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Note on holomorphically convex complex spaces

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Hirzebruch [4] proved that for any 2-dimensional complex space Y there exists a 2-dimensional complex manifold X which is obtained by a proper modification of Y in the inuniformisable points of Y. If Y is a Stein space, then X is obviously a holomorphically convex complex manifold. In the present paper we shall conversely consider the conditions that a holomorphically convex complex space can be obtained by a proper modification of a Stein space. (In the present paper we mean by a complex space an α space $=\beta_n$ space in Grauert-Remmert [3].)

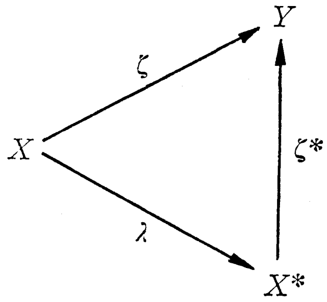
The following lemma is a special case of the theorem of factorization of Remmert-Stein [9].

LEMMA 1. *Let ζ be a proper holomorphic mapping of an n-dimensional connected complex space X onto an n-dimensional Stein space Y such that ζ induces an isomorphism of the integral domain $I(Y)$ of all holomorphic functions in Y onto the integral domain $I(X)$ in X. Then (X, ζ, Y) is a proper modification. Moreover, if each connected component of the set of degeneracy E of ζ is compact in X, then (X, ζ, Y) is a proper points-modification.*

PROOF. If $n=1$, ζ is biholomorphic. Therefore we may assume that $n\geqq 2$. Let x be any point of X. We denote by σ_x the connected component of $\zeta^{-1}\zeta(x)$ containing x. σ_x is a nowhere discrete connected compact analytic set in X if $x\in E$. We shall introduce an equivalence relation R in X as follows;

$$x \text{ and } y\in X \text{ are equivalent modulo } R \text{ if } \sigma_x=\sigma_y .$$

Let $X^*=X/R$ be the factor space of X by the equivalence relation R. If we consider the canonical mappings $\lambda: X\rightarrow X^*$ and $\zeta^*: X^*\rightarrow Y$, then the commutativity holds in the following diagram;

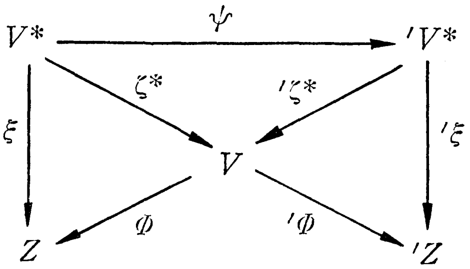


Since ζ is proper, λ and ζ^* are proper. We shall introduce a complex structure in the Hausdorff space X^* such that λ is holomorphic. Let y be any point of $\zeta(E)$. There exists a neighbourhood V of y with the following property;

There exists a proper holomorphic mapping Φ of V onto a domain Z of C^n such that (V, Φ, Z) is an analytic ramified covering of Z.

Then $(w_1, w_2, \cdots, w_n)=\Phi\circ\zeta$ is a proper mapping of $\zeta^{-1}(V)$ onto Z with the set of degeneracy $E\cap\zeta^{-1}(V)$. From Grauert-Remmert [3] the set L of all inuniformisable points of $\zeta^{-1}(V)$ is an $(n-2)$-dimensional analytic thin set in $\zeta^{-1}(V)$. Let M be the set of all points of x of $\zeta^{-1}(V)-L$ such that $\partial(w_1, w_2, \cdots, w_n)/\partial(t_1, t_2, \cdots, t_n)=0$ for some local coordinates $t_1, t_2, \cdots, t_n$ of x. Then M is a pure $(n-1)$-dimensional analytic set in $\zeta^{-1}(V)-L$. From Remmert-Stein [8] the closure $\bar{M}$ of M in $\zeta^{-1}(V)$ is an analytic set in $\zeta^{-1}(V)$. Since the set of branch points of an analytic ramified covering is a pure 1-codimensional analytic set from Grauert-Remmert [3], we have $L\subset\bar{M}$. It holds that $E\cap\zeta^{-1}(V)\subset\bar{M}\cup L=\bar{M}$. $N=\Phi\circ\zeta(\bar{M})$ is an analytic set in Z from Remmert [7] since $\Phi\circ\zeta$ is proper. We put $V^*=\lambda(\zeta^{-1}(V))$. $\Phi\circ\zeta$ induces naturally a nowhere degenerate proper mapping ξ of V^* onto Z. Since ξ is a local homeomorphism of $V^*-\xi^{-1}(N)$ onto $Z-N$ and N is an analytic set in Z such that $\xi^{-1}(N)$ nowhere separates V[1)], (V^*, ξ, Z) is an analytic ramified covering of Z with the critical set N. Obviously ζ^* is holomorphic. Let $'\Phi$ be another proper holomorphic mapping of V onto a domain $'Z$ of C^n such that $(V, '\Phi, 'Z)$ is an analytic ramified covering of $'Z$. The corresponding analytic sets in $\zeta^{-1}(V)$ and in $'Z$ and the corresponding analytic ramified covering of $'Z$ with the critical set $'N$ is denoted, respectively, by $'\bar{M}$, $'N$ and $('V^*, '\xi, 'Z)$. Let ψ be the canonical injective mapping of V^* onto $'V^*$. Then the commutativity holds in the following diagram;

1) We say that a set A nowhere separates a connected set B, if $B-A\cap B$ is connected and locally connected.



Therefore ψ is a homeomorphism of V^* onto $'V^*$ and $'\xi\circ\psi='\Phi\circ\zeta^*$ is a holomorphic mapping of V^* onto $'Z$. Hence ψ is a holomorphic mapping of V^* onto $'V^*$. The complex structure in $\lambda(\zeta^{-1}(V))$ does not depend on the special choice of Φ. Since $\xi\circ\lambda=\Phi\circ\zeta$ is holomorphic, λ is a holomorphic mapping of $\zeta^{-1}(V)$ onto V^*. Thus we can introduce a complex structure in X^* such that $\lambda: X\rightarrow X^*$ and $\zeta^*: X^*\rightarrow Y$ are holomorphic.

Suppose that there exists a nowhere discrete compact analytic set A in X^*. Then $\lambda^{-1}(A)$ is a nowhere discrete compact analytic set in X and $\zeta(\lambda^{-1}(A))$ is a compact analytic set in a Stein space Y. Therefore $\zeta(\lambda^{-1}(A))$ is a finite set in Y. Hence A itself is a finite set in X^*. But this is a contradiction. Therefore X^* contains no nowhere discrete compact analytic set and X^* is a Stein space. $(X, E, \lambda, X^*, \lambda(E))$ is a proper modification of a Stein space X^* in the analytic set $\lambda(E)$. Since λ induces an isomorphism of $I(X^*)$ onto $I(X)$, ζ^* is a biholomorphic mapping of X^* onto Y from Remmert [**6**]. Therefore $(X, E, \zeta, Y, \zeta(E))$ itself is a proper modification of a Stein space Y in the analytic set $\zeta(E)$. If each connected componenent of E is compact in X, then $\zeta(E)$ is a discrete set in Y. Hence $(X, E, \zeta, Y, \zeta(E))$ is a proper points-modification of Y in the discrete set $\zeta(E)$. q. e. d.

A nowhere discrete compact analytic set A in a complex space X is called an *exceptional analytic set* if there exists a proper points-modification (X, A, ζ, Y, D) of a complex space Y in a discrete subset D of Y. Grauert [**2**] proved that A is an exceptional analytic set, if and only if there exists a strongly pseudoconvex neighbourhood U of A such that A is a maximal compact analytic subset of $U\Subset X$. Similarly to Grauert [**2**] we shall prove the following theorem.

THEOREM 1. *A 2-dimensional connected holomorphically convex complex space X is obtained by a proper points-modification of a Stein space Y if and only if there exist two holomorphic functions in X which are analytically independent at each point of X.*

PROOF. The necessity of Theorem 1 follows immediately. Therefore it suffices to prove the sufficiency of Theorem 1 in case that X is not K-complete. From Remmert [**6**] there exists a Stein space Y and a proper holomorphic mapping ζ of X onto Y such that ζ induces naturally an isomorphism of $I(Y)$

onto $I(X)$. We shall prove that (X, ζ, Y) is the desired proper points-modification of Y.

Since there exist two holomorphic functions in X which are analytically independent at each point of X, Y is a 2-dimensional Stein space. From Remmert [**7**] the set of degeneracy E of ζ is a nowhere discrete analytic set in X and $\zeta(E)$ is also an analytic set in Y. Let x_0 be any point of E such that x_0 is a uniformisable point of X and that E is irreducible and regular in x_0. Then there exists a neighbourhood U of x_0 such that any point of U is uniformisable and that E is irreducible and regular in each point of $U \cap E$. Since $x_0 \in E$, $\zeta^{-1}\zeta(x_0)$ is 1-dimensional in x_0. Therefore for a sufficiently small neighbourhood U, $U \cap \zeta^{-1}\zeta(x_0)$ is a pure 1-dimensional analytic set in U. We have $U \cap \zeta^{-1}\zeta(x_0) \subset U \cap E$. Since E is irreducible at each point of $U \cap E$, we have $U \cap \zeta^{-1}\zeta(x_0) = U \cap E$.

This means $\zeta(U \cap E) = \zeta(x_0)$. We denote the connected components of E by E_i $(i=1, 2, \cdots)$. Then each $\zeta(E_i)$ consists of only a single point y_i $(i=1, 2, \cdots)$ and $\{y_i ; i=1, 2, \cdots\}$ is an analytic set in Y, that is, a discrete set in Y. Therefore each E_i is a pure 1-dimensional compact analytic set in X. From Lemma 1 (X, ζ, Y) is the desired proper points-modification of a Stein space Y. q. e. d.

We shall prove that Theorem 1 is a special case of a result in Grauert [**2**]. A complex space X is called to be *exhausted by strongly pseudoconvex domains* if there exists a sequence of strongly pseudoconvex domains X_n as follows;

(1) $X_n \Subset X_{n+1}$ $(n=1, 2, \cdots)$,

(2) $X = \bigcup_{n=1}^{\infty} X_n$.

THEOREM 2. *A connected holomorphically convex complex space X can be exhausted by strongly pseudoconvex domains if and only if X can be obtained by a proper points-modification of a Stein space Y.*

PROOF. If there exists a proper modification (X, E, ζ, Y, D) of a Stein space Y in a discrete set $D = \{y_i ; i=1, 2, \cdots\}$, then from Narasimhan [**5**] there exists a strongly plurisubharmonic function $p > 0$ in Y such that $\{y ; p(y) < c\}$ is relatively compact in Y for any $c > 0$. $p \circ \zeta$ is plurisubharmonic in X. There exists a sequence of positive numbers c_n as follows;

(1) $c_n < c_{n+1}$ and $c_n \to \infty$ as $n \to \infty$,

(2) $\{y ; p(y) = c_n\} \cap D = \phi$ for $n = 1, 2, \cdots$.

Since ζ is a biholomorphic mapping of $X-E$ onto $Y-D$, $p \circ \zeta$ is strongly plurisubharmonic in $X-E$. Therefore $B_n = \{x ; p \circ \zeta(x) < c_n\}$ is a relatively compact strongly pseudoconvex open set of X. If we take a suitable connected component X_n of B_n for any n, then X_n is a relatively compact strongly pseudoconvex domain of X such that $X_n \Subset X_{n+1}$ and $X = \bigcup_{n=1}^{\infty} X_n$. Hence X can be exhausted by strongly pseudoconvex domains.

Conversely, we shall suppose that X can be exhausted by strongly pseudoconvex domain X_n $(n=1, 2, \cdots)$. From Remmert [**6**] there exists a proper holomorphic mapping ζ of X onto a Stein space Y such that ζ induces naturally an isomorphism of $I(Y)$ onto $I(X)$. From Grauert [**2**] there exists a compact set $K_n \subset X_n$ such that any nowhere discrete compact analyic set $A \subset X_n$ is contained in K_n for any n as X_n is strongly pseudoconvex. Therefore the dimension of Y coincides with that of X. If we denote the set of degeneracy of ζ by E, then we have $E \cap X_n \subset K_n$. Each connected component E_i of E is a nowhere discrete compact analytic set in X. $\zeta(E_i)$ consists of a single point y_i. From Lemma 1 (X, E, ζ, Y, D) is a proper modification of Y in the discrete set D. q. e. d.

We can summerize the above results in the following theorem:

THEOREM 3. *If X is a 2-dimensional holomorphically convex complex space, the following three conditons are equivalent;*

(1) *X can be obtained by a proper points-modification of a Stein space.*

(2) *There exist two holomorphic functions in X which are analytically independent at each point of X.*

(3) *X can be exhausted by strongly pseudoconvex domains.*

We remark that in a similar way Theorem 1 can be generalized in the case of an n-dimensional complex space as follows;

THEOREM 4. *An n-dimensional connected holomorphically convex complex space X is obtained by a proper modification of a Stein space if and only if there exist n holomorphic functions in X which are analytically independent at each point of X.*

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References

[1] H. Grauert, On point modifications, Contributions to function theory, Tata Institute of Fundamental Research, Bombay, 1960, 139–144.

[2] H. Grauert, Über Modifikationen und exzeptionelle analytische Mengen, Math. Ann., **146** (1962), 331–368.

[3] H. Grauert und R. Remmert, Komplexe Räume, Math. Ann., **136** (1958), 243–316.

[4] H. Hirzebruch, Über vierdimensionale Riemannsche Flächen mehrdeutiger analytischer Funktionen von zwei komplexen Veränderlichen, Math. Ann., **126** (1953), 1–22.

[5] R. Narasimhan, The Levi problem for complex spaces, Math. Ann., **142** (1961), 355–365, II, ibid., **146** (1962), 195–216.

[6] R. Remmert, Reduction of complex spaces, Sem. on analytic func. I, Princeton, 1957, 190–205.

[7] R. Remmert, Holomorphe und meromorphe Abbildungen komplexer Räume, Math. Ann., **133** (1957), 328-370.

[8] R. Remmert und K. Stein, Über die wesentliche Singularitäten analytischer Mengen, Math. Ann., **126** (1953), 263-306.

[9] R. Remmert und K. Stein, Eigentliche holomorphe Abbildungen, Math. Z., **73** (1960), 150-189.

[10] K. Stein, Analytische Zerlegungen komplexer Räume, Math. Ann., **132** (1956), 63-93.

[11] K. Stein, Die Existenz komplexer Basen zu holomorphe Abbildungen, Math. Ann., **136** (1958), 1-8.