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Cross sections in locally compact groups

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Steenrod [**7**, p. 33] conjectured that the local cross section of a closed subgroup in a compact finite dimensional group will always exist. This conjecture has been solved for some special cases by Gleason [**1**], Mostert [**3**] and Karube [**2**]. The purpose of this paper is to show that the conjecture is true for more general groups, i. e. locally compact finite dimensional groups[1].

Throughout this paper the dimension means the covering dimension, a topological group means a T_1-group, a residue class means a left one and a factor space means a left one. Let G be a topological group, H a closed subgroup and ω the natural projection of G onto the factor space G/H. If there exist a neighborhood U of $\omega(H)$ and a continuous mapping σ of U into G such that $\omega\sigma$ is the identity mapping, we call that σ is a local cross section of H in G. When $U=G/H$, we call that σ is a cross section of H in G.

LEMMA 1 (Karube [**2**] or Nagami [**5**, Proof of Lemma 1.6]). *If any closed subgroup of a 0-dimensional compact group has a cross section, then any closed subgroup of a finite dimensional, locally compact group has a local cross section.*

LEMMA 2. *Let H be a closed subgroup of a 0-dimensional compact group G. Then H has a cross section in G.*

PROOF. By Pontrjagin [**6**, Theorem 68] there exists an inverse limiting system $\{G_\alpha, \pi^\alpha_\beta; 0 \leqq \alpha \leqq \tau\}$ for a suitable ordinal τ which satisfies the following conditions:

i) Every G_α is a compact 0-dimensional group.

1) It is to be noticed that Mostert [**4**] proved that any closed subgroup H of any locally compact group G has a local cross section if the factor space G/H is of finite dimension which is defined by him and somewhat different from the usual covering dimension. His dimension of a homogeneous space is finite if and only if the covering dimension is so. But his argument contains several gaps and is radically wrong. One of the most essential error is in the last part of the proof of Theorem 7 which cannot be concluded from the condition 4.2. Recall that f_γ in 4.2 depends upon a triple $f_\alpha, f_\beta, f_\delta$. So it seems that all of the essential part of his paper, which contains the above proposition, has not yet been correctly proved by anyone.

Let us take this opportunity to insist that Nagami [**5**, Lemma 1.6 and Theorem 2.1] are at the first sight special cases of the results in Mostert [**4**] but there is a good reason that we have given proofs of our propositions.

Mostert [**3**] contains also some gaps.

ii) Every π^α_β, $\beta<\alpha$, is an open, continuous homomorphism of G_α onto G_β.

iii) When α is an isolated ordinal, $N^{\alpha-1}_\alpha$, the kernel of $\pi^\alpha_{\alpha-1}$, is a finite group.

iv) When α is a limiting ordinal, $\cap\{N^\beta_\alpha\,;\beta<\alpha\}$, where N^β_α is the kernel of π^α_β, is the identity of G_α.

v) G_0 consists of only the identity.

vi) $G_\tau=G$.

Set $H_\alpha=\pi^\tau_\alpha(H)$, $\alpha\leqq\tau$, then we have a closed subgroup H_α of G_α. Let ω_α be the natural projection of G_α onto the factor space $K_\alpha=G_\alpha/H_\alpha$. Define $\rho^\alpha_\beta: K_\alpha\to K_\beta$, $\beta<\alpha$, by: $\rho^\alpha_\beta(k_\alpha)=\omega_\beta\pi^\alpha_\beta\omega^{-1}_\alpha(k_\alpha)$, $k_\alpha\in K_\alpha$. Then ρ^α_β is an open, onto, continuous mapping and $\{K_\alpha,\rho^\alpha_\beta\,;0\leqq\alpha\leqq\tau\}$ forms an inverse limiting system.

Let σ_0 be ω^{-1}_0. Now let $0\leqq\alpha\leqq\tau$ and $\mathrm{P}(\alpha)$ be the proposition that there exist cross sections $\sigma_\beta: K_\beta\to G_\beta$, $\beta\leqq\alpha$, which satisfy $\pi^\beta_\gamma\sigma_\beta=\sigma_\gamma\rho^\beta_\gamma$ for any ordered triple $\gamma<\beta\leqq\alpha$. Then $\mathrm{P}(0)$ is true.

When α is a limiting ordinal, suppose that $\mathrm{P}(\beta)$ is true for any $\beta<\alpha$, and define $\sigma_\alpha: K_\alpha\to G_\alpha$ as $\sigma_\alpha(k_\alpha)=\cap\{(\pi^\alpha_\beta)^{-1}\sigma_\beta\rho^\alpha_\beta(k_\alpha)\,;\beta<\alpha\}$, $k_\alpha\in K_\alpha$. Then $\sigma_\alpha(k_\alpha)$ consists of one and only one point by the condition iv). It can easily be seen that σ_α is a cross section of H_α in G_α which satisfies $\pi^\alpha_\beta\sigma_\alpha=\sigma_\beta\rho^\alpha_\beta$ for any $\beta<\alpha$. Thus $\mathrm{P}(\alpha)$ is true.

Assume that α is an isolated ordinal with $0<\alpha$ and that $\mathrm{P}(\alpha-1)$ is true. Since $N^{\alpha-1}_\alpha$ is finite by the condition iii), there exists an open normal subgroup L_α of G_α such that $L_\alpha\cap N^{\alpha-1}_\alpha$ consists of only the identity. Let $L_{\alpha-1}=\pi^\alpha_{\alpha-1}(L_\alpha)$, then $L_{\alpha-1}$ is an open normal subgroup of $G_{\alpha-1}$ and $\pi^\alpha_{\alpha-1}\,|\,L_\alpha$ is an isomorphism. Let $\{h_iL_{\alpha-1}\,;i=1,\cdots,s\}$ be the mutually disjoint collection of all the residue classes of $L_{\alpha-1}$ in $G_{\alpha-1}$. Choose $g_i\in G_\alpha$, $i=1,\cdots,s$, with $\pi^\alpha_{\alpha-1}(g_i)=h_i$. Then $\{g_iL_\alpha N^{\alpha-1}_\alpha\,;i=1,\cdots,s\}$ is the mutually disjoint collection of all the residue classes of $L_\alpha N^{\alpha-1}_\alpha$ in G_α. Let t be the number of all the residue classes of $H_\alpha\cap N^{\alpha-1}_\alpha$ in $N^{\alpha-1}_\alpha$ and u the order of $H_\alpha\cap N^{\alpha-1}_\alpha$. Let $N^{\alpha-1}_\alpha=\{n_{ij}\,;i=1,\cdots,t,j=1,\cdots,u\}$, where $\{n_{ij}\,;j=1,\cdots,u\}$ is a residue class of $H_\alpha\cap N^{\alpha-1}_\alpha$ for any i. Set $C_{ij}=(\pi^\alpha_{\alpha-1}\,|\,g_iL_\alpha n_{j1})^{-1}(\sigma_{\alpha-1}(K_{\alpha-1})\cap h_iL_{\alpha-1})$, $i=1,\cdots,s,j=1,\cdots,t$; then $\{C_{ij}\,;i=1,\cdots,s,j=1,\cdots,t\}$ forms a mutually disjoint collection of compact sets and hence $C=\cup\{C_{ij}\,;i=1,\cdots,s,j=1,\cdots,t\}$ is compact.

To show that $C^{-1}C\cap H_\alpha$ consists of only the identity, let c_1 and c_2 be arbitrary different points of C. a) When $c_1\in C_{ij}$, $c_2\in C_{i'j'}$, $i\neq i'$, then $\pi^\alpha_{\alpha-1}(c_1)\in\sigma_{\alpha-1}(K_{\alpha-1})\cap h_iL_{\alpha-1}$ and $\pi^\alpha_{\alpha-1}(c_2)\in\sigma_{\alpha-1}(K_{\alpha-1})\cap h_{i'}L_{\alpha-1}$ and hence $\pi^\alpha_{\alpha-1}(c_1^{-1}c_2)\notin H_{\alpha-1}$. Therefore $c_1^{-1}c_2\notin H_\alpha$. b) When $c_1\in C_{ij}$, $c_2\in C_{ij}$, we have $\pi^\alpha_{\alpha-1}(c_1)\neq\pi^\alpha_{\alpha-1}(c_2)$ and hence $\pi^\alpha_{\alpha-1}(c_1^{-1}c_2)\notin H_{\alpha-1}$. Therefore $c_1^{-1}c_2\notin H_\alpha$. c) When $c_1\in C_{ij}$, $c_2\in C_{ij'}$, $j\neq j'$, then there exist elements l_1,l_2 of L_α with $c_1=g_il_1n_{j1}$ and $c_2=g_il_2n_{j'1}$. If $l_1=l_2$, we have $c_1^{-1}c_2=n_{j1}^{-1}\,n_{j'1}\notin H_\alpha$. If $l_1\neq l_2$, then $\pi^\alpha_{\alpha-1}(l_1)\neq\pi^\alpha_{\alpha-1}(l_2)$. Since $\pi^\alpha_{\alpha-1}(c_1)$

$=h_i\pi^{\alpha}_{\alpha-1}(l_1)$ and $\pi^{\alpha}_{\alpha-1}(c_2)=h_i\pi^{\alpha}_{\alpha-1}(l_2)$, we know that $\pi^{\alpha}_{\alpha-1}(c_1)$ and $\pi^{\alpha}_{\alpha-1}(c_2)$ are different points of $\sigma_{\alpha-1}(K_{\alpha-1})$. Hence $\pi^{\alpha}_{\alpha-1}(c_1^{-1}c_2)\notin H_{\alpha-1}$ and we have $c_1^{-1}c_2\notin H_{\alpha}$.

To prove that $\omega_{\alpha}(C)=K_{\alpha}$, let g be an arbitrary element of G_{α} and consider the residue class gH_{α}. There exists an element h of $\sigma_{\alpha-1}(K_{\alpha-1})$ with $\pi^{\alpha}_{\alpha-1}(gH_{\alpha})=hH_{\alpha-1}$. Let g' be an element of gH_{α} with $\pi^{\alpha}_{\alpha-1}(g')=h$. Choose i with $h\in h_iL_{\alpha-1}$, $l_{\alpha-1}\in L_{\alpha-1}$ with $h=h_il_{\alpha-1}$ and $l_{\alpha}\in L_{\alpha}$ with $\pi^{\alpha}_{\alpha-1}(l_{\alpha})=l_{\alpha-1}$. Since $(\pi^{\alpha}_{\alpha-1})^{-1}(h)=g'N^{\alpha-1}_{\alpha}=\cup\{g_il_{\alpha}n_{jk}\,;j=1,\cdots,t,k=1,\cdots,u\}$, there exist j and k with $g'=g_il_{\alpha}n_{jk}$. Set $g''=g_il_{\alpha}n_{j1}$, then g'' is an element of C. Since $(g')^{-1}g''=n_{jk}^{-1}n_{j1}\in H_{\alpha}$, we have $g'H_{\alpha}=g''H_{\alpha}$ and hence $gH_{\alpha}=g''H_{\alpha}$. Thus $\omega_{\alpha}(C)=K_{\alpha}$ is proved.

Since $\omega_{\alpha}\,|\,C$ is a homeomorphism of a compact set C onto K_{α}, $\sigma_{\alpha}=(\omega_{\alpha}\,|\,C)^{-1}$ is a homeomorphism of K_{α} onto C. Consider the system $\{\sigma_{\beta}\,;\beta\leqq\alpha\}$, then we know that $\mathrm{P}(\alpha)$ is true. Thus the induction is completed and we have the desired cross section σ_{τ} of H in $G=G_{\tau}$.

By Lemmas 1 and 2 we have at once the following:

THEOREM. *Any closed subgroup of a locally compact, finite dimensional group has a local cross section.*

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References

[1] A. M. Gleason, Spaces with a compact Lie group of transformations, Proc. Amer. Math. Soc., **1** (1950), 35-43.
[2] T. Karube, On the local cross sections in locally compact groups, J. Math. Soc. Japan, **10** (1958), 343-347.
[3] P. S. Mostert, Local cross sections in locally compact groups, Proc. Amer. Math. Soc., **4** (1953), 645-649.
[4] P. S. Mostert, Sections in principal fibre spaces, Duke Math. J., **23** (1956), 57-71.
[5] K. Nagami, Dimension-theoretical structure of locally compact groups, J. Math. Soc. Japan, **14** (1962), 379-396.
[6] L. Pontrjagin, Continuous groups, Moscow, 1954.
[7] N. Steenrod, The topology of fibre bundles, Princeton University Press, 1951.