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Projective modules over polynomial rings

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Introduction. In this paper a ring means a commutative ring with a unit element. An integral domain R is said to be a PF domain if any finite projective R-module is free.

J. P. Serre raised in [**6**] the following question:

(S) Is the polynomial ring $K[X_1, X_2, \cdots, X_n]$ with a finite number of variables $X_1, X_2, \cdots, X_n$ over a field K a PF domain?

It is well-known (cf. [**3**]) that any principal ideal domain is a PF domain. Therefore, when $n=1$, the answer to (S) is affirmative. Also, when $n=2$, C. S. Seshadri gave in [**9**] an affirmative answer to (S). He proved in [**9**], more generally, that the polynomial ring $R[X]$ with a variable X over a principal ideal domain R is a PF domain. This was further generalized by Seshadri [**10**], Serre [**8**] and Bass [**1**] to the following form: If $R'=R[X]$ is the polynomial ring with a variable X over a Dedekind domain R, then every finite projective R'-module P' is expressible as $P' \cong R' \otimes_R P$ for some finite projective R-module P.

For $n \geqq 3$ the question (S) is still open. D. Lissner gave in [**4**] some results suggesting a negative answer to (S) in case $n \geqq 3$. Recently H. Bass and S. Schanuel showed in [2] that, for the polynomial ring $R'=R[X_1, X_2, \cdots, X_n]$ with a finite number of variables $X_1, X_2, \cdots, X_n$ over a semi-local principal ideal domain R, any projective R'-module of rank $>n$ is free.

Our main results in this paper are Theorems 4.7 and 5.4. In Theorem 4.7 we give a necessary and sufficient condition for a Noetherian integral domain R of dimension 1, whose derived normal ring (cf. [**5**]) is a finite R-module, to the effect that, for the polynomial ring $R'=R[X]$ with a variable X over R, every finite projective R'-module P' is expressible as $P' \cong R' \otimes_R P$ for some finite projective R-module P. This is a generalization of all the above-mentioned result in [**1**], [8], [**9**] and [**10**]. In Theorem 5.4, we give a necessary and sufficient condition for a semi-local integral domain R of dimension 1 to the effect that the polynomial ring $R[X, Y]$ with two variables X, Y over R is a PF domain. This is related with the above-mentioned result in [**2**].

In both cases, the conditions are expressed in the form that the ring R should be "weakly normal". Thus the concept of weakly normal rings will play an important part throughout this paper. This concept will be introduced in § 1. Another fundamental tool of this paper is the Proposition 2.2 in § 2, which seems to be the most general result proved by the method used by Seshadri, Serre and Bass.

Our notations and terminologies are the same as those in [**5**] for the ring-theoretical facts with some exceptions. The "dimension" of a ring means the "altitude" of it in [**5**]. An ideal $\mathfrak{a}$ of a ring R is said to be "unmixed" in it if, for any prime divisor $\mathfrak{p}$ of $\mathfrak{a}$ in R, we have $\mathrm{height}_R\mathfrak{p}=\mathrm{height}_R\mathfrak{a}$. A "multiplicative system" of a ring means a multiplicatively closed subset of it which does not contain 0. Let R be a ring, S be a multiplicative system of R and M be an R-module. Then we denote by R_S and M_S the quotient ring and the quotient module of R and M with respect to S, respectively. Especially, if S is the complementary set of a prime ideal $\mathfrak{p}$ in R, we use $R_{\mathfrak{p}}$ and $M_{\mathfrak{p}}$ instead of R_S and M_S, respectively. A projective R-module P is said to be of "rank n" in R if, for any maximal ideal $\mathfrak{m}$ of R, $P_{\mathfrak{m}}$ has an $R_{\mathfrak{m}}$-free base consisting of n elements. A projective R-module is called "quasi-free" if it is expressible as a direct sum of R-modules, each of which is isomorphic to an ideal generated by an idempotent of R. Let $R'=R[X_1, X_2, \cdots, X_n]$ be the polynomial ring with a finite number of variables $X_1, X_2, \cdots, X_n$ over a ring R. Then a finite R'-module M' is said to be "extended" if there exists a finite R-module M such that $M'\cong R'\otimes_R M$.

§ 1. Preliminaries.

First we refer to some well-known facts, which will be freely used throughout this paper.

LEMMA 1.1 (cf. [**3**]). *Let R be an integral domain and M be a finite R-module. Then M is R-projective if and only if, for any maximal ideal $\mathfrak{m}$ of R, $M_{\mathfrak{m}}$ is $R_{\mathfrak{m}}$-free.*

LEMMA 1.2 (cf. [**7**]). *A semi-local integral domain is a* PF *domain.*

LEMMA 1.3 (cf. [**5**]). *Let R be an integral domain and $\bar{R}$ be an integral extension of R. Then, for any prime ideal $\mathfrak{p}$ of R, there is a prime ideal $\bar{\mathfrak{p}}$ of $\bar{R}$ such that $\mathfrak{p}=\bar{\mathfrak{p}}\cap R$. Furthermore, $\bar{\mathfrak{p}}$ has height* 1 *in $\bar{R}$ if $\mathfrak{p}$ has height* 1 *in R, and $\mathfrak{p}$ is maximal in R if and only if $\bar{\mathfrak{p}}$ is maximal in $\bar{R}$.*

LEMMA 1.4 (cf. [**5**]). *Let R be a Noetherian integral domain of dimension* 1 *and $\bar{R}$ be an almost finite integral extension of R. Then $\bar{R}$ is also a Noetherian integral domain of dimension* 1, *and, if $\bar{R}$ is a normal ring, it is a Dedekind domain. Especially, if R is semi-local, then $\bar{R}$ is also semi-local, and, in case $\bar{R}$ is normal, it is a principal ideal domain.*

LEMMA 1.5 (cf. [**5**]). *Let R be a Noetherian ring and $R'=R[X_1, X_2, \cdots, X_n]$*

be the polynomial ring with a finite number of variables $X_1, X_2, \cdots, X_n$ over R. Then there exists a maximal ideal $\mathfrak{m}'$ in R' such that $\mathfrak{m}' \cap R$ is not maximal in R if and only if there is a prime ideal $\mathfrak{p}$ in R such that $R/\mathfrak{p}$ is a semi-local integral domain of dimension 1. If $\mathfrak{m}' \cap R$ is not maximal in R, $R/\mathfrak{m}' \cap R$ becomes, in fact, a semi-local integral domain of dimension 1.

We now introduce a notion of weakly normal rings.

DEFINITION. Let R be a local integral domain of dimension 1 with a maximal ideal $\mathfrak{m}$. Let $\bar{R}$ be the derived normal ring of R and $\bar{\mathfrak{n}}$ be the Jacobson radical of $\bar{R}$. Then R is said to be a weak (discrete) valuation ring if we have $\mathfrak{m} = \bar{\mathfrak{n}}$ in the set-theoretical sense.

In general, a Noetherian integral domain R is said to be a weakly normal ring if R satisfies the following two conditions:

1) For any prime ideal $\mathfrak{p}$ of height 1 in R, $R_{\mathfrak{p}}$ is a weak valuation ring.

2) Any principal ideal $(\neq 0)$ in R is unmixed, i.e., any prime divisor of it has height 1 in R.

Now we prove

PROPOSITION 1.6. *A Noetherian normal ring is a weakly normal ring. However, a weakly normal ring is not always a normal ring.*

PROOF. The first part of our proposition is obvious by our definition (cf. [**5**]). Hence we have only to give an example of a weak valuation ring which is not a valuation ring. Let Q be the field of all rational numbers and $Q[[X, Y]]$ be the formal power-series ring with two variables X, Y over Q. If we put $\mathfrak{p} = (X^2 + Y^2)Q[[X, Y]]$, then $\mathfrak{p}$ is a prime ideal of $Q[[X, Y]]$. Furthermore put $R_0 = Q[[X, Y]]/\mathfrak{p}$ and denote by a, b the residues of X, Y in R_0, respectively. Then R_0 is a local integral domain of dimension 1 with a maximal ideal $\mathfrak{m}_0 = aR_0 + bR_0$. Let $\bar{R}$ be the derived normal ring of R_0. As R_0 is complete, $\bar{R}$ is a valuation ring with a maximal ideal $\bar{\mathfrak{m}}$ (cf. [**5**]). Let R be an integral domain generated by $\bar{\mathfrak{m}}$ over R_0. Then R is obviously a weak valuation ring. If we put $\alpha = b/a$, then we have $\alpha^2 + 1 = 0$, hence $\alpha, \alpha^{-1} \in \bar{R}$, and so $\alpha \notin \bar{\mathfrak{m}}$. If we suppose $\alpha \in R$, then we have $\alpha = u + \beta$ for a unit u of R_0 and an element β of $\bar{\mathfrak{m}}$. From $\alpha^2 + 1 = 0$, we obtain $\beta(\beta + 2u) + u^2 + 1 = 0$. As $\beta \in \bar{\mathfrak{m}}$, we have $u^2 + 1 \in \bar{\mathfrak{m}} \cap R_0 = \mathfrak{m}_0$. Let $f(X, Y)$ be a representative of u in $Q[[X, Y]]$. Then we have $(f(X, Y))^2 + 1 \in XQ[[X, Y]] + YQ[[X, Y]]$. If we denoted by q the constant term of $f(X, Y)$, then we obtain $q^2 + 1 = 0$. As $q \in Q$, this is obviously a contradiction. Thus $\alpha \notin R$. This shows $R \neq \bar{R}$. Consequently R is a weak valuation ring which is not a valuation ring.

Here we show some basic properties of weakly normal rings.

LEMMA 1.7. *Let R be a weakly normal ring and S be a multiplicative system of R. Then R_S is also a weakly normal ring.*

Lemma 1.8. *Let R be a Noetherian integral domain of dimension* 1. *Then R is a weakly normal ring if and only if, for any prime ideal $\mathfrak{p}$ of R, $R_{\mathfrak{p}}$ is a weak valuation ring. Especially, if R is semi-local, and $\mathfrak{n}$ is the Jacobson radical of R, then R is a weakly normal ring if and only if we have $\mathfrak{n}=\bar{\mathfrak{n}}$ for the Jacobson radical $\bar{\mathfrak{n}}$ of the derived normal ring of R.*

These two results follow immediately from our definition.

Lemma 1.9. *Let R be a weakly normal ring. Then the polynomial ring $R'=R[X_1, X_2, \cdots, X_n]$ with a finite number of variables $X_1, X_2, \cdots, X_n$ over R is also a weakly normal ring.*

Proof. Let us show our lemma for $n=1$, i.e., $R'=R[X]$ with a variable X. The general case follows easily by induction on n. First we prove that, for any prime ideal $\mathfrak{p}'$ of height 1 in R', $R'_{\mathfrak{p}'}$ is a weak valuation ring. If $\mathfrak{p}'\cap R=0$, then we have $R'_{\mathfrak{p}'}=(K[X])_{\mathfrak{p}'K[X]}$ where K is the quotient field of R, and so $R'_{\mathfrak{p}'}$ is a discrete valuation ring. On the other hand, if $\mathfrak{p}'\cap R=\mathfrak{p}\neq 0$, then we have $\mathfrak{p}'=\mathfrak{p}R'$, hence $R'_{\mathfrak{p}'}=(R_{\mathfrak{p}}[X])_{\mathfrak{p}R_{\mathfrak{p}}[X]}$. Therefore we may suppose that R is a weak valuation ring and $\mathfrak{p}$ is a maximal ideal of it. Let $\bar{R}$ be the derived normal ring of R and $\bar{\mathfrak{n}}$ be the Jacobson radical of $\bar{R}$. Then we have $\mathfrak{p}=\bar{\mathfrak{n}}$. Since $\bar{R}[X]$ is the derived normal ring of $R[X]$, putting $S'=R[X]-\mathfrak{p}R[X]$, $(\bar{R}[X])_{S'}$ is also the derived normal ring of $(R[X])_{\mathfrak{p}R[X]}$. From $\mathfrak{p}=\bar{\mathfrak{n}}$ we see easily $\mathfrak{p}(R[X])_{\mathfrak{p}R[X]}=\bar{\mathfrak{n}}(\bar{R}[X])_{S'}$. As $\bar{\mathfrak{n}}(\bar{R}[X])_{S'}$ coincides with the Jacobson radical of $(\bar{R}[X])_{S'}$, this shows that $(R[X])_{\mathfrak{p}R[X]}$ is a weak valuation ring. Thus, in every case, $R'_{\mathfrak{p}'}$ is a weak valuation ring.

Let $\mathfrak{a}'$ be a non-zero principal ideal of R' and $\mathfrak{p}'$ be a prime divisor of it. Suppose $\mathrm{height}_{R'}\mathfrak{p}'\geqq 2$. Then we have $\mathfrak{p}'\cap R\neq 0$. Hence there exists a non-zero element a of R in $\mathfrak{p}'$. Then $\mathfrak{p}'$ is a prime divisor of aR' (cf. [**5**], 12.6). However, as R is weakly normal, aR is unmixed in R, and so aR' is unmixed in R'. This is a contradiction. Thus $\mathfrak{p}'$ must be of height 1 in R', i.e., $\mathfrak{a}'$ must be unmixed in R'. This completes our proof.

§ 2. Fundamental proposition.

We first prove a proposition, which gives in the most general form a reasoning which played essential parts in the proofs of the Seshadri-Serre-Bass's results.

Lemma 2.1. *Let R be an integral domain and $\mathfrak{p}$ be a prime ideal of R such that $R/\mathfrak{p}$ is Euclidean. Let P be a finite projective R-module and L be a finite projective R-submodule of P such that $L=L_1\oplus L_2\oplus\cdots\oplus L_n$ with each L_i a finite projective R-module of rank* 1. *Then there exists an automorphism φ of L and an integer r, $0\leqq r\leqq n$, such that $\varphi(L_1\oplus L_2\oplus\cdots\oplus L_r)\subset\mathfrak{p}P$ and $\varphi(L_{r+1}\oplus L_{r+2}\oplus\cdots\oplus L_n)\cap\mathfrak{p}P=\mathfrak{p}\cdot\varphi(L_{r+1}\oplus L_{r+2}\oplus\cdots\oplus L_n)$.*

The proof is omitted, because it runs in the same line as in [**1**], 2.2.

PROPOSITION 2.2. *Let R be a Noetherian integral domain and S be a multiplicative system of R consisting of elements $\{a_i\}$ such that a_iR is expressible as the product of invertible prime ideals $\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_s$ in R, for any $\mathfrak{p}_i$ of which $R/\mathfrak{p}_i$ is Euclidean. Let P be a finite projective R-module. If P_S is R_S-free, then P is expressible as a direct sum of projective R-modules of rank* 1. *Especially, if S is a multiplicative system of R generated by prime elements $\{p_i\}$ such that each R/p_iR is Euclidean, and if P_S is R_S-free, then P is R-free.*

PROOF (cf. [**1**], [**8**], [**9**] or [**10**]). As P_S is R_S-free, we can choose a free base $u_1, u_2, \cdots, u_t$ for P_S in P. If we put $L = u_1R + u_2R + \cdots + u_tR$, then we have $L \subset P$ and $sP \subset L$ for some $s \in S$. By our assumption on S we have $sR = \mathfrak{p}_1^{r_1}\mathfrak{p}_2^{r_2} \cdots \mathfrak{p}_s^{r_s}$ for invertible prime ideals $\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_s$ in R such that all $R/\mathfrak{p}_i$'s are Euclidean. Hence we established the existence of an R-submodule L of P satisfying the following conditions:

1) $L = L_1 \oplus L_2 \oplus \cdots \oplus L_n$ with each L_i a projective R-module of rank 1.

2) $\mathfrak{p}_1^{l_1}\mathfrak{p}_2^{l_2} \cdots \mathfrak{p}_k^{l_k} P \subset L$ with each $\mathfrak{p}_i$ an invertible prime ideal of R such that $R/\mathfrak{p}_i$ is Euclidean.

Since R is Noetherian, we may choose L maximal satisfying 1). Now we shall prove $P = L$. Suppose $L \subsetneqq P$. If we put $\mathfrak{p} = \mathfrak{p}_1$ and $\mathfrak{a} = \mathfrak{p}_1^{l_1 - 1}\mathfrak{p}_2^{l_2} \cdots \mathfrak{p}_k^{l_k} (l_1 \geqq 1)$, we may assume $\mathfrak{p}\mathfrak{a}P \subset L$ but $\mathfrak{a}P \not\subset L$. If $\mathfrak{p}\mathfrak{a}P \subset \mathfrak{p}L$, then $\mathfrak{a}P \subset \mathfrak{p}^{-1}\mathfrak{p}L = L$, contradiction. Therefore $\mathfrak{p}\mathfrak{a}P \not\subset \mathfrak{p}L$, however $\mathfrak{p}\mathfrak{a}P \subset \mathfrak{p}P \cap L$. This shows that $\mathfrak{p}L \subsetneqq \mathfrak{p}P \cap L$. By (2.1) we have an automorphism φ of L for which $\varphi(L_1 \oplus L_2 \oplus \cdots \oplus L_r) \subset \mathfrak{p}P$ and $\mathfrak{p}P \cap \varphi(L_{r+1} \oplus L_{r+2} \oplus \cdots \oplus L_n) = \mathfrak{p} \cdot \varphi(L_{r+1} \oplus L_{r+2} \oplus \cdots \oplus L_n)$. As $\mathfrak{p}P \cap L \neq \mathfrak{p}L$, we have $r > 0$. Hence, if we put $H = \mathfrak{p}^{-1} \cdot \varphi(L_1 \oplus L_2 \oplus \cdots \oplus L_r) \oplus \varphi(L_{r+1} \oplus L_{r+2} \oplus \cdots \oplus L_n)$, then H contradicts the maximality of L. Thus we obtain $P = L$. This completes the proof of the first part of our proposition. The second part is obvious from the above proof of the first part.

COROLLARY 2.3. *Let R be a Noetherian integral domain and S be a multiplicative system of R consisting of elements $\{a_i\}$ such that each a_iR is expressible as the product of maximal invertible ideals in R. Let $R' = R[X]$ be the polynomial ring with a variable X over R and P' be a finite projective R'-module. If P'_S is R'_S-free, then P' is expressible as a direct sum of projective R'-modules of rank* 1. *Especially, if S is a multiplicative system of R generated by prime elements $\{p_i\}$ such that all p_iR's are maximal in R and if P'_S is R'_S-free, then P' is also R'-free.*

LEMMA 2.4. *Let R be a ring and P be a finite quasi-free R-module. If P has its rank, then P is a free R-module.*

PROOF. Put $n = \operatorname{rank}_R P$. When $n = 0$, this is obvious. Hence we may assume $n \geqq 1$. We put $P = e_1R \oplus e_2R \oplus \cdots \oplus e_tR$ for idempotents, $e_1, e_2, \cdots, e_t$ $(t \geqq n)$ in R. If we put $e'_2 = (1-e_1)e_2, e'_3 = (1-e_1)(1-e_2)e_3, \cdots, e'_t = (1-e_1)(1-e_2) \cdots (1-e_{t-1})e_t$ and $e''_2 = e_2 - e'_2, \cdots, e''_t = e_t - e'_t$, then we have $P \cong e_1R \oplus e'_2R \oplus \cdots$

$\oplus e_t' R \oplus e_2'' R \oplus e_3'' R \oplus \cdots \oplus e_t'' R$. We see easily that $R \cong e_1 R \oplus e_2' R \oplus \cdots \oplus e_t' R$. Putting $P' = e_2'' R \oplus e_3'' R \oplus \cdots \oplus e_t'' R$, then P' is a quasi-free R-module of rank $n-1$ and we have $P \cong R \oplus P'$. By repeating this procedure to P', we conclude that P is a free R-module of rank n.

Proposition 2.5. *Let R be an integral domain, $\mathfrak{n}$ be an ideal contained in the Jacobson radical of R and P be a finite projective R-module. If $P/\mathfrak{n}P$ is a quasi-free $R/\mathfrak{n}$-module, then P is R-free.*

Proof. As R is an integral domain, P has its rank. If we put $n = \mathrm{rank}_R P$, $P_{\mathfrak{m}}$ is a free $R_{\mathfrak{m}}$-module of rank n, for any maximal ideal $\mathfrak{m}$ of R. Since $\mathfrak{n} \subset \mathfrak{m}$, $(P/\mathfrak{n}P)_{\mathfrak{m}/\mathfrak{n}}$ is also a free $(R/\mathfrak{n})_{\mathfrak{m}/\mathfrak{n}}$-module of rank n. Therefore we have $\mathrm{rank}_{R/\mathfrak{n}} P/\mathfrak{n}P = n$. As $P/\mathfrak{n}P$ is $R/\mathfrak{n}$-quasi-free, $P/\mathfrak{n}P$ is $R/\mathfrak{n}$-free by (2.4). Let $\{\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_n\}$ be a free base of $P/\mathfrak{n}P$ over $R/\mathfrak{n}R$, and $u_1, u_2, \cdots, u_n$ be the representatives of $\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_n$ in P, respectively. If we put $M = u_1 R + u_2 R + \cdots + u_n R$, we have $P = M + \mathfrak{n}P$. Then, by the Krull-Azumaya's lemma, we obtain $P = M$, i.e., P has a base $\{u_1, u_2, \cdots, u_n\}$ over R. As $P_{\mathfrak{m}}$ is of rank n, $\{u_1, u_2, \cdots, u_n\}$ is a free base of $P_{\mathfrak{m}}$ over $R_{\mathfrak{m}}$ for any maximal ideal $\mathfrak{m}$ of R. From this it follows that $\{u_1, u_2, \cdots, u_n\}$ is a free base of P over R. Thus P is R-free.

§ 3. Polynomial rings over semi-local integral domains of dimension 1, I.

Here we give

Proposition 3.1. *Let R be a semi-local integral domain of dimension* 1 *with the Jacobson radical $\mathfrak{n}$ and $R' = R[X_1, X_2, \cdots, X_n]$ be the polynomial ring with a finite number of variables $X_1, X_2, \cdots, X_n$ over R. Then the following conditions are equivalent:*

1) *R is a weakly normal ring.*

2) *Any prime ideal $\mathfrak{p}'$ of height* 1 *in R' such that $\mathfrak{p}' + \mathfrak{n}R' = R'$ is principal in R'.*

Proof. Let $\bar{R}$ be the derived normal ring of R and $\bar{\mathfrak{n}}$ be the Jacobson radical of $\bar{R}$. Denote by K the quotient field of R and put $\bar{R}' = \bar{R}[X_1, X_2, \cdots, X_n]$.

The implication 1)$\rightarrow$2): Suppose that R is weakly normal. Then, by (1.8), we have $\mathfrak{n} = \bar{\mathfrak{n}}$, hence $\mathfrak{n}R' = \bar{\mathfrak{n}}\bar{R}'$. Let $\mathfrak{p}'$ be a prime ideal of height 1 in R' such that $\mathfrak{p}' + \mathfrak{n}R' = R'$. As $\bar{R}'$ is integral over R', there exists a prime ideal $\bar{\mathfrak{p}}'$ of height 1 in $\bar{R}'$ such that $\mathfrak{p}' = \bar{\mathfrak{p}}' \cap R'$ by (1.3). Since $\bar{R}$ is a semi-local principal ideal domain by (1.4), $\bar{R}'$ is a unique factorization domain, as is well-known, hence $\bar{\mathfrak{p}}'$ is principal in $\bar{R}'$ (cf. [**5**], 13.1). As $\bar{\mathfrak{p}}' + \bar{\mathfrak{n}}\bar{R}' = \bar{R}'$, we have $\bar{\mathfrak{p}}' = (f+1)\bar{R}'$ for some $f \in \bar{\mathfrak{n}}\bar{R}' = \mathfrak{n}R'$. From $\mathfrak{p}' = \bar{\mathfrak{p}}' \cap R'$, we obtain $f+1 \in \mathfrak{p}'$. Hence we have also $\mathfrak{p}' K[X_1, X_2, \cdots, X_n] = (f+1)K[X_1, X_2, \cdots, X_n] = \bar{\mathfrak{p}}' K[X_1, X_2, \cdots, X_n]$. If $\mathfrak{m}'$ is a maximal ideal of R' containing $f+1$, then we have $\mathfrak{m}' \cap R = 0$

and so we have $R'_{\mathfrak{m}'}=(K[X_1, X_2, \cdots, X_n])_{\mathfrak{m}'K[X_1,\cdots,X_n]}$. Therefore we have $\mathfrak{p}'R'_{\mathfrak{m}'}=(f+1)R'_{\mathfrak{m}'}$. Thus we must have $\mathfrak{p}'=(f+1)R'$. This proves that $\mathfrak{p}'$ is principal in R'.

The implication 2)→1): Suppose that R is not weakly normal. Then, again by (1.8), we have a) $\mathfrak{n}\bar{R}\neq\mathfrak{n}$ or b) $\mathfrak{n}\bar{R}=\mathfrak{n}\neq\bar{\mathfrak{n}}$. In order to prove 2)→1) it suffices to show that there exists a non-principal prime ideal $\mathfrak{p}'$ of height 1 in R' such that $\mathfrak{p}'+\mathfrak{n}R'=R'$, in case a), b).

Case a): When $\mathfrak{n}\bar{R}\neq\mathfrak{n}$, we see easily $\mathfrak{n}\bar{R}\not\subset R$. Therefore we find an element b/a of $\bar{R}$, $a, b\in R$ such that $bc/a\notin R$ for a suitable element c of $\mathfrak{n}$. As $b/a\notin R$ but $b/a\in\bar{R}$, b/a is the root of an equation:

$$T^k+r_1T^{k-1}+\cdots+r_{k-1}T+r_k=0,\quad r_1, r_2, \cdots, r_k\in R,\quad k\geqq 2\,.$$

Hence bc/a is also the root of an eqation:

$$T^k+cr_1T^{k-1}+\cdots+c^{k-1}r_{k-1}T+c^kr_k=0\,.$$

If we put $d_i=c^ir_i$, $1\leqq i\leqq k$, then all d_i's are contained in $\mathfrak{n}$. Hence a/bc is the root of the following equation:

$$d_kT^k+d_{k-1}T^{k-1}+\cdots d_1T+1=0\,.$$

On the other hand, as is easily seen, a/bc is also the root of an equation $bcT-a=0$. Put $\mathfrak{p}'=(bcX_1-a)R'+(d_kX_1^k+d_{k-1}X_1^{k-1}+\cdots+d_1X_1+1)R'$. We shall now prove that $\mathfrak{p}'$ is as required. Let $\mathfrak{m}'$ be a maximal ideal of R'. If $\mathfrak{m}'\supset\mathfrak{p}'$, we have $\mathfrak{m}'\cap R=0$, as $d_1, d_2, \cdots, d_k\in\mathfrak{n}$, hence we obtain $R'_{\mathfrak{m}'}=(K[X_1, X_2, \cdots, X_n])_{\mathfrak{m}'K[X_1,\cdots,X_n]}$. Therefore we have $\mathfrak{p}'R'_{\mathfrak{m}'}=(X_1-a/bc)R'_{\mathfrak{m}'}$. If $\mathfrak{m}'\not\supset\mathfrak{p}'$, we have $\mathfrak{p}'R'_{\mathfrak{m}'}=R'_{\mathfrak{m}'}$. This shows that $\mathfrak{p}'$ is an invertible prime ideal of height 1 in R'. However, as $bc\notin aR$, $\mathfrak{p}'$ is not principal in R', and, as $d_1, d_2, \cdots, d_k\in\mathfrak{n}$, we have $\mathfrak{p}'+\mathfrak{n}R'=R'$. This shows that $\mathfrak{p}'$ is as required.

Case b): As $\mathfrak{n}\bar{R}=\mathfrak{n}\neq\bar{\mathfrak{n}}$, we find a positive integer $k\geqq 2$ such that $\bar{\mathfrak{n}}^k\subset\mathfrak{n}$. Then there exists an element b/a of $\bar{\mathfrak{n}}$ such that $b/a\notin\mathfrak{n}$ but $(b/a)^k=c\in\mathfrak{n}$. Hence a/b is the root of equations $cT^k-1=0$ and $bT-a=0$. If we put $\mathfrak{p}'=(bX_1-a)R'+(cX_1^k-1)R'$, we can prove, as in case a), that $\mathfrak{p}'$ is as required. This completes our proof.

COROLLARY 3.2. *Let R be a semi-local integral domain of dimension* 1 *and $R'=R[X_1, X_2, \cdots, X_n]$ be the polynomial ring with a finite number of variables $X_1, X_2, \cdots, X_n$ over R. If R is not weakly normal, then there exists an invertible ideal in R' which is not principal.*

PROOF. It is obvious from the proof of 2)→1) in (3.1).

PROPOSITION 3.3. *Let R be a semi-local integral domain of dimension* 1 *and let $R'=R[X]$ be the polynomial ring with a variable X over R. Then the following conditions are equivalent:*

1) *R is a weakly normal ring.*

2) R' *is a* PF *domain.*

PROOF. The implication 2)→1) was already proved by (3.2). Hence we have only to show the implication 1)→2). Suppose that R is weakly normal. Let $\mathfrak{n}$ be the Jacobson radical of R, and $\mathfrak{m}'$ be a maximal ideal of height 1 in R'. Then we have $\mathfrak{m}'+\mathfrak{n}R'=R'$ by (1.5). Therefore, by (3.1), $\mathfrak{m}'$ is principal in R'. Let S' be the multiplicative system of R' generated by all prime generators of maximal ideals of height 1 in R'. Then, for any maximal ideal $\mathfrak{m}'$ of height 2 in R', we have $\mathfrak{m}'\cap S'=\phi$. Accordingly, if we put $R''=R'_{S'}$, $\mathfrak{m}'R''$ is also maximal in R''. Conversely, for any maximal ideal $\mathfrak{m}''$ of R'', $\mathfrak{m}''\cap R'$ is a maximal ideal of height 2 in R'. As any maximal ideal of height 2 in R' contains $\mathfrak{n}R'$, any maximal ideal of R'' contains $\mathfrak{n}R''$ and so $\mathfrak{n}R''$ is contained in the Jacobson radical of R''. Now we have $R''/\mathfrak{n}R'' \cong (R/\mathfrak{n})[X] \cong (R/\mathfrak{m}_1)[X] \oplus \cdots \oplus (R/\mathfrak{m}_t)[X]$, where $\mathfrak{m}_1, \mathfrak{m}_2, \cdots, \mathfrak{m}_t$ are all maximal ideals of R. Since any $(R/\mathfrak{m}_i)[X]$ is a PF domain, any projective $R''/\mathfrak{n}R''$-module is quasi-free. Hence, by (2.5), R'' is a PF domain. By applying (2.2) to S', we see that R' is also a PF domain.

§4. Polynomial rings over Noetherian integral domains of dimension 1.

First we give two lemmas.

LEMMA 4.1. *Let R be an integral domain and $R'=R[X]$ be the polynomial ring with a variable X over R. Then any finite projective R'-module of rank* 1 *is isomorphic to an invertible ideal of R' containing a non-zero element of R.*

PROOF. Let P' be a finite projective R'-module of rank 1. If we denote by K the quotient field of R, we have $P'\subset K\otimes_R P'$. As $K\otimes_R P'$ is a free $K[X]$-module of rank 1, we may choose a generator α of $K\otimes_R P'$ such that $P'\subset \alpha R'$. If we consider $\alpha R'$ as R', we can also consider P' as an invertible ideal $\mathfrak{a}'$ of R'. Since, for some $a\neq 0$ in R, we have $a\alpha\in P'$, we have clearly $\mathfrak{a}'\cap R\neq 0$.

LEMMA 4.2. *Let R be a* PF *domain and $R'=R[X]$ be the polynomial ring with a variable X over R. Let P' be a finite projective R'-module of rank* 1. *If P' is extended, then P' is isomorphic to R'.*

PROOF. As this is easy, we omit it (cf. [3], X, Ex. 1).

Now we prove

THEOREM 4.3. *Let R be a Noetherian integral domain such that any principal ideal in it is unmixed, and $R'=R[X_1, X_2, \cdots, X_n]$ be the polynomial ring with a finite number of variables $X_1, X_2, \cdots, X_n$ over R. Then the following conditions are equivalent:*

1) *R is a weakly normal ring.*

2) *Any finite projective R'-module of rank* 1 *is extended.*

PROOF. It suffices to prove our theorem in case $n=1$, i.e., $R'=R[X]$ with a variable X, by the last part of the proof of (1.9) and by (1.9).

The implication 1)$\rightarrow$2): Suppose that R is weakly normal. According to (4.1), it suffices to show that, for an invertible ideal $\mathfrak{a}'$ of R' such that $\mathfrak{a}'\cap R=\mathfrak{a}\neq 0$, we have $\mathfrak{a}'=\mathfrak{a}R'$. As, for any maximal ideal $\mathfrak{m}'$ of R', $\mathfrak{a}'R'_{\mathfrak{m}'}$ is principal in $R'_{\mathfrak{m}'}$, $\mathfrak{a}'R'_{\mathfrak{m}'}$ is unmixed in $R'_{\mathfrak{m}'}$ by (1.7) and (1.9). So $\mathfrak{a}'$ must be unmixed in R'. Let $\mathfrak{p}'_1, \mathfrak{p}'_2, \cdots, \mathfrak{p}'_t$ be all prime divisors of $\mathfrak{a}'$ in R'. As all $\mathfrak{p}'_i$'s contain $\mathfrak{a}$, we have $\mathfrak{p}'_i\cap R=\mathfrak{p}_i\neq 0$ for any i. As $\text{height}_{R'}\mathfrak{p}'_i=1$ we have $\mathfrak{p}'_i=\mathfrak{p}_iR'$ for any i. Now let $\mathfrak{a}'=\mathfrak{q}'_1\cap\mathfrak{q}'_2\cap\cdots\cap\mathfrak{q}'_t$ be a primary decomposition of $\mathfrak{a}'$ in R' where each $\mathfrak{q}'_i$ is a primary ideal belonging to $\mathfrak{p}_iR'$. Then we have also $\mathfrak{q}'_i\cap R=\mathfrak{q}_i\neq 0$, for any i. Now we shall prove $\mathfrak{q}_iR'=\mathfrak{q}'_i$ for any i. Since $R_{\mathfrak{p}_i}$ is a weak valuation ring, $R_{\mathfrak{p}_i}[X]$ is a PF domain by (3.3). As we have $\mathfrak{a}'R_{\mathfrak{p}_i}[X]=\mathfrak{q}'_iR_{\mathfrak{p}_i}[X]$, and $\mathfrak{a}'R_{\mathfrak{p}_i}[X]$ is invertible in $R_{\mathfrak{p}_i}[X]$, $\mathfrak{q}'_iR_{\mathfrak{p}_i}[X]$ is principal in $R_{\mathfrak{p}_i}[X]$. From $\mathfrak{q}_i\neq 0$ $\mathfrak{q}'_iR_{\mathfrak{p}_i}[X]$ has a generator q_i in $R_{\mathfrak{p}_i}$. If we put $\mathfrak{q}^*_i=q_iR_{\mathfrak{p}_i}$, we have $\mathfrak{q}^*_iR_{\mathfrak{p}_i}[X]=\mathfrak{q}'_iR_{\mathfrak{p}_i}[X]$ and $\mathfrak{q}'_iR_{\mathfrak{p}_i}[X]\cap R_{\mathfrak{p}_i}=\mathfrak{q}^*_i$. Therefore $\mathfrak{q}^*_i$ is a primary ideal belonging to $\mathfrak{p}_iR_{\mathfrak{p}_i}$. Since $\mathfrak{q}_i=\mathfrak{q}^*_i\cap R$, we have $\mathfrak{q}_iR_{\mathfrak{p}_i}=\mathfrak{q}^*_i$, and so $\mathfrak{q}_i$ is also a primary ideal belonging to $\mathfrak{p}_i$. As $\mathfrak{q}_iR_{\mathfrak{p}_i}[X]=\mathfrak{q}'_iR_{\mathfrak{p}_i}[X]$, we have $\mathfrak{q}_iR'=\mathfrak{q}'_i$. From this we easily see $\mathfrak{a}=\mathfrak{q}_1\cap\mathfrak{q}_2\cap\cdots\cap\mathfrak{q}_t$ and $\mathfrak{a}'=\mathfrak{a}R'$. This proves 1)$\rightarrow$2).

The implication 2)$\rightarrow$1): Suppose that R is not weakly normal. Then, for some prime ideal $\mathfrak{p}$ of height 1 in R, $R_{\mathfrak{p}}$ is not a weak valuation ring. Let $\bar{R}$ be the derived normal ring of R, and put $S=R-\mathfrak{p}$. Then $\bar{R}_S$ is the derived normal ring of $R_{\mathfrak{p}}$. As $R_{\mathfrak{p}}$ is not a weak valuation ring, $\mathfrak{p}R_{\mathfrak{p}}$ does not coincide with the Jacobson radical $\bar{\mathfrak{n}}$ of $\bar{R}_S$ in the set-theoretical sense. Hence we have a) $\mathfrak{p}\bar{R}_S\neq\mathfrak{p}R_{\mathfrak{p}}$ or b) $\mathfrak{p}\bar{R}_S=\mathfrak{p}R_{\mathfrak{p}}\neq\bar{\mathfrak{n}}$. It suffices to show that there is an invertible ideal in R' which is not extended, in each case a), b).

Case a): There exists an element β of $\bar{R}_S$ such that $c\beta\notin R_{\mathfrak{p}}$ for some $c\in\mathfrak{p}R_{\mathfrak{p}}$. By multiplying β, c by a suitable element of S, we may assume $\beta\in\bar{R}$ such that $c\beta\notin R_{\mathfrak{p}}$ for some $c\in\mathfrak{p}$. Put $\beta=b/a$, $a, b\in R$. Then there exist only a finite number of prime divisors $\mathfrak{p}_1(=\mathfrak{p}), \mathfrak{p}_2, \cdots, \mathfrak{p}_t$ of aR, each of which is of height 1 in R. If we put $U=\bigcap_{i=1}^{t}(R-\mathfrak{p}_i)$, R_U is a semi-local integral domain of dimension 1 with maximal ideals $\mathfrak{p}_1R_U, \mathfrak{p}_2R_U, \cdots, \mathfrak{p}_tR_U$, which is not weakly normal. Again, by multiplying c by a suitable element of $\bigcap_{i=2}^{t}\mathfrak{p}_i$ which is not contained in $\mathfrak{p}$, we may assume that β is an element of $\bar{R}$ such that $c\beta\notin R_U$ for some $c\in\bigcap_{i=1}^{t}\mathfrak{p}_i$. If we put $\alpha=c\beta$, then, as in (3.3), α^{-1} is a root of an equation $cf(T)+1=0$, $f(T)\in R[T]$. Furthermore put $\mathfrak{p}'=(bcX-a)R'+(cf(X)+1)R'$. Let $\mathfrak{m}'$ be a maximal ideal of R'. If we put $\mathfrak{q}=\mathfrak{m}'\cap R$,

$\mathfrak{q}$ is a prime ideal of R. If $c \in \mathfrak{q}$, then $cf(X)+1 \notin \mathfrak{m}'$, hence we have $\mathfrak{p}'R'_{\mathfrak{m}'}=R'_{\mathfrak{m}'}$. If $c \notin \mathfrak{q}$, then $\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_t \not\subset \mathfrak{q}$, and so $a \notin \mathfrak{q}$, i. e., $a \notin \mathfrak{m}'$. Therefore we have $\mathfrak{p}'R'_{\mathfrak{m}'}=(\alpha X-1)R'_{\mathfrak{m}'}$. This shows that $\mathfrak{p}'$ is an invertible prime ideal of R'. However, $\mathfrak{p}'R_U[X]$ is not principal in $R_U[X]$ as in the proof of Case a) in (3.3). By (4.2), then, $\mathfrak{p}'R_U[X]$ is not extended in $R_U[X]$. Consequently $\mathfrak{p}'$ is not extended in R'. Thus $\mathfrak{p}'$ is as required.

Case b): There exists an element α of $\bar{R}_S$ such that $\alpha \notin R_{\mathfrak{p}}$ but $\alpha^k \in \mathfrak{p}R_{\mathfrak{p}}$ for some $k \geqq 2$. As in a) we may assume $\alpha \in \bar{R}$ such that $\alpha \notin R_{\mathfrak{p}}$ but $\alpha^k \in \mathfrak{p}R_{\mathfrak{p}}$. Put $\alpha=b/a$, $a, b \in R$. There exist a finite number of prime divisors $\mathfrak{p}_1(=\mathfrak{p}), \mathfrak{p}_2, \cdots, \mathfrak{p}_t$ of aR. If we put $U=\bigcap_{i=1}^{t}(R-\mathfrak{p}_i)$, then we may assume also that α is an element of $\bar{R}$ such that $\alpha \notin R_U$ and $\alpha^k \in \bigcap_{i=1}^{t}\mathfrak{p}_i$. If we put $c=\alpha^k$ and $\mathfrak{p}'=(bX-a)R'+(cX^k-1)R'$, then, as in a), we can show that $\mathfrak{p}'$ is as required. This completes our proof.

COROLLARY 4.4. *Let R be a Noetherian integral domain of dimension* 1 *and $R'=R[X_1, X_2, \cdots, X_n]$ be the polynomial ring with a finite number of variables $X_1, X_2, \cdots, X_n$ over R. Then the following conditions are equivalent:*

1) *R is a weakly normal ring.*
2) *Any finite projective R-module of rank* 1 *is extended.*

COROLLARY 4.5 (cf. [**1**]). *Let R be a Noetherian normal ring and $R'=R[X_1, X_2, \cdots, X_n]$ be the polynomial ring with a finite number of variables $X_1, X_2, \cdots, X_n$ over R. Then any finite projective R'-module of rank* 1 *is extended.*

LEMMA 4.6. *Let R be a Noetherian integral domain, whose derived normal ring, $\bar{R}$, is a finite R-module. Then there exist only a finite number of prime ideals $\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_t$ of height* 1 *in R such that $R_{\mathfrak{p}_i}$'s are not discrete valuation rings.*

PROOF. Let $\mathfrak{a}$ be the conductor of $\bar{R}$ to R, i. e., $\mathfrak{a}=\{a\,; a\bar{R} \subset R, a \in R\}$. By our assumption $\mathfrak{a}$ is a non-zero ideal of R. There exist only a finite number of prime ideals $\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_t$ of height 1 in R containing $\mathfrak{a}$. If $\mathfrak{p}$ is a prime ideal of height 1 in R which does not coincide with any of $\mathfrak{p}_i$'s, we have $\bar{R} \subset R_{\mathfrak{p}}$, as $\mathfrak{a}\bar{R} \subset R$. Putting $S=R-\mathfrak{p}$, we have $R_{\mathfrak{p}}=\bar{R}_S$. This shows that $R_{\mathfrak{p}}$ is a discrete valuation ring.

Our main result is given in the following:

THEOREM 4.7. *Let R be a Noetherian integral domain of dimension* 1, *whose derived normal ring, $\bar{R}$, is a finite R-module and let $R'=R[X]$ be the polynomial ring with a variable X over R. Then the following conditions are equivalent:*

1) *R is a weakly normal ring.*
2) *Any finite projective R'-module is extended.*

PROOF. The implication 2)→1) was proved in (4.3). Hence we have only to show the implication 1)→2). As R is one-dimensional, any prime ideal $\mathfrak{p}$ $(\neq 0)$ of R is of height 1 in R. According to (4.6), there exists only a finite number of prime ideals $\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_t$ in R such tnat $R_{\mathfrak{p}_i}$'s are not discrete valuation rings. Let $\mathfrak{p}$ be a prime ideal of R different from $\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_t$. Then $R_{\mathfrak{p}}$ is a discrete valuation ring and so $\mathfrak{p}$ is invertible in R. Now put $S=\bigcap_{i=1}^{t}(R-\mathfrak{p}_i)$. Then, for any element s of S, we have $s \notin \mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_t$, hence sR is expressible as the product of invertible prime ideals of R. Therefore S satisfies the condition in (2.3). Suppose that R is a weakly normal ring. Then R_S is also weakly normal. Hence, by (3.3), $R'_S=R_S[X]$ is a PF domain. So, by (2.3), any finite projective R'-module is expressible as a direct sum of projective R'-modules of rank 1. On the other hand, by (4.4), any finite projective R'-module of rank 1 is extended. Thus our proof is completed.

COROLLARY 4.8. *Let R be a Noetherian* PF *integral domain of dimension 1, whose derived normal ring, $\bar{R}$, is a finite R-module, and let $R'=R[X]$ be the polynomial ring with a variable X over R. Then the following conditions are equivalent:*

1) *R is a weakly normal ring.*
2) *R' is a* PF *domain.*

§ 5. Polynomial rings over semi-local integral domains of dimension 1, II.

We begin with

LEMMA 5.1. *Let R be a unique factorization domain and $R'=R[X_1, X_2, \cdots, X_n]$ be the polynomial ring with a finite number of variables $X_1, X_2, \cdots, X_n$ over R. Let $\mathfrak{a}'$ be an ideal of R' generated by $a_1X_1+b_1, a_2X_2+b_2, \cdots, a_nX_n+b_n$ such that, for any i, a_i, b_i are relatively prime non-zero elements of R. Then there exist relatively prime non-zero elements a_i, b_i of R and positive integers $l_2, \cdots, l_n$ such that $a(X_1+X_2^{l_2}+\cdots+X_n^{l_n})+b \in \mathfrak{a}'$ and $a \in \bigcap_{i=1}^{n} a_iR$.*

PROOF. This is obvious in case $n=1$. First we suppose $n=2$. If a_1, a_2 are relatively prime, a_1a_2 and $a_2b_1+a_1b_2$ are also relatively prime. We have now $a_1(a_2X_2+b_2)+a_2(a_1X_1+b_1)=a_1a_2(X_1+X_2)+a_2b_1+a_1b_2 \in \mathfrak{a}'$. Hence, if we put $a=a_1a_2$, $b=a_1b_2+a_2b_1$ and $l_2=1$, then our conclusion holds. If a_1, a_2 are not relatively prime, then there exist common prime divisors $p_1, p_2, \cdots, p_t$ of a_1, a_2 and we may write $a_1=p_1^{r_1}p_2^{r_2}\cdots p_t^{r_t}a_1'$ and $a_2=p_1^{s_1}p_2^{s_2}\cdots p_t^{s_t}a_2'$ for relatively prime elements a_1', a_2' of R such that $a_1', a_2' \notin p_iR$, $1 \leqq i \leqq t$ and positive integers r_i, s_i. Now, for a suitably large integer l, we have $ls_i > r_i$ for any i and $a_2^lX_2^l+b_2^l \in \mathfrak{a}'$. Then we have

$$\begin{aligned}&p_1^{ls_1-r_1}p_2^{ls_2-r_2}\cdots p_t^{ls_t-r_t}a_2'^l(a_1X_1+b_1)+a_1'(a_2^lX_2^l+b_2^l)\\&=p_1^{ls_1}p_2^{ls_2}\cdots p_t^{ls_t}a_1'\,a_2'^l(X_1+X_2^l)+p_1^{ls_1-r_1}p_2^{ls_2-r_2}\cdots p_t^{ls_t-r_t}a_2'^lb_1+a_1'\,b_2^l \in \mathfrak{a}'\,.\end{aligned}$$

If we put $a=p_1^{ls_1}p_2^{ls_2}\cdots p_t^{ls_t}$ and $b=p_1^{ls_1-r_1}p_2^{ls_2-r_2}\cdots p_t^{ls_t-r_t}a_2'^{l}b_1+a_1'b_2^l$, then a, b are relatively prime non-zero elements of R. Accordingly, for $a, b, l_2=l$, our conclusion holds.

In general case, we use the induction on n. Suppose that the conclusion holds for $n=k\ (\geqq 2)$. Let $\mathfrak{a}_{k+1}'$ be an ideal of $R[X_1, X_2, \cdots, X_k, X_{k+1}]$ generated by $a_1X_1+b_1, a_2X_2+b_2, \cdots, a_kX_k+b_k, a_{k+1}X_{k+1}+b_{k+1}$ such that for any i, a_i, b_i, are relatively prime non-zero elements of R. If we put $\mathfrak{a}_k'=(a_1X_1+b_1)R[X_1, X_2, \cdots, X_k]+\cdots+(a_kX_k+b_k)R[X_1, X_2, \cdots, X_k]$, then $\mathfrak{a}_k'$ is also an ideal of $R[X_1, X_2, \cdots, X_k]$ satisfying the condition in our lemma for $n=k$. By the assumption of induction, there is an element $a'(X_1+X_2^{l_2}+\cdots+X_k^{l_k})+b'$ in $\mathfrak{a}_k'$ such that $a'\in\bigcap_{i=1}^{k}a_iR$ and $a'R+b'R=R$, $a', b'\neq 0$. As $\mathfrak{a}_k'\subset\mathfrak{a}_{k+1}'$, $a'(X_1+X_2^{l_2}+\cdots+X_k^{l_k})+b'\in\mathfrak{a}_{k+1}'$. Put $Y=X_1+X_2^{l_2}+\cdots+X_k^{l_k}$ and let $\mathfrak{a}''$ be an ideal of $R[Y, X_{k+1}]$ generated by $a'Y+b'$, $a_{k+1}X_{k+1}+b_{k+1}$. If we apply the proof for $n=2$ to $a'Y+b'$, $a_{k+1}X_{k+1}+b_{k+1}$, we can find relatively prime elements a, b of R such that $a(Y+X_{k+1}^{l_{k+1}})+b\in\mathfrak{a}''$ and $a\in a'R\cap a_{k+1}R$. As $\mathfrak{a}''\subset\mathfrak{a}_{k+1}'$, we have $a(Y+X_{k+1}^{l_{k+1}})+b=a(X_1+X_2^{l_2}+\cdots+X_{k+1}^{l_{k+1}})+b\in\mathfrak{a}_{k+1}'$. Obviously we also have $a\in\bigcap_{i=1}^{k+1}a_iR$. This shows that our conclusion holds for $n=k+1$. Thus our proof is completed.

PROPOSITION 5.2. *Let R be a semi-local integral domain of dimension 1 and $R'=R[X_1, X_2, \cdots, X_n]$ be the polynomial ring with a finite number of variables $X_1, X_2, \cdots, X_n$ over R. Then any maximal ideal $\mathfrak{m}'$ of height n in R' contains a prime ideal $\mathfrak{p}'$ of height 1 in R' such that $R'/\mathfrak{p}'$ is isomorphic to the polynomial ring with $n-1$ variables over a field.*

PROOF. If $n=1$, this is trivial. Hence we may suppose $n\geqq 2$. As $\mathfrak{m}'$ has height n in R', at least one of X_i's is not contained in $\mathfrak{m}'$. Suppose that $X_{i_1}, X_{i_2}, \cdots, X_{i_k}\notin\mathfrak{m}'$ and $X_{j_1}, X_{j_2}, \cdots, X_{j_{n-k}}\in\mathfrak{m}'$, for some $k\geqq 1$. Then we have $\mathfrak{m}'=\mathfrak{m}''R'+X_{j_1}R'+X_{j_2}R'+\cdots+X_{j_{n-k}}R'$ for a maximal ideal $\mathfrak{m}''$ of height k in $R[X_{i_1}, X_{i_2}, \cdots, X_{i_k}]$.

If we use $\mathfrak{m}''$ instead of $\mathfrak{m}'$, we may assume $X_1, X_2, \cdots, X_n\notin\mathfrak{m}'$. If we put $\mathfrak{p}_i=\mathfrak{m}'\cap R[X_i]$ for any i, then $\mathfrak{p}_i$ is a prime ideal of height 1 in $R[X_i]$ and we have $\mathfrak{p}_i\cap R=0$. If at least one $\mathfrak{p}_{i_0}$ of $\mathfrak{p}_i$'s is maximal in $R[X_{i_0}]$, then we have $R'/\mathfrak{p}_{i_0}R'=(R[X_{i_0}]/\mathfrak{p}_{i_0})[X_1, X_2, \cdots, X_{i_0-1}, X_{i_0+1}, \cdots, X_n]$, and therefore our assertion is true. Hence we assume that any $\mathfrak{p}_i$ is not maximal in $R[X_i]$. Let K be the quotient field of R. Then $\mathfrak{p}_iK[X_i]$ is a prime ideal of $K[X_i]$, and so it is generated by an irreducible polynomial $f_i(X_i)$ in $K[X_i]$. As is easily seen, we can suppose $f_i(X_i)\in\mathfrak{p}_i$. Let $\alpha_1^{(i)}, \alpha_2^{(i)}, \cdots, \alpha_{t_i}^{(i)}$ be all the roots of the algebraic equation $f_i(T)=0$ over K for any i, and put $\bar{K}=K[\alpha_1^{(1)}, \alpha_2^{(1)}, \cdots, \alpha_{t_1}^{(1)}, \alpha_1^{(2)}, \cdots, \alpha_1^{(n)}, \cdots, \alpha_{t_n}^{(n)}]$. Let $\bar{R}$ be the integral closure of R in $\bar{K}$. Then, by (1.4), $\bar{R}$ is a semi-local principal ideal domain. If we put $\bar{R}'=\bar{R}[X_1, X_2,$

$\cdots, X_n]$, then $\bar{R}'$ is a unique factorization domain integral over R'. There exists a maximal ideal $\bar{\mathfrak{m}}'$ of $\bar{R}'$ such that $\mathfrak{m}'=\bar{\mathfrak{m}}'\cap R'$ by (1.3). Similarly, by (1.3), we have $\mathrm{height}_{\bar{R}'}\bar{\mathfrak{m}}'=n$. Since any $f_i(X_i)$ is expressible as the product of linear polynomials in $\bar{R}[X_i]$, and any X_i is not contained in $\bar{\mathfrak{m}}'$, we have

$$\bar{\mathfrak{m}}'=(\bar{a}_1X_1+\bar{b}_1)\bar{R}'+(\bar{a}_2X_2+\bar{b}_2)\bar{R}'+\cdots+(\bar{a}_nX_n+\bar{b}_n)\bar{R}',$$

for non-zero relatively prime elements $\bar{a}_i$, $\bar{b}_i$ of $\bar{R}$ for any i. Then, by (5.1), there is an element $\bar{a}(X_1+X_2^{l_2}+\cdots+X_n^{l_n})+\bar{b}$ in $\bar{\mathfrak{m}}'$ such that $\bar{a}\bar{R}+\bar{b}\bar{R}=\bar{R}$ and $\bar{a}\in\bigcap_{i=1}^{n}\bar{a}_i\bar{R}$. As $\bar{\mathfrak{m}}'$ is maximal in $\bar{R}'$, it is easily seen that $\prod_{i=1}^{n}\bar{a}_i$ is contained in the Jacobson radical $\bar{\mathfrak{n}}$ of $\bar{R}$. Hence $\bar{a}$ is also contained in $\bar{\mathfrak{n}}$. Since $\bar{a}\bar{R}+\bar{b}\bar{R}=\bar{R}, \bar{b}$ must be a unit of $\bar{R}$, and so we may suppose $\bar{b}=1$. If we put $U=X_1+X_2^{l_2}+\cdots+X_n^{l_n}$, then we have $R'=R[U, X_2, \cdots, X_n]$ and $\bar{R}'=\bar{R}[U, X_2, \cdots, X_n]$. Also we have $\bar{\mathfrak{m}}'\cap\bar{R}[U]=(\bar{a}U+1)\bar{R}[U]$. As $\bar{a}\in\bar{\mathfrak{n}}$, $(\bar{a}U+1)\bar{R}[U]$ is a maximal ideal of height 1 in $\bar{R}[U]$. If we put $\mathfrak{p}_U=(\bar{a}U+1)\bar{R}[U]\cap R[U]$, then $\mathfrak{p}_U$ is a prime ideal of height 1 in $R[U]$. Since $\bar{R}[U]$ is integral over $R[U]$, $\mathfrak{p}_U$ is also maximal in $R[U]$ by (1.3). If we put $\mathfrak{p}'=\mathfrak{p}_UR'$, then we have

$$R'/\mathfrak{p}'\cong(R[U]/\mathfrak{p}_U)[X_2, X_3, \cdots, X_n].$$

Obviously we have $\mathfrak{p}'\subset\mathfrak{m}'$. This completes our proof.

COROLLARY 5.3. *Let R be a semi-local principal ideal domain and $R'=R[X_1, X_2, \cdots, X_n]$ be the polynomial ring with a finite number of variables $X_1, X_2, \cdots, X_n$ over R. Then any maximal ideal $\mathfrak{m}'$ of R' is generated by elements in numbers equal to its height.*

Now we give

THEOREM 5.4. *Let R be a semi-local integral domain of dimension* 1 *and let $R'=R[X, Y]$ be the polynomial ring with two variables X, Y over R. Then the following conditions are equivalent:*

1) *R is a weakly normal ring.*

2) *R' is a* PF *domain.*

PROOF. As the implication 2)→1) was proved in (3.2), we have only to show the implication 1)→2). Suppose that R is weakly normal. Let $\mathfrak{m}'$ be a maximal ideal of height 2 in R'. Then, by (5.2), there exists a prime ideal $\mathfrak{p}'$ of height 1 in R' contained in $\mathfrak{m}'$ such that $R'/\mathfrak{p}'$ is isomorphic to the polynomial ring with a variable over a field, i. e., to a Euclidean ring. If we denote by $\mathfrak{n}$ the Jacobson radical of R, we see easily $\mathfrak{p}'+\mathfrak{n}R'=R'$. Hence, by (3.1), $\mathfrak{p}'$ is principal in R' and so we can put $\mathfrak{p}'=p'R'$ for a prime element p' of R'. Let S' be the multiplicative system of R' generated by all prime elements $\{p_i'\}$ in R' such that any $R'/p_i'R'$ is Euclidean. If a maximal ideal $\mathfrak{m}'$ of R' has height 2 in R', then we have $\mathfrak{m}'\cap S'\neq\phi$ by the preceding argument, and if $\mathfrak{m}'$ has height 3 in R', then we have $\mathfrak{m}' \cap S' = \phi$, as is easily seen. If we put $R'' = R'_{S'}$, there is no maximal ideal of height 2 in R''. Let $\mathfrak{m}''$ be a maximal ideal of height 1 in R''. If we put $\mathfrak{p}' = \mathfrak{m}'' \cap R'$, then $\mathfrak{p}'$ is a prime ideal of height 1 in R' such that $\mathfrak{p}' + \mathfrak{n}R' = R'$. Again, by (3.1), $\mathfrak{p}'$ is principal in R'. As $\mathfrak{m}'' = \mathfrak{p}'R''$, $\mathfrak{m}''$ is also principal in R''. Let S'' be the multiplicative system of R'' generated by all prime elements $\{p''_i\}$ in R'' such that any $R''/p''_i R''$ is a field. Then, for a maximal ideal $\mathfrak{m}''$ of height 1 in R'', we have $\mathfrak{m}'' \cap S'' \neq \phi$ by the preceding argument, and while, for a maximal ideal $\mathfrak{m}''$ of height 3 in R'', we have $\mathfrak{m}'' \cap S'' = \phi$. If we put $R^* = R''_{S''}$, then any maximal ideal $\mathfrak{m}^*$ of R^* has height 3 in R^* and is generated by a maximal ideal $\mathfrak{m}'$ of height 3 in R'. So we have $\mathfrak{m}^* \supset \mathfrak{n}R^*$. This shows that $\mathfrak{n}R^*$ is contained in the Jacobson radical of R^*. According to (2.2), if R^* is a PF domain, then R'' is so and if R'' is a PF domain, then R' is so. Furthermore, if any finite projective $R^*/\mathfrak{n}R^*$-module is quasi-free, then, by (2.5), R^* is a PF domain. Therefore it suffices to prove that $R^*/\mathfrak{n}R^*$ is a direct sum of PF domains. In fact, if we denote by $\mathfrak{m}_1, \mathfrak{m}_2, \cdots, \mathfrak{m}_t$ all maximal ideals of R, we have

$$R^*/\mathfrak{n}R^* \cong (R/\mathfrak{n})[X, Y] \cong (R/\mathfrak{m}_1)[X, Y] \oplus \cdots \oplus (R/\mathfrak{m}_t)[X, Y].$$

As any $R/\mathfrak{m}_i$ is a field, any $(R/\mathfrak{m}_i)[X]$ is a principal ideal domain. Then, by (4.8), any $(R/\mathfrak{m}_i)[X, Y]$ is a PF domain. This completes our proof.

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