J. Math. Soc. Japan
Vol. 14, No. 2, 1962

On hierarchies of predicates of ordinal numbers

By Gaisi TAKEUTI and Akiko KINO

(Received Sept. 4, 1961)

We shall assume the axiom of constructibility (' $V=L$ ', see Gödel [4]) throughout this paper.

In a former paper [17], we considered the class of ordinal numbers less than a certain cardinal number ($>\omega$) and defined the notion of semi-recursive and recursive functions. In this paper we shall consider the second number class and define the notion of primitive recursive functions and predicates by following Kleene's definition of natural numbers (§ 1). We shall also define the notion of recursive functions (sometimes called general recursive functions) analyzing Kleene's definition of natural numbers. The classes of both recursive functions defined in [17] and in this paper are seen to coincide (§ 2). We can construct a model of set theory in the primitive recursive predicates (§ 3). By restricting the usage of ' primitive recursion ' only to define functions neccessary to construct the model, we obtain elementary functions (§ 4). Then a predicate is general recursive if and only if it is expressible in both forms consisting of a universal and existential quantifiers prefixed to elementary predicates (§ 5). The class of predicates expressible in a given form consisting of a fixed succession of one or more quantifiers prefixed to a predicate is the same whether the predicate is allowed to be general recursive or primitive recursive (or elementary). We shall prove the enumeration theorem, the normal form theorem and the hierarchy theorem for the predicates described above (§ 6). Moreover we shall show that for $k\geqq 1$ the predicate expressible in the k-quantifier form in our sense is expressible in the $k+1$-function-quantifier form in Kleene's sense and vice versa. (This would also follow from the results of Kuratowski [13] and Spector [14].) An analytic predicate is expressible in both 2-function-quantifier forms in Kleene's sense, if and only if it can be expressible as a (general) recursive predicate in our sense (§§ 7–8). We shall characterize hyperarithmetical predicates in our hierarchy of ordinal numbers (§ 9). Let us call an ordinal number to be recursively expressible if it is expressible by means of (general) recursive functions, 0 and ω, and denote the least ordinal number not recursively expressible as ω^*. We shall show the predicate $a<\omega^*$ is not (general) recursive, i.e. not expressible in both 2-function-quantifier forms in Kleene's sense, but is expressible in Σ_2^1-form.

Moreover for any (general) recursive function $g(a)$ in our sense, we can find a primitive recursive function $p(a,b)$ and a recursively expressible ordinal number c^* such that

$$g(a)=p(a,c^*) \qquad \text{for} \quad a<\omega .$$

From the above arguments it seems that the investigation of properties of ω^* and primitive recursive functions in our sense may throw light on the study for finding $\Sigma_2^1 \cap \Pi_2^1$ hierarchy (§ 10).

There are many trials (e. g. [**12**]) for extending Church-Kleene's constructive ordinals ([**3**], [**5**], [**9**]). In each of these extensions the predicates representing that {a is a notation for an extended constructive ordinal} and {a is less than b in the sense of notations for the constructive ordinals} are expressible in both 2-function-quantifier forms. Then we see that the least ordinal not represented by notations in each of those systems is less than ω^*. It seems very interesting to define a system of notations not expressible in both 2-function-quantifier forms and compare the least ordinal not represented in the system to ω^*. For this, we shall give a candidate at the end of this paper (§ 11).

In the following we shall sometimes state or prove a proposition or theorem concerning functions or predicates only in the case that they are of one-argument, but it will be easily understood how to be extended to general cases. When the axiom of constructibility is not needed for a proposition or theorem or section, its number is marked with the symbol $\circ$.

The authors wish to express their indebtedness to Dr. T. Tugué for reading this paper in manuscript, and for suggesting a number of improvements.

§ 1°. Primitive recursive functions.

We say simply 'a is an ordinal number', if a is an ordinal number in the second number class. We use the concepts on ordinal numbers $=$, $<$, 0, ω, a' (successor of a) and $\max(a,b)$ as usual. We also use the functions defined as follows:

$$\mathrm{Iq}(a,b)=\begin{cases} 0 & \text{if} \quad a<b , \\ 1\ (=0') & \text{otherwise.} \end{cases}$$

$$j(g^1(a),g^2(a))=a, \quad g^1(j(a,b))=a, \quad g^2(j(a,b))=b .$$

$$j(a,b)<j(c,d) \rightleftarrows \max(a,b)<\max(c,d)$$
$$\vee (\max(a,b)=\max(c,d) \wedge (b<d \vee (b=d \wedge a<c))) .$$

$$\mu x_{x<a} f(a_1,\cdots,a_n,x)=\begin{cases} \text{the least } x \text{ such that } x<a \text{ and} \\ f(a_1,\cdots,a_n,x)=0 \quad \text{if such exists,} \\ 0 \quad \text{otherwise.} \end{cases}$$

$$\mu x f(a_1, \cdots, a_n, x) = \begin{cases} \text{the least } x \text{ such that } f(a_1, \cdots, a_n, x) = 0 \\ \text{if such exists,} \\ 0 \quad \text{otherwise.} \end{cases}$$

(In the following we shall sometimes abbreviate $\mu x_{x<a} f(a_1, \cdots, a_n, x)$ and $\mu x f(a_1, \cdots, a_n, x)$ as $\mu x_{x<a} A(a_1, \cdots, a_n, x)$ and $\mu x A(a_1, \cdots, a_n, x)$ respectively, if

$$\forall a_1 \cdots \forall a_n \forall x (f(a_1, \cdots, a_n, x) = 0 \leftrightarrow A(a_1, \cdots, a_n, x)).)$$

We can define the following functions by combining $0, \omega, a'$, Iq, max, j, g^1, g^2 and $\mu x_{x<a}$(bounded minimum):

$$N(a) = \begin{cases} 0 & \text{if } 0 < a, \\ 1 & \text{otherwise,} \end{cases}$$

which is defined to be $\mathrm{Iq}(0, a)$.

$$\mathrm{Dj}(a, b) = \begin{cases} 0 & \text{if } a = 0 \text{ or } b = 0, \\ 1 & \text{otherwise,} \end{cases}$$

which is defined to be $N(\max(N(a), N(b)))$.

$$\mathrm{Eq}(a,b) = \begin{cases} 0 & \text{if } a = b, \\ 1 & \text{otherwise,} \end{cases}$$

which is defined to be $\max(N(\mathrm{Iq}(a, b)), N(\mathrm{Iq}(b, a)))$.

$$\delta(a) = \begin{cases} b & \text{if } a = b', \\ a & \text{otherwise,} \end{cases}$$

which is defined to be $\mu x_{x<a'} \mathrm{Dj}(\mathrm{Eq}(a, x'), \mathrm{Eq}(a, x))$.

$$f^a(b, a_1, \cdots, a_n) = \begin{cases} f(b, a_1, \cdots, a_n) & \text{if } b < a, \\ 0 & \text{otherwise,} \end{cases}$$

which is defined to be $\mu x_{x<f(b,a_1,\cdots,a_n)'} \mathrm{Dj}(\max(\mathrm{Eq}(x, f(b, a_1, \cdots, a_n)), \mathrm{Iq}(b, a)), \max(\mathrm{Eq}(x, 0), N(\mathrm{Iq}(b, a))))$.

(Note that in the definition of f^a a is compared only with the leftmost argument of f. $f^a(b, a_1, \cdots, a_n)$ is also written as

$$\mathrm{Con}(\{x\} f(x, a_1, \cdots, a_n), a, b)$$

following the notation given in [**17**].)

$$S(f, g, h, a) = \begin{cases} g(a) & \text{if } f(a) = 0, \\ h(a) & \text{otherwise,} \end{cases}$$

which is defined to be

$$\mu x_{x<\max(g(a), h(a))'} \mathrm{Dj}(\max(f(a), \mathrm{Eq}(x, g(a))), \max(N(f(a)), \mathrm{Eq}(x, h(a)))).$$

DEFINITION. A function is called to be *primitive recursive*, if it can be defined by a series of applications of the following schemata:

(I) $f(a)=a'$.

(II) $\begin{cases} f(a)=0. \\ f(a)=\omega. \end{cases}$

(III) $f(a)=a$.

(IV) $f(a,b)=\mathrm{Iq}(a,b)$.

(V) $f(a,b)=\max(a,b)$.

(VI) $f(a,b)=j(a,b)$.

(VII) $\begin{cases} f(a)=g^1(a). \\ f(a)=g^2(a). \end{cases}$

(VIII) $f(h,a_1,\cdots,a_n)=h(a_1,\cdots,a_n)$,

where h is a function variable. In (IX)-(XII) $h_1,\cdots,h_m$ are function variables.

(IX) $f(h_1,\cdots,h_{m^+},a_1,\cdots,a_n)$
$=g(h_1,\cdots,h_m,g_1(h_1,\cdots,h_m,a_1,\cdots,a_n),\cdots,g_l(h_1,\cdots,h_m,a_1,\cdots,a_n)),\ m\leqq m^+$.

(X) $\begin{cases} f(h_1,\cdots,h_m,a_1,\cdots,a_n,a)=g(h_1,\cdots,h_m,a_1,\cdots,a_n). \\ f(h_1,\cdots,h_m,a,a_1,\cdots,a_n)=g(h_1,\cdots,h_m,a_1,\cdots,a_n). \end{cases}$

(XI) $f(h_1,\cdots,h_m,a_1,\cdots,a_n,a)=\mu x_{x<a}g(h_1,\cdots,h_m,a_1,\cdots,a_n,x)$.

(XII) $f(h_1,\cdots,h_m,a,a_1,\cdots,a_n)=C(f^a,h_1,\cdots,h_m,a,a_1,\cdots,a_n)$,

which is called 'primitive recursion'[1].

DEFINITION. A function is called to be *primitive recursive in the narrow sense*, if it can be defined by a series of applications of the schemata (I)-(VII), (IX)-(XII)[2].

DEFINITION. Let $F(h_1,\cdots,h_m,a_1,\cdots,a_n)$ be a predicate. A function $f(h_1,\cdots,h_m,a_1,\cdots,a_n)$ is called to be a *representing function* of $F(h_1,\cdots,h_m,a_1,\cdots a_n)$ if $f(h_1,\cdots,h_m,a_1,\cdots,a_n)$ takes only 0 and 1 as values and

$$\forall x_1\cdots\forall x_n(F(h_1,\cdots,h_m,x_1,\cdots,x_n)\mapsto f(h_1,\cdots,h_m,x_1,\cdots,x_n)=0).$$

DEFINITION. A predicate is called to be *primitive recursive*, if it has a primitive recursive *representing function*. A predicate is called to be *primitive recursive in the narrow sense*, if it has a representing function primitive recursive in the narrow sense.

We see easily the following propositions.

1) Consider a kind of function combination obtained from functions already defined by applications of the schemata and containing function variables $h,h_1,\cdots,h_m$. We write it as $C(h,h_1,\cdots,h_m)$. In application of (XII), $C(h,h_1,\cdots,h_m,a)$ is of this kind.

2) In this case, $h_1,\cdots,h_m$ are, of course, void in the schemata (IX)—(XII). Thus $C(h,a,a_1,\cdots,a_n)$ in (XII) is a function combination containing only one function variable h. The function defined by an application of (XII) should not contain any function variable.

PROPOSITION 1. *Let $F(f, \cdots, g, a, \cdots, b)$ be a predicate constructed from primitive recursive predicates, propositional connectives ($\neg, \wedge, \vee, \vdash$) and bounded quantifiers. Then $F(f, \cdots, g, a, \cdots, b)$ is primitive recursive.*

PROPOSITION 2. *Let $A_1(a), \cdots, A_n(a)$ be primitive recursive predicates such that $\forall x(A_1(x) \vee \cdots \vee A_n(x))$ and $\forall x \neg (A_i(x) \wedge A_j(x))$ for $i \neq j$ and $f_1(a), \cdots, f_n(a)$ primitive recursive functions. Then there exists a primitive recursive function $f(a)$ such that*

$$\forall x((A_1(x) \vdash f(x) = f_1(x)) \wedge \cdots \wedge (A_n(x) \vdash f(x) = f_n(x))) .$$

PROPOSITION 3. *Let $A_1(a), \cdots, A_n(a)$ be primitive recursive predicates such that $\forall x(A_1(x) \vee \cdots \vee A_n(x))$ and $\forall x \neg (A_i(x) \wedge A_j(x))$ for $i \neq j$ and $f_1(f, a), \cdots, f_n(f, a)$ primitive recursive functions. Then there exists a primitive recursive function $g(a)$ such that*

(1) $$\forall x((A_1(x) \vdash g(x) = f_1(g^x, x)) \wedge \cdots \wedge (A_n(x) \vdash g(x) = f_n(g^x, x))) .$$

In this case we say that the function $g(a)$ is defined by the predicate (1).

PROPOSITION 4. *Let $A_1(a), \cdots, A_n(a)$ be primitive recursive predicates such that $\forall x(A_1(x) \vee \cdots \vee A_n(x))$ and $\forall x \neg (A_i(x) \wedge A_j(x))$ for $i \neq j$ and $B_1(f, a), \cdots, B_n(f, a)$ primitive recursive predicates. Then there exists a primitive recursive function $g(a)$ such that*

$$\forall x((A_1(x) \vdash (g(x) = 0 \leftrightarrow B_1(g^x, x))) \wedge \cdots \wedge (A_n(x) \vdash (g(x) = 0 \leftrightarrow B_n(g^x, x)))) .$$

§ 2°. General recursive functions.

We defined the notion of 'recursive function' in [**17**]. In this paper we define 'recursive function' (which is also called 'general recursive' if we consider them in connection with primitive recursive functions and want to emphasize the circumstances) as follows. It is easily seen that a function is recursive in this sense if and only if it is recursive in the sense defined in [**17**].

DEFINITION. A function is called to be (*general*) *recursive*, if it can be defined by a series of applications of the schemata (I)-(XII) and the following additional schema

(XIII) $f(h_1, \cdots, h_m, a_1, \cdots, a_n) = \mu x g(h_1, \cdots, h_m, x, a_1, \cdots, a_n)$,

where g must satisfy the condition

$$\forall x_1 \cdots \forall x_n \exists x(g(h_1, \cdots, h_m, x_1, \cdots, x_n, x) = 0) .$$

DEFINITION. A function is called to be (*general*) *recursive in the narrow sense*, if it can be defined by a series of applications of the schemata (I)-(VII), (IX)-(XIII).

DEFINITION. A predicate is called to be (*general*) *recursive*, if it has a general recursive representing function. A predicate is called to be (*general*) *recursive in the narrow sense*, if it has a representing function general recursive in the narrow sense.

§ 3°. Construction of a model of the set theory.

In this section we shall show that we can construct a model of set theory in the theory of primitive recursive functions analyzing the construction given in [16]. In the following we shall use the notation $\{x\}$ instead of the usual notation λx or $\hat{x}$.

Let $\sup(f, a; b)$ be an abbreviation of

$$\mu x_{x<b} \forall y(y<a \vdash f(y)<x).$$

$a+b$ is defined by

$a+0=a \wedge \forall x(0<x \vdash a+x=\sup(\{z\}\mathrm{Con}(\{u\}(a+u), x, z), x; j(a, x)'))$.

$g^{ij}(a)$ is defined to be $g^i(g^j(a))$ where $i=1,2$; $j=1,2$. We define successively $0'=1,\ 1'=2,\ 2'=3,\ 3'=4,\ 4'=5,\ 5'=6,\ 6'=7,\ 7'=8,\ 8'=9$.

$J(a, b)$ is defined by

$J(0, b)=0 \wedge \forall x(\delta(x)<x \vdash J(x, b)=\mathrm{Con}(\{u\}J(u, b), x, \delta(x))+b)$

$\wedge \forall x(0<x \wedge x=\delta(x) \vdash J(x, b)=\sup(\{z\}\mathrm{Con}(\{u\}J(u, b), x, z), x; j(b+x', b)'))$.

$j(c, a, b)$ is defined to be $J(j(a, b), 9)+c$.

$g_0(a)$ is defined to be $\mu z_{z<9} \exists x \exists y(x<a' \wedge y<a' \wedge a=j(z, x, y))$.

$g_1(a)$ is defined to be $\mu z_{z<a'} \exists x \exists y(x<9 \wedge y<a' \wedge a=j(x, z, y))$.

$g_2(a)$ is defined to be $\mu z_{z<a'} \exists x \exists y(x<9 \wedge y<a' \wedge a=j(x, y, z))$.

Let $<(f, b, c)$ be $b>c \wedge f(j(b, c))=0$;

$=(f, b, c)$ be $(b \leqq c \wedge f(j(b, c))=0) \vee (b \geqq c \wedge f(j(c, b))=0)$;

$\lessdot(f, b, c)$ be $\exists x(=(f, x, c) \wedge <(f, b, x))$;

$=(f: b; \{c; d\})$ be $\forall x(x<b \vdash (\lessdot(f, b, x) \leftrightarrow =(f, x, c) \vee =(f, x, d)))$
$\wedge \exists x(x<b \wedge =(f, x, c)) \wedge \exists x(x<b \wedge =(f, x, d))$;

$\lessdot(f: b; \{c; d\})$ be $\exists x(x<b \wedge \lessdot(f, b, x) \wedge =(f: x; \{c; d\}))$;

$=(f: b; <c; d>)$ be

$\exists x \exists y(x<b \wedge y<b \wedge =(f: b; \{x; y\}) \wedge =(f: x; \{c; c\}) \wedge =(f: y; \{c; d\}))$;

$\lessdot(f: b; <c; d>)$ be $\exists x(x<b \wedge \lessdot(f, b, x) \wedge =(f: x; <c; d>))$;

$=(f: b; <c; d; e>)$ be $\exists x(x<b \wedge =(f: b; <c; x>) \wedge =(f: x; <d; e>))$;

$\lessdot(f: b; <c; d; e>)$ be $\exists x(x<b \wedge \lessdot(f, b, x) \wedge =(f: x; <c; d; e>))$.

Moreover let

$H_1(f, a)$ be $=(f, g^2(a), g_1(g^1(a))) \vee =(f, g^2(a), g_2(g^1(a)))$;

$H_2(f, a)$ be $\lessdot(f, g_1(g^1(a)), g^2(a))$
$\wedge \exists x \exists y(x<g^2(a) \wedge y<g^2(a) \wedge \lessdot(f, y, x) \wedge =(f: g^2(a); <x; y>))$;

$H_3(f, a)$ be $\lessdot(f, g_1(g^1(a)), g^2(a)) \wedge \neg \lessdot(f, g_2(g^1(a)), g^2(a))$;

$H_4(f, a)$ be $\leqslant(f, g_1(g^1(a)), g^2(a))$
$\wedge \exists x \exists y(x < g^2(a) \wedge y < g^2(a) \wedge =(f : g^2(a) ; <x ; y>) \wedge \leqslant(f, g_2(g^1(a)), y))$;
$H_5(f, a)$ be $\exists x(x < g_1(g^1(a)) \wedge \leqslant(f : g_1(g^1(a)) ; <x ; g_2(a)>))$;
$H_6(f, a)$ be $\leqslant(f, g_1(g^1(a)), g^2(a)) \wedge \exists x \exists y(x < g_2(g^1(a)) \wedge y < g_2(g^1(a))$
$\wedge \leqslant(f : g_2(g^1(a)) ; <x ; y>) \wedge =(f : g^2(a) ; <y ; x>))$;
$H_7(f, a)$ be $\leqslant(f, g_1(g^1(a)), g^2(a)) \wedge \exists x \exists y \exists z(x < g_2(g^1(a)) \wedge y < g_2(g^1(a)) \wedge z < g_2(g^1(a))$
$\wedge \leqslant(f : g_2(g^1(a)) ; <x ; y ; z>) \wedge =(f : g^2(a) ; <y ; z ; x>))$;
$H_8(f, a)$ be $\leqslant(f, g_1(g^1(a)), g^2(a)) \wedge \exists x \exists y \exists z(x < g_2(g^1(a)) \wedge y < g_2(g^1(a)) \wedge z < g_2(g^1(a))$
$\wedge \leqslant(f : g_2(g^1(a)) ; <x ; y ; z>) \wedge =(f : g^2(a) ; <x ; z ; y>))$;
$H_9(f, a)$ be $\forall x(x < g^2(a) \vdash (\leqslant(f, g^1(a), x) \vdash\dashv \leqslant(f, g^2(a), x)))$.
Then, according to Proposition 4, there exists a primitive recursive **function** $fn(a)$ with the following properties:
$g^1(a) > g^2(a) \wedge g_0(g^1(a)) = 0 \vdash fn(a) = 0$,
$g^1(a) > g^2(a) \wedge g_0(g^1(a)) = 1 \vdash (fn(a) = 0 \vdash\dashv H_1(fn^a, a))$,
$g^1(a) > g^2(a) \wedge g_0(g^1(a)) = 2 \vdash (fn(a) = 0 \vdash\dashv H_2(fn^a, a))$,
$g^1(a) > g^2(a) \wedge g_0(g^1(a)) = 3 \vdash (fn(a) = 0 \vdash\dashv H_3(fn^a, a))$,
$g^1(a) > g^2(a) \wedge g_0(g^1(a)) = 4 \vdash (fn(a) = 0 \vdash\dashv H_4(fn^a, a))$,
$g^1(a) > g^2(a) \wedge g_0(g^1(a)) = 5 \vdash (fn(a) = 0 \vdash\dashv H_5(fn^a, a))$,
$g^1(a) > g^2(a) \wedge g_0(g^1(a)) = 6 \vdash (fn(a) = 0 \vdash\dashv H_6(fn^a, a))$,
$g^1(a) > g^2(a) \wedge g_0(g^1(a)) = 7 \vdash (fn(a) = 0 \vdash\dashv H_7(fn^a, a))$,
$g^1(a) > g^2(a) \wedge g_0(g^1(a)) = 8 \vdash (fn(a) = 0 \vdash\dashv H_8(fn^a, a))$,
$g^1(a) \leqq g^2(a) \vdash (fn(a) = 0 \vdash\dashv H_9(fn^a, a))$.

We use the following abbreviations; $c \in b$ for $\leqslant(fn, b, c)$; $b \equiv c$ for $=(fn, b, c)$; $\{b, c\}$ for $j(1, b, c)$; $<c, d>$ for $\{\{c, c\}, \{c, d\}\}$; $<c, d, e>$ for $<c, <d, e>>$; Od(a) for $\mu x_{x<a'}(x \equiv a)$; $C(a)$ for $\mu x_{x<a}(x \in a)$; $b - a$ for $j(3, b, a)$; $b \cap c$ for $b-(b-c)$; $a \subseteqq b$ for $\forall x(x < a \wedge x \in a \vdash x \in b)$; $a \subset b$ for $a \subseteqq b \wedge \neg(a \equiv b)$.
Then we have the following properties: (For the proof, cf. [**15**, pp. 209-214].)
$g_0(b) = 0 \vdash (a \in b \vdash\dashv \exists x(x \equiv a \wedge x < b))$;
$d \in \{b, c\} \vdash\dashv b \equiv d \vee c \equiv d$;
$a \equiv b \wedge c \equiv d \vdash \{a, c\} \equiv \{b, d\}$;
$a \equiv c \wedge b \equiv d \vdash\dashv <a, b> \equiv <c, d>$;
$a \equiv b \wedge c \equiv d \wedge e \equiv f \vdash\dashv <a, c, e> \equiv <b, d, f>$;
$g_0(b) = 2 \vdash (c \in b \vdash\dashv c \in g_1(b) \wedge \exists x \exists y(x \in y \wedge c \equiv <x, y>))$;
$a \in (b-c) \vdash\dashv a \in b \wedge \neg(a \in c)$;
$a \in b \cap c \vdash\dashv a \in b \wedge a \in c$;
$g_0(b) = 4 \vdash (c \in b \vdash\dashv c \in g_1(b) \wedge \exists x \exists y(c \equiv <x, y> \wedge y \in g_2(b)))$;
$g_0(b) = 5 \vdash (c \in b \vdash\dashv \exists x(<x, c> \in g_1(b)))$;
$g_0(b) = 6 \vdash (c \in b \vdash\dashv c \in g_1(b) \exists x \exists y(<x, y> \in g_2(b) \wedge c \equiv <y, x>))$;
$g_0(b) = 7 \vdash (c \in b \vdash\dashv c \in g_1(b) \exists x \exists y \exists z(<x, y, z> \in g_2(b) \wedge c \equiv <y, z, x>))$;
$g_0(b) = 8 \vdash (c \in b \vdash\dashv c \in g_1(b) \exists x \exists y \exists z(<x, y, z> \in g_2(b) \wedge c \equiv <x, z, y>))$;

$\mathrm{Od}(a) \equiv a$;
$a \equiv b \vdash \mathrm{Od}(a) \leqq b$;
$\exists x(x \in a) \vdash C(a) \in a \wedge \forall x(x \in a \vdash C(a) \leqq x)$;
$\neg (a \in a)$;
$0 \in \omega \wedge \forall x(x \in \omega \vdash \exists y(y \in \omega \wedge x \subset y))$;
$\forall x \neg (x \in 0)$;
$\forall x \forall y(y \in x \wedge x \in a \vdash y \in j(0, a, 0))$.

Thus we can construct a model of set theory in our theory of primitive recursive functions.

§ 4°. Elementary functions.

Let $\tilde{a}$ be an ordinal number in the model of the set theory corresponding to the ordinal number a. Then we see easily that $a \leqq \tilde{a}$. Thus we can define a primitive recursive function $\mathrm{u}(a)$ by $\mathrm{u}(0) = 0 \wedge \forall x(x > 0 \vdash \mathrm{u}(x) = \mu z_{z<x'} \forall y(y < x \wedge y \in x \vdash \mathrm{Con}(\{v\}\mathrm{u}(v), x, y) < z))$. $\mathrm{u}(x)$ is a function by which an ordinal number $\tilde{b}$ in the model which corresponds to the ordinal number b is mapped to b.

DEFINITION. A function is called to be *elementary,* if it can be defined by a series of applications of the schemata (I)-(XI) in § 1 and the following schemata:
(XIV) $f(a, b) = a + b$.
(XV) $f(a, b) = J(a, b)$.
(XVI) $f(a) = fn(a)$.
(XVII) $f(a) = \mathrm{u}(a)$.

DEFINITION. A predicate is called to be *elementary,* if it has an elementary representing function.

Clearly the class of elementary functions is a proper subclass of the class of primitive recursive functions and we can construct a model of set theory in the theory of elementary functions.

We see easily the following proposition.

PROPOSITION 5. *Let $F(f, \cdots, g, a, \cdots, b)$ be a predicate constructed from elementary predicates, propositional connectives and bounded quantifiers. Then $F(f, \cdots, g, a, \cdots, b)$ is elementary.*

§ 5. Relations among elementary, primitive recursive and general recursive predicates and their quantified forms.

Consider the predicates constructed from elementary (or primitive recursive or general recursive) predicates, propositional connectives and quantifiers. For simplicity we call these predicates (*er*)- (or (*pr*)- or (*gr*)-, respectively) predicates. Let $P(f, \cdots, g, a, \cdots, b)$ be a predicate obtained from an elementary predicate by using a sequence of alternating k quantifiers. Then we call $P(f, \cdots, g, a, \cdots, b)$ a *k-er-predicate*. A *k-er-predicate* is called Σ_k^{er}- or Π_k^{er}*-predicate* according as the outermost quantifier is existencial or universal. We define *k-pr-predicate*, Σ_k^{pr}- and Π_k^{pr}*-predicates* by replacing 'elementary' by 'primitive recursive' in the above definition and *k-gr-predicate*, Σ_k^{gr}- and Π_k^{gr}*-predicates* by replacing 'elementary' by 'general recursive'. We use Σ_k^* or Π_k^* to denote the class of Σ_k^*- or Π_k^*-predicates (resp.) where $*$ stands for *er* or *pr* or *gr*. If a predicate is in both Σ_k^* and Π_k^* then it is called *both k-∗-predicate*. A predicate is said to be *expressible in* Σ_k^**-form* if it is equivalent to a Σ_k^*-predicate, *expressible in* Π_k^**-form* if it is equivalent to a Π_k^*-predicate, *expressible in both k-∗-forms* if it is equivalent to a both k-∗-predicate, ∗ being *er, pr* or *gr*. If we consider the similar concepts concerning with predicates primitive recursive or general recursive in the narrow sense, we write *prn* or *grn* instead of *pr* or *gr* respectively.

PROPOSITION 6. *For each of (er)-, (pr)- and (gr)-predicates, an unbounded quantifier can be advanced across a bounded quantifier of the opposite kind.*

PROOF. Let $a^{\dagger}b$ be $\mu y_{y<a}(<y, b> \in a)$ (which is elementary). Then we have

$$\forall x(x<a \vdash \exists y A(x, y)) \rightleftarrows \exists y \forall x(x<a \vdash A(x, \mathrm{u}(y^{\dagger}j(0, x, 0)))),$$

where $A(a, b)$ is any of (er)-, (pr)- and (gr)-predicates. The treatment is similar for the dual form.

PROPOSITION 7°. *For each of (er)-, (pr)- and (gr)-predicates, we can contract adjacent quantifiers of the same kind, i. e.*

$$\forall x_1 \forall x_2 A(x_1, x_2) \rightleftarrows \forall x A(g^1(x), g^2(x)),$$

$$\exists x_1 \exists x_2 A(x_1, x_2) \rightleftarrows \exists x A(g^1(x), g^2(x)).$$

PROPOSITION 8°. *Every (er)-predicate is expressible in one of k-er-form for some* $k \geqq 0$. *Every (pr)-predicate is expressible in one of k-pr-form for some* $k \geqq 0$. *Every (gr)-predicate is expressible in one of k-gr-form for some* $k \geqq 0$.

THEOREM 1°. *If a predicate* $P(h_1, \cdots, h_m, a_1, \cdots, a_n)$ *is expressible in both* 1*-er-forms (or both* 1*-pr- or both* 1*-gr-forms), then it is general recursive.*

PROOF. This is an analogy of Post's theorem and is proved in the same way as in Theorem VI, § 57 of [**6**]. Suppose

$$P(h_1, \cdots, h_m, a_1, \cdots, a_n) \rightleftarrows \exists x Q(h_1, \cdots, h_m, a_1, \cdots, a_n, x)$$
$$\rightleftarrows \forall x R(h_1, \cdots, h_m, a_1, \cdots, a_n, x),$$

where Q and R are elementary (or primitive recursive or general recursive). Then

$$P(h_1, \cdots, h_m, a_1, \cdots, a_n) \rightleftarrows Q(h_1, \cdots, h_m, a_1, \cdots, a_n, \mu y(Q(h_1, \cdots, h_m, a_1, \cdots, a_n, y) \vee \neg R(h_1, \cdots, h_m, a_1, \cdots, a_n, y))) .$$

THEOREM 2. *Every general recursive predicate is expressible in both* 1-*er-forms*.

PROOF. Let $f(h_1, \cdots, h_m, a_1, \cdots, a_n)$ be a general recursive function. We define $(b=f(h_1, \cdots, h_m, a_1, \cdots, a_n))^*$ for $b=f(h_1, \cdots, h_m, a_1, \cdots, a_n)$ by means of course-of-values induction on the number of the steps to construct f. If f is defined by one of the schemata (I)-(VIII), then $(b=f(h_1, \cdots, h_m, a_1, \cdots, a_n))^*$ is defined to be the predicate $b=f(h_1, \cdots, h_m, a_1, \cdots, a_n)$ itself. Let

$$f(h_1, \cdots, h_m, a_1, \cdots, a_n)=g(h_1, \cdots, h_{m^-}, g_1(h_1, \cdots, h_{m^-}, a_1, \cdots, a_n), \cdots, g_l(h_1, \cdots, h_{m^-}, a_1, \cdots, a_n))(m^- \leqq m) .$$

Then $(b=f(h_1, \cdots, h_m, a_1, \cdots, a_n))^*$ is defined to be

$$\exists x_1 \cdots \exists x_l((x_1=g_1(h_1, \cdots, h_{m^-}, a_1, \cdots, a_n))^* \wedge \cdots \wedge (x_l=g_l(h_1, \cdots, h_{m^-}, a_1, \cdots, a_n))^* \wedge (b=g(h_1, \cdots, h_{m^-}, x_1, \cdots, x_l))^*)$$

and equivalently

$$\forall x_1 \cdots \forall x_l((x_1=g_1(h_1, \cdots, h_{m^-}, a_1, \cdots, a_n))^* \wedge \cdots \wedge (x_l=g_l(h_1, \cdots, h_{m^-}, a_1, \cdots, a_n))^* \vdash (b=g(h_1, \cdots, h_{m^-}, x_1, \cdots, x_l))^*) .$$

Let $f(h_1, \cdots, h_m, a_1, \cdots, a_{n-1}, a_n)=g(h_1, \cdots, h_m, a_1, \cdots, a_{n-1})$. Then $(b=f(h_1, \cdots, h_m, a_1, \cdots, a_n))^*$ is defined to be $(b=g(h_1, \cdots, h_m, a_1, \cdots, a_{n-1}))^*$. Let

$$f(h_1, \cdots, h_m, a_1, \cdots, a_{n-1}, a_n)=\mu x_{x<a_n} g(h_1, \cdots, h_m, a_1, \cdots, a_{n-1}, x) .$$

Then $(b=f(h_1, \cdots, h_m, a_1, \cdots, a_n))^*$ is defined to be

$$((g(h_1, \cdots, h_m, a_1, \cdots, a_{n-1}, b)=0)^* \wedge \forall x(x<b \vdash \neg(g(h_1, \cdots, h_m, a_1, \cdots, a_{n-1}, x)=0)^* \wedge b<a_n)) \vee ((b=0)^* \wedge \forall x(x<a_n \vdash \neg(g(h_1, \cdots, h_m, a_1, \cdots, a_{n-1}, x)=0)^*)) .$$

Let $f(h_1, \cdots, h_m, a_1, \cdots, a_n)=\mu x g(h_1, \cdots, h_m, a_1, \cdots, a_n, x)$. Then $(b=f(h_1, \cdots, h_m, a_1, \cdots, a_n))^*$ is defined to be

$$(g(h_1, \cdots, h_m, a_1, \cdots, a_n, b)=0)^* \wedge \forall x(x<b \vdash \neg(g(h_1, \cdots, h_m, a_1, \cdots, a_n, x)=0)^*) .$$

Let $f(h_1, \cdots, h_m, a_1, a_2, \cdots, a_n)=C(f^{a_1}, h_1, \cdots, h_m, a_1, a_2, \cdots, a_n)$ and $(c=C(h, h_1, \cdots, h_m, a_1, \cdots, a_n))^*$ be $G(h, h_1, \cdots, h_m, c, a_1, \cdots, a_n)$. To define $(b=f(h_1, \cdots, h_m, a_1, \cdots, a_n))^*$, we first define some auxiliary notions. Let $O(a)$ be

$$\forall x \forall y(x<a \wedge y<a \wedge y \in x \wedge x \in a \vdash y \in a) \wedge \forall x \forall y(x<a \wedge y<a \wedge x \in a \wedge y \in a \vdash x \in y \vee x \equiv y \vee y \in x) ;$$

$F(b, c)$ be

$$\forall u \forall v \forall w(u < b \wedge v < b \wedge w < b \wedge < v, u > \in b \wedge < w, u > \in b \vdash v \equiv w)$$
$$\wedge \forall x(x < c' \vdash \exists y(y < b \wedge O(y) \wedge < y, j(0, x, 0) > \in b);$$

$S(a ; b, c)$ be

$$c < j(4, b, j(0, a, 0))'$$
$$\wedge \forall x(x < a \vdash \forall y(< y, j(0, x, 0) > \in c \mapsto < y, j(0, x, 0) > \in b)).$$

Then $(b = f(h_1, \cdots, h_m, a_1, \cdots, a_n))^*$ is defined to be

$$\exists x(F(x, a_1) \wedge \forall y \forall z(z < a_1' \wedge S(z ; x, y)$$
$$\vdash G(\{v\} \mathrm{u}(y^\dagger j(0, v, 0)), h_1, \cdots, h_m, \mathrm{u}(x^\dagger j(0, z, 0)), z, a_2, \cdots, a_n))$$
$$\wedge b = \mathrm{u}(x^\dagger j(0, a_1, 0)))$$

and equivalently

$$\forall x(F(x, a_1) \wedge \forall y \forall z(z < a_1' \wedge S(z ; x, y)$$
$$\vdash G(\{v\} \mathrm{u}(y^\dagger j(0, v, 0)), h_1, \cdots, h_m, \mathrm{u}(x^\dagger j(0, z, 0)), z, a_2, \cdots, a_n))$$
$$\vdash b = \mathrm{u}(x^\dagger j(0, a_1, 0))).$$

For every general recursive function f, we can prove that

$$(b = f(h_1, \cdots, h_m, a_1, \cdots, a_n))^* \rightleftarrows b = f(h_1, \cdots, h_m, a_1, \cdots, a_n)$$

and that $(b = f(h_1, \cdots, h_m, a_1, \cdots, a_n))^*$ is expressible in both 1-*er*-forms by induction on the number of steps to construct f. Since every general recursive predicate has a general recursive representing function, the theorem follows from the above, q. e. d.

By Theorems 1 and 2 we have easily the following.

THEOREM 3. *The class of general recursive predicates coincides to* $\Sigma_1^{er} \cap \Pi_1^{er}$. *For each* $k \geqq 1$; $\Sigma_k^{er} = \Sigma_k^{pr} = \Sigma_k^{gr}$ *and* $\Pi_k^{er} = \Pi_k^{pr} = \Pi_k^{gr}$.

In the following we shall sometimes denote $\Sigma_k^{er} (= \Sigma_k^{pr} = \Sigma_k^{gr})$ as Σ_k^{ord} and $\Pi_k^{er} (= \Pi_k^{pr} = \Pi_k^{gr})$ as Π_k^{ord} and say simply, a predicate is 'expressible in k-quantifier forms' instead of saying 'expressible in Σ_k^{ord}-form or expressible in Π_k^{ord}-form' for $k \geqq 1$.

COROLLARY. *For each* $k \geqq 0$; *the class of* Σ_k^{gr}*-predicates containing no function variable coincides with the class* Σ_k^{grn} *and the class of* Π_k^{gr}*-predicates containing no function variable coincides with the class* Π_k^{grn}.

§ 6. The enumeration theorem and hierarchy theorem.

In this section we shall prove the enumeration theorem for elementary functions, the normal form theorem for general recursive functions and the hierarchy theorem for any of (*er*)-, (*pr*)- and (*gr*)-predicates.

PROPOSITION 9°. *Let C be a class of functions of one variable satisfying the following conditions:*

(1) a', 0, ω, a, $\mathrm{Iq}(g^1(a), g^2(a))$, $\max(g^1(a), g^2(a))$, $g^1(a)+g^2(a)$, $J(g^1(a), g^2(a))$, $fn(a)$, $\mathrm{u}(a)$ *belong to C.*

(2) *If $f(a)$ and $g(a)$ belong to C, then $f(g(a)), j(f(a), g(a))$, $g^1(f(a))$, $g^2(f(a))$, $\mu x_{x<g^2(a)} f(j(g^1(a), x))$ belong to C.*

Then C contains all elementary functions of one variable.

PROOF. Let $j_n(a_1, \cdots, a_n)$ $(n \geqq 1)$ be defined as follows:

$$j_1(a_1) = a_1,$$
$$j_{n+1}(a_1, \cdots, a_n, a_{n+1}) = j(j_n(a_1, \cdots, a_n), a_{n+1}).$$

To prove the proposition it is sufficient to prove that for every elementary function $f(a_1, \cdots, a_n)$, there exists a function $\tilde{f}$ of one variable such that $\tilde{f}(a)$ belongs to C and $\tilde{f}(j_n(a_1, \cdots, a_n)) = f(a_1, \cdots, a_n)$. We prove this by induction on the number of steps to construct f. If f is an initial function it is clear. If

$$f(a_1, \cdots, a_n) = g(h_1(a_1, \cdots, a_n), \cdots h_m(a_1, \cdots, a_n)),$$

then $f(a_1, \cdots, a_n) = \tilde{g}(j_m(\tilde{h}_1(j_n(a_1, \cdots, a_n)), \cdots, \tilde{h}_m(j_n(a_1, \cdots, a_n))))$ by the hypothesis of induction. Then put $\tilde{f}(a) = \tilde{g}(j_m(\tilde{h}_1(a), \cdots, \tilde{h}_m(a)))$. If

$$f(a_1, \cdots, a_n) = g(a_1, \cdots, a_{n-1}),$$

then put $\tilde{f}(a) = \tilde{g}(g^1(a))$. If

$$f(a_1, \cdots, a_{n-1}, a_n) = \mu x_{x<a_n} g(a_1, \cdots, a_{n-1}, x),$$

then $f(a_1, \cdots, a_{n-1}, a_n) = \mu x_{x<a_n} \tilde{g}(j_n(a_1, \cdots, a_{n-1}, x))$ by the hypothesis of induction. Then put $\tilde{f}(a) = \mu x_{x<g^2(a)} \tilde{g}(j(g^1(a), x))$, q. e. d.

We can define a primitive recursive function $T(x, y)$ with the following properties:

$T(1, a) = a'$,
$T(2, a) = 0$,
$T(2^2, a) = \omega$,
$T(2^3, a) = a$,
$T(2^4, a) = \mathrm{Iq}(g^1(a), g^2(a))$,
$T(2^5, a) = \max(g^1(a), g^2(a))$,
$T(2^6, a) = g^1(a)+g^2(a)$,
$T(2^7, a) = J(g^1(a), g^2(a))$,
$T(2^8, a) = fn(a)$,
$T(2^9, a) = \mathrm{u}(a)$,
$T(2^i \cdot 3^j, a) = T(i, T(j, a))$,
$T(2^i \cdot 3^j \cdot 5, a) = j(T(i, a), T(j, a))$,
$T(3^i, a) = g^1(T(i, a))$,
$T(3^i \cdot 5, a) = g^2(T(i, a))$,

$T(3^i \cdot 5^2, a) = \mu y_{y<g^2(a)} T(i, j(g^1(a), y))$.

Then we have the following theorem.

THEOREM 4°. (The enumeration theorem on elementary functions.) *For any elementary function $f(a_1, \cdots, a_n)$ an ordinal number e less than ω can be found such that*

$$f(a_1, \cdots, a_n) = T(e, j_n(a_1, \cdots, a_n)).$$

PROOF. We have only to prove that for every function $f(a)$ in C an ordinal number e less than ω can be found such that

$$f(a) = T(e, a).$$

It is easily proved by induction on the number of steps to construct functions in C.

COROLLARY. (The enumeration theorem.) *For any general recursive predicate $R(a_1, \cdots, a_n, x)$ ordinal numbers e and f less than ω can be found such that*

$$\exists x R(a_1, \cdots, a_n, x) \rightleftarrows \exists x(T(e, j_n(a_1, \cdots, a_n, x)) = 0)$$

$$\forall x R(a_1, \cdots, a_n, x) \rightleftarrows \forall x(T(f, j_n(a_1, \cdots, a_n, x)) = 0).$$

Similarly with more quantifiers.

COROLLARY. (The normal form theorem.) *Let f be a general recursive function. Then there exists an ordinal number e less than ω such that*

$$f(a) = g^1(\mu x(T(e, j_3(a, g^1(x), g^2(x))))).$$

PROOF. For the general recursive predicate $b = f(a)$, an ordinal number e less than ω can be found such that

$$b = f(a) \rightleftarrows \exists y(T(e, j_3(a, b, y)) = 0),$$

by means of the enumeration theorem. Since $\exists x(x = f(a))$,

$$\exists x \exists y(T(e, j_3(a, x, y)) = 0),$$

which is equivalent to $\exists x(T(e, j_3(a, g^1(x), g^2(x))) = 0)$. Then

$$\mu x T(e, j_3(a, g^1(x), g^2(x)))$$

is general recursive and $b = g^1(\mu x T(e, j_3(a, g^1(x), g^2(x))))$.

THEOREM 5. (The hierarchy theorem.) *To each class of Σ_k^{ord} and Π_k^{ord} $(k \geqq 1)$, there exists a predicate which belongs to that class and is not expressible in the dual form, a fortiori not in any of h-quantifier forms with h less than k.*

PROOF. The proof is similar to the proof of Theorem V, § 57 of [**6**]. We shall show that the Π_k^{ord}-predicate

$$\forall x_1 \cdots \natural x_k(T(a_1, j_{k+n}(x_1, \cdots, x_k, a_1, \cdots, a_n)) \neq 0)$$

is not expressible in Σ_k^{ord}-form, where $\natural$ stands for $\exists$ or $\forall$ according as k is even or odd. Suppose it is expressible in Σ_k^{ord}-form, say,

$$\exists x_1 \cdots \natural' x_k P(a_1, \cdots, a_n, x_1, \cdots x_k),$$

where $\natural'$ is $\forall$ or $\exists$ according as $\natural$ is $\exists$ or $\forall$ and P is an elementary predicate (cf. Theorem 3). Then by using Theorem 4, there exists a number e less than ω such that

$$P(a_1, \cdots, a_n, x_1, \cdots, x_k) \rightleftarrows T(e, j_{k+n}(x_1, \cdots, x_k, a_1, \cdots, a_n)) = 0.$$

By the help of Proposition 7 we have

$$\forall x_1 \cdots \natural x_k (T(a_1, j_{k+n}(x_1, \cdots, x_k, a_1, \cdots, a_n)) \neq 0)$$
$$\rightleftarrows \exists x_1 \cdots \natural' x_k (T(e, j_{k+n}(x_1, \cdots, x_k, a_1, \cdots, a_n)) = 0).$$

Substituting e for $a_1, \cdots, a_n$ in the above equivalence we have a contradiction. Similarly for the dual form, q. e. d.

REMARK. We show here the enumeration and hierarchy theorems in the case that predicates contain no function variable. But we can prove those theorems in case that predicates contain a finite number of function variables similarly as above.

THEOREM 6. *The reduction principle holds for* Σ_k^{ord} $(k \geqq 1)$.

PROOF. Let X and Y belong to Σ_k^{ord} and

$$X = \hat{a} \exists x P(a, x)$$
$$Y = \hat{a} \exists x Q(a, x).$$

Then, as in [2], they are reduced to the following X_1, Y_1:

$$X_1 = \hat{a} \exists x (P(a, x) \wedge \forall y (y < x \vdash \neg Q(a, y)))$$
$$Y_1 = \hat{a} \exists x (Q(a, x) \wedge \forall y (y < x' \vdash \neg P(a, y))).$$

Since the newly occuring universal quantifiers are bounded, X_1 and Y_1 belong to the same class as X and Y, q. e. d.

If a predicate P is expressible in the form $A \wedge B$ where A and B are predicates in the classes Σ_k^{ord} and Π_k^{ord} respectively, we call P to be expressible in $\Sigma_k^{ord} \wedge \Pi_k^{ord}$-form. A set X is called a $\Sigma_k^{ord} \wedge \Pi_k^{ord}$-set, if $x \in X$ is expressible in $\Sigma_k^{ord} \wedge \Pi_k^{ord}$-form.

THEOREM 7. *The uniformization of a* Σ_k^{ord}*- or* $\Sigma_k^{ord} \cap \Pi_k^{ord}$*-set is obtained by a set belonging to the same class. The uniformization of a* Π_k^{ord}*-set is obtained by a set belonging to* $\Sigma_k^{ord} \wedge \Pi_k^{ord}$.

PROOF. Let $\hat{a}\hat{b}P(a, b)$ be a set to be uniformized. If $P(a, b)$ belongs to Σ_k^{ord}, it is of the form $\exists x Q(a, b, x)$ where $Q \in \Pi_{k-1}^{ord}$. Then we give the following set as a uniformizator of $\hat{a}\hat{b}P(a, b)$:

$$\exists x (Q(a, b, x) \wedge \forall y (y < j(b, x) \vdash \neg Q(a, g^1(y), g^2(y))))$$

which is in Σ_k^{ord} by means of the fact that $\forall y(y<j(b,x)\vdash\neg Q(a,g^1(y),g^2(y)))$ belongs to Σ_k^{ord} and by Proposition 7. If $P(a,b)$ belongs to Π_k^{ord} or $\Sigma_k^{ord}\frown\Pi_k^{ord}$ the result of uniformization of the set $\hat{a}\hat{b}P(a,b)$ is given by $\hat{a}\hat{b}(P(a,b)\wedge\forall x(x<b\vdash\neg P(a,x)))$. $\forall x(x<b\vdash\neg P(a,x))$ belongs to Σ_k^{ord} or $\Sigma_k^{ord}\frown\Pi_k^{ord}$, according that $P(a,b)$ belongs to Π_k^{ord} or $\Sigma_k^{ord}\frown\Pi_k^{ord}$. From this we see easily the theorem.

Analyzing the table given in 4 of [1], we have the following table by using Propositions 5, 6, 7 and Theorem 6, where $\mathfrak{a}$ stands for a list $h_1,\cdots,h_m$, $a_1,\cdots a_n$ and $k\geqq 1$.

P	$\hat{\mathfrak{a}}\hat{a}\exists x(x<a\wedge P(\mathfrak{a},x))$	uniformizator
$\Sigma_k^{ord}\frown\Pi_k^{ord}$	$\Sigma_k^{ord}\frown\Pi_k^{ord}$	$\Sigma_k^{ord}\frown\Pi_k^{ord}$
Σ_k^{ord}	Σ_k^{ord}	Σ_k^{ord}
Π_k^{ord}	Π_k^{ord}	$\Sigma_k^{ord}\wedge\Pi_k^{ord}$

§7. Expression of our k-quantifier forms in Kleene hierarchy.

Let $C(k_1,\cdots,k_j,a_1,\cdots,a_n)$ be a function combination obtained from function variables $k_1,\cdots,k_j$ and primitive recursive functions in the narrow sense. C is composed by combining 0, ω, variables of ordinal numbers, function symbols (initial functions of primitive recursive functions, function symbols introduced by the primitive recursion and $k_1,\cdots,k_j$) and the bounded minimum $\mu x_{x<a}$. Thus it is one of the following forms:

$$g(k_1,\cdots,k_j,C_1(k_1,\cdots,k_j,a_1,\cdots,a_n),\cdots,C_m(k_1,\cdots,k_j,a_1,\cdots,a_n))$$

with a function symbol g and

$$\mu x_{x<C_0(k_1,\cdots,k_j,a_1,\cdots,a_n)}C_1(k_1,\cdots,k_j,a_1,\cdots,a_n,x).$$

(Note that the primitive recursion does not introduce new kind of objects to the forms of primitive recursive functions.)

Let f be a function variable not contained in C and put $f(a_1,\cdots,a_n)=C(k_1,\cdots,k_j,a_1,\cdots,a_n)$. We define 'a system of equations' $[f(a_1,\cdots,a_n)=C(k_1,\cdots,k_j,a_1,\cdots,a_n)]$ and its auxiliary functions. Auxiliary functions of $[f(a_1,\cdots,a_n)=C(k_1,\cdots,k_j,a_1,\cdots a_n)]$ are function variables different from f and the function variables contained in C. The definition is given inductively as follows:

1) Let $C(k_1,\cdots,k_j,a_1,\cdots,a_n)$ be of the form

$$g(k_1,\cdots,k_j,C_1(k_1,\cdots,k_j,a_1,\cdots,a_n),\cdots,C_m(k_1,\cdots,k_j,a_1,\cdots,a_n))$$

where g is a function symbol. Then $[f(a_1,\cdots,a_n)=C(k_1,\cdots,k_j,a_1,\cdots,a_n)]$ is

$$f(a_1, \cdots, a_n) = g(k_1, \cdots, k_j, h_1(a_1, \cdots, a_n), \cdots, h_m(a_1, \cdots, a_n))$$
$$\wedge \forall x_1 \cdots \forall x_n([h_1(x_1, \cdots, x_n) = C_1(k_1, \cdots, k_j, x_1, \cdots, x_n)])$$
$$\wedge \cdots$$
$$\wedge \forall x_1 \cdots \forall x_n([h_m(x_1, \cdots, x_n) = C_m(k_1, \cdots, k_j, x_1, \cdots, x_n)]),$$

where auxiliary functions of $[f(a_1, \cdots, a_n) = C(k_1, \cdots, k_j, a_1, \cdots, a_n)]$ are $h_1, \cdots, h_m$ and auxiliary functions of $[h_i(x_1, \cdots, x_n) = C_i(k_1, \cdots, k_j, x_1, \cdots, x_n)]$ $(i = 1, 2, \cdots, m)$.
2) Let $C(k_1, \cdots, k_j, a_1, \cdots, a_n)$ be of the form $\mu x_{x<C_0(k_1,\cdots,k_j,a_1,\cdots,a_n)} C_1(k_1, \cdots, k_j, a_1, \cdots, a_n, x)$. Then $[f(a_1, \cdots, a_n) = C(k_1, \cdots, k_j, a_1, \cdots, a_n)]$ is

$$(f(a_1, \cdots, a_n) < h_0(a_1, \cdots, a_n) \wedge h_1(a_1, \cdots, a_n, f(a_1, \cdots, a_n)) = 0$$
$$\wedge \forall x(x < f(a_1, \cdots, a_n) \vdash h_1(a_1, \cdots, a_n, x) \neq 0))$$
$$\vee (f(a_1, \cdots, a_n) = 0 \wedge \forall x(x < h_0(a_1, \cdots, a_n) \vdash h_1(a_1, \cdots, a_n, x) \neq 0)))$$
$$\wedge \forall x_1 \cdots \forall x_n([h_0(x_1, \cdots, x_n) = C_0(k_1, \cdots, k_j, x_1, \cdots, x_n)])$$
$$\wedge \forall x_1 \cdots \forall x_n \forall x([h_1(x_1, \cdots, x_n, x) = C_1(k_1, \cdots, k_j, x_1, \cdots, x_n, x)]),$$

where auxiliary functions of $[f(a_1, \cdots, a_n) = C(k_1, \cdots, k_j, a_1, \cdots, a_n)]$ are h_0, h_1 and auxiliary functions of $[h_0(x_1, \cdots, x_n) = C_0(k_1, \cdots, k_j, x_1, \cdots, x_n)]$ and $[h_1(x_1, \cdots, x_n, x) = C_1(k_1, \cdots, k_j, x_1, \cdots, x_n, x)]$.
3) Let C be a function symbol, then $[f(a_1, \cdots, a_n) = C(k_1, \cdots, k_j, a_1, \cdots, a_n)]$ is $f(a_1, \cdots, a_n) = C(k_1, \cdots, k_j, a_1, \cdots, a_n)$ itself. It has no auxiliary function.

PROPOSITION 10. *For each function combination C obtained from $k_1, \cdots, k_j$ and primitive recursive functions in the narrow sense, there exists a system of equations $[f(a_1, \cdots, a_n) = C(k_1, \cdots, k_j, a_1, \cdots, a_n)]$. This is unique in the sense that*

$$[f(a_1, \cdots, a_n) = C(k_1, \cdots, k_j, a_1, \cdots, a_n)] \rightarrow f(a_1, \cdots, a_n) = C(k_1, \cdots, k_j, a_1, \cdots, a_n)$$

holds not depending on the choice of f and auxiliary functions.

DEFINITION. An ordinal number a is said to be *closed* with respect to functions $f_1, \cdots, f_n$ if the following conditions are satisfied:
1) $\omega < a$.
2) $a_1 < a, \cdots, a_{r_i} < a \rightarrow f_i(a_1, \cdots, a_{r_i}) < a$ $(1 \leqq i \leqq n)$.

We can prove the existence of such an ordinal number for given $f_1, \cdots, f_n$. Now let $C(k_1, \cdots, k_j, a_1, \cdots, a_n)$ be primitive recursive and a_0 an ordinal number closed with respect to the function symbols occurring in $C(k_1, \cdots, k_j, a_1, \cdots, a_n)$. We define *a system* $[f(a_1, \cdots, a_n) = C(k_1, \cdots, k_j, a_1, \cdots, a_n)]^{a_0}$ *of equations restricted by* a_0 inductively as follows (and auxiliary functions similarly as above);

$[f(a_1, \cdots, a_n) = g(k_1, \cdots, k_j, C_1(k_1, \cdots, k_j, a_1, \cdots, a_n), \cdots, C_m(k_1, \cdots, k_j, a_1, \cdots, a_n))]^{a_0}$ is

$$(a_1 < a_0 \wedge \cdots \wedge a_n < a_0 \vdash f(a_1, \cdots, a_n) = g(k_1, \cdots, k_j, h_1(a_1, \cdots, a_n), \cdots, h_m(a_1, \cdots, a_n)))$$
$$\wedge \forall x_1 \cdots \forall x_n([h_1(x_1, \cdots, x_n) = C_1(k_1, \cdots k_j, x_1, \cdots, x_n)]^{a_0})$$

$\wedge \cdots$

$\wedge \forall x_1 \cdots \forall x_n([h_m(x_1, \cdots, x_n) = C_m(k_1, \cdots, k_j, x_1, \cdots, x_n)]^{a_0})$.

$[f(a_1, \cdots, a_n) = \mu x_{x<C_0(k_1,\cdots,k_j,a_1,\cdots,a_n)} C_1(k_1, \cdots, k_j, a_1, \cdots, a_n, x)]^{a_0}$ is

$$(a_1 < a_0 \wedge \cdots \wedge a_n < a_0 \vdash (f(a_1, \cdots, a_n) < h_0(a_1, \cdots, a_n)$$
$$\wedge h_1(a_1, \cdots, a_n, f(a_1, \cdots, a_n)) = 0$$
$$\wedge \forall x(x < a_0 \wedge x < f(a_1, \cdots, a_n) \vdash h_1(a_1, \cdots, a_n, x) \neq 0))$$
$$\vee (f(a_1, \cdots, a_n) = 0$$
$$\wedge \forall x(x < a_0 \wedge x < h_0(a_1, \cdots, a_n) \vdash h_1(a_1, \cdots, a_n, x) \neq 0)))$$
$$\wedge \forall x_1 \cdots \forall x_n([h_0(x_1, \cdots, x_n) = C_0(k_1, \cdots, k_j, x_1, \cdots, x_n)]^{a_0})$$
$$\wedge \forall x \forall x_1 \cdots \forall x_n([h_1(x_1, \cdots, x_n, x) = C_1(x_1, \cdots, x_n, x)]^{a_0}).$$

$[f(a_1, \cdots, a_n) = g(a_1, \cdots, a_n)]^{a_0}$ for a function symbol g is

$$a_1 < a_0 \wedge \cdots \wedge a_n < a_0 \vdash f(a_1, \cdots, a_n) = g(a_1, \cdots, a_n).$$

PROPOSITION 11. *Let C be primitive recursive and a_0 closed with respect to the function symbols in C. Then*

$$[f(a_1, \cdots, a_n) = C(k_1, \cdots, k_j, a_1, \cdots, a_n)]^{a_0} \rightarrow f(a_1, \cdots, a_n) < a_0.$$

Let $C(a_1, \cdots, a_n)$ be a primitive recursive function in the narrow sense. There exists an ordinal number a_0 closed with respect to j and the functions in C defined by primitive recursions (It is easily seen that a_0 is closed with respect to any functions in C.) To translate $C(a_1, \cdots, a_n)$ in Kleene hierarchy we shall first illustrate the outline: a_0 corresponds to a function α_0 from natural numbers to natural numbers which gives a well ordering of natural numbers, the order-type being a_0'. Each ordinal number b less than a_0 corresponds to a natural number $\hat{b}$ such that it is in the domain of α_0 and the order type of $\alpha_0 \upharpoonright \hat{b}$ (cf. below) is b'. Then we shall translate the system of equations to a predicate of natural numbers in which $x \leqq y$ corresponds to $\alpha_0(\hat{x}, \hat{y}) = 0$. Next we define some auxiliary notions in Kleene's theory.

Let $D(\alpha, a, b)$ be $\alpha(a, b) = 0 \vee \alpha(b, a) = 0$;

$D(\alpha, a)$ be $\exists x D(\alpha, a, x)$;

$W(\alpha)$ be $\forall x \forall y(D(\alpha, x) \wedge D(\alpha, y) \vdash D(\alpha, x, y))$

$$\wedge \forall x \forall y(\alpha(x, y) = 0 \wedge \alpha(y, x) = 0 \vdash x = y)$$
$$\wedge \forall x \forall y \forall z(\alpha(x, y) = 0 \wedge \alpha(y, z) = 0 \vdash \alpha(x, z) = 0)$$
$$\wedge \forall \psi \exists y(\alpha(\psi(y+1), \psi(y)) \neq 0 \vee \psi(y+1) = \psi(y)),$$

which means that α is a well-ordering over a set of natural numbers and which is expressible in 1-function-quantifier form with a universal quantifier as the outermost quantifier;

$$=(\alpha,\beta) \text{ be } \exists\psi(\forall x(D(\alpha,x)\vdash D(\beta,\psi(x)))$$
$$\wedge\ \forall x(D(\beta,x)\vdash\exists y(D(\alpha,y)\wedge\psi(y)=x))$$
$$\wedge\ \forall x\forall y(D(\alpha,x)\wedge D(\alpha,y)$$
$$\vdash(\alpha(x,y)=0\wedge x\neq y\vdash\dashv\beta(\psi(x),\psi(y))=0\wedge\psi(x)\neq\psi(y)))),$$

which means under the assumption of $W(\alpha)$ or $W(\beta)$, that α and β are isomorphic. Though the meaning of $=(\alpha,\beta)$ itself may not be clear, we always use this notion in connection with $W(\alpha)$ or $W(\beta)$ in practice and understand the correct meaning;

$$\alpha\restriction a \text{ be } \lambda xy(\alpha(x,y)+\alpha(x,a)+\alpha(y,a));$$

and $Cl(\alpha;\theta_1,\cdots,\theta_m)$ be

$$\forall x_1\cdots\forall x_{i_1}(D(\alpha,x_1)\wedge\cdots\wedge D(\alpha,x_{i_1})\vdash D(\alpha,\theta_1(x_1,\cdots,x_{i_1})))$$
$$\wedge\cdots$$
$$\wedge\ \forall x_1\cdots\forall x_{i_m}(D(\alpha,x_1)\wedge\cdots\wedge D(\alpha,x_{i_m})\vdash D(\alpha,\theta_m(x_1,\cdots,x_{i_m}))),$$

which means the domain of α is closed with respect to functions $\theta_1,\cdots,\theta_m$. This notion is used only under the assumption that $W(\alpha)$.

We define further auxiliary notions which correspond to the definitions of initial functions. The notions are used only under the assumption that $W(\alpha_0)$:

$(\hat{\text{I}})$ $$\forall x\forall y(D(\alpha_0,x)\wedge D(\alpha_0,y)\vdash(y=\psi_0(x)\vdash\dashv\alpha_0(x,y)=0\wedge x\neq y$$
$$\wedge\ \forall u(\alpha_0(x,u)=0\wedge u\neq x\vdash\alpha_0(y,u)=0)))$$
$$\wedge\ \forall x(D(\alpha_0,x)\vdash D(\alpha_0,\psi_0(x)))$$

(abbr. $M_0(\alpha_0;\psi_0)$), where ψ_0 corresponds to the successor function in the sense of α_0.

$(\widehat{\text{II}_1})$ $$\forall x\forall y(D(\alpha_0,x)\wedge D(\alpha_0,y)\vdash(y=\psi_1(x)\vdash\dashv\urcorner\exists z(\alpha_0(z,y)=0\wedge z\neq y)))$$
$$\wedge\ \forall x(D(\alpha_0,x)\vdash D(\alpha_0,\psi_1(x))).$$

(abbr. $M_1(\alpha_0;\psi_1)$), where $\psi_1(x)$ stands for the first element of the domain of α_0.

$(\widehat{\text{II}_2})$ $$\forall x\forall y(D(\alpha_0,x)\wedge D(\alpha_0,y)\vdash(y=\psi_2(x)\vdash\dashv\exists z(\alpha_0(z,y)=0\wedge z\neq y)$$
$$\wedge\ \forall z(\alpha_0(z,y)=0\wedge z\neq y\vdash\alpha_0(\psi_0(z),y)=0\wedge\psi_0(z)\neq y)$$
$$\wedge\ \forall u(\exists z(\alpha_0(z,u)=0\wedge z\neq u)$$
$$\wedge\ \forall z(\alpha_0(z,u)=0\wedge z\neq u\vdash\alpha_0(\psi_0(z),u)=0\wedge\psi_0(z)\neq u)$$
$$\vdash\alpha_0(y,u)=0)))$$
$$\wedge\ \forall x(D(\alpha_0,x)\vdash D(\alpha_0,\psi_2(x)))$$

(abbr. $M_2(\alpha_0,\psi_0,\psi_2)$), which is used only under the assumption that $M_0(\alpha_0;\psi_0)$.

$\psi_2(x)$ corresponds to ω.

$(\widehat{\text{III}})$ $\quad \forall x\forall y(D(\alpha_0, x) \wedge D(\alpha_0, y) \vdash (y=\psi_3(x) \vdash\!\dashv y=x))$

$\wedge\ \forall x(D(\alpha_0, x) \vdash D(\alpha_0, \psi_3(x)))$

(abbr. $M_3(\alpha_0, \psi_3)$), where ψ_3 corresponds to the identity function.

$(\widehat{\text{IV}})$ $\quad \forall x\forall y\forall z(D(\alpha_0, x) \wedge D(\alpha_0, y) \wedge D(\alpha_0, z)$

$\vdash (z=\psi_4(x, y) \vdash\!\dashv (\alpha_0(x, y)=0 \wedge x \neq y \wedge z=\psi_1(x))$

$\vee (\alpha_0(y, x)=0 \wedge z=\psi_0(\psi_1(x)))))$

$\wedge\ \forall x\forall y(D(\alpha_0, x) \wedge D(\alpha_0, y) \vdash D(\alpha_0, \psi_4(x, y)))$

(abbr. $M_4(\alpha_0\,; \psi_0, \psi_1, \psi_4)$), which is used only under the assumption that $M_0(\alpha_0, \psi_0)$ and $M_1(\alpha_0, \psi_1)$. ψ_4 corresponds to Iq.

$(\widehat{\text{V}})$ $\quad \forall x\forall y\forall z(D(\alpha_0, x) \wedge D(\alpha_0, y) \wedge D(\alpha_0, z)$

$\vdash (z=\psi_5(x, y) \vdash\!\dashv (\alpha_0(x, y)=0 \wedge z=y) \vee (\alpha_0(y, x)=0 \wedge z=x)))$

$\wedge\ \forall x\forall y(D(\alpha_0, x) \wedge D(\alpha_0, y) \vdash D(\alpha_0, \psi_5(x, y)))$

(abbr. $M_5(\alpha_0\,; \psi_5)$). ψ_5 corresponds to max.

$(\widehat{\text{VI}})$ $\quad \forall x\forall y\forall z(D(\alpha_0, x) \wedge D(\alpha_0, y) \wedge D(\alpha_0, z)$

$\vdash (z=\psi_6(x, y) \vdash\!\dashv \underline{\forall}\varphi(\forall u\forall v(\hat{R}(\alpha_0, u, v, x, y) \vee (u=x \wedge v=y)$

$\vdash \exists w(\alpha_0(w, z)=0 \wedge \varphi(w, u, v)=0))$

$\wedge\ \forall u(\alpha_0(u, z)=0 \vdash \exists v\exists w((\hat{R}(\alpha_0, v, w, x, y) \vee (v=x \wedge w=y)) \wedge \varphi(u, v, w)=0))$

$\wedge\ \forall r\forall s\forall t\forall u\forall v\forall w(D(\alpha_0, r) \wedge D(\alpha_0, s) \wedge D(\alpha_0, t)$

$\wedge\ D(\alpha_0, u) \wedge D(\alpha_0, v) \wedge D(\alpha_0, w) \wedge \varphi(r, s, t)=0 \wedge \varphi(u, v, w)=0$

$\vdash (\alpha_0(r, u)=0 \wedge r \neq u \vdash\!\dashv \hat{R}(\alpha_0, s, t, v, w)))$

$\wedge\ \forall u(D(\alpha_0, u) \vdash \varphi(\psi_1(u), \psi_1(u), \psi_1(u))=0) \,\underline{\vdash}\, \varphi(z, x, y)=0)))$

$\wedge\ \forall x\forall y(D(\alpha_0, x) \wedge D(\alpha_0, y) \vdash D(\alpha_0, \psi_6(x, y)))$

where $\hat{R}(\alpha_0, a, b, c, d)$ is the abbreviation of

$D(\alpha_0, a) \wedge D(\alpha_0, b) \wedge D(\alpha_0, c) \wedge D(\alpha_0, d)$

$\wedge\ (\forall x(D(\alpha_0, x) \vdash \psi_4(\psi_5(a, b), \psi_5(c, d))=\psi_1(x))$

$\vee\ (\psi_5(a, b)=\psi_5(c, d) \wedge \forall x(D(\alpha_0, x) \vdash (\psi_4(b, d)=\psi_1(x) \vee (b=d \wedge \psi_4(a, c)=\psi_1(x)))))$

(abbr. $M_6(\alpha_0\,; \psi_0, \psi_1, \psi_4, \psi_5, \psi_6)$), which is used only under the assumption that $M_0(\alpha_0\,; \psi_0)$, $M_1(\alpha_0\,; \psi_1)$, $M_4(\alpha_0\,; \psi_0, \psi_1, \psi_4)$ and $M_5(\alpha_0\,; \psi_5)$. ψ_6 corresponds to j. Instead of the above predicate we sometimes use the predicate obtained from this by replacing the underlined logical symbols $\forall$ and $\vdash$ by $\exists$ and $\wedge$ respectively as $M_6(\alpha_0\,; \psi_0, \psi_1, \psi_4, \psi_5, \psi_6)$. We use them under the presupposition by which we can consider them to be equivalent.

$(\widehat{\text{VII}}_1)$ $\quad \forall x \forall y(D(\alpha_0, x) \wedge D(\alpha_0, y) \vdash (y = \psi_7(x) \dashv\vdash \exists z(x = \psi_6(y, z))))$
$\wedge \forall x(D(\alpha_0, x) \vdash D(\alpha_0, \psi_7(x)))$

(abbr. $M_7(\alpha_0 ; \psi_0, \psi_1, \psi_4, \psi_5, \psi_6, \psi_7)$), which is used only under the assumption that $M_0(\alpha_0 ; \psi_0)$, $M_1(\alpha_0 ; \psi_1)$, $M_4(\alpha_0 ; \psi_0, \psi_1, \psi_4)$, $M_5(\alpha_0 ; \psi_5)$ and $M_6(\alpha_0 ; \psi_0, \psi_1, \psi_4, \psi_5, \psi_6)$. ψ_7 corresponds to g^1.

$(\widehat{\text{VII}}_2)$ $\quad \forall x \forall y(D(\alpha_0 ; x) \wedge D(\alpha_0 ; y) \vdash (y = \psi_8(x) \dashv\vdash \exists z(x = \psi_6(z, y))))$
$\wedge \forall x(D(\alpha_0 ; x) \vdash D(\alpha_0 ;\psi_8(x)))$

(abbr. $M_8(\alpha_0 ; \psi_0, \psi_1, \psi_4, \psi_5, \psi_6, \psi_8)$), which is used only under the assumption that $M_0(\alpha_0 ; \psi_0)$, $M_1(\alpha_0 ; \psi_1)$, $M_4(\alpha_0 ; \psi_0, \psi_1, \psi_4)$, $M_5(\alpha_0 ; \psi_5)$ and $M_6(\alpha_0 ; \psi_0, \psi_1, \psi_4, \psi_5, \psi_6)$. ψ_8 corresponds to g^2.

We see easily that $M_i(\alpha_0 ; \psi_{i_1}, \cdots, \psi_i)$ is expressible in both 1-function-quantifier forms for each i $(0 \leqq i \leqq 8)$.

Let $C(a_1, \cdots, a_n)$ be a primitive recursive function in the narrow sense. We define 'a system of equations $[f(a_1, \cdots, a_n) = C(a_1, \cdots, a_n)]^{\hat{\alpha}_0}$ with respect to α_0' in Kleene's theory corresponding to $[f(a_1, \cdots, a_n) = C(a_1, \cdots, a_n)]^{a_0}$ where $W(\alpha_0)$ is presupposed and the order-type of α_0 is a_0'. The definition is given under the presupposition that $W(\alpha_0)$, $M_0(\alpha_0 ; \psi_0)$, $M_1(\alpha_0 ; \psi_1), \cdots$, $M_8(\alpha_0 ; \psi_0, \psi_1, \psi_4, \psi_5, \psi_6, \psi_8)$. In the definition, functions $\hat{f}, \hat{g}, \cdots$ from natural numbers to natural numbers and number variables $\hat{a}_1, \cdots, \hat{a}_n$ correspond to $f, g, \cdots$ and $a_1, \cdots, a_n$. If $C(a_1, \cdots, a_n)$ is of the form $g(C_1(a_1, \cdots, a_n), \cdots, C_m(a_1, \cdots, a_n))$ where g is a function symbol, then $[f(a_1, \cdots, a_n) = C(a_1, \cdots, a_n)]^{\hat{\alpha}_0}$ is

$$(D(\alpha_0, \hat{a}_1) \wedge \cdots \wedge D(\alpha_0, \hat{a}_n) \vdash \hat{f}(\hat{a}_1, \cdots, \hat{a}_n) = \hat{g}(\hat{h}_1(\hat{a}_1, \cdots, \hat{a}_n), \cdots, \hat{h}_m(\hat{a}_1, \cdots, \hat{a}_n)))$$
$$\wedge \forall \hat{x}_1 \cdots \forall \hat{x}_n([h_1(x_1, \cdots, x_n) = C_1(x_1, \cdots, x_n)]^{\hat{\alpha}_0}$$
$$\wedge \cdots$$
$$\wedge \forall \hat{x}_1 \cdots \forall \hat{x}_n([h_m(x_1, \cdots, x_n) = C_m(x_1, \cdots, x_n)]^{\hat{\alpha}_0}),$$

where $\hat{g}$ is ψ_i $(0 \leqq i \leqq g)$ or k_l $(0 \leqq l \leqq j)$. If $C(a_1, \cdots, a_n)$ is of the form $\mu x_{x < C_0(a_1, \cdots, a_n)} C_1(a_1, \cdots, a_n, x)$, then $[f(a_1, \cdots, a_n) = C(a_1, \cdots, a_n)]^{\hat{\alpha}_0}$ is

$$(D(\alpha_0, \hat{a}_1) \wedge \cdots \wedge D(\alpha_0, \hat{a}_n) \vdash (\alpha_0(\hat{f}(\hat{a}_1, \cdots, \hat{a}_n), \hat{h}_0(\hat{a}_1, \cdots, \hat{a}_n)) = 0$$
$$\wedge \hat{f}(\hat{a}_1, \cdots, \hat{a}_n) \neq \hat{h}_0(\hat{a}_1, \cdots, \hat{a}_n) \wedge \hat{h}_1(\hat{a}_1, \cdots, \hat{a}_n, \hat{f}(\hat{a}_1, \cdots, \hat{a}_n)) = \psi_1(\hat{a}_1)$$
$$\wedge \forall \hat{x}(D(\alpha_0, \hat{x}) \wedge \alpha_0(\hat{x}, \hat{f}(\hat{a}_1, \cdots, \hat{a}_n)) = 0 \wedge \hat{x} \neq \hat{f}(\hat{a}_1, \cdots, \hat{a}_n)$$
$$\vdash \hat{h}_1(\hat{a}_1, \cdots, \hat{a}_n, \hat{x}) \neq \psi_1(a_1)))$$
$$\vee (\hat{f}(\hat{a}_1, \cdots, \hat{a}_n) = \psi_1(\hat{a}_1)$$
$$\wedge \forall \hat{x}(D(\alpha_0, \hat{x}) \wedge \alpha_0(\hat{x}, \hat{h}_0(\hat{a}_1, \cdots, \hat{a}_n)) = 0 \wedge \hat{x} \neq \hat{h}_0(\hat{a}_1, \cdots, \hat{a}_n)$$
$$\vdash \hat{h}_1(\hat{a}_1, \cdots, \hat{a}_n, \hat{x}) \neq \psi_1(\hat{a}_1))))$$

$$\wedge \forall \hat{x}_1 \cdots \forall \hat{x}_n([h_0(x_1, \cdots, x_n) = C_0(x_1, \cdots, x_n)]^{\hat{\alpha}_0})$$
$$\wedge \forall \hat{x}_1 \cdots \forall \hat{x}_n \forall \hat{x}([h_1(x_1, \cdots, x_n, x) = C_1(x_1, \cdots, x_n, x)]^{\hat{\alpha}_0}).$$

$[f(a_1, \cdots, a_n) = g(a_1, \cdots, a_n)]^{\hat{\alpha}_0}$ for a function symbol g is $D(\alpha_0, \hat{a}_1) \wedge \cdots \wedge D(\alpha_0, \hat{a}_n) \vdash \hat{f}(\hat{a}_1, \cdots, \hat{a}_n) = \hat{g}(\hat{a}_1, \cdots, \hat{a}_n)$, where $\hat{g}$ is ψ_i or k_l according as g is introduced by one of (I)-(VII) or (XII).

Now let $C(a_1, \cdots, a_n)$ be a primitive recursive function in the narrow sense. We define the result of the translation of $b = C(a_1, \cdots, a_n)$ (write this as $A(a_1, \cdots, a_n, b)$) in Kleene hierarchy which is denoted by $(b = C(a_1, \cdots, a_n))^{\wedge}$ or $\hat{A}(\varphi_1, \cdots, \varphi_n, \varphi)$ (where $\varphi_1, \cdots, \varphi_n, \varphi$ correspond to $a_1, \cdots, a_n, b$). This has two forms $(b = C(a_1, \cdots, a_n))^{\hat{\forall}}$ $(\hat{A}^{\forall}(\varphi_1, \cdots, \varphi_n, \varphi))$ and $(b = C(a_1, \cdots, a_n))^{\hat{\exists}}$ $(\hat{A}^{\exists}(\varphi_1, \cdots, \varphi_n, \varphi))$. $\hat{A}^{\forall}(\varphi_1, \cdots, \varphi_n, \varphi)$ is

$$\underline{\forall}\alpha_0\, \underline{\forall}\hat{a}_1 \cdots \underline{\forall}\hat{a}_n\, \underline{\forall}\hat{b}\, \underline{\forall}\psi_0\, \underline{\forall}\psi_1 \cdots \underline{\forall}\psi_8\, \underline{\forall}\hat{h}_1 \cdots \underline{\forall}\hat{h}_m\, \underline{\forall}\hat{k}_1 \cdots \underline{\forall}\hat{k}_j\, \underline{\forall}\hat{f}$$
$$(W(\alpha_0) \wedge D(\alpha_0, \hat{a}_1) \wedge \cdots \wedge D(\alpha_0, \hat{a}_n) \wedge D(\alpha_0, \hat{b})$$
$$\wedge = (\varphi_1, \alpha_0 \upharpoonright \hat{a}_1) \wedge \cdots \wedge = (\varphi_n, \alpha_0 \upharpoonright \hat{a}_n) \wedge = (\varphi, \alpha_0 \upharpoonright \hat{b})$$
$$\wedge M_0(\alpha_0, \psi_0) \wedge \cdots \wedge M_8(\alpha_0, \psi_0, \psi_1, \psi_4, \psi_5, \psi_6, \psi_8)$$
$$\wedge Cl(\alpha_0\,; \psi_6, \hat{k}_1, \cdots, \hat{k}_j) \wedge [f(a_1, \cdots, a_n) = C(a_1, \cdots, a_n)]^{\hat{\alpha}_0}),$$
$$\underline{\vdash}\, \hat{b} = \hat{f}(\hat{a}_1, \cdots, \hat{a}_n)$$

where $h_1, \cdots, h_m$ are auxiliary functions of $[f(a_1, \cdots, a_n) = C(a_1, \cdots, a_n)]$ and $k_1, \cdots, k_j$ are functions introduced by primitive recursions (XII) applied in the construction of C. $\hat{A}^{\exists}(\varphi_1, \cdots, \varphi_n, \varphi)$ is obtained from $\hat{A}^{\forall}(\varphi_1, \cdots, \varphi_n, \varphi)$ by replacing the underlined $\forall$'s and $\vdash$ by $\exists$'s and $\wedge$ respectively. From the definition we have

PROPOSITION 12. *A primitive recursive predicate in the narrow sense is expressible in both 2-function-quantifier forms in Kleene hierarchy.*

Let $P(a_1, \cdots, a_n)$ be a predicate expressible in the k-*pr*-quantifier form and contain no function variable. There exists a primitive recursive function $C(a_1, \cdots, a_n, x_1, \cdots, x_k)$ such that it contains no other variable than $a_1, \cdots, a_n, x_1, \cdots, x_k$ and contains function symbols $f_1, \cdots, f_m$ besides initial functions and

$$P(a_1, \cdots, a_n) \rightleftarrows [x_1, \cdots, x_k](C(a_1, \cdots, a_n, x_1, \cdots, x_k) = 0)$$

where $[x_1, \cdots, x_k]$ stands for a sequence of alternate quantifiers. Then $A(a_1, \cdots, a_n, x_1, \cdots, x_k)$ standing for $C(a_1, \cdots, a_n, x_1, \cdots, x_k) = 0$, $P(a_1, \cdots, a_n)$ is translated in

$$[\xi_1^{W(\xi_1)} \cdots \xi_k^{W(\xi_k)}]\, \hat{A}^{\forall}(\varphi_1, \cdots, \varphi_n, \xi_1, \cdots, \xi_k)$$

or

$$[\xi_1^{W(\xi_1)} \cdots \xi_k^{W(\xi_k)}]\, \hat{A}^{\exists}(\varphi_1, \cdots, \varphi_n, \xi_1, \cdots, \xi_k)$$

(where $\varphi_1, \cdots, \varphi_n, \xi_1, \cdots, \xi_k$ are function variables corresponding to $a_1, \cdots, a_n, x_1, \cdots, x_k$ and $\xi_i^{W(\xi_i)}$ in $[\quad]$ means $\forall \xi_i(W(\xi_i) \vdash \quad)$ or $\exists \xi_i(W(\xi_i) \wedge \quad)$

according as the appearence of x_i in $[x_1, \cdots, x_k]$ is of the form $\forall x_i$ or $\exists x_i$ $(1 \leqq i \leqq k)$) according as innermost quantifier of P is universal or existential. We shall denote it as $(P(a_1, \cdots, a_n))^{\wedge}$ or $\hat{P}(\varphi_1, \cdots, \varphi_n)$.

THEOREM 8. *Each predicate containing no function variable and expressible in the k-pr-quantifier form* $(k \geqq 1)$ *is expressible in the* $k+1$*-function-quantifier form keeping the outermost quantifier in the same kind.*

PROOF. Let $P(a_1, \cdots, a_n)$ be a predicate expressible in the following form:

$$[x_1, \cdots, x_k] A(a_1, \cdots, a_n, x_1, \cdots, x_k)$$

where $[x_1, \cdots, x_k]$ is a sequence of alternate quantifiers and $A(a_1, \cdots, a_n, x_1, \cdots, x_k)$ stands for $C(a_1, \cdots, a_n, x_1, \cdots, x_k) = 0$ with a primitive recursive function C in our sense. Then $(P(a_1, \cdots, a_n))^{\wedge}$ is

$$[\xi_1^{W(\xi_1)} \cdots \xi_k^{W(\xi_k)}] \hat{A}^*(\varphi_1, \cdots, \varphi_n, \xi_1, \cdots, \xi_k),$$

where $*$ stands for $\forall$ or $\exists$ according as the innermost quantifier is universal or existential. Since $\hat{A^*}(\varphi_1, \cdots, \varphi_n, \xi_1, \cdots, \xi_k)$ is expressible in the 2-function quantifier form with a universal or existential quantifier as the outermost one according as $*$ stands for $\forall$ or $\exists$, which is of the same kind as the innermost quantifier of $[x_1, \cdots, x_k]$, these quantifiers are contracted by Proposition 7. Thus we see that $(P(a_1, \cdots, a_n))^{\wedge}$ is expressible in the $k+1$-function-quantifier form keeping the outermost quantifier in the same kind as that in the expression of the *k-pr*-quantifier form of $P(a_1, \cdots, a_n)$.

§ 8. Expression of the $k+1$-function-quantifier forms in our theory.

In this section we shall show that an analytic predicate in the $k+1$-function-quantifier form is expressible in the k-quantifier form in our sense keeping the outermost quantifier in the same kind.

For this purpose we first define several auxiliary functions. Let $\mathrm{P}(f, x)$ be defined by the following:

$$\mathrm{P}(f, 0) = 1 \wedge \forall x(0 < x \wedge x < \omega \vdash \mathrm{P}(f, x) = J(f(\delta(x)), \mathrm{Con}(\{u\}\mathrm{P}(f, u), x, \delta(x))))$$
$$\wedge \forall x(x \geqq \omega \vdash \mathrm{P}(f, x) = \sup(\{z\}\mathrm{Con}(\{u\}\mathrm{P}(f, u), x, z), x ; \omega)).$$

Moreover, let $\mathrm{m}(i)$ be $\mu x_{x<\omega}(\mathrm{u}(x) = i)$; $f^{\dagger\dagger}i$ be $\mu y_{y<\omega}(<\mathrm{m}(y), \mathrm{m}(i)> \in f \wedge \mathrm{m}(y) \in \omega)$; $f^{\sim}y$ be $\mathrm{P}(\{x\}p_x^{f^{\dagger\dagger}x}, y)$.

Now we define $\#$-operation from analytic predicates to predicates of ordinal numbers, by which a variable and a function variable turn a variable of ordinal number $< \omega$ and a variable of ordinal number being a function in the model constructed in § 3. If φ is a recursive function in Kleene's sense, then there exists a primitive recursive function $\ddot{\varphi}$ in our sense such that $\ddot{\varphi}$ is equal to

φ on the domain of natural numbers. The same assertion is true for recursive predicates. Without loss of generality we may assume each function variable is of one variable.

Let $F(f)$ be

$$\forall x(x<f \wedge x \in f \vdash \exists u \exists v(<u,v> \equiv x \wedge u<\omega \wedge v<\omega \wedge u \in \omega \wedge v \in \omega)$$
$$\wedge \forall u \forall v \forall w(u<\omega \wedge v<\omega \wedge w<\omega \wedge <v,u> \in f \wedge <w,u> \in f \vdash v \equiv w)$$
$$\wedge \forall x(x<\omega \vdash \exists y(y<\omega \wedge <\mathrm{m}(y), \mathrm{m}(x)> \in f)) \wedge \forall x(x<f \vdash \neg x \equiv f).$$

By means of the enumeration and hierarchy theorems for both arithmetical and analytic predicates, 2 of [7] and the relativization of Theorem 1 of [14], we define $\sharp$-operation by induction on the number of prenex quantifiers of numbers and of functions respectively.

$$(j=\alpha(i))^{\sharp} \quad \text{is} \quad F(f) \wedge j=f^{\dagger\dagger}i \wedge j<\omega \wedge i<\omega .$$

Each recursive function φ of m function variables $\alpha_1, \cdots, \alpha_m$ and n number variables $a_1, \cdots, a_n$ can be expressible as

$$U(\mu y\, T_n^{\overbrace{1,\cdots,1}^{m}}(\tilde{\alpha}_1(y), \cdots, \tilde{\alpha}_m(y), e, a_1, \cdots, a_n, y))$$

for some number e. (cf. [7]).) Then we define $(b=\varphi(\alpha_1, \cdots, \alpha_m, a_1, \cdots, a_n))^{\sharp}$ as

$$b=\ddot{U}(\mu y_{y<\omega}\, \ddot{T}_n^{\overbrace{1,\cdots,1}^{m}}(f_1^{\sim}y, \cdots, f_m^{\sim}y, e, a_1, \cdots, a_n, y)) \wedge F(f_1) \wedge \cdots \wedge F(f_m).$$

which is primitive recursive in our sense.

Let $R(\alpha_1, \cdots, \alpha_m, a_1, \cdots, a_n)$ be a recursive predicate with the representing function $\varphi(\alpha_1, \cdots, \alpha_m, a_1, \cdots, a_n)$. Then $(R(\alpha_1, \cdots, \alpha_m, a_1, \cdots, a_n))^{\sharp}$ is defined as $(0=\varphi(\alpha_1, \cdots, \alpha_m, a_1, \cdots, a_n))^{\sharp}$.

Let $A(\alpha_1, \cdots, \alpha_m, a_1, \cdots a_n)$ be an arithmetical predicate expressible in $j+1$ prenex quantifiers such as

$$\exists x A_0(\alpha_1, \cdots, \alpha_m, a_1, \cdots, a_n, x)$$
$$(\text{or } \forall x A_0(\alpha_1, \cdots, \alpha_m, a_1, \cdots, a_n, x))$$

and assume $(A_0(\alpha_1, \cdots, \alpha_m, a_1, \cdots, a_n, x))^{\sharp}$ is defined. Then $(A(\alpha_1, \cdots, \alpha_m, a_1, \cdots, a_n))^{\sharp}$ is

$$\exists x(x<\omega \wedge (A_0(\alpha_1, \cdots, \alpha_m, a_1, \cdots, a_n, x))^{\sharp})$$
$$(\text{or } \forall x(x<\omega \vdash (A_0(\alpha_1, \cdots, \alpha_m, a_1, \cdots, a_n, x))^{\sharp})).$$

Let $\psi(f_1, \cdots, f_m, c, a, b)$ be the primitive recursive function

$$\ddot{U}(\mu y_{y<\omega}(\ddot{T}_2^{1,\cdots,1}(f_1^{\sim}y, \cdots, f_m^{\sim}y, c, a, b, y))).$$

We define a primitive recursive function $B(f_1, \cdots, f_m, c, x)$ in the following (for simplicity we shall abbreviate a list $f_1, \cdots, f_m$ as $\mathfrak{f}$):

$$B(\mathfrak{f}, c, 0) = \mu z_{z<\omega}(\psi(\mathfrak{f}, c, z, z) = 0 \wedge \forall x(x < \omega \wedge \psi(\mathfrak{f}, c, x, x) = 0$$
$$\wedge\, x \neq z \vdash \psi(\mathfrak{f}, c, z, x) = 0 \wedge \psi(\mathfrak{f}, c, x, z) \neq 0))$$
$$\wedge\, \forall x(0 < x \vdash B(\mathfrak{f}, c, x) = \mu z_{z<\omega}(\psi(\mathfrak{f}, c, z, z) = 0$$
$$\wedge\, \forall y(y < x \vdash z \neq B(\mathfrak{f}, c, y))$$
$$\wedge\, \forall u(u < \omega \wedge \forall y(y < x \vdash u \neq B(\mathfrak{f}, c, y)) \wedge \psi(\mathfrak{f}, c, u, u) = 0$$
$$\wedge\, u \neq z \vdash \psi(\mathfrak{f}, c, z, u) = 0 \wedge \psi(\mathfrak{f}, c, u, z) \neq 0))\,.$$

Then $(c \in W^{\alpha_1,\cdots,\alpha_m})^\#$ is $\forall x \exists y(x < \omega \wedge \psi(\mathfrak{f}, c, x, x) = 0 \vdash x = B(\mathfrak{f}, c, y))$ where the scope of the quantifier is primitive recursive. Let $A(\alpha_1, \cdots, \alpha_m, a)$ be an analytic predicate of the form $\forall \beta \exists x R(\alpha_1, \cdots, \alpha_m, a, \beta, x)$ with recursive R. Then this is expressible in the form

$$\eta^{\alpha_1,\cdots,\alpha_m}(a) \in W^{\alpha_1,\cdots,\alpha_m}\,,$$

where $\eta^{\alpha_1,\cdots,\alpha_m}$ is a function recursive uniformly in $\alpha_1, \cdots, \alpha_m$. We define $(A(\alpha_1, \cdots, \alpha_m, a))^\#$ as

$$\exists x(x < \omega \wedge (x = \eta^{\alpha_1,\cdots,\alpha_m}(a))^\# \wedge (x \in W^{\alpha_1,\cdots,\alpha_m})^\#)\,,$$

which is expressible in the form $\exists x B(f_1, \cdots, f_m, a, x)$, where B is primitive recursive in our sense and $f_1, \cdots, f_m$ are variables of ordinal numbers corresponding to $\alpha_1, \cdots, \alpha_m$.

Let $A(\alpha_1, \cdots, \alpha_m, a)$ be an analytic predicate expressible in the form

$$\natural \beta_1 \cdots \exists \beta_k \forall \beta \exists x R(\alpha_1, \cdots, \alpha_m, a, \beta_1, \cdots, \beta_k, \beta, x)\,,$$

where $\natural$ is a quantifier, R is recursive and

$$(\forall \beta \exists x R(\alpha_1, \cdots, \alpha_m, a, \beta_1, \cdots, \beta_k, \beta, x))^\#$$

is expressible in the form $\exists x B(f_1, \cdots, f_m, g_1, \cdots, g_k, a, x)$ with primitive recursive B (in our sense). Then we define $(A(\alpha_1, \cdots, \alpha_m, a))^\#$ as

$$\natural g_1 \cdots \exists g_k \exists x B(f_1, \cdots, f_m, g_1, \cdots, g_k, a, x)\,.$$

The other cases are treated by the method of duality in logic.

THEOREM 9. *An analytic predicate expressible in the $k+1$-function-quantifier form is expressible in the k-quantifier form in our sense keeping the outermost function-quantifier in the same kind.*

PROOF. It is easily seen from the definition of $\#$-operation and Proposition 7.

Now let us consider the classical hierarchy. In this case predicates are considered to be obtained from predicates recursive in some function from natural numbers to natural numbers by quantification similar to the above (cf. [2]). If a predicate is of the form

$$\natural \beta_1 \cdots \exists \beta_n \forall \beta \exists x R^\rho(\alpha_1, \cdots, \alpha_m, a, \beta_1, \cdots, \beta_n, \beta, x)\,,$$

where $\natural$ is $\forall$ or $\exists$ and ρ is a specified function from natural numbers to natural numbers, it is expressible as

$$\natural\beta_1 \cdots \exists\beta_n \eta^{\rho,\alpha_1,\cdots,\alpha_m,\beta_1,\cdots,\beta_n}(a) \in W^{\rho,\alpha_1,\cdots,\alpha_m,\beta_1,\cdots,\beta_n},$$

where $\eta^{\rho,\alpha_1,\cdots,\alpha_m,\beta_1,\cdots,\beta_n}$ is a function recursive in ρ and recursive uniformly in $\alpha_1, \cdots, \alpha_m, \beta_1, \cdots, \beta_n$. In the translation given above, we took a variable of ordinal numbers with some conditions as a counterpart of a function variable. Besides this we take an ordinal number p as a counterpart of ρ such that $F(p)$ and $j=\rho(i)$ implies $j=p^{\dagger\dagger}i$ for each natural number i. Thus

$$\eta^{\rho,\alpha_1,\cdots,\alpha_m,\beta_1,\cdots,\beta_n}(a) \in W^{\rho,\alpha_1,\cdots,\alpha_m,\beta_1,\cdots,\beta_n}$$

can be expressible in the form $\exists x B(p, f_1, \cdots, f_m, g_1, \cdots, g_n, a, x)$, where B is a primitive recursive predicate in our sense and p is a parameter of an ordinal number in our hierarchy. So similarly as above,

THEOREM 10. *A predicate expressible with the $k+1$-quantifier form in the classical hierarchy is expressible in the k-quantifier form in our sense with an ordinal number as a parameter and keeping the outermost quantifier in the same kind.*

§ 9. On hyperarithmetical predicates in connection with our hierarchy.

In this section we shall characterize hyperarithmetical predicates ([8]) in our hierarchy.

THEOREM 11. *Let $P(x_1, \cdots, x_n)$ be a primitive recursive predicate in the narrow sense and contain no function variable and $c_1, \cdots, c_n$ elements of W. Then $P(|c_1|, \cdots, |c_n|)$ is expressible in both* 1*-function quantifier forms.*

PROOF. In §7 we defined several notions and notations, e.g. a system of equations for a primitive recursive function in our sense, an ordinal number closed with respect to some functions, $D(\alpha, a, b)$, $=(\alpha, \beta)$ and so on. We shall use them freely in this section and identify the Gödel number of a recursive function to the function itself in Kleene hierarchy.

LEMMA. *Let $a_1, \cdots, a_n$ be ordinal numbers less than Church-Kleene's ω_1. Then there exists an ordinal number a_0 satisfying the following conditions:*

(1) $a_1 < a_0, \cdots, a_n < a_0$,

(2) $a_0 < \omega_1$,

(3) *a_0 is closed with respect to j,*

(4) *$f(a_1, \cdots, a_n) < a_0$ for any primitive recursive function f in our sense.*

PROOF. Since the constructive ordinals are closed with respect to the operation corresponding to j (cf. [3]), the existence of a_0 satisfying (1)-(3) is easily seen. Then by induction on the number of stages to construct f, we see that such a_0 satisfies (4).

We call a_0 the *closure* of $a_1, \cdots, a_n$.

Let $C(x_1, \cdots, x_n)$ be a primitive recursive function in the narrow sense, $a_i < \omega_1$, for $1 \leqq i \leqq n$ and a_0 the closure of $a_1, \cdots, a_n$. Let us consider a system $[f(a_1, \cdots, a_n) = C(a_1, \cdots, a_n)]^{a_0}$ of equations restricted by a_0.

To translate $C(a_1, \cdots, a_n)$ in Kleene hierarchy, let a_0 correspond to an element f_0 of Spector's W such that

$$|f_0| = a_0',$$

and each ordinal number a less than a_0 to a natural number $\hat{a}$ such that it is in the domain of f_0 and $|f_0 \upharpoonright \hat{a}| = a'$. where $f_0 \upharpoonright \hat{a}$ is a Gödel number of the recursive function

$$\lambda xy(f_0(x, y) + f_0(x, \hat{a}) + f_0(y, \hat{a})).$$

Let $M_i(f_0 ; \psi_{i_1}, \cdots, \psi_i)$ be the predicate obtained from $M_i(\alpha_0 ; \psi_{i_1}, \cdots, \psi_i)$ defined in §7 by replacing every occurrence of α_0 by f_0 for each i $(0 \leqq i \leqq 8)$ and $M_6'(f_0 ; \psi_0, \psi_1, \psi_4, \psi_5, \psi_6)$ the predicate obtained from $M_6(f_0 ; \psi_0, \psi_1, \psi_4, \psi_5, \psi_6)$, by removing '$\wedge \forall x \forall y(D(f_0, x) \wedge D(f_0, y) \vdash D(f_0, \psi_6(x, y)))$', Moreover let $M(\varphi ; \alpha, \beta)$ be

$$\begin{aligned} &\forall x(D(\alpha, x) \vdash D(\beta, \varphi(x))) \\ &\wedge \forall x \forall y(D(\alpha, x) \wedge D(\alpha, y) \\ &\quad \vdash (\alpha(x, y) = 0 \wedge x \neq y \leftrightarrow \beta(\varphi(x), \varphi(y)) = 0 \wedge \varphi(x) \neq \varphi(y))) \\ &\wedge \forall x \forall y(\beta(y, \varphi(x)) = 0 \vdash \exists z(D(\alpha, z) \wedge y = \varphi(z))); \end{aligned}$$

$\cong(\alpha, \beta)$ be

$$W(\alpha) \wedge W(\beta) \wedge \forall \varphi(M(\varphi ; \alpha, \beta) \vdash \forall x \exists y(D(\beta, x) \vdash x = \varphi(y) \wedge D(\alpha, y)))$$

which means α and β are isomorphic and is in Π_1^1 (cf. Corollaire 1 of [**13**, p. 181]). We define 'a system $[f(a_1, \cdots, a_n) = C(a_1, \cdots, a_n)]^{\hat{f}_0}$' to be the predicate obtained from $[f(a_1, \cdots, a_n) = C(a_1, \cdots, a_n)]^{\hat{\alpha}_0}$ in §7 by replacing every occurrence of α_0 by f_0. Now we define each of the following predicates as the result of the translation of $b = C(a_1, \cdots, a_n)$ in Kleene hierarchy:

$$\begin{aligned} &\exists f_0 \exists \hat{a}_1 \cdots \exists \hat{a}_n \exists \hat{b}(f_0 \in W \\ &\wedge \forall \psi_0 \forall \psi_1 \forall \psi_4 \forall \psi_5 \forall \psi_6(M_0(f_0 ; \psi_0) \wedge \cdots \wedge M_5(f_0 ; \psi_5) \wedge M_6'(f_0, \psi_0, \cdots, \psi_6) \\ &\qquad\qquad \vdash Cl(f_0 ; \psi_6)) \\ &\wedge D(f_0, \hat{a}_1) \wedge \cdots \wedge D(f_0, \hat{a}_n) \wedge D(f_0, \hat{b}) \\ &\wedge \cong(\varphi_1, f_0 \upharpoonright \hat{a}_1) \wedge \cdots \wedge \cong(\varphi_n, f_0 \upharpoonright \hat{a}_n) \wedge \cong(\varphi, f_0 \upharpoonright \hat{b}) \\ &\wedge \forall \psi_0 \cdots \forall \psi_8 \forall \hat{h}_1 \cdots \forall \hat{h}_m \forall \hat{k}_1 \cdots \forall \hat{k}_j \forall \hat{f}(M_0(f_0 ; \psi_0) \wedge \cdots \wedge M_8(f_0, \psi_0, \cdots, \psi_8) \\ &\quad \wedge [f(a_1, \cdots, a_n) = C(a_1, \cdots, a_n)]^{\hat{f}_0} \vdash \hat{b} = \hat{f}(\hat{a}_1, \cdots, \hat{a}_n))); \end{aligned}$$

$$\forall f_0 \forall \hat{a}_1 \cdots \forall \hat{a}_n \forall \hat{b}(f_0 \in W$$
$$\wedge \forall \psi_0 \forall \psi_1 \forall \psi_4 \forall \psi_5 \forall \psi_6 (M_0(f_0, \psi_0) \wedge \cdots \wedge M_5(f_0 ; \psi_5) \wedge M_6'(f_0, \psi_0, \cdots, \psi_6) \vdash Cl(f_0 ; \psi_6))$$
$$\wedge D(f_0, \hat{a}_1) \wedge \cdots \wedge D(f_0, \hat{a}_n) \wedge D(f_0, \hat{b})$$
$$\wedge \cong (\varphi_1, f_0 \upharpoonright \hat{a}_1) \wedge \cdots \wedge \cong (\varphi_n, f_0 \upharpoonright \hat{a}_n) \wedge \cong (\varphi ; f_0 \upharpoonright \hat{b})$$
$$\vdash \exists \psi_0 \cdots \exists \psi_8 \exists \hat{h}_1 \cdots \exists \hat{h}_m \exists \hat{k}_1 \cdots \exists \hat{k}_j \exists \hat{f}(M_0(f_0 ; \psi_0) \wedge \cdots \wedge M_8(f_0 ; \psi_0, \cdots, \psi_8)$$
$$\wedge \hat{b} = \hat{f}(\hat{a}_1, \cdots, \hat{a}_n) \wedge [f(a_1, \cdots, a_n) = C(a_1, \cdots, a_n)]^{\hat{f}_0})),$$

where $h_1, \cdots, h_m$ are auxiliary functions of $[f(a_1, \cdots, a_n) = C(a_1, \cdots, a_n)]$ and $k_1, \cdots, k_j$ are functions introduced by primitive recursions applied in the construction of C.

It is easily seen that they are Π_1^1 and Σ_1^1-predicates (resp.). Thus $|c| = C(|c_1|, \cdots, |c_n|)$ with primitive recursive C in our sense and $c_1, \cdots, c_n, c \in W$ is expressible in both 1-function-quantifier forms in Kleene hierarchy. From this follows the theorem, q. e. d.

COROLLARY. *Let $P(a, b)$ be a predicate primitive recursive in the narrow sense. Then for each $b < \omega_1$, $\hat{a}(a < \omega \wedge P(a, b))$ is hyperarithmetical.*

PROOF. Let $b < \omega_1$ and f be a Gödel number of the representing function of $\hat{x}\hat{y}(x < y \wedge y < n)$. $P(|f|, b)$ is expressible as a hyperarithmetical predicate.

THEOREM 12. *Let $A(a_1, \cdots, a_n)$ be a hyperarithmetical predicate. Then there exist a primitive recursive predicate $P(x, x_1, \cdots, x_n)$ in the narrow sense and an ordinal number b less than Church-Kleene's ω_1 such that $A(a_1, \cdots, a_n)$ is expressed by $P(b, a_1, \cdots, a_n)$ in the hierarchy of predicates of ordinal numbers.*

PROOF. It is sufficient to show that every predicate $H_y(a)$ for $y \in O$ is expressible in the form $P(b, a)$ where P is primitive recursive in our sense and $b < \omega_1$.

Let $C(a, b)$ be $\exists x(x < \omega \wedge \ddot{V}(a, b, x))$ (cf. [**5** or **9**] for $V(a, b, x)$). $C(a, b)$ is primitive recursive in our sense. A function $E(b, x)$ is defined as follows:

$$E(b, 0) = \mu z_{z<\omega}(C(z, b) \wedge \forall y(y < \omega \wedge z \neq y \wedge C(y, b) \vdash C(z, y)))$$
$$\wedge \forall x(0 < x \vdash E(b, x) = \mu z_{z<\omega}(C(z, b)$$
$$\wedge \forall y(y < x \vdash z \neq E(b, y))$$
$$\wedge \forall u(u < \omega \wedge \forall y(y < x \vdash u \neq E(b, y)) \wedge C(u, b) \wedge u \neq z \vdash C(z, u)))).$$

Let $M(b, c)$ be $\forall x(x < c \vdash \exists z(0 < z \wedge z < \omega \wedge E(b, x) = z))$
$$\wedge E(b, c) = 0$$
and $\varphi(e, a)$ be $\ddot{U}(\mu y_{y<\omega} \ddot{T}_1(e, a, y))$.

Let $b \in O$. Then $|b| < \omega_1$. We can define a function H possessing the following properties by means of the primitive recursion

$$
\begin{aligned}
&\forall x(x \leqq |b| \vdash \\
&\qquad (x=0 \vdash \forall u(u<\omega \vdash (H(x,u)=0 \mapsto u=u))) \\
&\wedge (0<x \vdash (\delta(x)<x \vdash \forall u(u<\omega \vdash (H(x,u)=0 \\
&\qquad\qquad \mapsto \exists z(z<\omega \wedge \ddot{T}_1^1(\mathrm{P}(\{v\}p_v^{H(\delta(x),v)}, z), u, u, z))))) \\
&\wedge (\delta(x)=x \vdash \forall u(u<\omega \vdash (H(x,u)=0 \\
&\qquad\qquad \mapsto \exists z(z<\omega \wedge M(z,x) \\
&\qquad\qquad\quad \wedge H(\mu y_{y<|b|} M(\varphi((z)_2, (u)_{1_0}), y), (u)_0)=0)))))) .
\end{aligned}
$$

Then $H_b(a)$ in Kleene hierarchy is expressible by $H(|b|, a)=0$ in our hierarchy, where H is primitive recursive and $|b|<\omega_1$, q. e. d.

COROLLARY. *Let φ be a function from natural numbers to natural numbers and the predicate $\varphi(a)=b$ is hyperarithmetical. Then there exist a primitive recursive function $p(a,b)$ and an ordinal number c_0 less than ω_1 such that $p(a,c_0)$ is the value of $\varphi(a)$ for $a<\omega$.*

PROOF. By Theorem 12, there exist a primitive recursive predicate $P(x,a,b)$ and an ordinal number c_0 less than ω_1 such that $\varphi(a)=b$ is expressed by $P(c_0,a,b)$. Let $p(a,c)$ be $\mu x_{x<\omega} P(c,a,x)$. Then $p(a,c_0)$ supplies the value of $\varphi(a)$ for $a<\omega$.

§ 10°. Recursively expressible ordinal numbers.

a is called to be *recursively expressible,* if a is expressible by using recursive functions, 0 and ω. Let ω^* be the least ordinal number which is not recursively expressible. Since the class of recursive functions in the present sense coincides the class of recursive functions in [**17**], we shall follow the definition in [**17**] in the discussion concerning the recursively expressible ordinal numbers.

Let $\mathfrak{R}=\{f(a) \mid f$ is a recursive function of one variable and $a<\omega^*\}$ and φ be a one to one mapping from $\mathfrak{R}$ to $\{x \mid x<\mathfrak{r}\}$ satisfying

$$a_1 \in \mathfrak{R} \wedge a_2 \in \mathfrak{R} \wedge a_1<a_2 \vdash \varphi(a_1)<\varphi(a_2) .$$

Moreover let $\mathfrak{B}=\{\{x\}f(x,a) \mid f$ is a recursive function of two variables and $a \in \mathfrak{R}\}$.

If $f \in \mathfrak{B}$ and $\varphi(f(a))=g(\varphi(a))$ for every $a \in \mathfrak{R}$, then we say 'g is an f^{φ}'.

PROPOSITION 13. *If $a \in \mathfrak{R}$ and $b \in \mathfrak{R}$, then*

$$
\begin{aligned}
&\varphi(a')=\varphi(a)' , \\
&\varphi(0)=0 , \\
&\varphi(\omega)=\omega , \\
&\varphi(Iq(a,b))=Iq(\varphi(a), \varphi(b)) , \\
&\varphi(\max(a,b))=\max(\varphi(a), \varphi(b)) , \\
&\varphi(j(a,b))=j(\varphi(a), \varphi(b)) , \\
&\varphi(g^i(a))=g^i(\varphi(a)) , \qquad i=1,2 .
\end{aligned}
$$

PROPOSITION 14. *If g is an f^{φ} and $\mu x f(x)$ is recursive, then*

$$\varphi(\mu x f(x)) = \mu x g(x) .$$

PROOF. Let a be $\mu x f(x)$. Then $a \in \mathfrak{R}$ and $f(a)=0$, which implies $g(\varphi(a)) =0$. Suppose that there exists b such that $b<\varphi(a)$ and $g(b)=0$. Then there exists c such that $c \in \mathfrak{R}$ and $\varphi(c)=b$. Then $0=g(\varphi(c))=\varphi(f(c))$. Hence $f(c)=0$ and $c<a$, which is a contradiction.

PROPOSITION 15. *If g is an f^{φ} and $a \in \mathfrak{R}$, then*

$$\varphi(\mu y_{y<a} f(y)) = \mu y_{y<\varphi(a)} g(y) .$$

PROOF. If $\exists x(f(x)=0 \wedge x<a)$, the proof is performed in the same way as above. Let $\forall x(x<a \vdash 0<f(x))$. Then $\mu y_{y<a} f(y)=0$. We have only to prove that $\mu y_{y<\varphi(a)} g(y)=0$. Let b be $\mu y_{y<\varphi(a)} g(y)$ and $0<b$. Then $b<\varphi(a)$ and $g(b)=0$. Therefore there exists c such that $c \in \mathfrak{R}$, $b=\varphi(c)$ and $c<a$. Then

$$\varphi(f(c))=g(\varphi(c))=g(b)=0$$

whence follows $f(c)=0$, which is a contradiction.

THEOREM 13. *If $\boldsymbol{f}(f_1, \cdots, f_m, x_1, \cdots, x_n)$ is recursive, g_i is an f_i^{φ} for each i $(1 \leqq i \leqq m)$ and $a_j \in \mathfrak{R}$ for each j $(1 \leqq j \leqq n)$, then*

$$\varphi(\boldsymbol{f}(f_1, \cdots, f_m, a_1, \cdots, a_n)) = \boldsymbol{f}(g_1, \cdots, g_m, \varphi(a_1), \cdots, \varphi(a_n)) .$$

PROOF. We prove this by induction on the number of stages to construct $\boldsymbol{f}$. If the outermost function of $\boldsymbol{f}$ is not Rec, then the theorem is clearly proved by Propositions 13-15 and the hypothesis of induction. Then we have only to prove it in the case when $\boldsymbol{f}$ is of the form

$$\mathrm{Rec}(\{f, x\} h_1(f, f_1, \cdots, f_m, x, a_1, \cdots, a_n), h_2(f_1, \cdots, f_m, a_1, \cdots, a_n)) .$$

Let b be $h_2(f_1, \cdots, f_m, a_1, \cdots, a_n)$. Then, by the hypothesis of induction,

$$\varphi(b) = h_2(g_1, \cdots, g_m, \varphi(a_1), \cdots, \varphi(a_n)) .$$

Therefore, we have only to prove

$$\begin{aligned}&\varphi(\mathrm{Rec}(\{f, x\} h_1(f, f_1, \cdots, f_m, x, a_1, \cdots, a_n), b)) \\ &= \mathrm{Rec}(\{f, x\} h_1(f, g_1, \cdots, g_m, x, \varphi(a_1), \cdots, \varphi(a_n)), \varphi(b)) .\end{aligned}$$

We prove this by transfinite induction on $b \in \mathfrak{R}$. Let $f_0(y)$ be

$$\mathrm{Con}(\{z\} \mathrm{Rec}(\{f, x\} h_1(f, f_1, \cdots, f_m, x, a_1, \cdots, a_n), z), b, y) .$$

Then $f_0 \in \mathfrak{B}$ and

$$\mathrm{Rec}(\{f, x\} h_1(f, f_1, \cdots, f_m, x, a_1, \cdots, a_n), b) = h_1(f_0, f_1, \cdots, f_m, b, a_1, \cdots, a_n) .$$

Let $c \in \mathfrak{R}$, Then

$$\varphi(f_0(c)) = \begin{cases} \varphi(\mathrm{Rec}(\{f, x\} h_1(f, f_1, \cdots, f_m, x, a_1, \cdots, a_n), c)), & \text{if } c<b \\ 0 & \text{otherwise,} \end{cases}$$

by the hypothesis of induction on $b \in \mathfrak{R}$,

$$= \begin{cases} \mathrm{Rec}(\{f, x\} h_1(f, g_1, \cdots, g_m, x, \varphi(a_1), \cdots, \varphi(a_n)), \varphi(c)) & \text{if } c < b\,, \\ 0 & \text{otherwise.} \end{cases}$$

Now let $g_0(y)$ be

$$\mathrm{Con}(\{z\}\, \mathrm{Rec}(\{f, x\} h_1(f, g_1, \cdots, g_m, x, \varphi(a_1), \cdots, \varphi(a_n)), z), \varphi(b), y)\,.$$

Then

$$g_0(\varphi(c)) = \mathrm{Con}(\{z\}\, \mathrm{Rec}(\{f, x\} h_1(f, g_1, \cdots, g_m, x, \varphi(a_1), \cdots, \varphi(a_n)), z), \varphi(b), \varphi(c)) \quad \text{for } c \in \mathfrak{R}$$

$$= \begin{cases} \mathrm{Rec}(\{f, x\} h_1(f, g_1, \cdots, g_m, x, \varphi(a_1), \cdots, \varphi(a_n)), \varphi(c)) \\ \qquad\qquad \text{if } \varphi(c) < \varphi(b) \ (c < b)\,, \\ 0 \qquad\qquad \text{otherwise.} \end{cases}$$

Thus we see g_0 is an f_0^{φ}.

From this and the hypothesis of induction,

$$\varphi(h_1(f_0, f_1, \cdots, f_m, b, a_1, \cdots, a_n)) = h_1(g_0, g_1, \cdots, g_m, \varphi(b), \varphi(a_1), \cdots, \varphi(a_n))$$

$$= \mathrm{Rec}(\{f, x\} h_1(f, g_1, \cdots, g_m, x, \varphi(a_1), \cdots, \varphi(a_n)), \varphi(b))\,.$$

THEOREM 14. *If $f(x_1, \cdots, x_n)$ is recursive and $a_i \in \mathfrak{R}$ for each i $(1 \leqq i \leqq n)$, then $\varphi(f(a_1, \cdots, a_n)) = f(\varphi(a_1), \cdots, \varphi(a_n))$.*

THEOREM 15. *If $b < a$ and a is recursively expressible, then b is also recursively expressible, that is, an ordinal number is recursively expressible if and only if it is less than ω^*.*

PROOF. We have only to prove that

$$\varphi(a) = a \quad \text{for every} \quad a \in \mathfrak{R}\,.$$

Let $a \in \mathfrak{R}$. Then $a = f(b)$ for some recursive function f and b such that $b < \omega^*$. $b < \omega^*$ implies $\varphi(b) = b$.

Thus we have

$$\begin{aligned} \varphi(a) &= \varphi(f(b)) \\ &= f(\varphi(b)) \qquad \text{(by Theorem 14)} \\ &= f(b) \\ &= a. \end{aligned}$$

THEOREM 16. *The predicate $a < \omega^*$ is not expressible in both 1-quantifier forms.*

PROOF. If $a < \omega^*$ is expressible in both 1-quantifier forms, it is general recursive by Theorem 1. Then the function

$$\mu a(\neg\, a < \omega^*) \qquad \text{(which is } \omega^*\text{)}$$

is general recursive, i. e. ω^* is recursively expressible, which is a contradiction.

THEOREM 17. *The predicate $a<\omega^*$ is expressible in Σ_1^{ord}-form.*

PROOF. $a<\omega^*$ if and only if there exists a general recursive function f such that $f(0)=a$. Then, by means of the normal form theorem,

$$a<\omega^* \rightleftarrows \exists e(e<\omega \wedge \exists x(T(e, j_3(0, g^1(x), g^2(x)))=0 \wedge a=g^1(x)$$
$$\wedge \forall y(y<x \vdash \neg T(e, j_3(0, g^1(y), g^2(y)))=0))),$$

where the right side of this equivalence can easily be expressible in Σ_1^{ord}-form.

THEOREM 18. *The predicate $a=\omega^*$ is not expressible in Σ_1^{ord}-form, but is expressible in $\Sigma_1^{ord} \wedge \Pi_1^{ord}$-form.*

PROOF. Since

$$a<\omega^* \rightleftarrows \forall x(x=\omega^* \vdash a<x),$$

$a<\omega^*$ should be expressible in Π_1^{ord}-form, if $a=\omega^*$ were expressible in Σ_1^{ord}-form. This contradicts Theorem 16. On the other hand,

$$a=\omega^* \rightleftarrows \forall x(x<a \vdash x<\omega^*) \wedge \forall x(x<\omega^* \vdash x<a),$$

which implies that $a=\omega^*$ is expressible in $\Sigma_1^{ord} \wedge \Pi_1^{ord}$-form by means of Theorem 17.

We see that both predicates $a<\omega^*$ and $a \leqq \omega^*$ are expressible in Σ_1^{ord}-form and both predicates $a>\omega^*$ and $a \geqq \omega^*$ are expressible in Π_1^{ord}-form. But it remains open if the predicate $a=\omega^*$ is expressible in Π_1^{ord}-form or not.

THEOREM 19. *For every general recursive function $g(a)$, there exist a primitive recursive function $p(a, b)$ and a recursively expressible ordinal number c^* such that*

$$g(a)=p(a, c^*) \quad \textit{for each} \quad a<\omega.$$

PROOF. Let $p(a, b)$ be a function obtained from $g(a)$ by replacing every occurence of recursive μx in $g(a)$ by $\mu x_{x<b}$. Then $p(a, b)$ is clearly primitive recursive,

$$\forall a(a<\omega \vdash g(a)=p(a, b))$$

is general recursive and

$$\exists b \forall a(a<\omega \vdash g(a)=p(a, b)).$$

Let c^* be $\mu y \forall x(x<\omega \vdash g(x)=p(x, y))$, which satisfies the required conditions.

§11°. Additional note concerning constructive ordinals.

In the preceding section we defined ω^* to be the least ordinal number not recursively expressible and showed that the predicate $a<\omega^*$ is not expressible in the both 1-quantifier forms in our sense.

There are many trials for extending Church-Kleene's constructive ordinals (cf. [3]). In these the predicates representing that {a is a notation for an extended constructive ordinal} and {a is less than b in the sense of notations for extended constructive ordinals} are expressible in both 2-function-quantifier forms in Kleene's sense. Among them there are systems of notations C and $\tilde{C}$ given by Kreider and Rogers [12]. If ω_C is the first ordinal not in the segment represented by C (or $\tilde{C}$), then $\omega_C < \omega^*$. Because, since the predicate $a <_C b$ (or $a <_{\tilde{C}} b$) is expressible in both 2-function-quantifier forms in Kleene's sense, it is expressible in both 1-quantifier forms in our sense by Theorem 9 and then it is general recursive by Theorem 1. Thus we can find a general recursive function ψ in our sense such that $\psi(a, b)=0$ means $a <_C b$ (or $a <_{\tilde{C}} b$). In the same way as in Theorem 8 of [16], using $\psi(a, b)$ in place of $\varphi(c, a, b)$ there, we can represent ω_C as a recursively expressible ordinal number i.e. $\omega_C < \omega^*$.

In this section we shall show the definition of our candidate of a system of notations not expressible in both 2-function-quantifier forms using Putnam's notation. In the definition of the candidate small Greek letters stand for ordinal numbers and we denote a (partial) recursive function by its Gödel number. An ordinal number α is called to be *constructively accessible* if N_α can be defined to satisfy the following conditions.

(i) There exist two functions recursive uniformly in functions (or predicates) of one variable and two variables respectively, (we denote the uniform Gödel numbers as d and e respectively) satisfying the following condition (*).

(*) If A is a set and $\leqq_A$ is a well-ordering on A, then $\lambda xy\{e\}^{A, \leqq_A}(x, y)$ gives a well-ordering on the set $\hat{x}(\{d\}^{A, \leqq_A}(x)=0)$.

(ii) There exist partial recursive functions, (let the Gödel numbers be p and q).

(iii) To state the definition of N_α, we use the following auxiliary definitions:

$$C_\alpha = \bigcup_{\beta<\alpha} N_\beta .$$

$n \leqq_{C_\gamma} m$, if and only if {there exist ordinal numbers α and β less than γ such that $m \in N_\alpha$, $n \in N_\beta$ and $\alpha \leqq \beta$}. {f is an *o. p. c. m.* (to read: order preserving cofinal mapping) from C_β into C_α} if and only if the following conditions are satisfied:

(a) If $n \in C_\beta$, then $f(n) \in C_\alpha$.

(b) If $n \in C_\beta$ and $m \in C_\beta$, then $n \leqq_{C_\beta} m$ if and only if $f(n) \leqq_{C_\alpha} f(m)$.

(c) For every $n \in C_\alpha$, there exists m such that $m \in C_\beta$ and $n \leqq_{C_\alpha} f(m)$.

(iv) Definition of N_α:

Case 1. $N_0 = \{1\}$.

Case 2. $N_{\alpha+1} = \{2^x \mid x \in N_\alpha\}$.

Case 3. α is a limit number and there exists an ordinal number β satisfying the following conditions: $\beta < \alpha$; there exist a number a and a partial recursive o. p. c. m. g from C_β into C_α such that $a \in N_\beta$ and $p(a, g)$ does not belongs to C_α. Then

$N_\alpha = \{p(a,f) \mid a \in N_{\beta_0}$ where β_0 is the least ordinal number β satisfying the above conditions, a is the least number required to exist for β_0 in the sense of $\leqq_{N_{\beta_0}}$ and f is a partial recursive o. p. c. m. such that $p(a,f) \notin C_\alpha.\}$

Case 4. α is a limit number and there exists no β satisfying the conditions in Case 3. There exists a number a satisfying the following conditions: $\{d\}^{C_\alpha, \leqq C_\alpha}(a) = 0$; there exists a partial recursive o. p. c.m. g from C_α to C_α such that $q(a, g)$ does not belong to C_α. Then

$N_\alpha = \{q(a, f) \mid a$ is the least number satisfying the above conditions in the sense of $\lambda xy\{e\}^{C_\alpha, \leqq C_\alpha}(x,y) = 0$ and f is a partial recursive o. p. c. m. from C_α to C_α such that $q(a,f) \notin C_\alpha.\}$

It remains open if the predicate 'α is a constructively accessible ordinal number' can be expressible in both 2-function-quantifier forms. But we suspect that the condition (*) for d and e is not expressible in both 2-function-quantifier forms.

Tokyo University of Education

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Added in proof. In reading the proof, we found the book 'Essays on the foundations of mathematics' (Amsterdam, 1962). By the result of Shoenfield in this book, the following result of ours becomes absolute, i.e., it can be proved without using $V=L$:

A predicate of natural numbers expressible in Σ_1^{ord}- or Π_1^{ord}-form in our hierarchy is expressible in Σ_2^1- or Π_2^1-form in Kleene hierarchy respectively and vice versa. A predicate of natural numbers is general recursive in our sense, if and only if it is expressible as a predicate in $\Sigma_2^1 \cap \Pi_2^1$ in Kleene hierarchy.

Now we can also give the reference to Putnam's notation in § 11.

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