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Martin boundaries for certain Markov chains

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1. Introduction.

We shall be concerned with the theory of Martin's ideal boundary as adapted for Markov chains by J. L. Doob in [1]. This theory has been developed by Hunt in [4] and by Watanabe in [8]; some applications and illustrations are contained in works by Doob, Snell and Williamson [2], Dynkin and Malyutov [3], and Watanabe [9]. However, the specific cases to which the boundary theory has been applied always fall under the general heading of homogeneous processes (usually sums of independent lattice random variables). In the present paper we shall give some theorems describing and applying the Martin boundary theory associated with certain other classes of Markov chains. We assume the reader is familiar with the basic ideas of the Martin boundary; [1], [4], [8] contain detailed treatments, and short sketches of the theory can be found in [2] and [9].

It seems to us that there are three purposes which a study of this type can serve. The first is that the elegance and interest of the general Martin theory makes it desirable to have a variety of specific examples; as noted above, those so far studied are all of one general kind. The second thing is that the specific results may be of some analytical interest. The central Martin representation theorem provides, in our examples, the general positive solution to certain partial difference equations. In addition to these solutions and some of their properties we obtain (in § 4) amusing characterizations of sequences analogous to moment sequences obtained from orthogonal polynomials. Finally, some results of primarily probabilistic interest are found; these include a 0—1 law (§ 2) and some limit theorems for certain Markov processes.

The Martin boundary for a denumerable Markov chain is defined in [1] relative to a preferred state which we denote by 0. We denote by R the set of all states which can be reached from 0, and by P the matrix which gives the transition probabilities between states of R. Given such a chain we shall also consider a second one called the space-time chain for P. A state of this new chain represents the position and the time in the previous chain; we denote these states by (i, t). Thus the space-time chain is in state (i, t) if the space chain occupies state i on the t-th step. The transition probabilities for the space-time chain are

$$\hat{P}_{(i,t),(j,t+1)} = P_{ij}$$

and

$$\hat{P}_{(i,s),(j,t)} = 0 \quad \text{if} \quad t \neq s+1 .$$

When discussing the Martin boundary for the space-time chain we shall use the reference state $(0, 0)$. We denote the state space (i. e. the states (i, t) which can be reached from $(0, 0)$ with positive probability) by $\hat{R}$.

We shall also consider a modified space-time process defined as follows: A number α with $0 < \alpha < 1$ is specified and the transition probabilities are defined by

$$\hat{P}^{\alpha}_{(i,t),(j,t)} = (1-\alpha)P_{ij} ,$$

$$\hat{P}^{\alpha}_{(i,t),(i,t+1)} = \alpha ,$$

$$\hat{P}^{\alpha}_{(i,s),(j,t)} = 0 \quad \text{otherwise} .$$

We denote the transition matrix for this chain by $\hat{P}^{\alpha}$. The state space will be called $\hat{R}^{\alpha}$; it contains, as $\hat{R}$ may not, all pairs $(i, t), t \geqq 0$, where i can be reached from 0 in the original chain.

The principal class of Markov chains that we will consider will be random walks. These are chains which move on the non-negative integers in such a way that each step is at most one unit away from the state previously occupied. The transition probabilities are specified by numbers $p_i, r_i, q_i, i = 0, 1, 2, \cdots$ with $q_0 = 0$ and $p_i + r_i + q_i = 1$, as follows:

$$P_{i,i-1} = q_i, \quad i \geqq 1 ,$$

$$P_{i,i} = r_i, \quad i \geqq 0 ,$$

$$P_{i,i+1} = p_i, \quad i \geqq 0 .$$

We shall frequently assume in addition that $r_i = 0$ for all i. We always assume $q_i > 0, i \geqq 1$ and $p_i > 0$ for $i \geqq 0$. These random walks have been studied intensively by Karlin and McGregor in [5], and our work has some points of contact with theirs.

In studying the boundary of any of the above chains it is often useful to introduce new chains related to certain functions. We shall describe these processes in terms of the space chain determined by P. A function h on R is *regular* if $h = Ph$. For any non-negative regular function h with $h(0) \neq 0$ we can define a new chain with state space the set R^h of points where $h \neq 0$ and transition matrix given by

$$P_{ij}^{(h)}=\frac{P_{ij}h_j}{h_i}.$$

This new chain is called the h-process.

2. Minimality of the constant function.

A regular function h for P is *minimal* if it is non-negative, $h(0)\neq 0$, and if it has the property that any other such function g with $g\leqq h$ must be a constant multiple of h. With each Martin boundary point needed in the representation theorem there is associated a unique minimal regular function. Thus in determining the boundary it is useful to have methods of testing the minimality of a given regular function h. One method of doing this is to form the h process and to show that for this process the constant function is minimal. We shall therefore need to have conditions on a process which insure that the constant function is minimal. We shall develop several such criteria in this section.

We shall consider first a space chain P which is recurrent.

THEOREM 2.1. *The constant function is minimal for any space-time chain obtained from a recurrent chain.*

PROOF. Let $h(i, t)$ be a non-negative bounded function defined on $\hat{R}$ which is space-time regular. Assume that $h(0,0)=1$. Let x_t be the position after time t of the chain started in state 0 at time 0. Then the fact that h is space-time regular implies that $\{h(x_t, t), t=0,1,\cdots\}$ is a martingale. Let α_k be the time of the k-th return to 0. Since h is bounded, by a standard martingale system theorem $\{h(x_{\alpha_k}, \alpha_k), k=0,1,2,\cdots\}$ is again a martingale. This means that h, restricted to the set of points in $\hat{R}$ of the form $(0, s)$, is regular for the process $\{\alpha_k, k=0,1,2,\cdots\}$. But this process consists of sums of independent random variables, and it was proved in [2] that for such a process only the constant function is bounded and regular. Hence $h(0, s)$ is equal to 1 for all $(0, s)$ in $\hat{R}$. Assume now that (j, u) is any state in $\hat{R}$. Consider the process $\{h(x_t, t), t=0,1,2,\cdots\}$, stopped as soon as either (j, u) or else a state of the form $(0, s)$ with $s>u$ is reached. Let p be the probability that the process is stopped at (j, u); clearly $p>0$. But by the martingale system theorem

$$1=ph(j, u)+(1-p)$$

and hence $h(j, u)=1$.

We consider next a class of chains which are characterized by a coincidence property. This property is as follows. Consider a Markov chain with transition matrix P and state space R. Let $i, j\in R$ and let $\{x_t\}$ and $\{y_t\}$ be two independent processes determined by P with initial states i and j respectively. If, with probability one, the processes will at some time occupy the same state regardless of the choice of i and j we say that P has the coincidence property.

THEOREM 2.2. *Let P be a chain with the coincidence property. Then the constant function is minimal for the associated space-time chain $\hat{P}$.*

PROOF. Let h be any bounded space-time regular function with $h(0,0)=1$. Let (i, s) and (j, s) be two points in the space-time state space having different space coordinates i and j and the same time coordinate s. We shall prove that $h(i, s)=h(j, s)$. Form the process $\{x_t, y_t; t=s, s+1, \cdots\}$ which represents the positions of two independent particles started at time s in states i and j respectively and both having the transition matrix of the given chain.

Let F_t be the Borel field generated by the set of random variables x_r, y_r with $s \leqq r \leqq t$. Then $\{h(x_t, t), t=s, s+1, \cdots\}$ and $\{h(y_t, t), t=s, s+1, \cdots\}$ are both martingales with respect to the fields $\{F_t, t \geqq s\}$. Let τ be the time of coincidence of the two processes $\{x_t\}$ and $\{y_t\}$. Then by a standard martingale system theorem $h(i, s)=E[h(x_\tau, \tau)]$ and $h(j, s)=E[h(y_\tau, \tau)]$. Since $x_\tau=y_\tau$ this proves that $h(i, s)=h(j, s)$. Furthermore, because

$$1=\sum P_{0j}^{(s)} h(j, s)$$

and $h(\cdot, s)$ is a constant function we have $h(j, s)=1$ for all j; hence $h\equiv 1$.

We shall need to know when coincidence is certain for two particles executing independently the same random walk. This problem has been completely solved for birth-and-death processes by Karlin and McGregor in [**6**]. However we shall give a proof in the discrete case which has much the same general plan as that in [**6**] but is considerably simpler. We here assume that $r_n=0$ for all n, and that i and j are both even or both odd.

LEMMA. *Coincidence of two particles in the random walk fails to be certain if and only if*

$$\prod_{n=1}^{\infty} p_n = p > 0 . \tag{2.1}$$

PROOF. Suppose two walks start at states i and $j, i<j$. Clearly if the walk from j moves to the right at each step, the other can never overtake it so that the coincidence cannot occur. If (2.1) holds, the probability of this is positive and so coincidence is not certain.

Suppose now that (2.1) fails. Suppose particles A and B begin a walk at the time 0 in (for simplicity) states 0 and 2 respectively. Let $T_A(k)$ and $T_B(k)$ be their respective times of first reaching state k; we will show that

$$Pr[T_B(k)<T_A(k) \quad \text{for all } k]=0 \tag{2.2}$$

which implies coincidence. Note that

$$T_B(k)=\sum_{j=2}^{k-1}[T_B(j+1)-T_B(j)] \tag{2.3}$$

where the summands are independent and positive; similarly

$$T_A(k)=\sum_{j=2}^{k-1}[T_A(j+1)-T_A(j)]+T_A(2)\,. \tag{2.4}$$

The terms in the second sum (2.4) are independent of each other and of those in the first. Hence

$$\begin{aligned}(2.5)\qquad & Pr[T_B(k)<T_A(k)\quad\text{for all } k]\\ &=Pr\Big[\sum_{j=2}^{k-1}\{[T_B(j+1)-T_B(j)]-[T_A(j+1)-T_A(j)]\}<T_A(2)\quad\text{for all } k\Big].\end{aligned}$$

Now the terms in the braces are independent of each other and of $T_A(2)$; since $T_B(j+1)-T_B(j)$ is the passage time from j to $j+1$ it has the same law as $T_A(j+1)-T_A(j)$. Hence the probability in question is less than

$$Pr\Big(\sum_{j=2}^{k-1}T^*(j)\text{ is bounded above}\Big),$$

where $T^*(j)$ are integer valued, symmetric, independent random variables. This probability is either 0 or 1; it is easy to see that it cannot be 1 unless

$$Pr(T^*(j)\neq 0\quad\text{infinitely often})=0\,. \tag{2.6}$$

Now recalling that $T^*(j)$ is the difference of two independent random variables representing passage time from j to $j+1$, we have

$$Pr(T^*(j)=0)\leqq p_j^2+(1-p_j)^2p_j^2+(1-p_j)^4p_j^2+\cdots,$$

so that

$$Pr(T^*(j)\neq 0)\geqq 1-\frac{p_j^2}{1-(1-p_j)^2}>(1-p_j)\,.$$

However, $\sum(1-p_j)$ diverges because we are assuming the product (2.1) is 0, and this, together with the second Borel-Cantelli lemma, contradicts (2.6).

Combining this lemma with Theorem 2.2 we have at once the

COROLLARY: *If a random walk with matrix P has $r_n=0$ for all n and $\prod_{n=1}^{\infty}p_n$ divergent to* 0, *then the constant function is minimal for the associated space-time walk $\hat{P}$.*

As mentioned above, the constant function is minimal for any chain formed by adding independent random variables [2]. We shall now adapt the idea used in Dynkin and Malyutov's elementary proof of this fact [3] to obtain

THEOREM 2.3.[1] *Let P be the transition matrix of any Markov chain and $\hat{P}^\alpha$ the corresponding modified space-time matrix. Then the constant function is*

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1) The authors are indebted to the referee for noting that Theorem 2.3 was incorrect in an earlier version and for suggesting the correct statement of the theorem.

minimal for $\hat{P}^{\alpha}$ if and only if it is minimal for P.

PROOF. Let $f(i,t)$ be a bounded, non-negative function which is regular for $\hat{P}^{\alpha}$. Let $g(i,t)=f(i,t+1)-f(i,t)$; we shall now show that $g(i,t)=0$. Suppose instead that $0<\beta=\sup g(i,t)$, and choose a state (l,τ) such that $g(l,\tau)>\beta(1-\delta)$. Then for any integer n, we have (since g is also regular)

$$g(l,\tau)=\sum_{(i,t)\neq(l,\tau+n)}[\hat{P}^{\alpha}]^{n}_{(l,\tau),(i,t)}g(i,t)+\alpha^{n}g(l,\tau+n)$$
$$\leqq(1-\alpha^{n})\beta+\alpha^{n}g(l,\tau+n)\,.$$

It follows that

$$g(l,\tau+n)\geqq\beta\left(1-\frac{\delta}{\alpha^{n}}\right). \tag{2.7}$$

Now choose N large enough so that $N\beta>4\sup|f(i,t)|$, and let δ be so small that $1-\delta/\alpha^{N}>1/2$ (by proper choice of (l,τ)). By (2.7), $g(l,\tau), g(l,\tau+1),\cdots$, $g(l,\tau+N)$ are then each larger than $\beta/2$, and so their sum exceeds $(N+1)\beta/2>2\sup|f|$. But on the other hand

$$g(l,\tau)+\cdots+g(l,\tau+N)=f(l,\tau+N+1)-f(l,\tau)<2\sup|f|\,.$$

Thus assuming that $\beta>0$ leads to a contradiction. The assumption that $\inf g<0$ can be handled in the same way, and so we have proved that $g(i,t)\equiv 0$.

Consider the condition for regularity of f:

$$f(i,t)=(1-\alpha)\sum P_{ij}f(j,t)+\alpha f(i,t+1)\,.$$

Since we have just seen that $f(i,t+1)=f(i,t)$, the function $f(i,0)$ is regular for the space matrix P. If the constant is minimal for P, we then have $f=$ constant everywhere. If conversely there is a non-constant, non-negative bounded regular function $h(i)$ for P, the example $f(i,t)=h(i)$ shows that the constant function is not minimal for $\hat{P}^{\alpha}$.

It is obvious that if P is a random walk matrix, the constant function is minimal regular. Thus we obtain the

COROLLARY. *If P is the transition matrix of a random walk, the constant function is a minimal function for $\hat{P}^{\alpha}$.*

Whenever we can prove that the constant function is the only bounded regular function we have in fact proved a $0-1$ law for the probability of hitting any set infinitely often. To see this let S^{E}_{i} be the probability starting in i that E is entered infinitely often. Then S^{E} is bounded and regular and hence a constant, say $S^{E}_{i}=c$. Let B^{E}_{ij} be the probability starting in i that E is entered at j. Then

$$c=S^{E}_{i}=\sum_{j\in E}B^{E}_{ij}S^{E}_{j}=c\sum_{j\in E}B^{E}_{ij}=c\,Pr\ \text{(hit } E \text{ from } i)\,.$$

Thus either $c=0$ or the probability that E is entered from any i is 1. In the latter case $S_i^F = 1$ for any i.

Another consequence of proving that the constant function is minimal is the following. From general Martin boundary theory, if h is minimal the h process converges with probability one to the boundary point corresponding to h. Hence if the constant function is minimal the process itself converges with probability one to the boundary point corresponding to the constant function. This convergence is in terms of the Martin topology. For the space-time process, this convergence is equivalent to the statement that

$$\lim_{t\to\infty} \frac{P_{i,x_t}^{t-m}}{P_{0,x_t}^{t}} \tag{2.8}$$

exists with probability one for any m and i. If further the constant function is space-time minimal then this limit is 1 with probability one. Thus, for example, this is the case whenever the original chain was recurrent. While for such a chain $x_t = j$ infinitely often, it is not true in general that

$$\lim_{t\to\infty} \frac{P_{i,j}^{t-m}}{P_{0,j}^{t}} = 1\,. \tag{2.9}$$

Thus the sample sequences are in state j at times which make this limit exist with probability one. The question of whether (2.9) holds, with some of the consequences if it does, has been studied recently by Orey [7] and others.

3. Minimal regular functions.

In this section we shall consider the problem of determining all minimal functions for certain processes. The unifying feature of our work here is that in many cases the minimal functions for a space-time chain factor into a space part and a time part. This situation makes the construction of the boundary quite simple. Some examples of more complicated behavior are briefly discussed at the end of the section.

We consider first the modified space-time process defined in the introduction.

THEOREM 3.1. *Let P be any stochastic matrix, and let $\hat{P}^{\alpha}$ be the modified space-time transition matrix. Then a function $h(i,t)$ is minimal regular only if it is of the form*

$$h(i,t) = c^t g(i) \tag{3.1}$$

where $g(i) \geqq 0$, $\frac{1}{\alpha} > c \geqq 0$, and $Pg = \lambda g$ with $\lambda = \frac{1-\alpha c}{1-\alpha}$. Such a function is in fact minimal if and only if the constant function is minimal for the chain with matrix

$$\frac{1-\alpha}{1-\alpha c} P_{ij} \frac{g(j)}{g(i)} = q_{ij}\,.$$

PROOF. The function h will be $\hat{P}^{\alpha}$ regular if

$$h(i,t)=\alpha h(i,t+1)+(1-\alpha)\sum_j P_{ij}h(j,t)\,. \tag{3.2}$$

The function $h'(i,t)=h(i,t+1)$ also satisfies (3.2) and in addition $h'(i,t)\leqq\alpha^{-1}h(i,t)$. Thus if h is minimal, we must have $h'(i,t)=h(i,t+1)=ch(i,t)$, so that (3.1) holds with $1/\alpha>c\geqq 0$. Substituting (3.1) in (3.2) we see that

$$(1-\alpha c)g(i)=(1-\alpha)\sum_j P_{ij}g(j)\,, \tag{3.3}$$

and the asserted form for $h(i,t)$ is proved. Note that $c=\alpha^{-1}$ is impossible.

It remains to investigate when such functions are minimal. But it is easy to check that the h-process corresponding to such a function is again a modified space-time chain; the matrix of the underlying chain is $[q_{ij}]$. By Theorem 2.3, the constant is therefore minimal for the h-process if and only if it is minimal for the matrix $[q_{ij}]$, and so the proof is complete.

Let P be the matrix of a random walk as defined in the introduction. Following Karlin and McGregor [5], we define a system of polynomials $Q_i(x), i=0,1,2,\cdots$, by setting $Q_0(x)=1$ and solving recursively the equation

$$xQ_n(x)=p_nQ_{n+1}(x)+r_nQ_n(x)+q_nQ_{n-1}(x)\,. \tag{3.4}$$

$Q_n(x)$ is of degree n with positive leading coefficient. The polynomials $Q_n(x)$ are orthogonal with respect to a measure $\psi(x)$ on $[-1,1]$, so their zeros lie in this range. For each x the vector $\{Q_n(x)\}$ is the unique right eigenvector of P with eigenvalue x. We shall show now that $Q_n(x)\geqq 0$ for all n and fixed x if and only if $x\geqq\beta$ where

$$\beta=\sup\{\text{zeros of } Q_i(x)\}\leqq 1\,.$$

Under our assumptions, the zeros of $Q_n(x)$ and $Q_{n+1}(x)$ are simple and strictly interlace. Suppose that $x_0<\beta$; there is then a first value i such that $Q_i(x)$ has a zero greater than x_0. If $Q_i(x_0)\geqq 0$, then since $Q_i(1)=1>0, Q_i(x)$ must have at least two zeros greater than or equal to x_0. But then by the strict interlacing property $Q_{i-1}(x)$ would have a zero between these two and hence greater than x_0, which contradicts the choice of i. Hence $Q_i(x_0)<0$.

With the aid of these facts we can add to Theorem 3.1 the following

COROLLARY: *If P is the transition matrix of the above random walk, and $\hat{P}^{\iota}$ is that of the corresponding modified space-time process, then the minimal regular functions with $h(0,0)=1$ are obtained from the formula*

$$h(i,t)=\left[\frac{1-(1-\alpha)x}{\alpha}\right]^t Q_i(x) \tag{3.5}$$

by assigning to x any value in the range $[\beta,(1-\alpha)^{-1}]$.

PROOF. This follows immediately from the theorem and the facts above, plus the remark that the matrices $[q_{ij}]$ of Theorem 3.1 are again random walks, so that minimality is assured.

Consider next a space-time random walk. We shall assume $r_n=0$, so that the set $\hat{R}$ of points which can be reached from $(0,0)$ consists of all states (k,t) such that $k \leqq t$ and $k+t$ is even. (The latter reflects the periodicity of the underlying random walk.)

THEOREM 3.2. *Let P be a random walk such that for some $\varepsilon>0, p_{jj}^{(2)} \geqq \varepsilon$ for every j; assume that $r_n=0$ for all n. Then the space-time minimal functions with $h(0,0)=1$ are the functions*

$$\text{(3.6)} \qquad \text{(a)} \quad h(i,t)=\frac{Q_i(x)}{x^t} \quad \textit{for} \quad \beta \leqq x<\infty,$$

$$\text{(b)} \quad h_\infty(i,t) \ \textit{defined by}$$

$$h_\infty(0,0)=1$$

$$h_\infty(t,t)=\frac{1}{p_0 p_1 \cdots p_{t-1}}, \quad t \geqq 1$$

$$h_\infty(i,t)=0 \qquad \textit{otherwise.}$$

PROOF. Let h be minimal and consider the function h' defined by

$$h'(i,t)=h(i,t+2).$$

Then h' is regular on $\hat{R}$ and since

$$h(i,t) \geqq \varepsilon h(i,t+2),$$

$h' \leqq \frac{1}{\varepsilon} h$. Thus by the minimality of h, h' must be a constant multiple of h; i.e.,

$$\text{(3.7)} \qquad h(i,t+2)=c^2 h(i,t) \quad \text{for some} \quad c \geqq 0.$$

We must now consider two cases. First assume that $c \neq 0$. Then we can define a sequence $g(n)$ so that

$$h(n,n)=c^n g(n).$$

Applying (3.7), we obtain

$$h(n,t)=c^t g(n)$$

for all $n \leqq t$ such that $n+t$ is even; that is, all (n,t) in $\hat{R}$. Substituting in the condition for regularity we find that

$$g(n)=c(q_n g(n-1)+p_n g(n+1))$$

and so, since eigenvectors are unique, (3.6) (a) follows from (3.4) with $1/c=x$. As we have seen, $x \geqq \beta$ is the condition for non-negativity.

Next we will verify that the function $x^{-t}Q_i(x)$ is minimal. The h-process turns out to be the space-time process formed from a random walk with transition probabilities

$$\tilde{p}_j = \frac{p_j Q_{j+1}(x)}{x Q_j(x)}, \quad \tilde{q}_j = \frac{q_j Q_{j-1}(x)}{x Q_j(x)}.$$

In view of Theorem 2.2 and the subsequent lemma and corollary, if we can show that $\tilde{p}_j \not\to 1$ it will be more than enough to imply that h is minimal. Suppose $\tilde{p}_j \to 1$; then

$$\frac{Q_{j+1}(x)}{Q_j(x)} \sim \frac{x}{p_j}$$

and so

$$0 = \lim_{j\to\infty} \tilde{q}_j = \lim_{j\to\infty} \frac{p_{j-1}q_j}{x^2}.$$

However, the latter contradicts the assumption that

$$p_{jj}^{(2)} = p_j q_{j+1} + q_j p_{j-1} \geqq \varepsilon > 0.$$

Assume finally that $c=0$ in (3.7), so that $h(n,t) \neq 0$ only for those points in $\hat{R}$ with $n=t$. The assumption of regularity immediately shows h to be given by (3.6) (b). The h-process consists simply of deterministic motion along the line $n=t$; it is trivial that only the constant is regular for this "process" so the function h_∞ is minimal and the theorem proved.

We shall only construct the Martin boundary for some of the processes whose minimal functions have already been determined, but it may be interesting to point out a difficulty that arises in slightly different cases. For instance, suppose that P is a random walk as above with a finite number of $r_n > 0$, including $r_0 > 0$. Then the space-time process has state space $\hat{R}$ consisting of *all* (n, t) with $n \leqq t$. The difficulty which this causes is that *in general, the functions* $x^{-t}Q_n(x)$ *are no longer minimal.* Indeed, if $Pr(X_n \to \infty) = 1$ (if the space walk is transient), then

$$h(n,t) = Pr(X_k + k \text{ is even for all large } k \mid X_t = n) \tag{3.8}$$

is a bounded regular function which is not constant, and so $h \equiv 1$ is not minimal. This argument also applies to the h-processes formed from the functions $x^{-t}Q_n(x)$, and so the latter are usually not minimal either. It may be that in this situation each function $x^{-t}Q_n(x)$ is replaced by two others, corresponding in the case of the constant function $(x=1)$ to the function h of (3.8) and to $1-h$. There seem to be even more possibilities when infinitely many r_n may be positive. However, if $r_n \geqq \varepsilon > 0$ for all n the situation is once again simple, since essentially the proof and conclusion of Theorem 3.2 applies with $P_{ii}^{(2)}$ replaced by $r_i = P_{ii}$.

4. Martin representation theorems.

Using the results of § 3, it is easy to set up the Martin representation theorem for regular functions analogous to Poisson's integral formula. Let us consider first, as the somewhat simpler case, the modified space-time process associated with a random walk; our notation is the same as in § 2 and § 3. For the case we are considering, equation 3.2 becomes

$$(3.2^*)\qquad h(i,t)=\alpha h(i,t+1)+(1-\alpha)[q_i h(i-1,t)+r_i h(i,t)+p_i h(i+1,t)]$$
$$(q_0=0,\ q_i,r_i,p_i\geqq 0,\ p_i+q_i+r_i=1).$$

We have seen that for any (fixed) $\alpha\in(0,1)$, the minimal solutions of (3.2*) are in a one-to-one correspondence with the points of the interval $[\beta,(1-\alpha)^{-1}]$ given by

$$(4.1)\qquad h_x(n,t)=\left\{\frac{1-(1-\alpha)x}{\alpha}\right\}^t Q_n(x).$$

This interval is a compact set; the mapping $x\to h_x(n,t)$ is continuous from the interval onto the minimal part of the Martin boundary and hence it is a homeomorphism. Thus the general representation theorem [1], [4], [8] yields in this case.

THEOREM 4.1. *If $h(n,t)\geqq 0$ satisfies* (3.2*), *there is a unique Lebesgue-Stieltjes measure μ such that*

$$(4.2)\qquad h(n,t)=\int_\beta^{(1-\alpha)^{-1}}\left\{\frac{1-(1-\alpha)x}{\alpha}\right\}^t Q_n(x)d\mu(x).$$

Conversely, (4.2) *defines a non-negative solution of* (3.2*) *for every such measure.*

There is an analogous theorem for space-time random walks; it is necessary to make additional assumptions, however, since we will need to use Theorem 3.2. Suppose as always that $p_n>0$ and $q_m>0$ for $n\geqq 0, m>0$, that $r_n=0$ for all n, and that $p_nq_{n+1}+q_np_{n-1}\geqq\varepsilon$ for all n. By Theorem 3.2 the minimal functions then are

$$(4.3)\qquad h_x(n,t)=x^{-t}Q_n(x),\quad \beta\leqq x<\infty,$$

plus an additional function which is easily seen to be the limit of h_x as $x\to\infty$. (Recall that $n\leqq t$ for points in $\hat{R}$.) Including this extra function of (3.6) (b), we have again a one-to-one, continuous map from the compact interval $[\beta,\infty]$ to the minimal part of the boundary, so the map is in fact a homeomorphism. We therefore obtain

THEOREM 4.2. *Under the conditions above, every function $h(n,t)\geqq 0$ which satisfies the equation*

$$(4.4)\qquad h(n,t)=q_nh(n-1,t+1)+p_nh(n+1,t+1)$$

(for all points (n, t) with $n \leqq t$ and $n+t$ even) can be represented in the form

$$h(n, t)=\int_{\beta}^{\infty} x^{-t} Q_n(x) d\mu(x)+\mu(\infty) h_{\infty}(n, t). \tag{4.5}$$

Here $\mu(\infty) \geqq 0$ and $\mu(x)$ is a finite measure. Conversely, for every finite measure μ and non-negative number $\mu(\infty)$ (4.5) *defines such a solution to* (4.4).

These two theorems have interesting analytic corollaries, which can be deduced in much the spirit in which Watanabe obtained the solution of the Hausdorff moment problem from a coin-tossing process [**9**]. We realize that the results below can be proved more directly; in fact, S. Karlin has sent us a fairly simple non-probabilistic proof of Theorem 4.3. However, it seems to us amusing to present these things here as consequences of the Martin representation theorems.

THEOREM 4.3. *Let $Q_n(x)$ be the system of polynomials defined by* (3.4), *where* lub $\{$*zeros of* $Q_n(x)\} = \beta \leqq 1$. *Let $A>1$ be a constant, and define an operator*

$$(\psi h)_i = h_i - \frac{1}{A}(p_i h_{i+1} + r_i h_i + q_i h_{i-1}) \tag{4.6}$$

mapping sequences $\{h\}$ into sequences $\{\psi h\}$. Then the following three conditions are equivalent:

(I) *There is an increasing function G of bounded variation on $[\beta, A]$ such that*

$$f_n = \int_{\beta}^{A} Q_n(x) dG(x), \quad n=0,1,\cdots. \tag{4.7}$$

(II) *For every $n, k \geqq 0$, $(\psi^k f)_n \geqq 0$.*

(III) *The sequence $\left(\frac{A}{A-\beta}\right)^k (\psi^k f)_0$ is a Hausdorff moment sequence.*

PROOF. We shall use Theorem 4.1; accordingly, choose α so that $(1-\alpha)^{-1} = A$. Suppose (I) holds. Comparing (4.7) to (4.2), we see that $f_n = h(n, 0)$, where $h(n, t)$ is a non-negative regular function. From (3.2*), however, regularity implies that

$$\alpha h(n, t+1) = (\psi h(\cdot, t))_n,$$

and hence (II) follows. Given (II), it is easy to check that the function

$$h(n, t) = \left(\frac{A}{A-1}\right)^t (\psi^t f)_n, \quad t, n = 0, 1, \cdots,$$

is non-negative and regular; hence it can be written in the form (4.2). Putting $n=0$ and making the obvious change of variable in the integral, we obtain

$$\left(\frac{A}{A-1}\right)^k (\psi^k f)_0 = \int_0^{\frac{A-\beta}{A-1}} y^k d\tilde{\mu}(y) = \left(\frac{A-\beta}{A-1}\right)^k \int_0^1 z^k d\tilde{\tilde{\mu}}(z), \tag{4.8}$$

and therefore (III) holds. Finally, if (III) and so (4.8) holds, reversing the change of variable made above leads back to

$$\left(\frac{A}{A-1}\right)^k (\psi^k f)_0 = \int_\beta^A \left\{\frac{A-x}{A-1}\right\}^k dG(x),$$

for some non-decreasing function G. We compare the functions

$$\int_\beta^A \left\{\frac{A-x}{A-1}\right\}^k Q_n(x) dG(x) \quad \text{and} \quad \left(\frac{A}{A-1}\right)^k (\psi^k f)_n .$$

The two agree when $n=0$, and both satisfy (3.2*) with $A=1/\alpha$; it is very easy to see that they must be identical. But then putting $k=0$, we have

$$\int_\beta^A Q_n(x) dG(x) = f_n, \qquad \text{q. e. d.}$$

We shall obtain a corollary of Theorem 4.2 in much the same spirit. Assuming that all the conditions of the theorem are satisfied, we have

THEOREM 4.4. *Define an operator*

$$(\psi h)_i = \frac{1}{q_{i+1}}\{h_{i+1} - p_{i+1} h_{i+2}\}, \quad i>0; \quad (\psi h)_0 = 0. \tag{4.9}$$

Again let $\beta = \text{lub}\,\{\text{zeros of } Q_n(x)\}$. *The following three conditions are then equivalent:*

(I) *There is an increasing function G of bounded variation on* $[0, \beta^{-1}]$ *such that*

$$f_n = \int_0^{\beta^{-1}} x^n Q_n(x^{-1}) dG(x), \quad n=0,1,\cdots. \tag{4.10}$$

(II) *For every* $n, k \geqq 0$, $(\psi^k f)_n \geqq 0$.

(III) *The sequence* $\beta^{2k}(\psi^k f)_0$ *is a Hausdorff moment sequence.*

PROOF. If (I) holds, $f_n = h(n,n)$ where $h(n,t)$ is of the form (4.5) and hence non-negative regular. However, it is easy to see that

$$h(n, n+2k) = (\psi^k f)_n, \tag{4.11}$$

so the latter is $\geqq 0$. If (II) is assumed, $h(n,t)$ can be defined by (4.11) for $t=n+2k$; that is, for all states (n,t) which can be reached from $(0,0)$. As so defined h is non-negative regular and so of the form (4.5). Putting $n=0$, we have a characterization of $(\psi^k f)_0$ which is easily seen to give (III). Similarly (III) means that

$$(\psi^k f)_0 = \int_\beta^\infty x^{-2k} d\mu(x) + \mu(\infty)\delta_{k0} = h(0, 2k), \tag{4.12}$$

where h is defined by (4.5). The functions $h(n, n+2k)$ and $(\psi^k f)_n$ thus agree for $n=0$ and both (as functions of n and $t=n+2k$) are regular; they are therefore identical. But this means that $(\psi^k f)_n$ is of the form (4.5) with $t=n+2k$, and putting $k=0$ we obtain (I).

5. An example.

We shall discuss in this section a class of random walks for which we can make more explicit the connection between the boundary and the state space.

It follows from general Martin boundary theory that if $h(i)$ is a minimal function for P, there is a sequence of states j_n such that

$$\lim_{n\to\infty}\frac{G(i,j_n)}{G(0,j_n)}=h(i)$$

where $G(i,j)$ is the mean number of entries into j when the chain is started in state i. The sequence j_n then tends in the Martin topology to the boundary point corresponding to h. For the space-time chain this means that for each minimal function $h(i,t)$ there is a sequence (j_n, t_n) such that

$$\lim_{n\to\infty}\frac{P_{i,j_n}^{t_n-t}}{P_{0,j_n}^{t_n}}=h(i,t)\,. \tag{5.1}$$

In our example we shall be interested in determining the methods of approach to the boundary and giving for each minimal function the sequences which tend to the corresponding boundary points.

The example that we consider is the class of random walks determined by

$$\begin{aligned} p_n&=\frac{1}{2}\Big(1+\frac{a}{n+a}\Big)\,, \\ & \qquad\qquad\qquad\qquad n>0\,;\quad p_0=1\,, \\ q_n&=\frac{1}{2}\Big(1-\frac{a}{n+a}\Big)\,, \end{aligned} \tag{5.2}$$

where $a>-1/2$ is a constant. The polynomials $Q_n(x)$ associated with these random walks are the ultraspherical polynomials [5]; the spectral measure is given by

$$d\psi(x)=(1-x^2)^{a-1/2}dx\,.$$

We will use Karlin and McGregor's representation formula, according to which

$$P_{ij}^{(n)}=\frac{p_0\cdots p_{j-1}}{q_1\cdots q_j}\int_{-1}^{1}x^nQ_i(x)Q_j(x)d\psi(x)\,. \tag{5.3}$$

It is easily verified that the limit in (5.1) holds for a given sequence (j_n, t_n) for all (i,t) if it holds for points of the form $(0,t)$. Hence to determine the various methods of approaching a boundary point we need only find the sequences (j_n, t_n) such that

$$\lim_{n\to\infty}\frac{P_{0,j_n}^{t_n-t}}{P_{0,j_n}^{t_n}}$$

exists for all t. Let

$$I_{m,n}=\int_{-1}^{1} x^n Q_m(x)d\psi(x) \tag{5.4}$$

be the n-th moment of the m-th polynomial.

The ultraspherical polynomials satisfy the differential equation

$$\frac{d}{dx}\{(1-x^2)^{a+\frac{1}{2}}Q_m'(x)\}+m(m+2a)(1-x^2)^{a-\frac{1}{2}}Q_m(x)=0\,. \tag{5.5}$$

We multiply (5.5) by x^n and integrate from -1 to 1 as follows:

$$\begin{aligned}
m(m+2a)&\int_{-1}^{1} x^n Q_m(x)(1-x^2)^{a-\frac{1}{2}}dx \\
&=-\int_{-1}^{1} x^n\Big[\frac{d}{dx}\{(1-x^2)^{a+\frac{1}{2}}Q_m'(x)\}\Big]dx \\
&=-[x^n(1-x^2)^{a+\frac{1}{2}}Q_m'(x)]_{-1}^{1}+n\int_{-1}^{1} x^{n-1}(1-x^2)^{a+\frac{1}{2}}Q_m'(x)dx \\
&=n[x^{n-1}(1-x^2)^{a+\frac{1}{2}}Q_m(x)]_{-1}^{1} \\
&\qquad -n\int_{-1}^{1}[(n-1)x^{n-2}(1-x^2)-(2a+1)x^n](1-x^2)^{a-\frac{1}{2}}Q_m(x)dx\,.
\end{aligned}$$

Hence we obtain

$$m(m+2a)I_{m,n}=-n(n-1)I_{m,n-2}+n(n+2a)I_{m,n}\,,$$

or for $n>1$

$$\frac{I_{m,n-2}}{I_{m,n}}=\frac{(n-m)(n+m+2a)}{n(n-1)}\,.$$

Using the representation theorem (5.3) this gives

$$\frac{P_{0,a_n}^{t_n-2}}{P_{0,a_n}^{t_n}}=\frac{(t_n-a_n)(t_n+a_n+2a)}{t_n(t_n-1)}\,. \tag{5.6}$$

It is easy to see that the ratio (5.6) has a limit for a sequence (a_n, t_n) of points in $\hat{R}$ with $t_n\to\infty$ if and only if $\lim_{n\to\infty} a_n/t_n=\alpha$ exists with $0\leqq\alpha\leqq 1$. For such a sequence it follows from (5.6) that

$$\lim_{n\to\infty}\frac{P_{0,a_n}^{t_n-2t}}{P_{0,a_n}^{t_n}}=(\sqrt{1-\alpha^2})^{2t}\,.$$

Thus the sequences (a_n, t_n) with $\lim_{n\to\infty}\frac{a_n}{t_n}=\alpha$ and $\alpha<1$ converge to the boundary point determined by the minimal function $h(i,t)=Q_i(x)/x^t$ with $x=(1-\alpha^2)^{-1/2}$. For the sequences such that $\alpha=1$, we have

$$\lim_{n\to\infty}\frac{P_{0,a_n}^{t_n-2t}}{P_{0,a_n}^{t_n}}=h(0,2t)=0$$

for $t>0$; of course if $t=0$ the limit is 1. It is not hard to check that in this case, $h(j, t)$ is the function h_∞ defined in (3.6) (b). Incidently, by showing that all sequences for which (5.1) exists approach minimal boundary points we have established (for this example) that *the Martin boundary has no non-minimal part.*

It is interesting to observe that such a gross difference in the nature of the (space) random walk as that between recurrence $\left(a \leqq \frac{1}{2}\right)$ and transience $\left(a > \frac{1}{2}\right)$ has no effect upon the matter of which sequences converge to the boundary, since the latter is independent of a. In all cases, recurrent, or transient, the random variables x_n are ultimately of smaller magnitude than n, so that (x_n, n) a. s. approaches the boundary point corresponding to $\alpha = 0$ or $x = 1$. (We knew already that it must, since the constant function is minimal.) Of course the values of the minimal functions depend on a even though the form of the boundary does not.[2)]

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