J. Math. Soc. Japan
Vol. 14, No. 2, 1962

On stable processes with boundary conditions

By Shinzo WATANABE

(Received Aug. 31, 1961)

§ 0. Introduction.

Let $x_t(w)$, $t \geqq 0$, be the symmetric stable process with exponent α and I be the open interval $(-1, 1)$. For any right continuous path function $x_t(w)$ starting at some point $x \in I$, let $\sigma(w)$ be the first time $x_t(w)$ leaves I. The absorbing barrier stable process with exponent α is derived from $x_t(w)$ by killing it at time $\sigma(w)$. This process, which proves to be Markovian, was investigated by M. Kac [**9**] and J. Elliott [**3**]. Kac discovered the formal expression of the infinitesimal generator of the semi-group attached to this process and Elliott determined the domain of the generator in case $0 < \alpha < 1$. The first purpose of this paper is to determine this generator for every α $(0 < \alpha < 2)$, and this will be done in §§ 1-2.

In § 3 we shall compute the distribution of the first exit place x_σ and shall obtain the following results

$$P_x(x_\sigma \in [1, \infty)) = 2^{1-\alpha} \frac{\Gamma(\alpha)}{\left[\Gamma\left(\frac{\alpha}{2}\right)\right]^2} \int_{-1}^{x} (1-y^2)^{\frac{\alpha}{2}-1} dy$$

$$P_x(x_\sigma \in d\xi) = \frac{\sin \frac{\alpha\pi}{2}}{\pi} \left(\frac{1-x^2}{\xi^2-1}\right)^{\frac{\alpha}{2}} \frac{d\xi}{|\xi - x|}, \qquad |\xi| > 1.$$

These results have been obtained recently by H. Widom [**14**] in a somewhat different way. Our method consists in deriving the integro-differential equations governing these quantities and solving them.

In § 4 we shall determine the generator of the semi-group of the stable process on the space of continuous functions and shall also determine the generator of the absorbing barrier stable process on $I^- = (-\infty, 0)$.

Elliott [**2**] determined the most general boundary conditions by which the operator

$$\tilde{\Omega} u(x) = P \int_{-1}^{1} \frac{u'(y)}{y-x} dy$$

becomes a generator of a Markov process on $[-1, 1]$. In § 5 we extend this result to the case with general α. Our boundary conditions are obtained immediately from Feller's boundary conditions for the one-dimensional diffusion by replacing $u^+(-1)$ and $u^-(1)$ with

$$\delta_{-1}u=\lim_{\varepsilon\downarrow 0}\frac{u(-1+\varepsilon)-u(-1)}{\varepsilon^{\frac{\alpha}{2}}},$$

$$\delta_1 u=\lim_{\varepsilon\downarrow 0}\frac{u(1)-u(1-\varepsilon)}{\varepsilon^{\frac{\alpha}{2}}}$$

respectively. We have the same boundary conditions at $x=0$ for the stable process on the half line $\bar{I}^-=(-\infty, 0]$. Now path functions of these processes can be constructed from those of the ordinary stable process. The local time of the "reflecting barrier process" on $\bar{I}^-$ at $x=0$ is defined and its inverse function is a one-sided stable process of exponent $\frac{1}{2}$ for any α.

In § 6 we shall discuss the properties of the path functions of the stable process. In particular, we shall prove that if $\boldsymbol{Z}(w)$ denotes the set of zero points of the path function $x_t(w)$, then, with probability one, $\boldsymbol{Z}(w)\frown(0, t]$ is empty if $0<\alpha\leqq 1$, while a non-countable Borel set of the Hausdorff-Besicovitch dimension $1-\frac{1}{\alpha}$ if $1<\alpha\leqq 2$.

The auther wishes to express his hearty thanks to Prof. K. Ito and Prof. N. Ikeda for their kind suggestions and encouragement.

§ 1. The semi-group on L^1.

The symmetric stable process with exponent α $(0<\alpha\leqq 2)$ is a temporally homogeneous Lévy process $x_t(w)$ $(x_0=0)$ with the characteristic function

$$E(e^{i\xi x_t})=e^{-t|\xi|^\alpha}. \tag{1.1}$$

In the sequel, we shall assume that all path functions are right continuous, as we can by taking an appropriate version. A stable process induces a Markov process if we define the probability law governing the path starting at x by

$$P_x(B)=P(x+x.(w)\in B)^{1)}. \tag{1.2}$$

Its semi-group is

$$T_t f(x)=E_x(f(x_t))=\int_{-\infty}^{\infty} f(y)p(t, x-y)dy \qquad (t\geqq 0), \tag{1.3}$$

with

$$p(t, x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ix\xi}e^{-t|\xi|^\alpha}d\xi=\frac{1}{\pi}\int_0^{\infty}\cos x\xi e^{-t\xi^\alpha}d\xi. \tag{1.4}$$

Its resolvent operator is

---

1) Here B denotes a subset of the space of path functions.

$$(1.5)\qquad G_\lambda f(x)=\int_0^\infty e^{-\lambda t}T_t f(x)dt=\int_{-\infty}^\infty f(y)g_\lambda(x-y)dy \qquad (\lambda>0),$$

with

$$(1.6)\qquad g_\lambda(x)=\int_0^\infty e^{-\lambda t}p(t,x)dt=\frac{1}{\pi}\int_0^\infty \frac{\cos x\xi}{\lambda+\xi^\alpha}d\xi \qquad (x\neq 0, \lambda>0).$$

Hereafter we shall consider T_t as the semi-group of integral operators (1.3) acting on L^1, and shall determine its infinitesimal generator.

First, if $f\in L^1$, then $T_t f\in L^1$ and

$$(1.7)\qquad \|T_t f\|_1 \leqq \|f\|_1$$

$$(1.8)\qquad \|T_t f-f\|_1\to 0 \qquad (t\to 0).$$

(1.7) is obvious and so we shall check (1.8) only. Estimating $\|T_t f-f\|_1$ as

$$\|T_t f-f\|_1=\int_{-\infty}^\infty\Big|\int_{-\infty}^\infty p(t,z)(f(x+z)-f(x))dz\Big|dx$$

$$\leqq \int_{-\infty}^\infty p(t,z)\int_{-\infty}^\infty |f(x+z)-f(x)|dx\cdot dz$$

$$\leqq \int_{|z|\leqq\delta} p(t,z)\int |f(x+z)-f(x)|dx\cdot dz+2\|f\|_1\int_{|z|>\delta}p(t,z)dz,$$

taking δ sufficiently small and then letting $t\downarrow 0$, we obtain (1.8). Hence T_t is a semi-group on L^1 in the Hille-Yosida sense.

THEOREM 1.1. *The infinitesimal generator Ω_1 of T_t is given as follows.*

$$(1.9)\qquad \Omega_1 u(x)=\frac{c(\alpha)}{\alpha(\alpha-1)}\frac{d^2}{dx^2}\int_{-\infty}^\infty u(y)\frac{1}{|x-y|^{\alpha-1}}dy \qquad 1<\alpha<2$$

$$=\lim_{N\to\infty}\frac{1}{\pi}\frac{d^2}{dx^2}\int_{-N}^N u(y)\log\frac{1}{|x-y|}dy \qquad \alpha=1$$

$$=\frac{c(\alpha)}{\alpha}\frac{d}{dx}\int_{-\infty}^\infty u(y)\frac{\operatorname{sgn}(y-x)}{|x-y|^\alpha}dy \qquad 0<\alpha<1,$$

where

$$(1.10)\qquad c(\alpha)=\frac{1}{\pi}\Gamma(\alpha+1)\sin\frac{\alpha\pi}{2},$$

with the domain

$$(1.11)\qquad D(\Omega_1)=\{u\,;u\in L^1, \Omega_1 u\in L^1\}, \qquad 0<\alpha<2 \quad \alpha\neq 1$$

$$=\{u\,;u\in L^1, \exists f\in L^1\ \lim_{N\to\infty}\frac{1}{\pi}\frac{d^2}{dx^2}\int_{-N}^N u(y)\log\frac{1}{|x-y|}dy=f(x)$$

in the distribution sense$\}$, $\alpha=1$.

REMARK. If $u\in L^1$, $\int_{-\infty}^\infty \frac{u(y)}{|x-y|^\beta}dy$, $0<\beta<1$ is the sum of a bounded function and a function in L^1. So we can define $\frac{d^n}{dx^n}\int_{-\infty}^\infty\frac{u(y)}{|x-y|^\beta}dy$ in the distribution sense. Hence $\Omega_1 u$ is always defined if $u \in L^1$, $\alpha \neq 1$.

PROOF. Suppose $1 < \alpha < 2$, the proof of the other cases being similar. Put $u(x) = G_\lambda f(x)$ for $f \in L^1$. Taking the Fourier transforms of both sides, we have

$$\hat{u}(\sigma) = \frac{\hat{f}(\sigma)}{\lambda + |\sigma|^\alpha}. \tag{1.12}$$

Put $T_1(x) = \frac{1}{|x|^{\alpha-1}}$, and $T_2(x) = \frac{1}{|x|^{\alpha+1}}$.

Then $\frac{1}{\alpha(\alpha-1)} \frac{d^2}{dx^2} T_1 * u = T_2 * u$ $\quad$ ($*$: convolution).

Using the fact that $\hat{T}_2(\sigma) = -\frac{1}{c(\alpha)} |\sigma|^\alpha$, we get

$$\begin{aligned}(\widehat{T_2 * u}, \psi(\sigma)) &= (T_2 * u(x), \hat{\psi}(x))^{2)} \\ &= (u(x), T_{2.y}(\hat{\psi}(x+y))) = (u(x), T_{2.y}(\widehat{e^{-ix\sigma}\psi}(y))) \\ &= (u(x), \hat{T}_{2.\sigma}(e^{-ix\sigma}\psi(\sigma))) = (u(x), -\frac{1}{c(\alpha)} \int |\sigma|^\alpha \psi(\sigma) e^{-ix\sigma} d\sigma) \\ &= \left(-\frac{1}{c(\alpha)} \hat{u}(\sigma) |\sigma|^\alpha, \psi(\sigma)\right).\end{aligned}$$

This, combined with (1.12), implies

$$\widehat{\lambda u - \frac{c(\alpha)}{\alpha(\alpha-1)} \frac{d^2}{dx^2} T_1 * u} = \hat{f},$$

namely

$$\lambda u - \frac{c(\alpha)}{\alpha(\alpha-1)} \frac{d^2}{dx^2} T_1 * u = f.$$

Thus we have $\Omega_1 u = \lambda u - f \in L^1$, so that u belongs to $D(\Omega_1)$ by (1.11).

Conversely let u belong to $D(\Omega_1)$ defined by (1.11). Then $f = \lambda u - \Omega_1 u$ belongs to L^1 and if we define $v(x)$ by $v = G_\lambda f$, we have $\lambda v - \Omega_1 v = f$ from the fact obtained above. Put $w = u - v$. Then $w \in L^1$ and $\lambda w - \Omega_1 w = 0$. Taking the Fourier transforms, we have as above

$$\widehat{\lambda w - \Omega_1 w}(\sigma) = (\lambda + |\sigma|^\alpha)\hat{w}(\sigma) = 0.$$

Hence $\hat{w}(\sigma) = 0$ i.e. $w = 0$, this means that $u = G_\lambda f$. This proves the theorem.

2. The absorbing barrier process.

Let I be the open interval $(-1, 1)$. We consider the symmetric stable pro-

2) $(T, \phi) \equiv T(\phi)$ is the value of the functional T for a testing function $\phi \in (\mathcal{S})$, and $\hat{\phi}$ is the Fourier transform of ϕ i.e. $\hat{\phi}(x) = \int_{-\infty}^{\infty} e^{-ix\sigma} \phi(\sigma)\, d\sigma$.

cess starting at $x \in I$ which is killed as soon as it leaves I. Then we have a Markov process on I. We define $\sigma(w)$ by

$$(2.1) \qquad \sigma(w) = \inf(t\,;\, x(w) \notin I)\,.$$

Then transition probability of this process is given by

$$(2.2) \qquad \bar{P}(t, x, E) = P_x(x_t(w) \in E, \sigma(w) > t) \qquad x \in I, \quad E \subset I.$$

$\bar{P}(t, x, E)$ is absolutely continuous with respect to Lebesgue measure:

$$(2.3) \qquad \bar{P}(t, x, E) = \int_E \bar{p}(t, x, y)\, dy^{3)} \qquad E \subset I$$

Define $\bar{g}_\lambda(x, y)$ by

$$(2.4) \qquad \bar{g}_\lambda(x, y) = \int_0^\infty e^{-\lambda t} \bar{p}(t, x, y)\, dt \qquad x \in I.$$

We often use the following lemma due to Pólya-Szegö [**11**].

LEMMA 2.1 (Pólya-Szegö). *Let $P_n^\nu(x)$ be the ultra-spherical polynomials defined by*

$$\frac{1}{(1-2xw+w^2)^\nu} = P_0^{(\nu)}(x) + P_1^{(\nu)}(x)w + P_2^{(\nu)}(x)w^2 + \cdots + P_n^{(\nu)}(x)w^n + \cdots.$$

Then if $0 < \alpha < 2$, $\alpha \neq 1$, $x \in I$

$$(2.5) \qquad \int_{-1}^1 |x-y|^{1-\alpha} P_m^{\left(\frac{\alpha-1}{2}\right)}(y)(1-y^2)^{\frac{\alpha}{2}-1} dy = \lambda_m P_m^{\left(\frac{\alpha-1}{2}\right)}(x)\,, \qquad m = 0, 1, 2, \cdots$$

where

$$(2.6) \qquad \lambda_m = \frac{\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(1-\frac{\alpha}{2}\right)}{\Gamma(\alpha-1)} \cdot \frac{\Gamma(m+\alpha-1)}{\Gamma(m+1)}\,.$$

In particular, taking $m = 0$

$$(2.7) \qquad \int_{-1}^1 |x-y|^{1-\alpha}(1-y^2)^{\frac{\alpha}{2}-1} dy = \frac{\pi}{\sin\frac{\alpha\pi}{2}} \qquad x \in I.$$

First we prove the following theorem which was proved by Elliott [**3**] in case $0 < \alpha < 1$.

THEOREM 2.1. *If σ is defined by* (2.1), *then*

$$(2.8) \qquad E_x(\sigma) = \frac{(1-x^2)^{\frac{\alpha}{2}}}{\Gamma(\alpha+1)}\,, \qquad x \in I, \quad 0 < \alpha \leqq 2\,.$$

PROOF.[4] Define $u(x)$ by

3) $\bar{p}(t, x, y)$ is defined to be zero if $y \notin I$.

4) In case $\alpha = 2$, the above proof does not apply but in this case the result is well known.

$$(2.9)\qquad u(x)=\frac{(1-x^2)^{\frac{\alpha}{2}}}{\Gamma(\alpha+1)} \qquad |x|<1$$
$$=0 \qquad |x|\geqq 1\,.$$

From (2.7), we have

$$(2.10)\qquad \frac{c(\alpha)}{\alpha(\alpha-1)}\frac{d^2}{dx^2}\int_{-1}^{1}\frac{u(y)}{|x-y|^{\alpha-1}}dy=-1 \qquad (|x|<1)$$

while it is obvious that

$$(2.11)\qquad \frac{c(\alpha)}{\alpha(\alpha-1)}\frac{d^2}{dx^2}\int_{-1}^{1}\frac{u(y)}{|x-y|^{\alpha-1}}dy=c(\alpha)\int_{-1}^{1}\frac{u(y)}{|x-y|^{\alpha+1}}dy \qquad (|x|>1)\,.$$

Let $F(x)$ be equal to -1 if $|x|<1$ and to the right side of (2.11) if $|x|>1$. Then $F(x)$ is in L^1 and in order to prove that $\Omega_1 u=F$ in the distribution sense, it is enough to prove that $\frac{d}{dx}\int_{-1}^{1}\frac{u(y)}{|x-y|^{\alpha-1}}dy$ is continuous at $x=\pm 1$. This can be done by simple calculations so $u\in D(\Omega_1)$. Then we have [7]

$$u(x)=G_\lambda[\lambda u-F](x)=E_x\Big(\int_0^\infty e^{-\lambda t}[\lambda u(x_t)-F(x_t)]\,dt\Big)$$
$$=E_x\Big(\int_0^\sigma e^{-\lambda t}[\lambda u(x_t)-F(x_t)]\,dt\Big)+E_x(e^{-\lambda\sigma}u(x_\sigma))$$
$$=E_x\Big(\int_0^\sigma e^{-\lambda t}[\lambda u(x_t)+1]\,dt\Big)$$

since $x_t\in I$ for $t<\sigma$, and $x_\sigma\notin I$.

Now

$$E_x\Big(\int_0^\sigma e^{-\lambda t}dt\Big)\leqq E_x\Big(\int_0^\sigma e^{-\lambda t}[\lambda u(x_t)+1]\,dt\Big)\leqq(\lambda\,\|u\|_\infty+1)E_x(\sigma)\,.$$

Letting $\lambda\downarrow 0$, we have $u(x)=E_x(\sigma)$.

LEMMA 2.2.

$$\bar p(t,x,y)=\bar p(t,y,x) \qquad \bar g_\lambda(x,y)=\bar g_\lambda(y,x)$$

PROOF. We prove this lemma by using the method of Hunt [4] and Bochner's theory of subordination.

Let $W_1(\boldsymbol{B}_1,P_1)$, $W_2(\boldsymbol{B}_2,P_2)$ be two probability spaces and $W(\boldsymbol{B},P)$ be their product probability space. Let $\theta_t(w_1)$, $w_1\in W_1$, be a temporally homogeneous Lévy process ($\theta_0(w_1)\equiv 0$) with increasing paths given by

$$E_1(e^{-\xi\theta_t})=e^{-t\xi^{\frac{\alpha}{2}}}\,, \qquad \xi>0\,,\quad t\geqq 0\,.$$

Let $B_t(w_2)$, $w_2\in W_2$, be a Wiener process given by

$$E_2(e^{-i\xi B_t})=e^{-t|\xi|^2}\,, \qquad \xi\in R\,,\quad t\geqq 0\,.$$

Then $x_t(w)=B_{\theta_t(w_1)}(w_2)$, $w=(w_1,w_2)\in W$, gives a version of the symmetric stable process with exponent α.

Define $\bar{B}_s(w_2)$, $0 \leqq s \leqq t$, by

$$\bar{B}_s(w_2) = B_s(w_2) - \frac{s}{t} B_t(w_2) \qquad 0 \leqq s \leqq t.$$

Then [4]

(i) the process $\{\bar{B}_s(w_2)\}$ is independent of $B_t(w_2)$,

(ii) the process $\{\bar{B}_s'(w_2)\}$ defined by

$$\bar{B}_s'(w_2) = \bar{B}_{t-s}(w_2), \qquad 0 \leqq s \leqq t$$

is a version of $\{\bar{B}_s(w_2)\}$.

Now[5)]

$$P_x(x_t \in E, \sigma > t) = P(x + x_s(w) \in I,\ 0 \leqq s \leqq t,\ x_t(w) \in E - x)$$

$$= P_1 \times P_2(x + B_{\theta_s(w_1)}(w_2) \in I,\ 0 \leqq s \leqq t,\ B_{\theta_t(w_1)}(w_2) \in E - x)$$

$$= \int_{W_1} P_2(x + B_{\theta_s(w_1)}(w_2) \in I,\ 0 \leqq s \leqq t,\ B_{\theta_t(w_1)}(w_2) \in E - x) P_1(dw_1)$$

$$= \int_{W_1} \cdot \int_E P_2(x + B_{\theta_s(w_1)}(w_2) \in I,\ 0 \leqq s \leqq t, \mid B_{\theta_t(w_1)}(w_2) = y - x) p_B(\theta_t(w_1), x, y)\, dy \cdot P_1(dw_1)$$

$$= \int_E \cdot \int_{W_1} P_2(x + B_{\theta_s(w_1)}(w_2) \in I,\ 0 \leqq s \leqq t, \mid B_{\theta_t(w_1)}(w_2) = y - x) p_B(\theta_t(w_1), x, y)\, P_1(dw_1) \cdot dy$$

where $p_B(t, x, y) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{(x-y)^2}{4t}}$.

Hence we have by the definition of $\bar{p}(t, x, y)$

$$\bar{p}(t, x, y) = \int_{W_1} P_2(x + B_{\theta_s} \in I,\ 0 \leqq s \leqq t \mid B_{\theta_t} = y - x) p_B(\theta_t, x, y) P_1(dw_1)$$

Now using (i) and (ii), we get

$$P_2(x + B_{\theta_s} \in I,\ 0 \leqq s \leqq t \mid B_{\theta_t} = y - x)$$

$$= P_2(x + \bar{B}_{\theta_s} + \frac{\theta_s}{\theta_t} B_{\theta_t} \in I,\ 0 \leqq s \leqq t \mid B_{\theta_t} = y - x)$$

$$= P_2(x + \bar{B}_{\theta_s} + \frac{\theta_s}{\theta_t}(y - x) \in I,\ 0 \leqq s \leqq t \mid B_{\theta_t} = y - x)$$

$$= P_2(x + \bar{B}_{\theta_s} + \frac{\theta_s}{\theta_t}(y - x) \in I,\ 0 \leqq s \leqq t)$$

$$= P_2(x + \bar{B}_{\theta_t - \theta_s} + \frac{\theta_s}{\theta_t}(y - x) \in I,\ 0 \leqq s \leqq t)$$

$$= P_2(y + \bar{B}_{\theta_t - \theta_s} + \frac{\theta_t - \theta_s}{\theta_t}(x - y) \in I,\ 0 \leqq s \leqq t)$$

$$= P_2(y + B_{\theta_t - \theta_s} \in I,\ 0 \leqq s \leqq t \mid B_{\theta_t} = x - y).$$

5) $E - x = \{y\,; y = z - x, z \in E\}$.

But the following equality holds in general[6];

$$(2.12)\qquad E_1(f[\theta_t(w_1)-\theta_s(w_1)\,;\ 0\leqq s\leqq t])$$
$$=E_1(f[\theta_{t-s}(w_1)\,;\ 0\leqq s\leqq t]).$$

Thus we have

$$\bar{p}(t,x,y)=\int_{W_1}P_2(x+B_{\theta_s}\in I,\ 0\leqq s\leqq t\mid B_{\theta_t}=y-x)p_B(\theta_t,x,y)P_1(dw_1)$$
$$=\int_{W_1}P_2(y+B_{\theta_t-\theta_s}\in I,\ 0\leqq s\leqq t\mid B_{\theta_t}=x-y)p_B(\theta_t,x,y)P_1(dw_1)$$
$$=\int_{W_1}P_2(y+B_{\theta_{t-s}}\in I,\ 0\leqq s\leqq t\mid B_{\theta_t}=x-y)p_B(\theta_t,y,x)P_1(dw_1)$$
$$=\bar{p}(t,y,x)\,.$$

LEMMA 2.3.

$$(2.13)\qquad E_x(e^{-\lambda\sigma}\,;\ x_\sigma\in E)=c(\alpha)\int_E\int_I\frac{\bar{g}_\lambda(x,y)}{|y-\xi|^{\alpha+1}}dy\,d\xi\qquad E\subset I^c\,.$$

PROOF. Put $\pi_\lambda(x,E)=E_x(e^{-\lambda\sigma}\,;\ x_\sigma\in E)$. We first prove that $\pi_\lambda(x,E)$ is absolutely continuous with respect to the Lebesgue measure.

The function $u(x)$ in (2.9) belongs to $D(\Omega_1)$, as we have seen above, and satisfies $u(x)=\int_{-1}^1\bar{g}_\lambda(x,y)(\lambda u(y)+1)\,dy$. Now

$$g_\lambda(x-y)=\bar{g}_\lambda(x,y)+\int_{I^c}\pi_\lambda(x,d\xi)g_\lambda(\xi-y)$$

holds for every $x\in I$ and almost every y. Then noting the symmetry of $\bar{g}_\lambda(x,y)$, we have

$$u(y)=\int_{-1}^1\bar{g}_\lambda(x,y)(\lambda u(x)+1)\,dx$$
$$=\int_{-1}^1 g_\lambda(x-y)(\lambda u(x)+1)\,dx-\int_{-1}^1\Big\{\int_{I^c}\pi_\lambda(x,d\xi)\,g_\lambda(\xi-y)\Big\}(\lambda u(x)+1)\,dx\,.$$

On the other hand

$$u(y)=\int_{-\infty}^{\infty}g_\lambda(x-y)(\lambda u(x)-\Omega_1u(x))\,dx\,.$$

Comparing these two equations, we get

$$\int_{I^c}g_\lambda(\xi-y)(\lambda u(\xi)-\Omega_1u(\xi))\,d\xi=\int_{I^c}g_\lambda(\xi-y)\int_{-1}^1(\lambda u(x)+1)\pi_\lambda(x,d\xi)\,dx\,.$$

Since the potential determines its measure uniquely, we have

6) It is easy to prove (2.12) if f is a tame function and then taking limits we have (2.12).

$$(\lambda u(\xi)-\Omega_1 u(\xi))\,d\xi=\int_I(\lambda u(x)+1)\pi_\lambda(x,d\xi)\,dx\,.$$

Taking λ to be $-\lambda\|u\|_\infty+1>0$, we see that $\pi_\lambda(x,d\xi)$ is absolutely continuous with respect to Lebesgue measure $d\xi$. Hence we can wright

$$\pi_\lambda(x,d\xi)=\pi_\lambda(x,\xi)\,d\xi\,.$$

Now[7] it is easy to see that if $u\in\mathcal{D}^2=\{u\in C^2,$ with compact support$\}$ then

$$\begin{aligned}\Omega_1 u(x)&=\frac{c(\alpha)}{\alpha(\alpha-1)}\frac{d^2}{dx^2}\int_{-\infty}^{\infty}\frac{u(y)}{|x-y|^{\alpha-1}}dy\\&=\int_{-\infty}^{\infty}[u(y)-u(x)-(y-x)u'(x)]\,c(\alpha)\frac{dy}{|x-y|^{\alpha+1}}\,.\end{aligned}$$

For any element u of $\mathcal{D}^2$ such that $u(x)\equiv 0$ for $x\in\bar{I}$, we have

$$\begin{aligned}u(x)&=\int_{-1}^{1}\bar{g}_\lambda(x,y)(\lambda u(y)-\Omega_1 u(y))\,dy+\int_{I^c}\pi_\lambda(x,\xi)\,u(\xi)\,d\xi\\&=-\int_{-1}^{1}\bar{g}_\lambda(x,y)\Omega_1 u(y)\,dy+\int_{I^c}\pi_\lambda(x,\xi)\,u(\xi)\,d\xi\,.\end{aligned}$$

Hence if $x\in I$,

$$0=-\int_{-1}^{1}\bar{g}_\lambda(x,y)\Omega_1 u(y)\,dy+\int_{I^c}\pi_\lambda(x,\xi)\,u(\xi)\,d\xi\,.$$

So we have

$$\begin{aligned}\int_{I^c}\pi_\lambda(x,\xi)\,u(\xi)\,d\xi&=\int_{-1}^{1}\bar{g}_\lambda(x,y)\Omega_1 u(y)\,dy\\&=\int_I\bar{g}_\lambda(x,y)\int_{-\infty}^{\infty}[u(\xi)-u(y)-(\xi-y)\,u'(y)]\frac{c(\alpha)}{|y-\xi|^{\alpha+1}}d\xi\,dy\\&=\int_I\bar{g}_\lambda(x,y)\int_{I^c}u(\xi)\frac{c(\alpha)}{|y-\xi|^{\alpha+1}}d\xi\,dy\\&=\int_{I^c}u(\xi)\,c(\alpha)\int_I\frac{\bar{g}_\lambda(x,y)}{|y-\xi|^{\alpha+1}}dy\,d\xi\,,\end{aligned}$$

i. e.

$$\pi_\lambda(x,\xi)=c(\alpha)\int_I\frac{\bar{g}_\lambda(x,y)}{|y-\xi|^{\alpha+1}}dy\,.$$

REMARK. It is natural to conjecture that if we put $x_{\sigma-}=\lim_{n\uparrow\infty}x_{\sigma-\frac{1}{n}}$ then $E_x(e^{-\lambda\sigma};x_{\sigma-}\in E,x_\sigma\in F)=\int_F\cdot\int_E c(\alpha)\frac{\bar{g}_\lambda(x,y)}{|y-\xi|^{\alpha+1}}dy\,d\xi$. In fact this is true and we can see from this that σ and x_σ are independent under the condition that $x_{\sigma-}$ be given.

LEMMA 2.4. *Let $f\in\mathcal{B}(I)$[8], then*

---

7) The following argument is due to N. Ikeda.

8) $\mathcal{B}(I)=\{u$; bounded and measurable on $I\}$.

$$u(x)=\bar{G}_\lambda f(x)\equiv\int_I \bar{g}_\lambda(x,y)f(y)\,dy$$

belongs to $C(I)$[9] *and satisfies*

$$\lambda u(x)-\bar{\Omega}u(x)=f(x)$$

where

$$(2.14)\qquad \bar{\Omega}u(x)=\frac{c(\alpha)}{\alpha(\alpha-1)}\frac{d^2}{dx^2}\int_{-1}^{1}\frac{u(y)}{|x-y|^{\alpha-1}}dy^{10)}, \qquad 1<\alpha<2$$

$$=\frac{1}{\pi}\frac{d^2}{dx^2}\int_{-1}^{1}u(y)\log\frac{1}{|x-y|}dy$$

$$\left(=\frac{1}{\pi}\frac{d}{dx}P\int_{-1}^{1}u(y)\frac{1}{y-x}dy\right), \qquad \alpha=1$$

$$=\frac{c(\alpha)}{\alpha(\alpha-1)}\frac{d^2}{dx^2}\int_{-1}^{1}\frac{u(y)}{|x-y|^{\alpha-1}}dy$$

$$\left(=\frac{c(\alpha)}{\alpha}\frac{d}{dx}\int_{-1}^{1}u(y)\frac{\operatorname{sgn}(y-x)}{|x-y|^{\alpha}}dy\right), \qquad 0<\alpha<1.$$

PROOF. From Lemmas 2.2 and 2.3, we have

$$u(y)=\bar{G}_\lambda f(y)=\int_{-1}^{1}\bar{g}_\lambda(x,y)f(x)\,dx$$

$$=\int_{-1}^{1}g_\lambda(x-y)f(x)\,dx-\int_{-1}^{1}c(\alpha)\int_{|z|>1}\int_{|u|<1}\frac{\bar{g}_\lambda(x,u)}{|z-u|^{\alpha+1}}g_\lambda(z-y)\,du\,dz\,f(x)\,dx$$

$$=\int_{-1}^{1}g_\lambda(x-y)f(x)\,dx-c(\alpha)\int_{|z|>1}g_\lambda(z-y)\int_{|u|<1}\frac{\bar{G}_\lambda f(u)}{|z-u|^{\alpha+1}}du\,dz\,.$$

The first term is continuous in y since f is bounded and g_λ is in L^1. As for the second term, we have, by Theorem 2.1,

$$|\bar{G}_\lambda f(u)|\leqq\frac{\|f\|_\infty}{\Gamma(\alpha+1)}(1-u^2)^{\frac{\alpha}{2}}$$

so if we put $F(z)=c(\alpha)\int_{|u|<1}\frac{\bar{G}_\lambda f(u)}{|z-u|^{\alpha+1}}du,\ |z|>1$ then

$$F(z)=0\left(\frac{1}{(|z|-1)^{\frac{\alpha}{2}}}\right)\qquad\text{near}\quad |z|=1$$

$$=0\left(\frac{1}{|z|^{\alpha+1}}\right)\qquad\text{near}\quad |z|=\infty\,.$$

Now let $y\in I$ and y_n tend to y. We may assume $|y_n|<1-\varepsilon$ for some $\varepsilon>0$. Then, since $g_\lambda(x)$ is bounded and continuous in $|x|>\varepsilon$,

---

9) $C(I)=\{u\,;$ bounded and continuous on $I\}$.

10) The second derivative is understood in the Radon-Nikodyum sense or what is the same in the distribution sense.

$$\lim_{n\to\infty}\int_{|z|>1} g_\lambda(z-y_n)F(z)\,dz=\int_{|z|>1} g_\lambda(z-y)F(z)\,dz$$

by Lebesgue convergence theorem. This proves that $u(y)$ is continuous on I.

Now $u(y)=\bar{G}_\lambda f(y)=G_\lambda\varphi(y)$, where

$$\begin{aligned}\varphi(x)&=f(x) && x\in I\\ &=-c(\alpha)\int_{|u|<1}\frac{\bar{G}_\lambda f(u)}{|x-u|^{\alpha+1}}du && x\in I^c.\end{aligned}$$

This equality holds for all y if we define $u(y)$ to be 0 for $y\in I$. Since $\varphi\in L^1$, it follows from Theorem 1.1 that $u\in D(\Omega_1)$ and satisfies

$$\lambda u(x)-\Omega_1 u(x)=\varphi(x).$$

In particular, we have on I

$$\lambda u(x)-\bar{\Omega}u(x)=f(x).$$

LEMMA 2.5. *Let $u\in C(I)$ and $\bar{\Omega}u=0$ a.e. on I. Then $u\equiv 0$ on I.*

PROOF.[11] (i) Let $\mathcal{L}^2=L^2(I,dm)$ where $dm(y)=(1-y^2)^{\frac{\alpha}{2}-1}dy$. For $f\in\mathcal{L}^2$, define Kf by

$$Kf(x)=\int_{-1}^{1}\frac{f(y)}{|x-y|^{\alpha-1}}dm(y)=\int_{-1}^{1}\frac{f(y)}{|x-y|^{\alpha-1}}(1-y^2)^{\frac{\alpha}{2}-1}dy.$$

Lemma 2.1 means that $P_m^{(\frac{\alpha-1}{2})}(x)$ $m=0,1,2,\cdots$, form in $\mathcal{L}^2$ a complete orthogonal system of eigenfunctions of the operator K. Since the eigenvalues are bounded, K is a bounded symmetric operator on $\mathcal{L}^2$.

(ii) Let $f\in\mathcal{L}^2$ and $Kf=0$ in $\mathcal{L}^2$. Then[12]

$$Kf=\sum_{m=0}^{\infty}(Kf,\bar{P}_m)\bar{P}_m=\sum_{m=0}^{\infty}(f,K\bar{P}_m)\bar{P}_m=\sum_{m=0}^{\infty}\lambda_m(f,\bar{P}_m)\bar{P}_m=0.$$

Since $\lambda_m\neq 0$, $(f,\bar{P}_m)=0$, $m=0,1,2,\cdots$. This means $f=0$ in $\mathcal{L}^2$.

(iii) Let u be such that $u(x)(1-x^2)^{1-\frac{\alpha}{2}}\in\mathcal{L}^2$ and $\int_{-1}^{1}\frac{u(y)}{|x-y|^{\alpha-1}}dy=0$ on I. Then $u(x)=0$ a.e. on I.

If we put $f(x)=u(x)(1-x^2)^{1-\frac{\alpha}{2}}$, then $f\in\mathcal{L}^2$ and

$Kf(x)=\int_{-1}^{1}\frac{u(y)}{|x-y|^{\alpha-1}}dy=0$ $x\in I$. From (ii) $u(x)(1-x^2)^{1-\frac{\alpha}{2}}=0$ a.e. on I. Hence $u(x)=0$ a.e. on I.

11) We prove this lemma only in the case $\alpha\neq 1$. If $\alpha=1$, this lemma can be proved easily using the theory of finite Hilbert transforms [**13**, pp. 178–179].

12) $\bar{P}_m=\dfrac{P_m^{(\frac{\alpha-1}{2})}}{\left\|P_m^{(\frac{\alpha-1}{2})}\right\|}$

(iv) Let $u \in C(I)$ be such that for some a and b

$$\int_{-1}^{1} \frac{u(y)}{|x-y|^{\alpha-1}} dy = ax+b \qquad \text{a.e. on } I.$$

Then $u \equiv 0$ on I.

From Lemma 2.1 we can take some a', b' such that $v(y) = (1-y^2)^{\frac{\alpha}{2}-1}(a'y+b')$ satisfies $\int_{-1}^{1} \frac{v(y)}{|y-x|^{\alpha-1}} dy = ax+b$.

Then if we put $w(x) = u(x)-v(x)$, $w(x)(1-x^2)^{1-\frac{\alpha}{2}} \in \mathcal{L}^2$ and

$$\int_{-1}^{1} \frac{w(y)}{|y-x|^{\alpha-1}} dy = 0.$$

Hence from (iii), $w(x)=0$ on I: that is $u(x)=v(x)$ on I. On the other hand, $v(x)$ is bounded only when $a'=b'=0$ and $u(x)$ is bounded by assumption. So we have $a'=b'=0$, $u(x)\equiv 0$ and $a=b=0$.

Now the lemma follows immediately from (iv).

LEMMA 2.6. *Let $u \in C(I)$. If $\lambda u - \bar{\Omega} u = 0$ on I then $u \equiv 0$.*

PROOF. Put $F(x) = \sum_{n=0}^{\infty} (-\lambda)^n \bar{G}_\lambda^n u(x)$. Since $\| \bar{G}_\lambda u \|_\infty < \frac{1}{\lambda} \| u \|_\infty$, this series converges uniformly on any compact set in I. Hence F is bounded and continuous on I, and satisfies $F(x) - \lambda \bar{G}_\lambda F(x) = u(x)$. Then, from Lemma 2.4, $\bar{\Omega} F = 0$. This, in view of Lemma 2.5, implies $F \equiv 0$ on I. Hence $u \equiv 0$ on I.

Now we can determine the generator $\mathfrak{G}$ in the sense of [7] of the absorbing barrier stable process:

$$\mathfrak{G} = (\lambda - \bar{G}_\lambda^{-1}) \quad : \quad D(\mathfrak{G}) \equiv \bar{G}_\lambda(\mathscr{B}(I)) \longrightarrow \mathscr{B}(I)/\mathfrak{N}$$

where $\mathfrak{N} = \{f\,;\, \bar{G}_\lambda f = 0\}$ [7].

THEOREM 2.2. *The generator $\mathfrak{G}$ of the absorbing barrier stable process is given by*

$$\mathfrak{G} u(x) = \bar{\Omega} u(x)$$

where $\bar{\Omega} u(x)$ is defined in (2.14) *with the domain*

$$D(\mathfrak{G}) = D\bar{\Omega} \equiv \{u\,;\, u \in C(I),\, \bar{\Omega} u \in \mathscr{B}(I)\}$$

and

$$\mathfrak{N} = \{f\,;\, f=0 \text{ a.e. on } I\}.$$

PROOF. Let $u(x) = \bar{G}_\lambda f(x)$ for $f \in \mathscr{B}(I)$. From Lemma 2.4, $u \in C(I)$ and $\lambda u - \bar{\Omega} u = f$.

On the other hand, if $u \in D(\bar{\Omega})$ we put $v = \bar{G}_\lambda(\lambda u - \bar{\Omega} u)$. Then from Lemma 2.4, $v \in C(I)$ and $\lambda v - \bar{\Omega} v = \lambda u - \bar{\Omega} u$. This means that $w = u - v$ satisfies $\bar{\Omega} w = \lambda w$ and from Lemma 2.6, $w \equiv 0$ on I: that is $u = v = \bar{G}_\lambda(\lambda u - \bar{\Omega} u)$.

Finally if $\bar{G}_\lambda f = 0$, then $f = \lambda \bar{G}_\lambda f - \bar{\Omega} \bar{G}_\lambda f = 0$ a.e. on I.

COROLLARY. *If $u \in D(\bar{\Omega})$, then for some constant $M > 0$*

$$|u(x)| < M(1-x^2)^{-\frac{\alpha}{2}} \qquad x \in I.$$

This follows immediately from Theorems 2.1 and 2.2.

§ 3. Integro-differential equations for some quantities.

DEFINITION 3.1.

$$\xi_1^\lambda(x) = E_x(e^{-\lambda\sigma}; x_\sigma \in [1, \infty))$$

$$\xi_{-1}^\lambda(x) = E_x(e^{-\lambda\sigma}; x_\sigma \in (-\infty, -1]).$$

DEFINITION 3.2.

$$\tilde{\Omega}u(x) = \bar{\Omega}u(x) + \frac{c(\alpha)}{\alpha}\frac{u(1)}{(1-x)^\alpha} + \frac{c(\alpha)}{\alpha}\frac{u(-1)}{(1+x)^\alpha}.$$

$$D(\tilde{\Omega}) = \{u \in C(\bar{I})^{13)}, \tilde{\Omega}u(x) \in \mathscr{B}(I)\}.$$

REMARK. If $u \in C(\bar{I})$ and u' exists such that $u' \in L^1(I)$, then

$$\tilde{\Omega}u(x) = \frac{c}{\alpha(\alpha-1)}\frac{d}{dx}\int_{-1}^1 \frac{u'(y)}{|x-y|^{\alpha-1}}dy \qquad \alpha \neq 1$$

$$= \frac{1}{\pi}P\int_{-1}^1 \frac{u'(y)}{y-x}dy \qquad \alpha = 1.$$

THEOREM 3.1. *$\xi_1^\lambda(x)$, (resp. $\xi_{-1}^\lambda(x)$), is the unique solution of*

$$\lambda u - \tilde{\Omega}u = 0 \tag{3.1}$$

with the boundary conditions $u(-1) = 0$, $u(1) = 1$, (resp. $u(-1) = 1$, $u(1) = 0$).

PROOF. It is easy to check that $\xi_1^\lambda(x)$ is continuous on $\bar{I}$ and $\xi_1^\lambda(-1) = 0$, $\xi_1^\lambda(1) = 1$. We now prove that $\xi_1^\lambda(x)$ is the weak solution of the equation (3.1). From Lemma 2.3, we have

$$\xi_1^\lambda(x) = \int_1^\infty \pi_\lambda(x, \xi)\,d\xi = \frac{c(\alpha)}{\alpha}\int_{-1}^1 \frac{\bar{g}_\lambda(x, y)}{(1-y)^\alpha}dy.$$

Denoting by $T(x)$ the function $\frac{c(\alpha)}{\alpha(\alpha-1)}\frac{1}{|x|^{\alpha-1}}$ in case $\alpha \neq 1$, we have for every $\varphi \in \mathscr{D}(I)^{14)}$,

$$(\bar{\Omega}\xi_1^\lambda(x), \varphi(x)) = \left(\frac{d^2}{dx^2}T * [\xi_1^\lambda], \varphi\right) = (T * [\xi_1^\lambda], \varphi'') = (\xi_1^\lambda, T * \varphi'')$$

$$= (\xi_1^\lambda, (T * \varphi)'') = (\xi_1^\lambda, \bar{\Omega}\varphi) = \left(\frac{c(\alpha)}{\alpha}\int_{-1}^1 \frac{\bar{g}_\lambda(x, y)}{(1-y)^\alpha}dy, \bar{\Omega}\varphi(x)\right)$$

---

13) $C(\bar{I}) = \{u$; bounded and continuous on $\bar{I} = [-1, 1]\}$.

14) $\mathscr{D}\{(I) = \{\varphi; \varphi \in C^\infty$ and $S(\varphi) \subset I\}$. Note that $\mathscr{D}(I) \subset D(\bar{\Omega})$. For $f \in C(\bar{I})$, we define $[f] \in L^1(R^1)$ by

$$[f] = f \quad \text{on} \quad I$$
$$= 0 \quad \text{on} \quad I^c$$

$$=\left(\frac{c(\alpha)}{\alpha}\frac{1}{(1-y)^{\alpha}},\ \bar{G}_{\lambda}\bar{\Omega}\varphi(y)\right)=\left(\frac{c(\alpha)}{\alpha}\frac{1}{(1-y)^{\alpha}},\ \lambda\bar{G}_{\lambda}\varphi(y)-\varphi(y)\right)$$

$$=\left(\frac{c(\alpha)}{\alpha}\frac{1}{(1-y)^{\alpha}},\ \lambda\bar{G}_{\lambda}\varphi(y)\right)-\left(\frac{c(\alpha)}{\alpha}\frac{1}{(1-y)^{\alpha}},\ \varphi(y)\right)$$

$$=(\lambda\xi_1^{\lambda}(x),\varphi(x))-\left(\frac{c(\alpha)}{\alpha}\frac{1}{(1-x)^{\alpha}},\ \varphi(x)\right).$$

This proves that

$$\bar{\Omega}\xi_1^{\lambda}(x)=\lambda\xi_1^{\lambda}(x)-\frac{c(\alpha)}{\alpha}\frac{1}{(1-x)^{\alpha}}$$

$$=\lambda\xi_1^{\lambda}(x)-\frac{c(\alpha)}{\alpha}\frac{\xi_1^{\lambda}(1)}{(1-x)^{\alpha}}-\frac{c(\alpha)}{\alpha}\frac{\xi_1^{\lambda}(-1)}{(1+x)^{\alpha}},$$

i. e. $\tilde{\Omega}\xi_1^{\lambda}(x)=\lambda\xi_1^{\lambda}(x)$.

The Uniqueness follows immediately from Lemma 2.6.

COROLLARY.

$$\xi_1(x)=P_x(x_\sigma\in[1,\infty))=2^{1-\alpha}\frac{\Gamma(\alpha)}{\left[\Gamma\left(\frac{\alpha}{2}\right)\right]^2}\int_{-1}^{x}(1-y^2)^{\frac{\alpha}{2}-1}dy. \tag{3.2}$$

$$\xi_{-1}(x)=P_x(x_\sigma\in(-\infty,-1])=1-\xi_1(x). \tag{3.3}$$

PROOF. $\xi_1(x)$ is the unique solution in $D(\tilde{\Omega})$ of $\tilde{\Omega}u=0$, with boundary conditions $u(-1)=0$, $u(1)=1$. We can easily solve this equation using (2.7) and obtain (3.2).

It is, in fact, possible to do more than the corollary and we can obtain the density $\pi(x,\xi)$ of the measure $P_x(x_\sigma\in d\xi)$.

THEOREM 3.2.

$$\pi(x,\xi)=\frac{\sin\frac{\alpha\pi}{2}}{\pi}\left(\frac{1-x^2}{\xi^2-1}\right)^{\frac{\alpha}{2}}\frac{1}{|\xi-x|}\qquad x\in I,\quad \xi\notin I. \tag{3.4}$$

PROOF. From Lemma 2.3, $\pi(x,\xi)=c(\alpha)\int_I\frac{\bar{g}_0(x,y)}{|y-\xi|^{\alpha+1}}dy$. Hence $\pi(\cdot,\xi)$ is the unique solution in $C(I)$ of

$$\bar{\Omega}u(x)=-c(\alpha)\frac{1}{|x-\xi|^{\alpha+1}}.$$

Take, for instance, $\xi>1$ and applying Cauchy's theorem to the function $f(z)=\frac{(z^2-1)^{\frac{\alpha}{2}}}{\xi-z}\frac{1}{(z-x)^{\alpha-1}}$ (real if $z>1$) which is holomorphic in the domain bounded by Γ_1, Γ_2 and Γ_3 (Fig. 1), we have

$$\int_{\Gamma_1}f(z)dz+\int_{\Gamma_2}f(z)dz=\int_{\Gamma_3}f(z)dz,$$

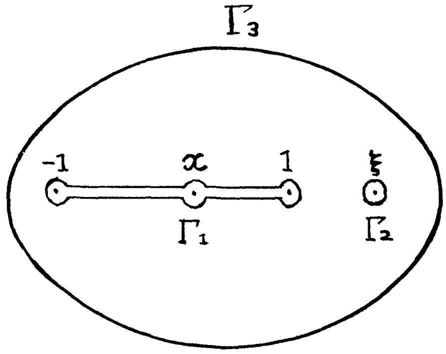

Fig. 1.

and also

$$\frac{1}{2\pi i}\int_{\Gamma_1} f(z)\,dz = -\frac{\sin\frac{\alpha\pi}{2}}{\pi}\int_{-1}^{1}\frac{(1-y^2)^{\frac{\alpha}{2}}}{|y-x|^{\alpha-1}}\frac{dy}{\xi-y}$$

$$\frac{1}{2\pi i}\int_{\Gamma_2} f(z)\,dz = -\frac{(\xi^2-1)^{\frac{\alpha}{2}}}{(\xi-x)^{\alpha-1}}$$

$$\frac{1}{2\pi i}\int_{\Gamma_3} f(z)\,dz = -(\xi+(\alpha-1)x)\,.$$

From this we have

$$\frac{c(\alpha)}{\alpha(\alpha-1)}\frac{d^2}{dx^2}\int_{-1}^{1}\left\{\frac{\sin\frac{\alpha\pi}{2}}{\pi}\left(\frac{1-y^2}{\xi^2-1}\right)^{\frac{\alpha}{2}}\frac{1}{\xi-y}\right\}\frac{1}{|x-y|^{\alpha-1}}\,dy = -c(\alpha)\frac{1}{(\xi-x)^{\alpha+1}}$$

Using (3.4), we can prove the following theorem which has been proved recently by H. Widom [14].

THEOREM 3.3. *The 0-th order Green function $\bar{g}_0(x,y)$ is given by*

$$\bar{g}_0(x,y) = F(|x-y|) - \int_{|\xi|>1}\pi(x,\xi)F(|\xi-y|)\,d\xi$$

where

$$F(\eta) = \frac{1}{2\cos\frac{\pi\alpha}{2}\Gamma(\alpha)}\eta^{\alpha-1} \qquad 0<\alpha<2,\quad \alpha\neq 1$$

$$= \frac{1}{\pi}\log\frac{1}{\eta} \qquad \alpha=1\,.$$

DEFINITION 3.3.

(3.5) $$\eta_1(x) = \frac{2^{1-\frac{\alpha}{2}}}{\alpha\left[\Gamma\left(\frac{\alpha}{2}\right)\right]^2}\frac{(1+x)^{\frac{\alpha}{2}}}{(1-x)^{1-\frac{\alpha}{2}}}$$

(3.6) $$\eta_{-1}(x) = \eta_1(-x)$$

(3.7) $$\eta_1^\lambda(x) = \eta_1(x) - \lambda\int_{-1}^{1}\bar{g}_\lambda(x,y)\,\eta_1(y)\,dy$$

(3.8) $$\eta_{-1}^\lambda(x) = \eta_{-1}(x) - \lambda\int_{-1}^{1}\bar{g}_\lambda(x,y)\eta_{-1}(y)\,dy\,.$$

THEOREM 3.4. *For $\lambda\geqq 0$*

(i) $$\lim_{\varepsilon\downarrow 0}\frac{\bar{g}_\lambda(x,1-\varepsilon)}{\varepsilon^{\frac{\alpha}{2}}} = \eta_1^\lambda(x)$$

$$\lim_{\varepsilon\downarrow 0}\frac{\bar{g}_\lambda(x,\varepsilon-1)}{\varepsilon^{\frac{\alpha}{2}}} = \eta_{-1}^\lambda(x)\,.$$

(ii) *If* $u(x)=\int_{-1}^{1}\bar{g}_\lambda(x,y)f(y)\,dy$, *then*

$$\delta_1 u=\lim_{\varepsilon\downarrow 0}\frac{u(1)-u(1-\varepsilon)}{\varepsilon^{\frac{\alpha}{2}}} \text{ exists and is given by}$$

$$\delta_1 u=-\int_{-1}^{1}\eta_\lambda(y)f(y)\,dy.$$

Similarly

$$\delta_{-1}u=\lim_{\varepsilon\downarrow 0}\frac{u(\varepsilon-1)-u(-1)}{\varepsilon^{\frac{\alpha}{2}}}=\int_{-1}^{1}\eta_{-1}^{\lambda}(y)f(y)\,dy.$$

We can prove this theorem by deriving the integro-differential equation for $\int_{-1}^{x}\eta_1^\lambda(y)\,dy$ and also even more directly by using a recent result of H. Kesten.[15]

§ 4. The generator of the semi-group on $C(R^1)$ and the half interval case.

In this section we consider the case of the half interval $I^-=(-\infty,0)$. First of all, we determine the generator of the semi-group (1.3) of the symmetric stable process acting on $C(R^1)$.[16] It is easy to see that if $f\in C(R^1)$, then $T_tf\in C(R^1)$. Here the generator is the operator $\lambda-G_\lambda^{-1}$.

THEOREM 4.1. *The generator is given as follows:*

$$\Omega u(x)=\lim_{A\uparrow\infty}\frac{c(\alpha)}{\alpha(\alpha-1)}\frac{d^2}{dx^2}\int_{-A}^{A}\frac{u(y)}{|x-y|^{\alpha-1}}\,dy, \qquad u\in D(\Omega),$$

where

$$D(\Omega)=\{u\in C(R^1)\,;\,{}^\forall A>0,\ \int_{-A}^{A}\frac{u(y)}{|x-y|^{\alpha-1}}\,dy\in C^2(-A,A) \text{ and }$$
$$\frac{c(\alpha)}{\alpha(\alpha-1)}\frac{d^2}{dx^2}\int_{-A}^{A}\frac{u(y)}{|x-y|^{\alpha-1}}\,dy \text{ converges at every point } x \text{ to a}$$
$$\text{function } f(x)\in C(R^1) \text{ when } A\uparrow\infty,\}$$

and

$$\mathfrak{N}\equiv\{f\,;\,G_\lambda f=0\}=\{0\}.$$

PROOF.

Let $u=G_\lambda f$, $f\in C(R^1)$. We have for every $A>0$ and $x\in I_A=(-A,A)$[17]

15) H. Kesten, Random walks with absorbing barriers and Toeplitz forms, Illinois J. Math., 5 (1961), 267-290.

16) $C(R^1)=\{f\,;$ bounded and continuous on $R^1.\}$

17) If we consider the absorbing barrier process on I_A, we denote the generator, green function, etc. as $\bar{\Omega}_A$, $\bar{g}_\lambda^A(x,y)$, etc.

$$u(x)=\int_{-\infty}^{\infty}g_{\lambda}(x-y)f(y)\,dy$$
$$=\int_{|y|<A}\bar{g}_{\lambda}^{A}(x,y)f(y)\,dy+\int_{|\xi|>A}\pi_{\lambda}^{A}(x,\xi)u(\xi)\,d\xi$$
$$=\int_{|y|<A}\bar{g}_{\lambda}^{A}(x,y)f(y)\,dy+\int_{|\xi|>A}c(\alpha)\int_{|y|<A}\frac{\bar{g}_{\lambda}^{A}(x,y)}{|y-\xi|^{\alpha+1}}dy\cdot u(\xi)\,d\xi$$
$$=\int_{|y|<A}\bar{g}_{\lambda}^{A}(x,y)\Big(f(y)+c(\alpha)\int_{|\xi|>A}\frac{u(\xi)}{|y-\xi|^{\alpha+1}}d\xi\Big)dy\,.$$

Just as the proof of Theorem 3.1, we can show

$$\lambda u(x)-\bar{\Omega}_{A}u(x)=f(x)+c(\alpha)\int_{|\xi|>A}\frac{u(\xi)}{|x-\xi|^{\alpha+1}}d\xi\,,\qquad x\in I_{A}\,. \tag{4.1}$$

Hence $\int_{-A}^{A}\frac{u(y)}{|x-y|^{\alpha-1}}dy\in C^{2}(A,A)$ and

$$\lim_{A\uparrow\infty}\bar{\Omega}_{A}u(x)=\lim_{A\uparrow\infty}\frac{c(\alpha)}{\alpha(\alpha-1)}\frac{d^{2}}{dx^{2}}\int_{-A}^{A}\frac{u(y)}{|x-y|^{\alpha-1}}dy=\lambda u(x)-f(x)\,.$$

This proves that $u\in D(\Omega)$ and $\lambda u-\Omega u=f$. In particular, if $u=G_{\lambda}f=0$ then $f=\lambda u-\Omega u=0$.

Conversely let $u\in D(\Omega)$ and put $f=\lambda u-\Omega u$. We first show that u satisfies (4.1). If $B>A$ and $x\in I_{A}$

$$\lambda u(x)-\bar{\Omega}_{B}u(x)=\lambda u(x)-\frac{c(\alpha)}{\alpha(\alpha-1)}\frac{d^{2}}{dx^{2}}\int_{-B}^{B}\frac{u(y)}{|x-y|^{\alpha-1}}dy$$
$$=\lambda u(x)-\frac{c(\alpha)}{\alpha(\alpha-1)}\frac{d^{2}}{dx^{2}}\int_{-A}^{A}\frac{u(y)}{|x-y|^{\alpha-1}}dy$$
$$-c(\alpha)\int_{A<|y|<B}\frac{u(y)}{|x-y|^{\alpha+1}}dy\,,$$

letting $B\uparrow\infty$, we have (4.1). Put

$$v(x)=\int_{-A}^{A}\bar{g}_{\lambda}^{A}(x,y)\Big(f(y)+c(\alpha)\int_{|\xi|>A}\frac{u(\xi)}{|y-\xi|^{\alpha+1}}d\xi\Big)dy\qquad x\in I_{A}\,.$$

Then it is easy to see that $v(x)$ is bounded and continuous on $(-A,A)$. Also it satisfies

$$\lambda v(x)-\bar{\Omega}_{A}v(x)=f(x)-c(\alpha)\int_{|\xi|>A}\frac{u(\xi)}{|x-\xi|^{\alpha+1}}d\xi\qquad\text{on}\quad I_{A}\,.$$

Hence if we put $w=u-v$, we have $\lambda w-\bar{\Omega}_{A}w=0$ and in view of Lemma 2.6, it follows that $w\equiv 0$, i.e.

$$u(x)=v(x)=\int_{-A}^{A}\bar{g}_{\lambda}^{A}(x,y)\Big\{f(y)+c(\alpha)\int_{|\xi|>A}\frac{u(\xi)}{|y-\xi|^{\alpha+1}}d\xi\Big\}dy\qquad\text{for}\quad x\in I_{A}\,.$$

Now if we put $\tilde{u}(x)=\int_{-\infty}^{\infty}g_{\lambda}(x-y)f(y)\,dy$, then $\tilde{u}$ satisfies

$$\tilde{u}(x)=\int_{-A}^{A}\bar{g}_{\lambda}^{A}(x,y)\Big\{f(y)+c(\alpha)\int_{|\xi|>A}\frac{\tilde{u}(\xi)}{|y-\xi|^{\alpha+1}}d\xi\Big\}dy\,.$$

Hence putting $\tilde{w}=u-\tilde{u}$, we have

$$\tilde{w}(x)=c(\alpha)\int_{-A}^{A}\bar{g}_{\lambda}^{A}(x,y)\int_{|\xi|>A}\frac{\tilde{w}(\xi)}{|y-\xi|^{\alpha+1}}d\xi\,.$$

Then

$$|\tilde{w}(x)|\leqq\|\tilde{w}\|_{\infty}c(\alpha)\int_{-A}^{A}\bar{g}_{\lambda}^{A}(x,y)\int_{|A|>\xi}\frac{1}{|y-\xi|^{\alpha+1}}d\xi=\|\tilde{w}\|_{\infty}E_{x}(e^{-\lambda\sigma_{A}})^{18)}$$

Letting $A\uparrow\infty$, we have $E_{x}(e^{-\lambda\sigma_{A}})\rightarrow 0$, proving $\tilde{w}=0$, namely

$$u(x)=\tilde{u}(x)=\int_{-\infty}^{\infty}g_{\lambda}(x-y)f(y)\,dy\,.$$

This proves the theorem.

LEMMA 4.1. *Let u and f be in $C(R^{1})$. Then $u\in D(\Omega)$ and $\Omega u=f$ if and only if*

$$\Big(u(x),\ \frac{c(\alpha)}{\alpha(\alpha-1)}\int_{-\infty}^{\infty}\frac{\varphi''(y)}{|x-y|^{\alpha-1}}dy\Big)=(f(x),\varphi(x))\tag{4.2}$$

for every $\varphi\in\mathcal{D}$.[19]

PROOF. First suppose that (4.2) holds for u and f in $C(R^{1})$. Then by simple caluculations, we have for every $\varphi\in\mathcal{D}(-A,A)$

$$\Big(\frac{c(\alpha)}{\alpha(\alpha-1)}\int_{-A}^{A}\frac{u(x)}{|x-y|^{\alpha-1}}dx,\ \varphi''(y)\Big)=(f(y),\varphi(y))-\Big(c(\alpha)\int_{|x|>A}\frac{u(x)}{|x-y|^{\alpha+1}}dx,\ \varphi(y)\Big)\,.$$

We see at once from this that

$$\frac{c(\alpha)}{\alpha(\alpha-1)}\int_{-A}^{A}\frac{u(y)}{|x-y|^{\alpha-1}}dy\in C^{2}(-A,A)\qquad\text{and}$$

$$\lim_{A\uparrow\infty}\frac{c(\alpha)}{\alpha(\alpha-1)}\frac{d^{2}}{dx^{2}}\int_{-A}^{A}\frac{u(y)}{|x-y|^{\alpha-1}}dy=f(x)\,.$$

Conversely suppose $u\in D(\Omega)$ and $\Omega u=f$. Then we have just as the proof of Theorem 4.1,

$$\frac{c(\alpha)}{\alpha(\alpha-1)}\frac{d^{2}}{dy^{2}}\int_{-A}^{A}\frac{u(x)}{|x-y|^{\alpha-1}}dx=f(y)+c(\alpha)\int_{|x|>A}\frac{u(x)}{|y-x|^{\alpha+1}}dx\qquad y\in(-A,A).$$

Hence for $\varphi\in\mathcal{D}(-A',A')$, $A'<A$

$$\Big(\frac{c(\alpha)}{\alpha(\alpha-1)}\int_{-A}^{A}\frac{u(x)}{|x-y|^{\alpha-1}}dx,\ \varphi''(y)\Big)=(f,\varphi)+\Big(c(\alpha)\int_{|x|>A}\frac{u(x)}{|y-x|^{\alpha+1}}dx,\ \varphi(y)\Big)$$

Letting $A\uparrow+\infty$

$$\Big(u(x),\ \frac{c(\alpha)}{\alpha(\alpha-1)}\int_{-\infty}^{\infty}\frac{\varphi''(y)}{|x-y|^{\alpha-1}}dy\Big)=(f,\varphi)$$

18) $\sigma_{A}(w)=\inf\{t\,;\,x_{t}\notin I_{A}\}$.

19) $\mathcal{D}=\{\varphi\,;\,\varphi\in C^{\infty}$, with compact support$\}$.

and since A' is arbitrary we have (4.2).

Now let I^- be the half line $(-\infty, 0)$ and consider the absorbing barrier process on I^-. Then we can prove quite similarly as above the following:

THEOREM 4.2. *The generator of the semi-group on $C(I^-)$ of the absorbing barrier process on I^- derived from the symmetric stable process is given as follows:*

$$\bar{\Omega}^- u(x) = \lim_{A \uparrow \infty} \frac{c(\alpha)}{\alpha(\alpha-1)} \frac{d^2}{dx^2} \int_{-A}^{0} \frac{u(y)}{|x-y|^{\alpha-1}} dy \qquad u \in D(\bar{\Omega}^-)$$

where

$$D(\bar{\Omega}^-) = \{u \in C(I^-) ; \forall A > 0, \int_{-A}^{0} \frac{u(y)}{|x-y|^{\alpha-1}} dy \in C^2(-A, 0) \text{ and }$$
$$\frac{c(\alpha)}{\alpha(\alpha-1)} \frac{d^2}{dx^2} \int_{-A}^{0} \frac{u(y)}{|x-y|^{\alpha-1}} dy \text{ converges to a function } f(x)$$
$$\in C(I^-) \text{ at every point } x \in I^-\},$$

and

$$\mathfrak{N} \equiv \{f ; \bar{G}_\lambda^- f = 0\} = \{f = 0\}.$$

In particular, it follows from this theorem that if $u \in D(\bar{\Omega}^-)$ and $\lambda u - \bar{\Omega}^- u = 0$ then $u \equiv 0$.

Corresponding to the Lemma 4.1, we have

LEMMA 4.2. *Let u and f be in $C(I^-)$. Then $u \in D(\bar{\Omega}^-)$ and $\bar{\Omega}^- u = f$ if and only if*

$$\left(u(x), \frac{c(\alpha)}{\alpha(\alpha-1)} \int_{-\infty}^{0} \frac{\varphi''(y)}{|x-y|^{\alpha-1}} dy\right) = (f, \varphi) \tag{4.3}$$

for every $\varphi \in \mathfrak{D}(I^-)$.

Now define $D(\tilde{\Omega}^-)$ by

$$D(\tilde{\Omega}^-) = \{u \in C(\bar{I}^-) ; \forall A > 0, \int_{-A}^{0} \frac{u(y)}{|x-y|^{\alpha-1}} dy \in C^2(-A, 0) \text{ and}$$
$$\frac{c(\alpha)}{\alpha(\alpha-1)} \frac{d^2}{dx^2} \int_{-A}^{0} \frac{u(y)}{|x-y|^{\alpha-1}} dy + \frac{c(\alpha)}{\alpha} \frac{u(0)}{(-x)^\alpha} \text{ converges to a}$$
$$\text{function } f(x) \in C(I^-) \text{ on } I^-,\}$$

and for $u \in D(\tilde{\Omega}^-)$, define $\tilde{\Omega}^- u$ by

$$\tilde{\Omega}^- u(x) = \lim_{A \uparrow \infty} \frac{c(\alpha)}{\alpha(\alpha-1)} \frac{d^2}{dx^2} \int_{-A}^{0} \frac{u(y)}{|x-y|^{\alpha-1}} dy + \frac{c(\alpha)}{\alpha} \frac{u(0)}{(-x)^\alpha}.$$

THEOREM 4.3. *Define $\sigma^-(w)$ by*

$$\sigma^-(w) = \inf \{t ; x_t(w) \notin I^-\}. \tag{4.4}$$

Then

$$\xi_\lambda(x) = E_x(e^{-\lambda\sigma^-}) \tag{4.5}$$

is the unique solution in $D(\tilde{\Omega}^-)$ of

$$\lambda u - \tilde{\Omega}^- u = 0 \tag{4.6}$$

with boundary condition $u(0)=1$.

The explicit formula of the 0-th order green function is obtained by D. Ray [12]:

$$\bar{g}_0^-(x,y)=\frac{1}{\left[\Gamma\left(\frac{\alpha}{2}\right)\right]^2}\int_0^{(-y)\wedge(-x)}\xi^{\frac{\alpha}{2}-1}(\xi+|y-x|)^{\frac{\alpha}{2}-1}d\xi. \tag{4.7}$$

Put

$$\eta_\lambda(x)=\eta(x)-\lambda\int_{-\infty}^0\bar{g}_\lambda^-(x,y)\eta(y)\,dy \tag{4.8}$$

where

$$\eta(x)=\frac{1}{\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha}{2}+1\right)}(-x)^{\frac{\alpha}{2}-1}. \tag{4.9}$$

THEOREM 4.4.

(i) $\lim\limits_{\varepsilon\downarrow 0}\frac{\bar{g}_\lambda(-\varepsilon,y)}{\varepsilon^{\frac{\alpha}{2}}}=\eta_\lambda(y)\qquad \lambda\geqq 0$

(ii) *If* $f\in C(I^-)$ *and* $u=\bar{G}_\lambda^-f$,

$$\delta u\equiv\lim_{\varepsilon\downarrow 0}\frac{u(0)-u(-\varepsilon)}{\varepsilon^{\frac{\alpha}{2}}}=-\int_{-\infty}^0 f(y)\eta_\lambda(y)\,dy\qquad \lambda>0.$$

§ 5. The boundary conditions for $\tilde{\Omega}$ and $\tilde{\Omega}^-$.

We determine the most general boundary conditions for $\tilde{\Omega}$ and $\tilde{\Omega}^-$ under which these operators become the infinitesimal generators of Markov processes. Elliott [2] determined them in the case $\alpha=1$ and obtained the corresponding resolvent operators. This can be extended to the case with general α in the same way. We consider also the construction of the path functions of these processes.

For simplicity we assume the left boundary condition $u(-1)=0$.

DEFINITION 5.1.[20,21] For given constants $\sigma\geqq 0$, $p\geqq 0$, $\gamma\geqq 0$, and a given measure $n(dx)\geqq 0$ such that $\int_{-1}^1(1-x)^{\frac{\alpha}{2}}n(dx)<+\infty$, define Σ as the set of all $u\in D_0(\tilde{\Omega})$ for which

$$pu(1)=\int_I[u(x)-u(1)]\,n(dx)-\sigma\tilde{\Omega}u(1)-\gamma\delta_1 u. \tag{5.1}$$

THEOREM 5.1. *The operator* $\tilde{\Omega}$ *with the domain* Σ *is the infinitesimal generator of a contraction semi-group with range dense in* $C_0(\bar{I})$ *or in the subspace defined by*

20) $C_0(\bar{I})=\{u\in C(\bar{I}),\,u(-1)=0\}$.
21) $D_0(\tilde{\Omega})=\{u\in C_0(\bar{I}),\,\tilde{\Omega}u\in C_0(\bar{I})\}$.

$$pu(1)=\int[u(x)-u(1)]n(dx)$$

according as $\sigma+\gamma>0$ *or* $\sigma+\gamma=0$.

Its resolvent is given by

$$u(x)=\int_{-1}^{1}\bar{g}_\lambda(x,y)f(y)\,dy+\xi_1^\lambda(x)Q(f)\qquad \lambda>0 \tag{5.2}$$

where

$$Q(f)=\frac{\int_{-1}^{1}\bar{G}_\lambda f(x)\,n(dx)+\sigma f(1)+\gamma\int_{-1}^{1}f(x)\eta_1^\lambda(x)\,dx}{p+\int_{-1}^{1}(1-\xi_\lambda(x))\,n(dx)+\lambda\sigma+\gamma\cdot\delta_1\xi_1^\lambda}.$$

PROOF. Using Theorems 3.1, 3.4 and Lemma 2.1, proof can be done in the same way as [2].

Similarly for the interval $\bar{I}^-=(-\infty,0]$ we have the following boundary condition for the operator $\tilde{\Omega}^-$:

$$pu(0)=\int_{-\infty}^{0}[u(x)-u(0)]\,n(dx)-\sigma\tilde{\Omega}^-u(0)-\gamma\delta u \tag{5.3}$$

where

$p\geqq 0$, $\sigma\geqq 0$, $\gamma\geqq 0$ and $n(dx)$ is a positive measure such that

$\int_{-1}^{0}(-x)^{\frac{\alpha}{2}}\,n(dx)<+\infty$ and $\int_{-\infty}^{-1}n(dx)<+\infty$.

The corresponding resolvent is given by the similar formula as (5.2). In particular, if the boundary condition is reflecting, i.e.

$$\delta u=0 \tag{5.4}$$

its resolvent is given by

$$u(x)\equiv\tilde{G}_\lambda f(x)=\bar{G}_\lambda f(x)+\frac{\xi_\lambda(x)}{\delta\xi_\lambda}\int_{-\infty}^{0}f(y)\eta_\lambda(y)\,dy,\qquad \lambda>0. \tag{5.5}$$

Now[22] the path functions of this process can be constructed from those of the ordinary symmetric stable process. Let $\boldsymbol{M}=(W,P_x,R^1)$ be the symmetric stable process defined in § 1. For any path function $x_t(w)$, $w\in W$, define $\tilde{x}_t(w)$ by

$$\begin{aligned}\tilde{x}_t(w)&=x_t(w) & t<\sigma^-(w)\\ &=x_t(w)-\sup_{\sigma^-\leqq s\leqq t}x_s(w) & t\geqq\sigma^-(w)\end{aligned}$$

where $\sigma^-(w)$ is defined in (4.4).

Put

$$\tilde{P}_x(B)=P_x(w\,;\tilde{x}.(w)\in B)^{23)}\qquad\text{for}\quad x\in\bar{I}^-.$$

22) This was suggested to me by Prof. K. Ito.

23) B is a (Borel) subset of the space of path functions.

THEOREM 5.2. *The process $\tilde{\boldsymbol{M}}=(W, \tilde{P}_x, \bar{I}^-)$ obtained in this way is a strict Markov process and its resolvent coincides with* (5.5), *i.e. the process $\tilde{\boldsymbol{M}}$ is the reflecting barrier process on $\bar{I}^-$ determined by $\tilde{\Omega}^-$ and* (5.4).

PROOF. We can easily check the strict Markov property of $\tilde{\boldsymbol{M}}$ and so we have only to prove that

$$\tilde{E}_x\Big(\int_0^\infty e^{-\lambda t}f(x_t)\,dt\Big)\equiv E_x\Big(\int_0^\infty e^{-\lambda t}f(\tilde{x}_t(w))dt\Big)$$

$$=\bar{G}_\lambda f(x)+\frac{\xi_\lambda(x)}{\delta\xi_\lambda}\int_{-\infty}^0 f(y)\eta_\lambda(y)\,dy\,.$$

Now

$$E_x\Big(\int_0^\infty e^{-\lambda t}f(\tilde{x}_t(w))\Big)=E_x\Big(\int_0^{\sigma^-} e^{-\lambda t}f(\tilde{x}_t(w))\,dt\Big)+E_x\Big(\int_{\sigma^-}^\infty e^{-\lambda t}f(\tilde{x}_t(w))\,dt\Big)$$

$$=\bar{G}_\lambda^- f(x)+E_x\Big(e^{-\lambda\sigma^-}\int_0^\infty e^{-\lambda t}f(\tilde{x}_{t+\sigma^-})\,dt\Big)$$

$$=\bar{G}_\lambda^- f(x)+E_x\Big(e^{-\lambda\sigma^-}E_{x_{\sigma^-}}\Big(\int_0^\infty e^{-\lambda t}f(\tilde{x}_t)\,dt\Big)\Big)$$

$$=\bar{G}_\lambda^- f(x)+E_x(e^{-\lambda\sigma^-})\,E_0\Big(\int_0^\infty e^{-\lambda t}f(\tilde{x}_t)\,dt\Big)$$

since if $a\geqq 0$ the probability law of $\tilde{x}_t$ with respect to P_a is the same as that with respect to P_0. Hence it is enough to prove that

$$E_0\Big(\int_0^\infty e^{-\lambda t}f(\tilde{x}_t)\,dt\Big)=\int_{-\infty}^0 \frac{\eta_\lambda(y)}{\delta\xi_\lambda}f(y)\,dy\,.$$

Now

$$\bar{P}^-(t, x, E)\equiv P_x(x_t\in E, \sigma^->t)$$

$$=P_x(x_t\in E, \sup_{0\leqq s\leqq t} x_s<0)\,.^{24)}$$

We have from this and the spatial homogeneity of the stable process, that

$$P_0(x_t\in E, \sup_{0\leqq s\leqq t} x_s<a)=P_{-a}(x_t\in E-a, \sup_{0\leqq s\leqq t} x_s<0)$$

$$=\bar{P}^-(t, -a, E-a)$$

$$=\int_E \bar{p}^-(t, -a, y-a)\,dy\,.$$

Hence, using the symmetry of $\bar{p}^-(t, x, y)$[25)]

24) For the rigorous justification, we may use Theorem 6.4 below.

25) This can be proved in the same way as Lemma 2.2.

$$\begin{aligned}P_0(\tilde{x}_t > b) &= P_0(x_t - \sup_{0\leqq s\leqq t} x_s > b)\\ &= P_0(\sup_{0\leqq s\leqq t} x_s < x_t - b)\\ &= \int_b^\infty \bar{p}^-(t, b-\xi, b)\,d\xi\\ &= \int_{-\infty}^0 \bar{p}^-(t, b, \xi)\,d\xi\,.\end{aligned}$$

Now if $\chi_{(b,0)}(x)$ is the characteristic function of the interval $(b, 0)$, $b<0$, then we have

$$E_0\Big(\int_0^\infty e^{-\lambda t}\chi_{(b,0)}(\tilde{x}_t)\,dt\Big) = \int_{-\infty}^0 \bar{g}_\lambda(b, \xi)\,d\xi\,.$$

By Theorem 4.2, this function of b is the unique solution in $D(\bar{\Omega}^-)$ of

$$\lambda u - \bar{\Omega}^- u = 1\,.$$

On the other hand, putting $u_\varepsilon(x) = \frac{1}{\varepsilon^{\frac{\alpha}{2}}}\int_x^0 \bar{g}_\lambda^-(-\varepsilon, y)\,dy$ for $\varepsilon > 0$ we have, for any testing function φ in $\mathscr{D}(I^-)$, that

$$\begin{aligned}&\Big(u_\varepsilon(x),\ \frac{c(\alpha)}{\alpha(\alpha-1)}\int_{-\infty}^0 \frac{\varphi''(y)}{|x-y|^{\alpha-1}}\,dy\Big)\\ &= \frac{1}{\varepsilon^{\frac{\alpha}{2}}}\int_{-\infty}^0\Big\{\int_x^0 \bar{g}_\lambda^-(-\varepsilon, y)\,dy\cdot\frac{c(\alpha)}{\alpha(\alpha-1)}\int_{-\infty}^0\frac{\varphi''(y)}{|x-y|^{\alpha-1}}\,dy\Big\}\,dx\\ &= \frac{1}{\varepsilon^{\frac{\alpha}{2}}}\int_{-\infty}^0 \bar{g}_\lambda^-(-\varepsilon, x)\frac{c(\alpha)}{\alpha(\alpha-1)}\int_{-\infty}^0\frac{\varphi'(y)}{|x-y|^{\alpha-1}}\,dy\,dx\\ &= -\frac{1}{\varepsilon^{\frac{\alpha}{2}}}\int_{-\infty}^0 \bar{g}_\lambda^-(-\varepsilon, x)\bar{\Omega}^-\psi(x)\,dx\\ &= -\frac{\lambda}{\varepsilon^{\frac{\alpha}{2}}}\int_{-\infty}^0 \bar{g}_\lambda^-(-\varepsilon, x)\psi(x)\,dx + \frac{\psi(-\varepsilon)}{\varepsilon^{\frac{\alpha}{2}}}\end{aligned}$$

since $\psi(x) \equiv \int_x^0 \varphi(y)\,dy \in D(\bar{\Omega}^-)$.

By an integration by parts, the last expression is equal to

$$-\frac{\lambda}{\varepsilon^{\frac{\alpha}{2}}}\int_{-\infty}^0 \bar{g}_\lambda^-(-\varepsilon, y)\,dy\cdot\int_{-\infty}^0\varphi(y)\,dy + \lambda\int_{-\infty}^0\frac{1}{\varepsilon^{\frac{\alpha}{2}}}\int_x^0\bar{g}_\lambda^-(-\varepsilon, y)\,dy\,\varphi(x)\,dx + \frac{\psi(-\varepsilon)}{\varepsilon^{\frac{\alpha}{2}}}\,.$$

Putting $u(x) = \lim_{\varepsilon\downarrow 0} u_\varepsilon(x) = \int_x^0 \eta_\lambda(y)\,dy$

and noting $\qquad \int_{-\infty}^0 \bar{g}_\lambda^-(-\varepsilon, y)\,dy = \frac{1-\xi_\lambda(-\varepsilon)}{\lambda}\,,$

we have, by letting $\varepsilon \downarrow 0$, that

$$\left(u(x), \frac{c(\alpha)}{\alpha(\alpha-1)}\int_{-\infty}^{0}\frac{\varphi''(y)}{|x-y|^{\alpha-1}}dy\right)=-\delta\xi_\lambda\cdot\int_{-\infty}^{0}\varphi(y)\,dy+\lambda\int_{-\infty}^{0}u(x)\varphi(x)\,dx\,.$$

In view of Lemma 4.2, it follows that $u(x)\in D(\bar{\Omega}^-)$ and satisfies

$$\lambda u-\bar{\Omega}^- u=\delta\xi_\lambda\,.$$

Hence we have

$$\int_{-\infty}^{0}\bar{g}_\lambda^-(b,y)\,dy=\frac{1}{\delta\xi_\lambda}\int_{b}^{0}\eta_\lambda(y)\,dy\,,\quad \text{i.e.} \tag{5.6}$$

$$E_0\left(\int_0^\infty e^{-\lambda t}\chi_{(b,0)}(\tilde{x}_t)\,dt\right)=\frac{1}{\delta\xi_\lambda}\int_{-\infty}^{0}\eta_\lambda(y)\cdot\chi_{(b,0)}(y)\,dy\,.$$

Now from this we have for every bounded function f

$$E_0\left(\int_0^\infty e^{-\lambda t}f(\tilde{x}_t)\,dt\right)=\frac{1}{\delta\xi_\lambda}\int_{-\infty}^{0}\eta_\lambda(y)f(y)\,dy$$

and the proof is complete.

Now we can define the local time at $x=0$ of this process. First we require the following lemma.

LEMMA 5.1. *If* $\lambda>0$, *then*

$$\lim_{\varepsilon\downarrow 0}\varepsilon^{1-\frac{\alpha}{2}}\int_{-\infty}^{0}\bar{g}_\lambda^-(-\varepsilon,y)\eta(y)\,dy=0\,.$$

PROOF. Suppose $\alpha\geqq\frac{2}{3}$, then

$$\int_{-\infty}^{0}\bar{g}_\lambda^-(-\varepsilon,y)\eta(y)\,dy=\int_{-\infty}^{-1}\bar{g}_\lambda^-(-\varepsilon,y)\eta(y)\,dy+\int_{-1}^{0}\bar{g}_\lambda^-(-\varepsilon,y)\eta(y)\,dy$$

and the first term is bounded in $\varepsilon>0$.
As for the second we have

$$\int_{-1}^{0}\bar{g}_\lambda^-(-\varepsilon,y)\eta(y)\,dy=k\cdot\int_{-1}^{0}\frac{\bar{g}_\lambda^-(-\varepsilon,y)}{(-y)^\alpha}(-y)^{\frac{3}{2}\alpha-1}dy$$

and this is also bounded in $\varepsilon>0$, since

$$\frac{c(\alpha)}{\alpha}\int_{-\infty}^{0}\frac{\bar{g}_\lambda^-(-\varepsilon,y)}{(-y)^\alpha}dy=E_{-\varepsilon}(e^{-\lambda\sigma^-})\leqq 1\,.$$

The proof of the case of $\alpha<\frac{2}{3}$ is omitted.

From (4.8), (4.9) and this lemma we have

$$\lim_{y\uparrow 0}\eta_\lambda(y)(-y)^{1-\frac{\alpha}{2}}=\lim_{y\uparrow 0}\eta(y)(-y)^{1-\frac{\alpha}{2}}=\frac{1}{\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha}{2}+1\right)} \tag{5.7}$$

$$\text{(5.8)} \qquad \lim_{\varepsilon\downarrow 0}\frac{\int_{-\varepsilon}^{0}\eta_\lambda(y)\,dy}{\varepsilon^{\frac{\alpha}{2}}}=\lim_{\varepsilon\downarrow 0}\frac{\int_{-\varepsilon}^{0}\eta(y)\,dy}{\varepsilon^{\frac{\alpha}{2}}}=\frac{1}{\left[\Gamma\left(\frac{\alpha}{2}+1\right)\right]^2}.$$

Let $\tilde{g}_\lambda(x,y)$ be the density of the resolvent kernel $\tilde{G}_\lambda(x,dy)$ with respect to the measure

$$\text{(5.9)} \qquad dm(y)=\frac{\alpha}{2}(-y)^{\frac{\alpha}{2}-1}dy\,.$$[26]

Then we have from (5.5), (5.7) and (5.8)

$$\text{(5.10)} \qquad \tilde{g}_\lambda(x,0)=\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon^{\frac{\alpha}{2}}}\int_{-\varepsilon}^{0}\tilde{G}_\lambda(x,dy)=\frac{1}{\left[\Gamma\left(\frac{\alpha}{2}+1\right)\right]^2}\frac{\xi_\lambda(x)}{\delta\xi_\lambda}.$$

$\tilde{g}_\lambda(x,0)$ is a λ-excessive function and also bounded and continuous. Hence from a theorem of H. Tanaka [6],[27] there exists an additive functional $s(t,w)$ such that

(i) $s(t,w)$ is continuous and increasing in $t\geqq 0$

(ii) $s(t,w)=0$ if $t<\sigma^-(w)$

(iii) $\tilde{E}_x\left(\int_0^\infty e^{-\lambda t}ds(t,w)\right)=\tilde{g}_\lambda(x,0)$.

Now the inverse function $t(u,w)=\max\{t\,;\,u=s(t,w)\}$ is a Lévy process with respect to $\tilde{P}_0$.

THEOREM 5.3. *$t(u,w)$ is a one-sided stable process of exponent $\frac{1}{2}$ given by*

$$\tilde{E}_0(e^{-\lambda t(u,w)})=e^{-\Gamma\left(\frac{\alpha}{2}+1\right)\sqrt{\lambda}\,u}.$$

PROOF. We have from (iii)

$$\tilde{E}_0(e^{-\lambda t(u,w)})=e^{-\frac{1}{\tilde{g}_\lambda(0,0)}u}.$$

On the other hand, by (5.6)

$$\int_{-\infty}^{0}\tilde{g}_\lambda(-\varepsilon,y)\,dy=\frac{1-E_{-\varepsilon}(e^{-\lambda\sigma^-})}{\lambda}=\frac{1}{\delta\xi_\lambda}\int_{-\varepsilon}^{0}\eta_\lambda(y)\,dy\,.$$

Using (5.8) we have

$$\frac{1}{\lambda}\delta\xi_\lambda=\lim_{\varepsilon\downarrow 0}\frac{1-E_{-\varepsilon}(e^{-\lambda\sigma^-})}{\lambda\varepsilon^{\frac{\alpha}{2}}}=\frac{1}{\delta\xi_\lambda}\frac{1}{\left[\Gamma\left(\frac{\alpha}{2}+1\right)\right]^2}.$$

Hence $\delta\xi_\lambda=\sqrt{\lambda}\cdot\dfrac{1}{\Gamma\left(\frac{\alpha}{2}+1\right)}$, this, in view of (5.10), implies

---

26) This is the invariant measure of the process $\tilde{\boldsymbol{M}}$.

27) Also cf. H. P. McKean & H. Tanaka, Additive functionals of the Brownian path. Mem. Fac. Sci. Univ. Kyoto Ser. A. Math., (1961).

$$\tilde{g}_\lambda(0, 0) = \frac{1}{\Gamma\left(\frac{\alpha}{2}+1\right)\sqrt{\lambda}} .$$

We can construct, for instance, the process determined by $\tilde{\Omega}^-$ and the boundary condition $\delta u = -\frac{\gamma u(0)}{\left[\Gamma\left(\frac{\alpha}{2}+1\right)\right]^2}$ by random killing defined by the multiplicative functional $e^{-\gamma s(t,w)}$ $(\gamma > 0)$, cf. [8].

REMARK 1. We can construct the paths of the reflecting barrier process on $\bar{I}$ just as the case of Brownian motion but we do not discuss of it here. We remark also that

$$2^{1-\alpha} \frac{\Gamma(\alpha)}{\left[\Gamma\left(\frac{\alpha}{2}\right)\right]^2} (1-x^2)^{\frac{\alpha}{2}-1} dx$$

is the invariant distribution of this process.

REMARK 2. There is another kind of the reflecting barrier process on $\bar{I}^-$ whose paths are defined as $-|x_t(w)|$ from the paths of the symmetric stable process.[28] This process, which is of course Markovian, has, as its invariant measure, Lebesgue measure dx and local time at $x = 0$ can be defined only in the case $1 < \alpha \leqq 2$ whose inverse function is a one-sided stable process with exponent $1 - \frac{1}{\alpha}$.[29]

§ 6. Some properties of the path functions of the stable process.

Define for a closed set F,

$$\sigma_F(w) = \inf\{t > 0 \,;\, x_t(w) \in F\} .$$

THEOREM 6.1. *For* $x \in I_A = (-A, A)$[30]

(6.1) $$P_x(\sigma_{\{y\}} < \sigma_A) = 0 \qquad 0 < \alpha \leqq 1$$

(6.2) $$P_x(\sigma_{\{y\}} < \sigma_A) = \frac{\bar{g}_0^A(x, y)}{\bar{g}_0^A(y, y)} \qquad 1 < \alpha \leqq 2 .$$

PROOF. Noting $\bar{g}_0^A(y, y) < +\infty$ if and only if $1 < \alpha \leqq 2$, this theorem can be proved using Hunt's potential theory [5] and details are omitted.

It is known [10] that if $1 \leqq \alpha \leqq 2$, the process is recurrent. Using this fact and letting $A \uparrow \infty$ we have the following:

THEOREM 6.2. *For* $x, y \in R^1$

28) It is well known that if $\alpha = 2$ these two processes coincide.

29) Cf. Theorem 6.3 below.

30) $\sigma_A = \inf\{t \,;\, x_t \notin I_A\}$.

(6.3) $$P_x(\sigma_{\{y\}} < +\infty) = 0 \qquad 0 < \alpha \leqq 1$$

(6.4) $$P_x(\sigma_{\{y\}} < +\infty) = 1 \qquad 1 < \alpha \leqq 2 .$$

(6.3) was proved by H. P. McKean [**10**].

Now consider a path $x_t(w)$ of the symmetric stable process and let $\boldsymbol{Z}(w)$ be the set of the zero points of $x_t(w)$.
Theorem 6.2 means that for $T > 0$

$$P_0(\boldsymbol{Z}(w) \cap (0, T] = \phi) = 1 \qquad 0 < \alpha \leqq 1$$

$$P_0(\boldsymbol{Z}(w) \cap (0, T] \neq \phi) = 1 \qquad 1 < \alpha \leqq 2 .$$

THEOREM 6.3. *With probability one,* $\boldsymbol{Z}(w) \cap (0, T]$ *is a non-countable Borel set of Hausdorff-Besicovitch dimension* $1 - \frac{1}{\alpha}$ *in the case* $1 < \alpha \leqq 2$.

PROOF. We define the local time of the symmetric stable process at $x = 0$. Put $s_\varepsilon(t, w) = \frac{1}{\varepsilon} \int_0^t \chi_{(0,\varepsilon)}(x_t) dt$ for $\varepsilon > 0$, then we can show that there exists some sequence $\{\varepsilon_m\}$ tending to zero and a function $s(t, w)$ such that

(6.5) $P_0(s_{\varepsilon_m}(t, w) \to s(t, w)$ uniformly on any compact in $[0, +\infty)) = 1$.
We give here the outline of the proof only.[31] Put

$$e_m(t, x) = E_x(s_{\varepsilon_m}(t, w)) = \frac{1}{\varepsilon_m} \int_0^{\varepsilon_m} \int_0^t p(s, x-y) ds\, dy .$$

Then, noting the fact $p(t, x) < K t^{-\frac{1}{\alpha}}$, we can show that

$$| e_m(t, x) - e(t, x) | \to 0 \quad m \uparrow \infty, \text{ uniformely on any compact}$$
$$\text{set in } R^1 \times [0, \infty) \text{ where } e(t, x) = \int_0^t p(s, x) ds .$$

Using this we can prove that $s_{\varepsilon_m}(t, w)$ converges in the mean, i. e. there exists $s(t, w)$ such that $E | s_{\varepsilon_m}(t, w) - s(t, w) |^2 \to 0$ $m \uparrow \infty$. Next, noting $E_x(s_{\varepsilon_m}(T, w) \mid \boldsymbol{B}_t)$ is a martingale, we can obtain (6.5).

$s(t, w)$ is continuous and non-decreasing in $t > 0$ and we can easily check that if $x_t(w) \neq 0$, then there exist t', $t < t'$ such that $s(t, w) = s(t', w)$. Hence if we put

$$t(u, w) = \max \{t \,;\, u = s(t, w)\}$$

then $x_{t(u,w)}(w) = 0$. We can prove from this and the fact that $s(t, w)$ is a additive functional that $t(u, w)$ is a Lévy process with respect to P_0. Its characteristic function is given by

$$E_0(e^{-\lambda t(u,w)}) = e^{-\frac{u}{g_\lambda(0,0)}}$$

31) The following method was given by K. Sato in the case of the multi-dimensional diffusion [**6**].

where

$$g_\lambda(0, 0)=\frac{1}{\pi}\int_0^\infty \frac{d\xi}{\lambda+\xi^\alpha}=\frac{1}{\lambda^{1-\frac{1}{\alpha}}}\frac{1}{\pi}\int_0^\infty \frac{d\eta}{1+\eta^\alpha}.$$

Hence $t(u, w)$ is a one-sided stable process with exponent $1-\frac{1}{\alpha}$. Now we can also check that (with probability one) if $(t', t'')\cap \boldsymbol{Z}(w)$ is not empty then $s(t', w)< s(t'', w)$, and from this we have

$$P_0(\boldsymbol{Z}(w)\subset \{t; t=t(u, w) \text{ or } t=t(u-, w) \text{ for some } u\geqq 0\})=1.$$

The theorem follows now from a theorem of Blumenthal-Getoor [1, Theorem 3.2].

Now consider the interval $I^-=(-\infty, 0)$ and $\sigma^-(w)$ be defined by (4.4).

THEOREM 6.4. *For* $0<\alpha<2$ $x\in I^-$

$$P_x(^\exists t\leqq \sigma^-, x_{t-}=0)=0.$$

PROOF. The function $\eta(x)=\lim_{\varepsilon\downarrow 0}\frac{\bar{g}_0^-(x, -\varepsilon)}{\varepsilon^\alpha}$ in (4.9) is an exessive function for the absorbing barrier stable process on I^- and

$$\eta(0)=+\infty.$$

Since $\eta(x_t)$ is a lower semi-martingale it is bounded on any interval $0\leqq t\leqq T$ with probability one and the theorem follows immediately from this.

This theorem means that, though in the case $1<\alpha<2$ particles of the symmetric stable process hit a given point almost surely, they can not remain in one of the half lines cut by the point up to this hitting time.

University of Kyoto

References

[1] R. M. Blumenthal and R. K. Getoor, Some theorems on stable processes, Trans. Amer. Math. Soc., 95 (1960), 263-273.

[2] J. Elliott, The boundary value problems and semi-groups associated with certain integro-differential operators, Trans. Amer. Math. Soc., 76 (1954), 300-331.

[3] J. Elliott, Absorbing barrier processes connected with the symmetric stable densities, Illinois J. Math., 3 (1959), 200-216.

[4] G. A. Hunt, Some theorems concerning Brownian motion, Trans. Amer. Math. Soc., 81 (1956), 294-319.

[5] G. A. Hunt, Markov processes and potentials, Illinois J. Math., 1 (1957), 44-93.

[6] N. Ikeda, T. Ueno, H. Tanaka and K. Sato, On the boundary problems of multi-dimensional diffusions (Japanese), Seminar on Prob., vol. 6.

[7] K. Ito, Stochastic process, Tata Institute notes, to appear.

[8] K. Ito and H. P. McKean, Diffusion, to appear.

[9] M. Kac, On some connections between probability theory and differential and integral equations, Proceedings of the second Berkeley symposium, Berkeley, 1950, 189-215.

[10] H. P. McKean, Sample functions of stable processes, Annals of Math., **61** (1955), 564-579.
[11] G. Pólya and G. Szegö, Über den transfiniten Durchmesser (Kapzitätskonstante) von ebenen und räumlichen Punktmengen, J. Reine Angew. Math., **165** (1931), 4-49.
[12] D. Ray, Stable processes with an absorbing barrier, Trans. Amer. Math. Soc. **87** (1958), 187-197.
[13] F. G. Tricomi, Integral equations, New York, 1957.
[14] H. Widom, Stable processes and integral equations, Trans. Amer. Math. Soc., **98** (1961), 430-448.