i)}{u}^{\nu}\underset{(j)}{u}^{\mu}\underset{(i)}{u}^{\lambda}\underset{(j)}{u}^{\kappa}\,,$$

and thus from (1.4) we obtain

$$R(S)=F^{(p)}(u). \tag{1.5}$$

If we take another orthonormal base $(\underset{(1)}{w}, \cdots, \underset{(p)}{w})$ of S and make

$$w=\underset{(1)}{w}\wedge\cdots\wedge\underset{(p)}{w},$$

we have

$$w=\varepsilon u,$$

where $\varepsilon=+1$ or -1, and consequently

$$F^{(p)}(w)=F^{(p)}(u). \tag{1.6}$$

From (1.5) and (1.6) we can conclude that $R(S)$ is independent of the choice of the orthonormal base. We call $R(S)$ defined above *the extended Ricci curvature with respect to* S.

If a skew-symmetric p-tensor v has a form

$$v=\underset{(1)}{v}\wedge\cdots\wedge\underset{(p)}{v}$$

where $\underset{(1)}{v}, \cdots, \underset{(p)}{v}$ are p linearly independent vectors, we call v to be *simple*. Then, v determines a p-dimensional subspace spanned by $\underset{(1)}{v}, \cdots, \underset{(p)}{v}$ of the tangent space. It does not depend on the decomposition of v. When v' is another simple skew-symmetric p-tensor, v and v' determine the same subspace if and only if

$$v'=a\cdot v$$

where a is a non-zero scalar.

If we consider an orthonormal base $(\underset{(1)}{u}, \cdots, \underset{(p)}{u})$ of the subspace S determined by v, we have

$$v=\pm\|v\|\cdot u$$

where

$$u=\underset{(1)}{u}\wedge\cdots\wedge\underset{(p)}{u},$$

and therefore from (1.5)

$$F^{(p)}(v)=\|v\|^2F^{(p)}(u)=\|v\|^2R(S).$$

Thus we have

THEOREM 1.1. *If v is a simple skew-symmetric p-tensor at a point of V_m and S is a p dimensional subspace determined by v of the tangent space of V_m at the point, the value $F^{(p)}(v)$ coincides with the Ricci curvature with respect to S up to a factor i.e. we have*

$$F^{(p)}(v)=\|v\|^2R(S).$$

If v is simple, $*v$ is also simple and the subspace S' determined by $*v$ is the orthogonal complement of the subspace S determined by v.

From the definition of $R(S)$ it is easily found that

$$R(S)=R(S').$$

Therefore we can get from Theorem 1.1

$$F^{(m-p)}(*v)=F^{(p)}(v) \tag{1.7}$$

for a simple skew-symmetric tensor v.

But this identity is satisfied by any skew-symmetric tensor, that is,

THEOREM 1.2. *For any skew-symmetric p-tensor v we have*

$$F^{(m-p)}(*v)=F^{(p)}(v).$$

The proof of Theorem 1.2 can be obtained by the straight-forward calculation, but we shall give another simple proof later.

In the remainder of this section we introduce the operators applied to skew-symmetric tensor fields.

Unless otherwise stated, every tensor field is of class C^∞ throughout the paper.

For a p-tensor field v, we denote by ∇v the covariant derivative of v. If we take a coordinate system (ξ^κ) and denote by $\left\{ {\kappa \atop \mu\lambda} \right\}$ the Christoffel symbols, the covariant components of ∇v is given by

$$\begin{aligned}(\nabla v)_{\lambda\lambda_1\cdots\lambda_p} &= \nabla_\lambda v_{\lambda_1\cdots\lambda_p} \\ &= \frac{\partial v_{\lambda_1\cdots\lambda_p}}{\partial \xi^\lambda} - \left\{ {\alpha \atop \lambda\lambda_1} \right\} v_{\alpha\lambda_2\cdots\lambda_p} - \cdots - \left\{ {\alpha \atop \lambda\lambda_p} \right\} v_{\lambda_1\cdots\lambda_{p-1}\alpha}.\end{aligned}$$

Consider a skew-symmetric p-tensor field v.

The differential dv of v is given by

$$(dv)_{\lambda_1\lambda_2\cdots\lambda_{p+1}}=(p+1)\nabla_{[\lambda_1}v_{\lambda_2\cdots\lambda_{p+1}]}$$

in the coordinate system (ξ^κ). dv is a skew-symmetric tensor field of order $p+1$.

The divergence δv of v is given by

$$\delta v=(-1)^{m(p-1)}*d*v.$$

δv is the skew-symmetric tensor field of order $p-1$. In the coordinate system the covariant components of δv are given by

$$(\delta v)_{\lambda_1\cdots\lambda_{p-1}}=g^{\beta\alpha}\nabla_\beta v_{\alpha\lambda_1\cdots\lambda_{p-1}}.$$

Therefore, we have

$$\delta v=2g \lrcorner \nabla v.$$

For a skew-symmetric tensor field v, ∇v, Dv and $\square v$ are given respectively by

$$\triangle v=\delta dv+d\delta v,$$

$$Dv=2g \lrcorner \nabla\nabla v$$

and

$$\Box v=(p+1)Dv-\triangle v\,.$$

The covariant components of Dv are

$$(Dv)_{\lambda_1\cdots\lambda_p}=g^{\beta\alpha}\nabla_\beta\nabla_\alpha v_{\lambda_1\cdots\lambda_p}\,.$$

$\triangle v$ has the covariant components of the form

$$(\triangle v)_{\lambda_1\cdots\lambda_p}=g^{\beta\alpha}\nabla_\beta\nabla_\alpha v_{\lambda_1\cdots\lambda_p}-pK_{\alpha[\lambda_1}v^\alpha{}_{\lambda_2\cdots\lambda_p]}$$
$$-\frac{p(p-1)}{2}K_{\beta\alpha[\lambda_1\lambda_2}v^{\beta\alpha}{}_{\lambda_3\cdots\lambda_p]}\,.$$

Thus we have

$$\triangle v=Dv-Fv$$

or

$$Fv=Dv-\triangle v\,.$$

From the definition of $\triangle$, and D we can easily see that

$$\triangle *v=*\triangle v \quad\text{and}\quad D*v=*Dv\,.$$

Therefore we get

$$\Box *v=*\Box v \quad\text{and}\quad F*v=*Fv\,.$$

If v is a skew-symmetric p-tensor at a point, we can extend v to a skew-symmetric tensor field in a neighborhood of the point and we have at the point

$$F*v=*Fv\,.$$

Thus we have

$$F^{(m-p)}(*v)=\langle F*v,*v\rangle=\langle *Fv,*v\rangle=\langle Fv,v\rangle=F^{(p)}(v)\,.$$

This gives a proof of Theorem 1.2.

§ 2. Riemannian manifolds with boundary.

A compact Riemannian manifold M with boundary B is a compact subdomain of V_m satisfying the following condition; At each point of the boundary B of M there is a neighborhood U of the point in V_m and a coordinate system $(\xi^1,\cdots,\xi^m)$ (called *a boundary coordinate system*) of V_m such that $U\cap M$ is represented by an inequality $\xi^1\leqq 0$.

Since we can always choose as a boundary coordinate system a positively ordered system, we shall assume that every boundary coordinate system considered is positively ordered.

If $(\xi^1,\cdots,\xi^m)$ is a boundary coordinate system, B is locally represented by $\xi^1=0$ and $(\xi^2,\cdots,\xi^m)$ becomes a coordinate system of B. Thus B is an $m-1$ dimensional submanifold of V_m. If $(\bar{\xi}^1,\cdots,\bar{\xi}^m)$ is another boundary coordinate system, it is easily seen that we have on B

$$\frac{\partial\bar{\xi}^1}{\partial\xi^1}>0 \quad \text{and} \quad \frac{\partial\bar{\xi}^1}{\partial\xi^\lambda}=0 \qquad \text{for } \lambda=2,\cdots,m.$$

Since the Jacobian of the coordinate transformation of (ξ^κ) and $(\bar{\xi}^\kappa)$ is positive, the Jacobian of the coordinate transformation of the coordinate systems $(\xi^2,\cdots,\xi^m)$ and $(\bar{\xi}^2,\cdots,\bar{\xi}^m)$ of B is also positive. Therefore the coordinate systems of B obtained from the boundary coordinate systems define an orientation of B. In the following we assume that B is oriented by this orientation.

At each point of the boundary B, there are two unit vectors normal to B. In each boundary coordinate system the first contravariant component of the one is positive and that of the other is negative. We denote the former by N.

In the sequel we use the following notations.

If $(\xi^1,\cdots,\xi^m)$ is a coordinate system (not necessarily boundary) of V_m at a point of B and $(\eta^1,\cdots,\eta^{m-1})$ is a coordinate system of B whose domain is contained in that of (ξ^κ), B is locally represented by

$$\xi^\kappa=\xi^\kappa(\eta^i).$$

We denote by B_i^κ the derivative of ξ^κ with respect to η^i, i.e. we put

$$B_i^\kappa=\frac{\partial\xi^\kappa}{\partial\eta^i},$$

and

$$B_{i_1\cdots i_p}^{\kappa_1\cdots\kappa_p}=B_{i_1}^{\kappa_1}\cdots B_{i_p}^{\kappa_p}.$$

Then the covariant components of the fundamental tensor field $'g$ of B are given by

$$'g_{ji}=g_{\mu\lambda}B_{ij}^{\mu\lambda}$$

and the determinant $'\mathfrak{g}$ formed with $'g_{ji}$ is given by

$$'\mathfrak{g}=\mathfrak{g}\det(N^\lambda,B_1^\lambda,\cdots,B_{m-1}^\lambda)^2.$$

In the following a tensor field of V_m defined in a neighborhood of M will be called simply a tensor field on M.

The following theorem is well known.

STOKES' THEOREM. *For an arbitrary vector field w on M, we have*

$$\int_M(\delta w)d\sigma=\int_B\langle N,w\rangle d'\sigma \tag{2.1}$$

where $d\sigma$ and $d'\sigma$ are the volume elements of V_m and B respectively.

PROOF. First we remark that, in any coordinate system (ξ^κ) of V_m, $(\delta w)d\sigma$ is written in the form

$$(\delta w)d\sigma=\Big(\sum_{\alpha=1}^m\frac{\partial\sqrt{\mathfrak{g}}\,w^\alpha}{\partial\xi^\alpha}\Big)d\xi^1\wedge\cdots\wedge d\xi^m.$$

Using the partition of unity, it is easily seen that, for the proof of the theorem, it is sufficient to prove it in the following two case: One is the case in which the carrier of w is contained in the interior of M and in a coordinate neighborhood U, and the other is the case in which the carrier of w is contained in a domain W of a boundary coordinate system.

Proof in the first case: We take a coordinate system (ξ^κ) in U. We may assume that the carrier of w is contained in a domain $|\xi^\lambda|<a$.

Then we have

$$\int_M (\delta w)d\sigma=\sum_{\alpha=1}^m\int_{-a}^a\cdots\int_{-a}^a\frac{\partial\sqrt{\mathfrak{g}}\,w^\alpha}{\partial\xi^\alpha}d\xi^1\cdots d\xi^m=0\,.$$

On the other hand, since w is zero on B, the right hand member of (2.1) is clearly zero.

Proof in the second case: Let $(\xi^1,\cdots,\xi^m)$ be a boundary coordinate system in w and $(\eta^1,\cdots,\eta^{m-1})$ be a coordinate system of B defined by (ξ^κ) i.e. $\eta^i=\xi^{i+1}$ $(i=1,\cdots,m-1)$. Then in these coordinate systems (ξ^κ) and (η^i) we have

$$\sqrt{'\mathfrak{g}}=\sqrt{\mathfrak{g}}N^1\,,$$

and the covariant components of N are $(N_1,0,\cdots,0)$. N_1 and N^1 satisfy

$$N_1N^1=1\,.$$

We may assume that the carrier of w is contained in a domain $|\xi^\kappa|<a$. Then we have

$$\begin{aligned}\int_M(\delta w)d\sigma&=\int_{-a}^a\cdots\int_{-a}^a\Big\{\int_{-a}^0\frac{\partial\sqrt{\mathfrak{g}}\,w^1}{\partial\xi^1}d\xi^1\Big\}d\xi^2\cdots d\xi^m\\&\quad+\int_{-a}^0\Big\{\int_{-a}^a\cdots\int_{-a}^a\sum_{\alpha=2}^m\frac{\partial\sqrt{\mathfrak{g}}\,w^\alpha}{\partial\xi^\alpha}d\xi^2\cdots d\xi^m\Big\}d\xi^1\\&=\int_{-a}^a\cdots\int_{-a}^a\sqrt{\mathfrak{g}(0,\eta^1,\cdots,\eta^{m-1})}\,w^1(0,\eta^1,\cdots,\eta^{m-1})d\eta^1\cdots d\eta^{m-1}\\&=\int_{-a}^a\cdots\int_{-a}^a w^1N_1\sqrt{\mathfrak{g}}N^1d\eta^1\cdots d\eta^{m-1}\\&=\int_{-a}^a\cdots\int_{-a}^a\langle N,w\rangle\sqrt{'\mathfrak{g}}\,d\eta^1\cdots d\eta^{m-1}\\&=\int_B\langle N,w\rangle d'\sigma\,.\end{aligned}$$

Thus we have proved the theorem. q.e.d.

We denote by H the second fundamental tensor field of B with respect to the normal N. In the local coordinate systems (ξ^κ) and (η^i) the covariant components of H are given by

$$H_{ji}=\Big[\frac{\partial B_i^\kappa}{\partial\eta^j}+\Big\{{\kappa\atop\mu\lambda}\Big\}B_{ji}^{\mu\lambda}\Big]N_\kappa\,.$$

The equations of Gauss and of Weingarten can be written respectively in the form

$$\frac{\partial B_i^\kappa}{\partial \eta^j}+\left\{ {\kappa \atop \mu\lambda} \right\} B_{ji}^{\mu\lambda}-'\left\{ {a \atop ji} \right\} B_a^\kappa = H_{ji}N^\kappa$$

and

$$\frac{\partial N^\kappa}{\partial \eta^j}+\left\{ {\kappa \atop \mu\lambda} \right\} B_j^\mu N^\lambda = -H_j^a B_a^\kappa .$$

For any skew-symmetric p-tensor u of B at a point, skew-symmetric p-tensors Hu and $\hat{H}u$ are defined respectively by

$$(Hu)_{i_1\cdots i_p} = pH_{a[i_1}u^a{}_{i_2\cdots i_p]}$$

and

$$(\hat{H}u)_{i_1\cdots i_p} = (g^{ba}H_{ba})u_{i_1\cdots i_p} - pH_{a[i_1}u^a{}_{i_2\cdots i_p]} .$$

By a straightforward calculation, we get

$$H*u = *\hat{H}u \qquad \text{and} \qquad \hat{H}*u = *Hu . \tag{2.2}$$

The quadratic forms $H^{(p)}$ and $\hat{H}^{(p)}$ on the vector space consisting of all skew-symmetric p-tensors of B at a point is defined respectively by

$$H^{(p)}(u) = \langle Hu, u\rangle = \langle u, Hu\rangle$$

and

$$\hat{H}^{(p)}(u) = \langle \hat{H}u, u\rangle = \langle u, \hat{H}u\rangle .$$

From (2.2) we obtain

THEOREM 2.1. *For any skew-symmetric p-tensor u of B at a point we have*

$$\hat{H}^{(m-p-1)}(*u) = H^{(p)}(u) \qquad \text{and} \qquad H^{(m-p-1)}(*u) = \hat{H}^{(p)}(u) .$$

The quadratic form $H^{(1)}$ is the so-called fundamental quadratic form of B.

If M satisfies the following condition, we say that *M has a convex (or concave) boundary: At each point x of B any geodesic of V_m through x and tangent to B at x does not intersect the interior (exterior) of M near x.*

THEOREM 2.2. *If M has a convex (or concave) boundary B, the quadratic forms $H^{(p)}$ and $\hat{H}^{(p)}$ ($p=0, 1, \cdots, m-1$) is negative (or positive) semi-definite. If the second fundamental form $H^{(p)}$ of B is negative (or positive) definite, M has a convex (or concave) boundary B.*

PROOF. At a point x of B, take a boundary coordinate system $(\xi^1, \cdots, \xi^m)$ and an arbitrary coordinate system $(\eta^1, \cdots, \eta^{m-1})$ of B. Since B is locally represented by $\xi^1 = 0$, we have

$$B_j^1 = \partial\xi^1/\partial\eta^j = 0 \qquad (j=1, \cdots, m-1) .$$

Therefore from the equations of Gauss, we get,

$$H_{ji}N^1 = \left\{ {1 \atop \mu\lambda} \right\} B_{ji}^{\mu\lambda} . \tag{2.3}$$

Let u be an arbitrary tangent vector to B at x, and (u^i) be its components in the system (η^i). The components of u in the coordinate system of (ξ^κ) are $(B_j^\kappa u^j)$. If $(\xi^\kappa(t))$ is a geodesic of V_m such that

$$(\xi^\kappa(0)) = x\,,$$

and

(2.4) $$\frac{d\xi^\kappa}{dt} = B_j^\kappa u^j\,, \qquad \text{at} \quad t=0\,.$$

Since $\xi^\kappa(t)$ satisfies the equation

$$\frac{d^2\xi^\kappa}{dt^2} + \left\{ {\kappa \atop \mu\lambda} \right\} \frac{d\xi^\mu}{dt}\frac{d\xi^\lambda}{dt} = 0\,,$$

we have at $t=0$,

$$\frac{d^2\xi^\kappa}{dt^2} = -\left\{ {\kappa \atop \mu\lambda} \right\} \frac{d\xi^\mu}{dt}\frac{d\xi^\lambda}{dt} = -\left\{ {\kappa \atop \mu\lambda} \right\} B_{ji}^{\mu\lambda} u^j u^i\,.$$

Thus from (2.3) we get

(2.5) $$\frac{d^2\xi^1}{dt^2} = -H^{(1)}(u)N^1\,, \qquad \text{at} \quad t=0\,.$$

If we suppose that M has a convex (or concave) boundary, then

$$\begin{cases} \xi^1(0)=0 \\ \text{and} \quad \xi^1(t) \geqq 0 \ (\text{or} \ \leqq 0) \ \text{for all } t \text{ near } 0\,. \end{cases}$$

Therefore $\xi^1(t)$ takes a minimal (or maximal) value at $t=0$, and we have

$$\frac{d^2\xi^1}{dt^2} \geqq 0 \quad (\text{or} \ \leqq 0) \qquad \text{at} \quad t=0\,.$$

From (2.5) this means that $H^{(1)}(u)$ is non positive (or negative), for N^1 is positsve. This proves the first part of the theorem.

If we suppose that $H^{(1)}$ is negative (or positive) definite, then from (2.4) and (2.5) we have

$$\frac{d\xi^1}{dt} = 0 \quad \text{and} \quad \frac{d^2\xi^1}{dt^2} > 0\,, \qquad (\text{or} < 0) \quad \text{at} \quad t=0\,.$$

Thus, $\xi^1(t)$ takes a minimal (or maximal) value at $t=0$, and therefore we have

$$\xi^1(t) \geqq 0 \quad (\text{or} \ \leqq 0) \qquad \text{for all } t \text{ near } 0\,.$$

This means that $(\xi^\kappa(t))$ does not lie in the interior (or exterior) of M, and we have proved the second part of the theorem. q. e. d.

Denote by f the injection of B into V_m and f^* the dual map of the differential map df of f.

For an arbitrary skew-symmetric p-tensor v of V_m at a point of B, considering v as a covariant tensor, we define a skew-symmetric p-tensor tv and a skew-symmetric $(p-1)$-tensor nv of B by

$$tv = f^*(v) \quad \text{and} \quad nv = f^*(N \lrcorner v) = t(N \lrcorner v)\,. \tag{2.6}$$

If we consider tv and nv as contravariant tensors, we have

$$v = df(tv) + N \wedge df(nv)\,. \tag{2.7}$$

Thus v is zero if and only if tv and nv is zero. We call *tv the tangential part* of v and *nv the normal part of v*. If tv (or nv) is zero, v is said *to be normal* (*or tangential*) *to B*.

If w is another skew-symmetric p-tensor of V_m at a point, we have the following formula.

$$\langle v, w \rangle = \langle tv, tw \rangle + \langle nv, nw \rangle\,.$$

In the coordinate systems, tv and nv are respectively represented by

$$\left\{\begin{aligned} (tv)_{i_1\cdots i_p} &= B^{\lambda_1\cdots\lambda_p}_{i_1\cdots i_p} v_{\lambda_1\cdots\lambda_p} \\ (nv)_{i_1\cdots i_{p-1}} &= N^\lambda B^{\lambda_1\cdots\lambda_{p-1}}_{i_1\cdots i_{p-1}} v_{\lambda\lambda_1\cdots\lambda_{p-1}}\,. \end{aligned}\right. \tag{2.8}$$

If v is a skew-symmetric p-tensor field on M, tv and nv are skew-symmetric tensor fields on B.

Differentiating (2.8), we find

$$'\nabla_j (tv)_{i_1\cdots i_p} = -pH_{j[i_1}(nv)_{i_2\cdots i_p]} + B^{\mu\lambda_1\cdots\lambda_p}_{ji_1\cdots i_p} \nabla_\mu v_{\lambda_1\cdots\lambda_p}\,,$$

$$'\nabla_j (nv)_{i_1\cdots i_{p-1}} = -H_{aj}(tv)^a_{i_1\cdots i_{p-1}} + N^\lambda B^{\mu\lambda_1\cdots\lambda_{p-1}}_{ji_1\cdots i_{p-1}} \nabla_\mu v_{\lambda\lambda_1\cdots\lambda_{p-1}}$$

where $'\nabla$ denotes the covariant derivation with respect to the metric of B. From these equations we can obtain the formulas:

$$\left\{\begin{aligned} dtv &= tdv \\ \delta tv &= t\delta v + \hat{H} nv - n(\nabla_N v) \end{aligned}\right. \tag{2.9}$$

and

$$\left\{\begin{aligned} dnv &= ndv - Htv + t(\nabla_N v) \\ \delta nv &= -n\delta v\,, \end{aligned}\right. \tag{2.10}$$

where $\nabla_N v = N \lrcorner \nabla v$.

§ 3. Necessary and sufficient conditions.

In this section we shall consider necessary and sufficient conditions for a skew-symmetric tensor field to be a harmonic field or a Killing field in M and also for a vector field to be a conformal vector field in M.

If the differential and the divergence of a skew-symmetric tensor field v of M vanish at each point of M, we call v a harmonic field in M.

From this definition, it is easy to see that a harmonic field v in M satisfies

$$\triangle v = 0 \tag{3.1}$$

in M.

In order to obtain necessary and sufficient conditions for v to be a harmonic field in M, we shall introduce an integral formula.

For an arbitrary skew-symmetric tensor field v of M we have

$$\langle \delta dv, v\rangle + \| dv \|^2 = \delta(dv \llcorner v)$$

and

$$\langle d\delta v, v\rangle + \| \delta v \|^2 = \delta(v \llcorner \delta v)\,.$$

Adding these equations we get

$$\langle \triangle v, v\rangle + \| dv \|^2 + \| \delta v \|^2 = \delta(dv \llcorner v + v \llcorner \delta v)\,. \tag{3.2}$$

Integrating (3.2) on M and applying Stokes' theorem to the vector field $dv \llcorner v + v \llcorner \delta v$, we obtain

$$\int_M [\langle \triangle v, v\rangle + \| dv \|^2 + \| \delta v \|^2] d\sigma = \int_B \langle N, dv \llcorner v + v \llcorner \delta v\rangle d'\sigma\,. \tag{3.3}$$

Using the formulas (1.1), (2.6) and (2.8) we have

$$\langle N, dv \llcorner v + v \llcorner \delta v\rangle = \langle ndv, tv\rangle + \langle t\delta v, nv\rangle\,.$$

Therefore we can write (3.4) in the form

$$\int_M [\langle \triangle v, v\rangle + \| dv \|^2 + \| \delta v \|^2] d\sigma = \int_B [\langle ndv, tv\rangle + \langle t\delta v, nv\rangle] d'\sigma\,. \tag{3.4}$$

If v satisfies (3.1) in M and moreover, satisfies

$$ndv = 0 \tag{3.5}$$

and

$$t\delta v = 0 \tag{3.6}$$

on B, we can find from the integral formula (3.4) that v is a harmonic field in M.

Conversely, if v is a harmonic field in M, it satisfies (3.5) and (3.6) on B.

Thus we have

THEOREM 3.1.[3] *In order that a skew-symmetric tensor field v of M is a harmonic field in M it is necessary and sufficient that v satisfies*

$$\begin{cases} \triangle v = 0 & \text{in } M \\ ndv = 0\,, \quad t\delta v = 0 & \text{on } B\,. \end{cases}$$

If a skew-symmetric tensor field v of M satisfies (3.1) in M and (3.5) on B, and v is tangential to B, we can find from (3.4) that v is a harmonic field in M.

Conversely, if v is a harmonic field in M and tangential to B, v satisfies (3.1) in M, (3.5) on B, and $nv = 0$ on B.

Thus we have

3) This follows from Duff's lemma also (Algebraic geometry and topology, 1957, p. 133).

THEOREM 3.2. *In order that a skew-symmetric tensor field v of M is a harmonic field in M tangential to B it is necessary and sufficient that v satisfies*

$$\begin{cases} \Delta v=0 & \text{in } M \\ ndv=0, \quad nv=0 & \text{on } B. \end{cases}$$

From the formula (2.10) we have for an arbitrary skew-symmetric tensor field v

$$ndv=-dnv-Htv+t(\nabla_N v). \tag{3.7}$$

We can easily find that v satisfies (3.5) and $nv=0$ on B if and only if it satisfies $Htv=t(\nabla_N v)$ and $nv=0$ on B.

Thus we have

COROLLARY. *In order that a skew-symmetric tensor field v of M is a harmonic field in M tangential to B it is necessary and sufficient that v satisfies*

$$\begin{cases} \Delta v=0 & \text{in } M \\ Htv=t(\nabla_N v), \quad nv=0 & \text{on } B. \end{cases}$$

If a skew-symmetric tensor field v of M satisfies (3.1) in M and (3.6) on B and v is normal to B, we can find from (3.4) that v is a harmonic field in M.

Conversely if v is a harmonic field in M and normal to B, it satisfies (3.1) in M, (3.6) on B and $tv=0$ on B.

Thus we have

THEOREM 3.3. *In order that a skew-symmetric tensor field v of M is a harmonic field in M normal to B it is necessary and sufficient that v satisfies*

$$\begin{cases} \nabla v=0 & \text{in } M \\ t\delta v=0, \quad tv=0 & \text{on } B. \end{cases}$$

From the formula (2.9) for an arbitrary skew-symmetric tensor field v of M, we have

$$t\delta v=\delta tv-\hat{H}nv+n(\nabla_N v). \tag{3.8}$$

We can easily find that v satisfies (3.6) and $tv=0$ on B if and only if it satisfies $\hat{H}nv=n(\nabla_N v)$ and $tv=0$ on B.

Thus we have

COROLLARY. *In order that a skew-symmetric tensor field v of M is a harmonic field in M normal to B it is necessary and sufficient that v satisfies*

$$\begin{cases} \nabla v=0 & \text{in } M \\ \hat{H}nv=n(\nabla_N v), \quad tv=0 & \text{on } B. \end{cases}$$

Next we consider Killing fields in M.

In the covariant derivative ∇v of a skew-symmetric tensor field v of M is *skew-symmetric at each point of M, we call v a Killing field in M.* v is a Killing field if and only if it satifies

$$dv=(p+1)\nabla v \qquad (p \text{ is the order of } v). \tag{3.9}$$

If v is a Killing field in M it satisfies

$$\Box v=0 \tag{3.10}$$

and

$$\delta v=0 \tag{3.11}$$

in M.

In order to obtain necessary and sufficient conditions for v to be Killing field, we shall introduce an integral formula.

For an arbitrary skew-symmetric tensor field v of M we have

$$\langle Dv, v\rangle+(p+1)\|\nabla v\|^2=\delta(\nabla v \llcorner v). \tag{3.12}$$

Forming $(p+1)\times(3.12)-(3.3)$, we have

$$\begin{aligned}\langle \Box v, v\rangle+\|(p+1)\nabla v\|^2-\|dv\|^2-\|\delta v\|^2 \\ =\delta[\{(p+1)\nabla v-dv\}\llcorner v-v\llcorner\delta v].\end{aligned}$$

Since

$$\langle (p+1)\nabla v, dv\rangle=\|dv\|^2$$

we have

$$\|(p+1)\nabla v-dv\|^2=\|(p+1)\nabla v\|^2-\|dv\|^2. \tag{3.13}$$

Therefore we get

$$\begin{aligned}\langle \Box v, v\rangle+\|(p+1)\nabla v-dv\|^2-\|\delta v\|^2 \\ =\delta[\{(p+1)\nabla v-dv\}\llcorner v-v\llcorner\delta v].\end{aligned} \tag{3.14}$$

Integrating (3.14) and applying Stokes' theorem to the right hand member we obtain an integral formula

$$\begin{aligned}\int_M[\langle \Box v, v\rangle+\|(p+1)\nabla v-dv\|^2-\|\delta v\|^2]d\sigma \\ =\int_B\langle N, \{(p+1)\nabla v-dv\}\llcorner v-v\llcorner\delta v\rangle d'\sigma.\end{aligned} \tag{3.15}$$

Therefore if v satisfies (3.10) and (3.11) in M and (3.9) only on B, we find that v is a Killing field in M.

Thus we have

THEOREM 3.4. *In order that a skew-symmetric tensor field v of M of order p is a Killing field in M, it is necessary and sufficient that v satisfies*

$$\begin{cases}\Box v=0, \quad \delta v=0 & \text{in } M \\ dv=(p+1)\nabla v & \text{on } B.\end{cases}$$

Since $N \lrcorner dv-(p+1)N \lrcorner \nabla v=N \lrcorner dv-(p+1)\nabla_N v$ is a skew-symmetric tensor at each point of B, we can make the tangential part and the normal part of the tensor, that is, the tangential part is given by $ndv-(p+1)t(\nabla_N v)$ and the normal part is given by $-(p+1)n(\nabla_N v)$. Therefore v satisfies (3.9) on B, if and only if it satisfies

(3.16) $$ndv=(p+1)t(\nabla_N v)$$

and

(3.17) $$n(\nabla_N v)=0$$

on B.

Since

$$\begin{aligned}\langle N, \{(p+1)\nabla v-dv\} \llcorner v-v \llcorner \delta v\rangle \\ =\langle (p+1)t(\nabla_N v)-ndv, tv\rangle+\langle (p+1)n(\nabla_N v)-t\delta v, nv\rangle ,\end{aligned}$$

we can write the integral formula (3.15) in the form

(3.18) $$\begin{aligned}&\int_M [\langle \Box v, v\rangle+\|(p+1)\nabla v-dv\|^2-\|\delta v\|^2]d\sigma \\ &=\int_B [\langle (p+1)t(\nabla_N v)-ndv, tv\rangle+\langle (p+1)n(\nabla_N v)-t\delta v, nv\rangle]d'\sigma .\end{aligned}$$

Thus we have

COROLLARY. *In order that a skew-symmetric tensor field v of M of order p is a Killing field in M, it is necessary and sufficient thal v satisfies*

$$\begin{cases} \Box v=0, \quad \delta v=0 & \text{in } M \\ ndv=(p+1)t(\nabla_N v), \quad n(\nabla_N v)=0 & \text{on } B . \end{cases}$$

Also we have from (3.18)

THEOREM 3.5. *In order that a skew-symnetric tensor field v of M of order p is a Killing field in M tangential to B, it is necessary and sufficient that v satisfies*

$$\begin{cases} \Box v=0, \quad \delta v=0 & \text{in } M \\ ndv=(p+1)t(\nabla_N v), \quad nv=0 & \text{on } B . \end{cases}$$

From (3.7) we can easily see that v satisfies (3.16) and is tangential to B if and only if it satisfies $pt(\nabla_N v)=-Htv$ and $nv=0$.

Thus we have

COROLLARY. *In order that a skew-symmetric tensor field v of M of order p is a Killing field in M tangential to B, it is necessary and sufficient that v satisfies*

$$\begin{cases} \Box v=0, \quad \delta v=0 & \text{in } M \\ pt(\nabla_N v)=-Htv \quad nv=0 & \text{on } B . \end{cases}$$

We have easily from (3.18)

THEOREM 3.6. *In order that a skew-symmetric tensor field v of M is a Killing field in M normal to B it is necessary and sufficient that v satisfies*

$$\begin{cases} \square v=0\,, \quad \delta v=0 & \text{in } M \\ n(\nabla_N v)=0\,, \quad tv=0 & \text{on } B\,. \end{cases}$$

It is easy to see from (3.8) that v satisfies (3.11) in M and (3.17) on B and is normal to B if and only if it satisfies (3.11) in M and $\hat{H}nv=0$ on B and is normal to B.

Thus we have

COROLLARY. *In order that a skew-symmetric tensor field v of M is a Killing field normal to B it is necessary and sufficient that v satisfies*

$$\begin{cases} \square v=0\,, \quad \delta v=0 & \text{in } M \\ \hat{H}nv=0\,, \quad tv=0 & \text{on } B\,. \end{cases}$$

In the rest of this section we consider conformal vector fields in M.

If a vector field v satisfies at each point of M

$$2\nabla v=dv+\frac{2}{m}\delta v\cdot g\,, \tag{3.19}$$

we call v a conformal vector field in M.[4]

A conformal vector field v in M satisfies

$$\square v+\frac{m-2}{m}d\delta v=0\,. \tag{3.20}$$

On the other hand for an arbitrary vector field v of M we have

$$\begin{aligned} &\langle \square v+\frac{m-2}{m}d\delta v, v\rangle+\| 2\nabla v-dv-\frac{2}{m}\delta v\cdot g\|^2 \\ &\qquad =\delta[\{2\nabla v-dv-\frac{2}{m}\delta v\cdot g\}\llcorner v\}\,. \end{aligned} \tag{3.21}$$

Therefore we get an integral formula

$$\begin{aligned} &\int_M[\langle \square v+\frac{m-2}{m}d\delta v, v\rangle+\| 2\nabla v-dv-\frac{2}{m}\delta v\cdot g\|^2]d\sigma \\ &\qquad =\int_B\langle N, \{2\nabla v-dv-\frac{2}{m}\delta v\cdot g\}\llcorner v\rangle d'\sigma\,. \end{aligned} \tag{3.22}$$

Thus we have

THEOREM 3.7. *In order that a vector field v of M is a conformal vector field in M, it is necessary and sufficient that v satisfies*

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4) It is easily seen that a conformal tensor field of order >1 defined in [18] is necessarily a Killing field.

$$\begin{cases} \square v + \dfrac{m-2}{m} d\delta v = 0 & \text{in } M \\ 2\nabla v = dv + \dfrac{2}{m} \delta v \cdot g & \text{on } B. \end{cases}$$

If v is a conformal vector field in M, contracting (3.19) with N we have

$$2\nabla_N v = N \lrcorner dv + \frac{2}{m} \delta v \cdot N. \tag{3.23}$$

Thus the tangential part and normal part of $\nabla_N v$ are given respectively by

$$2t(\nabla_N v) = ndv, \tag{3.24}$$

$$n(\nabla_N v) = \frac{1}{m} \delta v. \tag{3.25}$$

Since for an arbitrary vector field v of M we have

$$\begin{aligned} &\langle N, \{2\nabla v - dv - \frac{2}{m} \delta v \cdot g\} \llcorner v \rangle \\ &= \langle 2t(\nabla_N v) - n(dv), tv \rangle + 2\{n(\nabla_N v) - \frac{2}{m} \delta v\} \cdot nv \end{aligned}$$

the integral formula (3.22) can be written in the form

$$\begin{aligned} &\int_M [\langle \square v + \frac{m-2}{m} d\delta v, v \rangle + \| 2\nabla v - dv - \frac{2}{m} \delta v \cdot g \|^2] d\sigma \\ &= \int_B [\langle 2t(\nabla_N v) - ndv, tv \rangle + 2\{n\nabla_N v - \frac{1}{m} \delta v\} \cdot nv] d'\sigma. \end{aligned} \tag{3.26}$$

Thus we have

COROLLARY. *In order that a vector field v of M is a conformal vector field in M, it is necessary and sufficient that v satisfies*

$$\begin{cases} \square v + \dfrac{m-2}{m} d\delta v = 0 & \text{in } M \\ 2t(\nabla_N v) = ndv, \quad n\nabla_N v = \dfrac{1}{m} \delta v & \text{on } B. \end{cases}$$

Also we have from (3.26)

THEOREM 3.8. *In order that a vector field v of M is a conformal vector field in M tangential to B, it is necessary and sufficient that v satisfies*

$$\begin{cases} \square v + \dfrac{m-2}{m} d\delta v = 0 & \text{in } M \\ 2t(\nabla_N v) = ndv, \quad nv = 0 & \text{on } B. \end{cases}$$

From the formula (3.7) we can find that a vector field v satisfies (3.24) on B and is tangential to B if and only if it satisfies $t(\nabla_N v) = -Htv$, and $nv = 0$ on B.

Thus we have

COROLLARY. *In order that a vector field v of M is a conformal vector field in M tangential to B, it is necessary and sufficient that v satisfies*

$$\begin{cases} \square v+\dfrac{m-2}{m}d\delta v=0 & \text{in } M \\ t(\nabla_N v)=-Htv & \text{on } B. \end{cases}$$

From (3.26) we have

THEOREM 3.9. *In order that a vector field v of M is a conformal vector field in M normal to B it is necessary and sufficient that v satisfies*

$$\begin{cases} \square v+\dfrac{m-2}{m}d\delta v=0 & \text{in } M \\ n(\nabla_N v)=\dfrac{1}{m}\delta v\,, \quad tv=0 & \text{on } B. \end{cases}$$

From the formula (3.8), it is easily seen that a vector field v of M satisfies (3.25) and is normal to B if and only if it satisfies $(m-1)n(\nabla_N v)=-\hat{H}nv$, and $tv=0$ on B.

Thus we have

COROLLARY. *In order that a vector field v of M is a conformal vector field in M normal to B it is necessary and sufficient that v satisfies*

$$\begin{cases} \square v+\dfrac{m-2}{m}d\delta v=0 & \text{in } M \\ (m-1)n(\nabla_N v)=-\hat{H}nv\,, \quad tv=0 & \text{on } B. \end{cases}$$

§ 4. Non existence of harmonc and Kililng fields and conformal fields in M.

In this section we shall consider the conditions for non existence of harmonic fields, Killing fields or conformal vector fields.

Let v be an arbitrary skew-symmetric p-tensor field in M.

Forming (3.12)-(3.2) we have

$$\langle Dv-\Delta v, v\rangle+(p+1)\|\nabla v\|^2-\|dv\|^2-\|\delta v\|^2 \\ =\delta(\nabla v \llcorner v-dv\llcorner v-v\llcorner \delta v)\,. \tag{4.1}$$

Since

$$Dv-\Delta v=Fv$$

we can reduce (4.1) to the form

$$F^{(p)}(v)+(p+1)\|\nabla v\|^2-\|dv\|^2-\|\delta v\|^2=\delta(\nabla v\llcorner v-dv\llcorner v-v\llcorner \delta v)\,.$$

Integrating this equation and applying Stokes' theorem we have

$$\int_M [F^{(p)}(v)+(p+1)\|\nabla v\|^2-\|dv\|^2-\|\delta v\|^2]d\sigma$$

$$=\int_B \langle N, \nabla v \llcorner v - dv \llcorner v - v \llcorner \delta v\rangle d'\sigma .$$

Since we have

$$\langle N, \nabla v \llcorner v - dv \llcorner v - v \llcorner \delta v\rangle$$
$$=\langle t\nabla_N v - ndv, tv\rangle + \langle n\nabla_N v - t\delta v, nv\rangle$$

we get, using the formula (2.9) and (2.10),

$$\langle N, \nabla v \llcorner v - dv \llcorner v - v \llcorner \delta v\rangle$$
$$=H^{(p)}(tv)+\hat{H}^{(p-1)}(nv)+\langle dnv, tv\rangle - \langle \delta tv, nv\rangle .$$

On the other hand we find by virtue of compactness of B that

$$\int_B [\langle dnv, tv\rangle - \langle \delta tv, nv\rangle] d'\sigma = 2\int_B \langle dnv, tv\rangle d'\sigma .$$

Thus we obtain

$$\int_M [F^{(p)}(v)+(p+1)\|\nabla v\|^2-\|dv\|^2-\|\delta v\|^2]d\sigma \tag{4.2}$$

$$=\int_B [H^{(p)}(tv)+\hat{H}^{(p-1)}(nv)+2\langle dnv, tv\rangle] d'\sigma .$$

If v is a harmonic field in M, we have

$$\int_M [F^{(p)}(v)+(p+1)\|\nabla v\|^2] d'\sigma = \int_B [H^{(p)}(tv)+\hat{H}^{(p-1)}(nv)+2\langle dnv, tv\rangle] d'\sigma .$$

Thus we have

THEOREM 4.1. *If a harmonic field v in M of order p tangential* (*normal*) *to B satisfies $F^{(p)}(v)\geqq 0$ in M and $H^{(p)}(tv)\leqq 0$ $(\hat{H}^{(p-1)}(nv)\leqq 0)$ on B, v satisfies $F^{(p)}(v)=0$ and $\nabla v=0$ in M and $H^{(p)}(tv)=0$ $(\hat{H}^{(p-1)}(nv)=0)$ on B.*

Assume that $F^{(p)}$ is positive semi-definite at each point in M and $H^{(p)}(\hat{H}^{(p-1)})$ is negative semi-definite at each point on B, and let v be a harmonic field in M of order p tangential (normal) to B. Then v satisfies the conditions in Theorem 4.1, and therefore v satisfies $F^{(p)}(v)=0$ and $\nabla v=0$ in M and also $H^{(p)}(tv)=0$ $(\hat{H}^{(p-1)}(nv)=0)$ on B. Here we assume moreover that either $F^{(p)}$ is positive definite at one point at least in M or $H^{(p)}(\hat{H}^{(p-1)})$ is negative definite at one point at least on B. Under the former assumption, we have $v=0$ at the point and thus we can conclude that v vanishes at every point in M, because v satisfies $\nabla v=0$. Under the later one, we have $tv=0$ $(nv=0)$ at the point, and thus from the formula (2.7) we have $v=0$ at the point. Therefore we can also conclude that v vanishes at every point in M.

Thus we have

THEOREM 4.2. *There exists no harmonic field in M of order p tangential (normal) to B other than the zero tensor field, if $F^{(p)}$ is positive semi-definite everywhere in M, $H^{(p)}(\hat{H}^{(p-1)})$ is negative semi-definite everywhere on B, and there exists at least one point x in M such that $F^{(p)}$ is positive definite at x, or such that x is on B and $H^{(p)}(\hat{H}^{(p-1)})$ is negative definite at x.*

In particular, we have

COROLLARY 1. *There exists no harmonic field in M of order p tangential (normal) to B other than the zero tensor field, if one of the following conditions is satisfied in M:* (1) *$F^{(p)}$ is positive definite everywhere in M and $H^{(p)}(\hat{H}^{(p-1)})$ is negative semi-definite everywhere on B.* (2) *$F^{(p)}$ is positive semi-definite everywhere in M and $H^{(p)}(\hat{H}^{(p-1)})$ is negative definite everywhere on B.*

The condition (1) in the corollary has been obtained by Yano in [20].

From Theorem 1.2 we find that if $F^{(p)}$ is positive semi-definite (or definite) at a point then $F^{(m-p)}$ is also positive semi-definite (or definite) at the point. Also, from Theorem 2.1 we find that $H^{(p)}$ is negative semi-definite (or definite) if and only if $\hat{H}^{(m-p-1)}$ is negative semi-definite (or definite).

Thus we have

COROLLARY 2. *Under the assumptions in Theorem* 4.2, *there exists no harmonic field in M of order p tangential (normal) to B and no harmonic field in M of order $m-p$ normal (tangential) to B other than the zero tensor field.*

We denote by $R_p(M)$ ($R_p(M, B)$) the absolute p-th Betti number of M (the relative p-th Betti number of M modulo B).

By Duff and Spencer [7] $R_p(M)$ ($R_p(M, B)$) is equal to the number of linearly independent harmonic fields in M of order p tangential (normal) to B.

Then from Corollary 2 of Theorem 4.2. we have

COROLLARY 3. *Under the assumption in Theorem* 4.2 *we have*

$$R_p(M) = R_{m-p}(M, B) = 0 \qquad (R_{m-p}(M) = R_p(M, B) = 0)\,.$$

In particular, applying the result in the parentheses of the corollary to $p=1$, we have $R_1(M, B) = 0$.

On the other hand, because of connectedness of M, we find that $R_0(M) = 1$ and $R_0(M, B) = 0$.

Therefore using the homology sequence of the pair (M, B) (see [8]) we find $R_0(B) = R_0(M) = 1$. This means that B is connected.

Thus we have

COROLLARY 4. *If the Ricci curvature of V_m is non-negative at each point of M, the mean curvature of B is non-positive at each point of B, and moreover if there is one point at least in M such that the Ricci curvature is positive for all directions at the point or such that the point is on B and the mean curvature is negative at the point, then the boundary B of M is connected.*

Next we consider the Killing fields in M.

In the case of the Killing fields, the non existence conditions for the field tangential to B and for the field normal to B do not go in parallel, because from the corollary of Theorem 3.6, a Killing field v normal to B satisfies $\hat{H}nv=0$ and so $\hat{H}^{(p-1)}(nv)=0$.

First we consider the integral formula (4.2).

From (3.13) we have for an arbitrary skew-symmetric p-tensor field v,

$$(p+1)\|\nabla v\|^2-\|dv\|^2=\|(p+1)\nabla v-dv\|^2-p(p+1)\|\nabla v\|^2.$$

Then we can write the integral formula (4.2) in the form

$$\int_M [F^{(p)}(v)+\|(p+1)\nabla v-dv\|^2-p(p+1)\|\nabla v\|^2-\|\delta v\|^2]\,d\sigma$$
$$=\int_B [H^{(p)}(tv)+\hat{H}^{(p-1)}(nv)+2\langle dnv, tv\rangle]\,d'\sigma.$$

When v is in particular a Killing field in M, we have

$$\int_M [F^{(p)}(v)-p(p+1)\|\nabla v\|^2]\,d\sigma \tag{4.3}$$
$$=\int_B [H^{(p)}(tv)+\hat{H}^{(p-1)}(nv)+2\langle dnv, tv\rangle]\,d'\sigma.$$

Thus we have

THEOREM 4.3. *If a Killing field v in M of order p tangential to B satisfies $F^{(p)}(v)\leqq 0$ in M and $H^{(p)}(tv)\geqq 0$ on B, v satisfies $F^{(p)}(v)=0$ and $\nabla v=0$ in M and $H^{(p)}(tv)=0$ on B.*

By a similar argument as in Theorem 4.2, we have

THEOREM 4.4. *There exists no Killing field in M of order p tangential to B other than the zero tensor field, if $F^{(p)}$ is negative semi-definite everywhere in M, $H^{(p)}$ is positive semi-definite everywhere on B and if there exists one point at least in M such that $F^{(p)}$ is negative definite at the point, or such that the point is on B and $H^{(p)}$ is positive definite at the point.*

In particular

COROLLARY. *There exists no Killing field in M of order p tangential to B other than the zero tensor field, if one of the following conditions is satisfied in M: (1) $F^{(p)}$ is negative definite everywhere in M and $H^{(p)}$ is positive semi-definite everywhere on B. (2) $F^{(p)}$ is negative semi-definite everywhere in M and $H^{(p)}$ is positive definite everywhere on B.*

The condition (1) in the corollary has been obtained by Yano in [**20**].

If v is a Killing field in M of order p normal to B, then v satisfies

$$Hnv=0 \tag{4.4}$$

and the integral formula can be reduced to the form

$$\int_M [F^{(p)}(v)-p(p+1)\|\nabla v\|^2]\,d\sigma=0. \tag{4.5}$$

Thus we have

THEOREM 4.5. *If a Killing field in M of order p normal to B satisfies $F^{(p)}(nv)\leqq 0$, then v satisfies $F^{(p)}(v)=0$ and $\nabla v=0$ in M.*

If $F^{(p)}$ is negative semi-definite everywhere in M and there is one point at least in M where $F^{(p)}$ is negative definite, then any Killing field v in M of order p normal to B satisfies $F^{(p)}(v)=0$ and $\nabla v=0$ in M, and from this we have $v=0$ at the point. Therefore we can conclude that v vanishes everywhere in M.

If $F^{(p)}$ is negative semi-definite, the operator $\hat{H}$ is non-degenerate at one point at least on B as an endomorphism of the vector space of skew-symmetric $(p-1)$-tensors of B at the point, then, we have $\nabla v=0$ in M and $nv=0$ at the point and so we have $v=0$ at the point. Therefore we can find that v vanishes everywhere in M.

Thus we have

THEOREM 4.6. *There exists no Killing field in M of order p normal to B other than the zero tensor field, if one of the following conditions is satisfied in M:* (1) *$F^{(p)}$ is negative semi-definite everywhere in M and there is one point at least in M where $F^{(p)}$ is negative definite.* (2) *$F^{(p)}$ is negative semi-definite everywhere in M and there is one point at least on B where the operator $\hat{H}$ is non-degenerate as an endomorphism of the vector space of all skew-symmetric $(p-1)$-tensors of B at the point.*

In particular

COROLLARY. *If $F^{(p)}$ is negative definite everywhere in M, there exists no Killing field in M of order p normal to B other than the zero tensor field.*

This corollary has been obtained by Yano and the present author in [**23**].

Now, in order to obtain the non-existence conditions for conformal vector field, we take an arbitrary vector field of M. Then by an easy calculation we can write (4.2) in the form

$$\int_M [F^{(1)}(v)+\|2\nabla v-dv-\frac{2}{m}\delta v\cdot g\|^2-2\|\nabla v\|^2-\frac{m-2}{m}\|\delta v\|^2]\,d\sigma$$
$$=\int_B [H^{(1)}(tv)+\hat{H}^{(0)}(nv)+2\langle dnv, tv\rangle]\,d'\sigma\,,$$

where v is a conformal vector, we have

$$\int_M [F^{(1)}(v)-2\|\nabla v\|^2-\frac{m-2}{m}\|\delta v\|^2]\,d\sigma \tag{4.6}$$
$$=\int_B [H^{(1)}(tv)+\hat{H}^{(0)}(nv)+2\langle dnv, tv\rangle]\,d'\sigma\,.$$

Thus we have

THEOREM 4.7. *If a conformal vector field v in M tangential to B satisfies $F^{(1)}(v)\leqq 0$ in M and $H^{(1)}(tv)\geqq 0$ on B, v satisfies $F^{(1)}(v)=0$ and $\nabla v=0$ in M*

and $H^{(1)}(tv)=0$ *on* B.

THEOREM 4.8. *There exists no conformal vector field in* M *tangential to* B *other than the zero vector field, if the Ricci curvature of* V_m *is negative semi-definite everywhere in* M, *the second fundamental form of* B *is positive semi-definite everywhere on* B *and moreover there exists one point at least in* M *such that the Ricci curvature is negative definite at the point or such that the point is on* B *and there the second fundamental form is positive definite.*

In particular,

COROLLARY. *There exists no conformal vector field in* M *tangential to* B *other than the zero vector field if one of the following conditions is satisfied in* M: (1) *The Ricci curvature is negative definite everywhere in* M *and the second fundamental form of* B *is positive semi-definite everywhere on* B. (2) *The Ricci curvature is negative semi-definite everywhere in* M *and the second fundamental form of* B *is positive definite everywhere on* B.

The condition (1) in the corollary has been obtained by Yano in [20].

If we denote by Q the mean curvature of B, i.e.

$$Q=g^{ij}H_{ij}\,,$$

then $\hat{H}^{(0)}(nv)$ is written in the form

$$\hat{H}^{(0)}(nv)=Q\cdot(nv)^2\,.$$

Remark here that nv is a function on B for v is a vector field.

Then if v is a conformal vector field in M normal to B, (4.6) is reduced to the form

$$\int_M [F^{(1)}(v)-2\,\|\nabla v\|^2-\frac{m-2}{m}(\delta v)^2]\,d\sigma$$
$$=\int Q\cdot(nv)^2 d'\sigma\,.$$

Thus we have

THEOREM 4.9. *When the mean curvature of* B *is non-negative everywhere on* B, *a conformal vector field in* M *normal to* B *satisfying* $F^{(1)}(v)\leqq 0$ *satisfies* $F^{(1)}(v)=0$ *and* $\nabla v=0$ *in* M *and* $Q\cdot(nv)^2=0$ *on* B.

THEOREM 4.10. *There exists no conformal vector field in* M *normal to* B *other than the zero vector field, if the Ricci curvature is negative semi-definite everywhere in* M, *the mean curvature of* B *is non-negative everywhere on* B, *and moreover there exists one point at least in* M *such that the Ricci curvature is negative definite at the point or such that the point is on* B *and there the mean curvature does not vanish.*

In particular

COROLLARY 1. *There exists no conformal vector field on* M *normal to* B *other than the zero vector field, if one of the following conditions is satisfied in*

M: (1) *The Ricci curvature is negative definite everywhere in M and the mean curvature is non-negative everywhere on B.* (2) *The Ricci curvature is negative semi-definite everywhere in M and the mean curvature is positive everywhere on B.*

The condition (1) in the corollary has been obtained by Yano [**19**].

If the second fundamental form of B is positive semi-definite (or definite), then the mean curvature is non-negative (or positive).

Thus we have

COROLLARY 2. *Under the condition in Theorem* 4.5, *there exists no conformal vector field in M either tangential or normal to B other than the zero vector field.*

§ 5. Some applications.

We denote by K the curvature tensor field of M, by P the projective curvature tensor field of M, by Z the concircular curvature tensor field of M and by C the conformal curvature tensor field of M.

If in a local coordinate system (ξ^κ) the covariant components of the curvature tensor field K are denoted by $K_{\nu\mu\lambda\kappa}$ and those of the Ricci tensor field are denoted $K_{\mu\lambda}$, the covariant components of P, Z and C are given respectively by

$$P_{\nu\mu\lambda\kappa}=K_{\nu\mu\lambda\kappa}-\frac{1}{m-1}(g_{\nu\kappa}K_{\mu\lambda}-g_{\mu\kappa}K_{\nu\lambda})$$

$$Z_{\nu\mu\lambda\kappa}=K_{\nu\mu\lambda\kappa}-\frac{R}{m(m-1)}(g_{\nu\kappa}g_{\mu\lambda}-g_{\mu\kappa}g_{\nu\lambda})$$

$$C_{\nu\mu\lambda\kappa}=K_{\nu\mu\lambda\kappa}-\frac{1}{m-2}(g_{\nu\kappa}K_{\mu\lambda}-g_{\mu\kappa}K_{\nu\lambda}+g_{\mu\lambda}K_{\nu\kappa}-g_{\nu\lambda}K_{\mu\kappa})$$
$$+\frac{R}{(m-1)(m-2)}(g_{\nu\kappa}g_{\mu\lambda}-g_{\mu\kappa}g_{\nu\lambda}),$$

where R is the curvature scalar of M.

We denoe by L the smallest eigenvalue of the matrix $(K_{\mu\lambda})$ and also by L' the largest eigenvalue.

Now we introduce the quantities $\hat{P}$, $\hat{Z}$ and $\hat{C}$, respectively given by

$$\hat{P}=\operatorname*{Sup}_{w}\frac{2\langle P\llcorner w, w\rangle}{\|w\|^2}, \tag{5.1}$$

$$\hat{Z}=\operatorname*{Sup}_{w}\frac{2\langle Z\llcorner w, w\rangle}{\|w\|^2}, \tag{5.2}$$

$$\hat{C}=\operatorname*{Sup}_{w}\frac{2\langle C\llcorner w, w\rangle}{\|w\|^2}, \tag{5.3}$$

at each point of M, where w is a skew-symmetric tensor of order 2.

First we consider the curvature tensor field K.

If K satisfies

$$0<\frac{1}{2}A\leqq-\frac{2\langle K\llcorner w,w\rangle}{\|w\|^2}\leqq A, \tag{5.4}$$

for any skew-symmetric tensor of order 2 and for some positive constant A at each point of M, then the quadratic form $F^{(p)}$ is positive definite for $p=1,2,\cdots,[m/2]$. (See [18] or [22].) From Theorem 1.2, we find that $F^{(p)}$ is positive definite for $p=1,2,\cdots,m-1$.

Similary, if K satisfies

$$A\geqq\frac{2\langle K\llcorner w,w\rangle}{\|w\|^2}\geqq\frac{1}{2}A\geqq 0 \tag{5.5}$$

for any skew-symmetric tensor of order 2 and for some positive constant A at each point of M, then $F^{(p)}$ is negative definite for $p=1,2,\cdots,[m/2]$ (see [18] or [22]), and thus $F^{(p)}$ is negative definite for $p=1,2,\cdots,m-1$.

Next we consider P, Z and C.

Using $\hat{P}$, L and L' we can find that for any skew-symmetric tensor v of order p, we have (see [18] or [22])

$$F^{(p)}(v)\geqq p\Big(\frac{m-p}{m-1}L-\frac{p-1}{2}\hat{P}\Big)\|v\|^2 \tag{5.6}$$

and also

$$F^{(p)}(v)\leqq p\Big(\frac{p-1}{2}\hat{P}-\frac{m-p}{m-1}L'\Big)\|v\|^2. \tag{5.7}$$

Therefore if L and $\hat{P}$ satisfy

$$\frac{m-p}{m-1}L>\frac{p-1}{2}\hat{P}, \qquad p=1,\cdots,m-1 \tag{5.8}$$

$F^{(p)}$ is positive definite for $p=1,\cdots,m-1$ and if they satisfy

$$\frac{m-p}{m-1}L\geqq\frac{p-1}{2}\hat{P}, \qquad p=1,\cdots,m-1 \tag{5.9}$$

then $F^{(p)}$ is positive semi-definite for $p=1,\cdots,m-1$. Also if L' and $\hat{P}$ satisfy

$$\frac{m-p}{m-1}L'>\frac{p-1}{2}\hat{P}, \qquad p=1,\cdots,m-1 \tag{5.10}$$

$F^{(p)}$ is negative definite for $p=1,\cdots,m-1$ and if they satisfy

$$\frac{m-p}{m-1}L'\geqq\frac{p-1}{2}\hat{P}, \qquad p=1,\cdots,m-1 \tag{5.11}$$

$F^{(p)}$ is negative semi-definite for $p=1,\cdots,m-1$.

Using L, L', R and $\hat{Z}$, we have for any skew-symmetric tensor v of order p (see [18] or [22])

$$F^{(p)}(v)\geqq p\Big(L-\frac{(p-1)}{m(m-1)}R-\frac{p-1}{2}\hat{Z}\Big)\|v\|^2 \tag{5.12}$$

and

$$F^{(p)}(v) \leqq p\Big(-L' - \frac{(p-1)}{m(m-1)}R + \frac{p-1}{2}\hat{Z}\Big)\|v\|^2. \tag{5.13}$$

Thus if L, R and $\hat{Z}$ satisfy

$$L - \frac{(p-1)}{m(m-1)}R > \frac{p-1}{2}\hat{Z}, \qquad (p=1, \cdots, m-1) \tag{5.14}$$

$F^{(p)}$ is positive definite for $p=1, \cdots, m-1$ and if they satisfy

$$L - \frac{(p-1)}{m(m-1)}R \geqq \frac{p-1}{2}\hat{Z}, \qquad (p=1, \cdots, m-1) \tag{5.15}$$

then $F^{(p)}$ is positive semi-definite for $p=1, \cdots, m-1$. Also if L', R, $\hat{Z}$ satisfy

$$L' + \frac{p-1}{m(m-1)}R > \frac{p-1}{2}\hat{Z}, \qquad (p=1, \cdots, m-1) \tag{5.16}$$

then $F^{(p)}$ is negative definite for $p=1, \cdots, m-1$ and if they satisfy

$$L' + \frac{p-1}{m(m-1)}R \geqq \frac{p-1}{2}\hat{Z}, \qquad (p=1, \cdots, m-1) \tag{5.17}$$

$F^{(p)}$ is negative semi-definite for $p=1, \cdots, m-1$.

Using L, L', R and $\hat{C}$ we have for any skew-symmetric tensor v of order $p(\leqq m/2)$ (see [**18**] or [**22**])

$$F^{(p)}(v) \geqq p\Big(\frac{m-2p}{m-2}L + \frac{p-1}{(m-1)(m-2)}R - \frac{p-1}{2}\hat{C}\Big)\|v\|^2 \tag{5.18}$$

and

$$F^{(p)}(v) \leqq p\Big(-\frac{m-2p}{m-2}L' + \frac{p-1}{(m-1)(m-2)}R + \frac{p-1}{2}\hat{C}\Big)\|v\|^2. \tag{5.19}$$

Thus if L, R and $\hat{C}$ satisfy

$$\frac{m-2p}{m-2}L + \frac{p-1}{(m-1)(m-2)}R > \frac{p-1}{2}\hat{C} \qquad (p=1, \cdots, [m/2]), \tag{5.20}$$

$F^{(p)}$ is positive definite for $p=1, \cdots, [m/2]$ and therefore from Theorem **1.2** we find that $F^{(p)}$ is positive definite for $p=1, \cdots, m-1$, and if they satisfy

$$\frac{m-2p}{m-2}L + \frac{p-1}{(m-1)(m-2)}R \geqq \frac{p-1}{2}\hat{C} \qquad (p=1, \cdots, [m/2]) \tag{5.21}$$

then $F^{(p)}$ is positive semi-definite for $p=1, 2, \cdots, m-1$. Also if L', R, $\hat{C}$ satisfy

$$\frac{m-2p}{m-2}L' - \frac{p-1}{(m-1)(m-2)}R > \frac{p-1}{2}\hat{C} \qquad (p=1, \cdots, [m/2]) \tag{5.22}$$

then $F^{(p)}$ is negative definite for $p=1, \cdots, m-1$, and if they satisfy

$$\frac{m-2p}{m-2}L' - \frac{p-1}{(m-1)(m-2)}R \geqq \frac{p-1}{2}\hat{C} \qquad (p=1, \cdots, [m/2]) \tag{5.23}$$

then $F^{(p)}$ is negative semi-definite for $p=1, \cdots, m-1$.

Thus we have

THEOREM 5.1. *If, in a compact Riemannian manifold M with convex boundary B, one of the inequalities* (5.8), (5.14) *or* (5.20) *is satisfied, or the curvature tensor field K satisfies* (5.4) *for any skew-symmetric tensor of order* 2 *and for some postive constant A, then we have* $R_p(M)=R_p(M, B)=0$ *for* $p=1, \cdots, m-1$.

THEOREM 5.2. *If, in a compact Riemannian manifold M with convex boundary B, one of the inequalities* (5.9), (5.15) *or* (5.21) *is satisfied and there exists one point at least on B where the second fundamental form of B is negative definite, then we have* $R_p(M)=R_p(M, B)=0$.

THEOREM 5.3. *If, in a compact Riemannian manifold M with boundary B, one of the inequalities* (5.10), (5.16) *or* (5.22) *is satisfied or the curvature tensor field K satisfies* (5.5) *for any skew-symmetric tensor of order* 2 *and for some positve constant A, then there exists no Killing field in M of order* $p\,(=1, 2, \cdots, m-1)$, *normal to B other than the zero tensor field.*

THEOREM 5.4. *If, in a compact Riemannian manifold M with boundary B, one of the inequalities* (5.11), (5.17) *or* (5.23) *is satisfied and there exists one point at least on B where the operator* $\hat{H}$ *is non-degenerate, then there exists no Killing field in M of order* $p\ (=1, 2, \cdots, m-1)$ *normal to B other than the zero tensor field.*

THEOREM 5.5. *If, in a Riemannian manifold M with concave boundary B, one of the inequalities* (5.10), (5.16) *or* (5.22) *is satisfied or the curvature tensor field K satisfies* (5.5) *for any skew-symmetric tensor of order* 2 *and for some positive constant A, there exists neither Killing field in M of order* $p\ (=1, \cdots, m-1)$ *tangential to B nor conformal vector field in M tangential or normal to B other than the zero tensor field.*

THEOREM 5.6. *If, in a Riemannian manifold M with concave boundary B, one of the inequalities* (5.11), (5.17) *or* (5.23) *is satisfied and there exists one point at least on B where the second fundamental form of B is positive definite, then there exists neither Killing field in M of order* $p\ (=1, 2, \cdots, m-1)$ *tangential to B nor conformal vector field in M tangential or normal to B other than the zero tensor field.*

Musashi Institute of Technology

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