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On the Partial Sums of Certain Laurent Expansions

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Let $f(z)=z+\sum_{1}^{\infty}\frac{a_k}{z^k}$ be analytic and univalent for $|z|>1$. Kung Sun [**3**; p. 111] has demonstrated the existence of a constant R_0 such that each partial sum $S_n(z)=z+\sum_{1}^{n}\frac{a_k}{z^k}$, is univalent for $|z|>R_0$; he showed that $\left[\frac{4}{3}\right]^{\frac{1}{3}}\leqq R_0<\frac{3}{2}$.

In this note we offer a slight improvement of Kung Sun's result in the form of the following

THEOREM. *If $f(z)=z+\sum_{1}^{\infty}\frac{a_k}{z^k}$ is analytic and univalent for $|z|>1$, then there exists a constant R_0, $\left[\frac{3}{2}\right]^{\frac{1}{4}}\leqq R_0\leqq\sqrt{2}$, such that each partial sum $S_n(z)=z+\sum_{1}^{n}\frac{a_k}{z^k}$, is univalent for $|z|>R_0$.*

PROOF: First we consider the analytic and univalent function $z\left(1+\frac{1}{z^4}\right)^{\frac{1}{2}}=z+\frac{1}{2z^3}+\cdots$, which maps the domain where $|z|>1$ onto a domain whose complement is "star-shaped" with respect to the origin (in the image plane). It is easy to see that the partial sum $\sigma_1(z)=z+\frac{1}{2z^3}$ has a derivative that vanishes for $z=\left[\frac{3}{2}\right]^{\frac{1}{4}}$. Hence Kung Sun's constant R_0 must satisfy $R_0\geqq\left[\frac{3}{2}\right]^{\frac{1}{4}}>\left[\frac{4}{3}\right]^{\frac{1}{3}}$.

Now we consider a general univalent function $f(z)=z+\sum_{1}^{\infty}\frac{a_k}{z^k}$, $|z|>1$. If $z_1\neq z_2$, $|z_1|=|z_2|=R>1$, then an obvious calculation yields

$$
\begin{aligned}
\left|\frac{S_n(z_2)-S_n(z_1)}{z_2-z_1}\right| &= \left|1-\sum_{k=1}^{n}a_k\sum_{m=0}^{k-1}\frac{z_1^m z_2^{k-m-1}}{z_1^k z_2^k}\right| \\
&\geqq 1-\sum_{k=1}^{n}\frac{k|a_k|}{R^{k+1}}\geqq 1-\sum_{1}^{\infty}\frac{k|a_k|}{R^{k+1}}.
\end{aligned}
\tag{1}
$$

The Bieberbach "area principle" [**1**; p. 73] gives us $\sum_{k=1}^{\infty}k|a_k|^2\leqq 1$, while the Cauchy-Schwarz inequality gives us

$$\sum_{1}^{\infty}\frac{k|a_k|}{R^{k+1}}\leqq\Big[\sum_{1}^{\infty}k|a_k|^2\Big]^{\frac{1}{2}}\Big[\sum_{1}^{\infty}\frac{k}{R^{2k+2}}\Big]^{\frac{1}{2}}.$$

If we apply these last two inequalities to (1), we obtain

$$\left|\frac{S_n(z_2)-S_n(z_1)}{z_2-z_1}\right|>1-\Big[\sum_{1}^{\infty}\frac{k}{R^{2k+2}}\Big]^{\frac{1}{2}}=1-\frac{1}{R^2-1}. \tag{2}$$

From (2) we conclude that the derivative $S_n'(z)$ does not vanish where $|z|>\sqrt{2}$, and we conclude that each circle $|z|=R>\sqrt{2}$ is mapped by $S_n(z)$ onto a simple closed curve. The theorem now follows.

We note that by using the finite partial sums in (1) and (2), we can obtain slightly sharper results, at least for small values of n. For example, if $n=1$, we have

$$\left|\frac{S_1(z_2)-S_1(z_1)}{z_2-z_1}\right|\geqq 1-\frac{1}{R^2},$$

so that $S_1(z)$ is univalent for $|z|>1$. Again, for $n=2$, we have

$$\left|\frac{S_2(z_2)-S_2(z_1)}{z_2-z_1}\right|\geqq 1-\Big[\frac{1}{R^4}+\frac{2}{R^6}\Big]^{\frac{1}{2}},$$

from which we conclude that $S_2(z)$ is univalent (at least) for $|z|>[3]^{\frac{1}{4}}$.

We can also refine Kung Sun's method a bit to obtain slightly sharper results as follows. Set $Q_n(z)=f(z)-S_n(z)$, for fixed n, $n\geqq 1$. Then another application of the "area principle" and the Cauchy-Schwarz inequality yields, for $z_1\neq z_2$, $|z_1|=|z_2|=R>1$,

$$\left|\frac{Q_n(z_2)-Q_n(z_1)}{z_2-z_1}\right|\leqq\frac{[(n+1)R^2-n]^{\frac{1}{2}}}{R^{n+1}(R^2-1)}.$$

But it is known [**2**; p. 127] that for $z_1\neq z_2$, $|z_1|=|z_2|=R>1$, we have

$$\left|\frac{f(z_2)-f(z_1)}{z_2-z_1}\right|\geqq 1-\frac{1}{R^2}$$

Hence $S_n(z_1)\neq S_n(z_2)$ whenever

$$1-\frac{1}{R^2}>\frac{[(n+1)R^2-n]^{\frac{1}{2}}}{R^{n+1}(R^2-1)}. \tag{3}$$

Elementary calculations show that (3) holds for $R>[3]^{\frac{1}{4}}$, provided $n\geqq 7$. Hence we have shown that the partial sums $S_n(z)$ are univalent for $|z|>[3]^{\frac{1}{4}}$, for $n\geqq 7$.

We close with the remark that Kung Sun has shown, using (3) above, that $S_n(z)$ is univalent for $|z|>\left[1-\frac{1}{n}5\log n\right]^{-\frac{1}{2}}$, provided $n\geqq 13$.

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Bibliography

[1] L. Bieberbach, Lehrbuch der Funktionentheorie, New York, 1945.
[2] G. W. Golusin, Geometrische Funktionentheorie, Berlin, 1957.
[3] Kung Sun, The Sections of Schlicht Functions, Acta Math. Sinica, **4** (1954), 105-112.

Added in proof: The crude result $R_0<\sqrt{2}$ can be refined a bit by using a recent result due to Robertson (Notices, Amer. Math. Soc., 8 (1961), p. 516) and Pommerenke (Math. Z., **78** (1962), p. 274) to show that each partial sum $S_n(z)$ of $f(z)$ yields a star-like mapping, $|z|>\sqrt{2}$. In addition, an examination of a certain extremal mapping due to Garabedian and Schiffer (Ann. of Math., **61** (1955), p. 133) casts serious doubt on the lower bound $4\sqrt{3/2}$ given above. Additional results of the same general nature, for other maps (star-like, convex, etc.) were announced at the Conference on Analytic Functions held in Cracow, Poland, 28 August—5 September 1962. The Proceedings of this Conference will be published soon.