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On the moduli of Abelian varieties with multiplications

Dedicated to Professor Y. Akizuki on his 60th birthday

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§ 1. Introduction. Reduction of the problem.

In a previous paper [2] we have shown how the orbits of the Siegel modular group in the generalized upper half-plane, $H_n=\{Z \mid Z \text{ an } n\times n \text{ complex, symmetric matrix, } Z=X+iY,\ Y \text{ positive definite}\}$, form the points of a Q-open subset V of an algebraic variety V^* defined over the rational numbers Q. It is well-known that these orbits are in a natural one-to-one correspondence with the isomorphism classes of normally polarized Abelian varieties of dimension n, and in the paper cited, we have shown that if $x\in V$, then the field of moduli of the corresponding isomorphism class of polarized Abelian varieties is just $Q(x)$. In the present paper we generalize this result to cover not only the case of the moduli of Abelian varieties with arbitrary polarization, but also the case of the moduli of polarized Abelian varieties with prescribed endomorphism ring in certain cases. Namely, we let k be a totally real algebraic number field F_0 of degree n_0 over Q or a purely imaginary quadratic extension of such a field, let $[k:Q]=n$, denote by $\mathfrak{o}$ the ring of integers in k, and consider an Abelian variety (A,ι) of type $\mathfrak{o}$ [12] and of dimension m satisfying the following conditions:

(1) $\iota(1)=$ identity endomorphism on A, and therefore $n\mid m$, or $m=np$;

(2) if $\alpha\in\mathfrak{o}$, then $\iota(\alpha)$ is represented by an $m\times m$ matrix among whose eigenvalues each conjugate of α appears exactly p times;

(3) if * is the positive involution of the endomorphism algebra of A defined by the polarization attached to a hyperplane section of A, then $\iota(\alpha)^*=\iota(\bar{\alpha})$ for all $\alpha\in\mathfrak{o}$.

We consider the space H_p of complex $p\times p$ matrices Z such that $i({}^t\bar{Z}-Z)$ is positive Hermitian, and in case k is totally real (i. e., $n=n_0$) we assume ${}^tZ=Z$, ^{t} denoting transpose. For data satisfying the above hypotheses, we may choose a system of coordinates in C^m such that for some $Z=(Z_1,\cdots,Z_{n_0})\in H_p^{n_0}$,

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and some lattice L in k^{2p} (i. e., a maximal free Abelian subgroup of k^{2p}, having $2np$ independent generators), A is isomorphic to C^m/L_Z, where

$$L_Z=\{v_\lambda=((Z_1E)\lambda^{\sigma_1}, \cdots, (Z_{n_0}E)\lambda^{\sigma_{n_0}}, ({}^tZ_1E)\lambda^{\sigma_1\tau}, \cdots, ({}^tZ_{n_0}E)\lambda^{\sigma_{n_0}\tau}) \mid \lambda\in L\},$$

E denoting the $p\times p$ identity matrix, (Z_iE) or $({}^tZ_iE)$ denoting a $p\times 2p$ matrix written in $p\times p$ blocks, and τ denoting complex conjugation on k—the last n_0 terms with ${}^tZ_i$ appear only if k is a purely imaginary quadratic extension of F_0 (so $n=2n_0$) and $\tau\neq$ identity on k; the polarization is defined by the skew Hermitian form

$$(v_\lambda, v_\kappa)=tr({}^t\bar{\lambda}J\kappa),$$

where $tr=tr_{k/Q}$ and $J=\begin{pmatrix}0 & E\\ -E & 0\end{pmatrix}$; and $\iota(\alpha)$, $\alpha\in\mathfrak{o}$, is represented by the diagonal matrix whose diagonal entries are the n conjugates of α, each repeated p times, in an appropriate order. The details of this coordinatization of the problem are described in a paper of Shimura [**13**]. The group

$$Sp(p, k)=\{M \mid {}^t\bar{M}JM=J\}$$

acts naturally on $H_p^{n_0}$. Let

$$\Gamma_L=\{M\in Sp(p, k) \mid {}^t\bar{M}L=L\},$$

the elements of L being viewed as column vectors. Then for $Z_1, Z_2\in H_p^{n_0}$ the corresponding isomorphism classes, for the above data, are the same if and only if $Z_1\in\Gamma_L Z_2$.

We now introduce the following definitions (see [**3**, § 2]): An analytic family of Abelian varieties is a triple $(\mathfrak{A}, \lambda, \mathfrak{B})$, where $\mathfrak{A}$ and $\mathfrak{B}$ are irreducible complex analytic spaces and λ is a proper complex analytic mapping of $\mathfrak{A}$ onto $\mathfrak{B}$ having the following properties:

(i) there is an analytic subset $\mathcal{E}$ of $\mathfrak{B}$ such that for $b\in\mathfrak{B}-\mathcal{E}$, the fiber $A_b=\lambda^{-1}(b)$ is an Abelian variety of fixed dimension m.

(ii) we define $\mathfrak{A}^{(l)}=(\lambda^l)^{-1}(\text{diagonal of } \mathfrak{B}^l)$, $\lambda^{(l)}=\lambda\mid\mathfrak{A}^{(l)}$ (for any positive integer l), and identify $\mathfrak{B}$ with the diagonal of $\mathfrak{B}^l$; then the group law of A_b is cut out on $A_b^{(3)}=\lambda^{(3)-1}(b)$ by a fixed analytic subset $\mathfrak{G}$ of $\mathfrak{A}^{(3)}$ for all $b\in\mathfrak{B}-\mathcal{E}$.

If in addition $\mathfrak{r}$ is some ring with unit 1, we say that $(\mathfrak{A}, \lambda, \mathfrak{B})$ admits $\mathfrak{r}$ as an endomorphism ring if we have further:

(iii) for each $\rho\in\mathfrak{r}$ we are given an analytic subset $\iota(\rho)$ of $\mathfrak{A}^{(2)}$ such that for each $b\in\mathfrak{B}-\mathcal{E}$, $\iota_b(\rho)=\iota(\rho)\cap A_b^{(2)}$ is the graph of an endomorphism of A_b and such that

$$\iota_b:\rho\longrightarrow\iota_b(\rho), \qquad b\in\mathfrak{B}-\mathcal{E},$$

is an isomorphism of $\mathfrak{r}$ into $\mathcal{A}(A_b)$, and $\iota_b(1)$ is the identity on A_b. We say that $b\in\mathfrak{B}-\mathcal{E}$ is a regular point of the fibering $(\mathfrak{A}, \lambda, \mathfrak{B})$ if each point of $\lambda^{-1}(b)$ is a non-singular point of $\mathfrak{A}$, if b is a non-singular point of $\mathfrak{B}$, and if the rank of the Jacobian matrix of λ at each point of $\lambda^{-1}(b)$ is equal to $\dim B$. If that is so, then for a suitable small neighborhood N of b, $(\lambda^{-1}(N), \lambda, N)$ is a complex analytic family in the sense of [**7**, p. 335, Def. 1.4]. If N is suitably small, we can find a basis of holomorphic 1-forms on $\lambda^{-1}(N)$ such that the periods of these with respect to a basis of 1-cycles on $\lambda^{-1}(b')$ for each $b' \in N$ are holomorphic functions of b' on N (see [**8**, 163-165]). We assume each polarized Abelian variety $\lambda^{-1}(b')$ with endomorphism ring is of the type described in (1)-(3) above. It is then easy to see that the above coordinatization of the problem of moduli of Abelian varieties with endomorphism ring can be carried out complex analytically on N and on $\lambda^{-1}(N)$. This shows that in an appropriately defined sense, $H_p^{n_0}$ is a natural universal space for analytic fiber systems of Abelian varieties with the given data. It is therefore natural if we restrict ourselves to fiber systems of polarized Abelian varieties with the given data whose base space is some quotient space of $H_p^{n_0}$ (or of some part of $H_p^{n_0}$). When we do this, we obtain a result analogous to that of [**2**, p. 363, Theorem 2], and which is stated in precise form at the end of § 3.

In the above definitions and remarks it is natural to replace complex analytic objects by the corresponding objects of algebraic geometry. When this is done, we should expect, following the ideas suggested by [**2**, 376-379] and using the techniques of [**11**] to be able to transfer the statements about universality of complex analytic fiber systems of Abelian varieties into statements about similar algebraic fiber systems, conserving as far as possible fields of definition. However, we shall dwell mainly on the special types of algebraic fiber systems to be dealt with here in § 3, since our main concern is to relate the problem of moduli for Abelian varieties with the theory of automorphic functions.

Finally, regarding the compactification of the orbit spaces $H_p^{n_0}/\Gamma_L$ as projective varieties, this has already been achieved for the case when k is totally real [**1**]. The result when k is a purely imaginary quadratic extension of a totally real field is a special case of results [**4**] for which it is anticipated proofs may be published before long.

At this point we should like to acknowledge our indebtedness to K. Katayama and G. Shimura for making available to us manuscripts of articles [**6**, **13**] which at the time of their being available to us had not yet appeared in published form. We should also like to acknowledge our indebtedness to these colleagues for numerous valuable conversations which helped to clear up our ideas on some of the matters touched on in this paper.

§ 2. The theory of θ-functions.

Let F_0 be a totally real number field and let F be a purely imaginary quadratic extension of F_0; put $n_0=[F_0:Q]$. Let $\sigma_1, \cdots, \sigma_{n_0}$ be distinct isomorphisms of F into C such that $i\neq j \Rightarrow \sigma_i \neq \bar{\sigma}_j$, and let τ denote complex conjugation on F. Let k be F_0 or F, which we call case I or case II, respectively. Let p be a positive integer, denote by L a lattice in k^{2p}, and denote by $\mathfrak{r}=\mathfrak{r}(L)$ the order of L, $\mathfrak{r}=\{\alpha\in k\,|\,\alpha L\subset L\}$. Denote by $Z_1, \cdots, Z_{n_0}$ complex $p\times p$ matrices and let $\zeta^{(1)}, \cdots, \zeta^{(n)}$, where $n=[k:Q]$, be complex p-vectors, $\zeta^{(j)}=(\zeta_1^{(j)}, \cdots, \zeta_p^{(j)})\in C^p$. For convenience of notation we let Z be a $p\times p$ matrix of indeterminates, let ζ be a p-vector with indeterminate components, and define $Z^{\sigma_j}=Z_j$, $\zeta^{\sigma_j}=\zeta^{(j)}$, and $\zeta^{(i)\tau}=\zeta^{(j)}$ with $j\equiv i+n_0 \pmod n$, $Z^{(j)\tau}={}^tZ^{(j)}$. In case I, we assume ${}^tZ_j=Z_j$, and in either case we assume

$$i(\overline{{}^tZ_j}-Z_j)\gg 0\,, \qquad j=1, \cdots, n_0\,.$$

We let $g=(g^{(1)}, \cdots, g^{(n)})$ and $h=(h^{(1)}, \cdots, h^{(n)})$ be n-tuples of complex p-vectors such that

$$g^{(i)}=\overline{g^{(j)}}=g^{(i)\tau}\,, \qquad h^{(i)}=\overline{h^{(j)}}=h^{(i)\tau}\,,$$

where $j\equiv n_0+i \pmod n$. We let L_Z denote the lattice in C^{2np} defined by

$$\{(\xi^{\sigma_1}, \cdots, \xi^{\sigma_{n_0}}, \xi^{\sigma_1\tau}, \cdots, \xi^{\sigma_{n_0}\tau})\,|\,\xi=(ZE)\lambda,\ \lambda\in L\}\,,$$

where (ZE) is a $p\times 2p$-matrix written in $p\times p$ blocks and E the identity matrix. G_Q will denote the group of $M\in GL(2p, k)$ such that ${}^t\bar{M}JM=J$, where $J=\begin{pmatrix} 0 & E \\ -E & 0\end{pmatrix}$. From the Corollary p. 267 of [6], we deduce, by an obvious modification of the proof of that corollary the following (see also [**16**], Prop. 1.3).

LEMMA 1. *Let L be an $\mathfrak{o}$-lattice in $V(2p, k)$ $(=k^{2p})$, where $\mathfrak{o}$ is the principal order of (all integers in) k. Let P denote a non-degenerate, skew-Hermitian matrix such that ${}^t\bar{x}Py\in\mathfrak{c}$ for x, y L, where $\mathfrak{c}$ is an ideal in k. Then, there exists*

$$T\in GL(2p, k) \text{ such that } L=T\begin{pmatrix} \left.\begin{matrix}\mathfrak{o}\\ \vdots \\ \mathfrak{o}\end{matrix}\right\} p \\ \left.\begin{matrix}\mathfrak{a}_1\\ \vdots \\ \mathfrak{a}_p\end{matrix}\right\} p\end{pmatrix} \text{ and such that } {}^t\bar{T}PT=J.$$

Hence, we may assume without loss of generality in the case $\mathfrak{r}=\mathfrak{o}$ that $L=L_1\oplus L_2$, where

$$L_1=\left\{\begin{pmatrix}\lambda_1\\ 0\end{pmatrix}\,\middle|\, \lambda_1\in\mathfrak{o}^p\right\}$$

and

$$L_2=\left\{\begin{pmatrix}0\\ \alpha\end{pmatrix}\middle|\ \alpha\in\begin{pmatrix}\mathfrak{a}_1\\ \vdots\\ \mathfrak{a}_p\end{pmatrix}\right\}$$

for $\mathfrak{a}_1, \cdots, \mathfrak{a}_p$ suitable lattices in k. We henceforth make this assumption. Using the above notation, we put

$$\theta_{L_1}[g, h](\zeta, Z)=\theta[g, h](\zeta, Z)=\sum_{\lambda_1\in L_1}\varepsilon\Big(\frac{1}{2}Z[\lambda_1+g]+{}^t(\overline{\lambda_1+g})(\zeta+h)\Big),$$

where $\varepsilon(\ )=e^{2\pi i\, tr(\ )}$ and $Z[a]={}^t\bar{a}Za$.
It is easy to see that $\theta[g, h]$ converges uniformly on any compact subset of the region $\{Z_j\in H_p,\ j=1, \cdots, n_0;\ \zeta^{(i)}\in C^p,\ i=1, \cdots, n\}$. An entire function in this region satisfying the functional equation

$$\theta(\zeta+Z\lambda_1+\lambda_2, Z)=\varepsilon\Big(m\Big(-\frac{1}{2}Z[\lambda_1]-{}^t\bar{\lambda}_1\zeta\Big)+{}^t\bar{g}\lambda_2-{}^t\bar{h}\lambda_1\Big)\theta(\zeta, Z),$$
$$\lambda_i\in L_i, \quad i=1, 2,$$

is called a θ-function of m^{th} order and characteristic (g, h) with respect to L. It is easy to see that $\theta_{L_1}[g, h]$ is a θ-function of the first order with respect to $L^*=L_1\oplus L_1'$, L_1 and L_1' being identified with lattices in k^p in an obvious manner, and L_1' being the complementary lattice to L_1: $L_1'=\{\lambda\in k^p\mid tr({}^t\bar{\lambda}\lambda)\in Z$: the rational integers$\}$. Let $M=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in G_Q$, define $Z'=MZ=(AZ+B)\cdot(CZ+D)^{-1}$ and $\zeta'={}^t(\bar{C}{}^tZ+\bar{D})^{-1}\zeta$. The main problem of the transformation theory of θ-functions of this kind is to find an expression for $\theta[g, h](\zeta, Z)$ as a linear combination of θ-functions of the form $\theta[g', h'](\zeta', Z')$ for suitable (g', h'). For this purpose let $\mathcal{A}$ be the group of $M=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in G_Q$ such that $C=0$, and consider the double coset decomposition

$$G_Q=\bigcup \mathcal{A}T_i\mathcal{A}. \qquad \text{(Bruhat decomposition)}$$

It is easy to show that there are $p+1$ such double cosets and as a set of representatives, we may choose $T_0, \cdots, T_p$, where

$$T_q=\begin{pmatrix}0 & 0 & E_q & 0\\ 0 & E_{p-q} & 0 & 0\\ -E_q & 0 & 0 & 0\\ 0 & 0 & 0 & E_{p-q}\end{pmatrix}$$

E_ν denoting the $\nu\times\nu$ identity matrix; if $M=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in G_Q$, then $M\in\mathcal{A}T_q\mathcal{A}$ if and only if the rank of C is q. It is obvious that if $\begin{pmatrix}A & B\\ 0 & D\end{pmatrix}\in\mathcal{A}$, we may write $\begin{pmatrix}A & B\\ 0 & D\end{pmatrix}=\begin{pmatrix}\rho^{-1}A' & \sigma^{-1}B'\\ 0 & \sigma^{-1}D'\end{pmatrix}$, where σ and ρ are positive integers and A', B', $D'\in M_p(\mathfrak{r})$, $\mathfrak{r}=\mathfrak{r}(L)$; then we must have $\rho^{-1}{}^t\bar{A}'=\sigma D'^{-1}$, and so

$$M=\begin{pmatrix} {}^t\bar{D}'^{-1} & 0 \\ 0 & D' \end{pmatrix}\begin{pmatrix} E & {}^t\bar{D}'B' \\ 0 & E \end{pmatrix}\begin{pmatrix} \sigma E & 0 \\ 0 & \sigma^{-1}E \end{pmatrix}$$

E denoting the $p\times p$ identity. Transformations of the form $\begin{pmatrix} {}^t\bar{D} & 0 \\ 0 & D^{-1} \end{pmatrix}$ are said to be of the first kind, those of the form $\begin{pmatrix} E & S \\ 0 & E \end{pmatrix}$, with $S\in M_p(\mathfrak{r})$, of the second kind, while $T_0, \cdots, T_p$ are said to be of the third kind. We now consider these types in turn. First let

$$M=\begin{pmatrix} \sigma^{-1}\,{}^t\bar{D} & 0 \\ 0 & \sigma D^{-1} \end{pmatrix}$$

σ a positive integer, and assume D has coefficients in $\mathfrak{r}=\mathfrak{r}(L)$. Let $\det D=\delta$. We have

$$\theta(\zeta, Z)=\sum_{m\in L_1}\varepsilon\Big(\frac{1}{2}Z[m]+{}^t\bar{m}\zeta\Big).$$

If A is any matrix, denote by A^* the adjoint matrix of A (so that $A^* =(\det A)A^{-1}$). Put, for $m\in L_1$, $n=\sigma D^{-1}m$, or $\delta n=\sigma D^*m$. Then

$$\theta(\zeta, Z)=\sum_n{}'\,\varepsilon\Big(\frac{1}{2}Z'[n]+{}^t\bar{n}\zeta'\Big),$$

where $Z'=\sigma^{-2}Z[D]$, and $\zeta'=\sigma^{-1}\,{}^t\bar{D}\zeta$, and where Σ' means to sum over all $n\in\sigma D^{-1}L_1$. Now let

$$n=\hat{n}+\sigma D^{-1},$$

where $\hat{n}\in L_1$ is such that $D\hat{n}\in\sigma L_1$. Then

$$\theta(\zeta, Z)=\sum_\rho\sum_{\hat{n}}{}^*\varepsilon\Big(\frac{1}{2}Z'[\hat{n}+\sigma D^{-1}\rho]+{}^t(\bar{\hat{n}}+\overline{\sigma D^{-1}\rho})\zeta'\Big),$$

where $\sum^*$ denotes summation over $\hat{n}\in L_1\cap\sigma D^{-1}L_1$. Put

$$F(\hat{n}, \rho)=\frac{1}{N(\sigma)^p}\sum_{\gamma\in L_1'/\sigma L_1'}\varepsilon(\sigma^{-1}\,{}^t(\bar{\hat{n}}+\sigma\bar{D}^{-1}\bar{\rho}){}^t\bar{D}\gamma).$$

Then

$$F=\begin{cases} 1 & \text{if } D\hat{n}\in\mathfrak{o}L_1, \\ 0 & \text{otherwise.}\end{cases}$$

(This follows from elementary properties of group characters.) Hence,

$$\begin{aligned} N(\sigma)^p\theta(\zeta, Z)&=\sum_{\rho,\gamma}\sum_{\hat{n}\in L_1}\varepsilon\Big(\frac{1}{2}Z'[\hat{n}+\sigma D^{-1}\rho]+{}^t(\hat{n}+\sigma D^{-1})(\zeta+\sigma^{-1}\,{}^t\bar{D}\gamma)\Big) \\ &=\sum_{\rho\in L_1/DL_1}\sum_{\gamma\in L_1'/\sigma L_1'}\theta[\sigma D^{-1}\rho, \sigma^{-1}D\gamma](\zeta', Z'); \end{aligned}$$

if $\sigma=\delta=1$, this sum contains just one term. By making a suitable change of variables (replacing ζ by $\zeta+Zg+h$), we obtain (with $L_1^*=L_1\cap\sigma D^{-1}L_1$)

$$N(\sigma)^p\theta_{L_1}[g, h](\zeta, Z)=\sum_{\rho}\sum_{\gamma}\theta_{L_1^*}[\sigma D^{-1}(\rho+g), \sigma^{-1}D(\gamma+h)](\zeta', Z').$$

We now consider transformations of type II. Here we have

$$\theta_{L_1}[g, h](\zeta, Z)=\sum_{m\in L_1}\varepsilon\Big(\frac{1}{2}Z[m+g]+{}^t(\overline{m+g})(\zeta+h)\Big).$$

Put $Z'=Z+S$, $\zeta'=\zeta$, where $S=(S_{ij})$ is Hermitian and $S_{ij}\in\mathfrak{a}=\{\alpha\in k\,|\,\alpha L_1\subset L_1'\}$. We see from our earlier considerations that we may take S_{ij} in any pre-assigned lattice $\mathfrak{a}$ in k. Then for some fixed basis $\{\omega_i\}_{i=1}^{pn}$ of L_1, we solve the equations

$$\sum_{\substack{j=1\\ \alpha=1}}^{\substack{\alpha=p\\ j=n}}\omega_{i\alpha}^{\rho_j}\bar{a}_{\alpha j}=\sum_{\substack{j=1\\ \alpha=1}}^{\substack{\alpha=p\\ j=n}}(S_{\alpha\alpha}\omega_{i\alpha}\omega_{i\alpha})^{\rho_j}, \qquad (S_{\alpha\alpha}\text{ is totally real})$$

where $\omega_i=(\omega_{i1},\cdots,\omega_{ip})\in k^p$ and $\rho_1,\cdots,\rho_n$ are all the isomorphisms of k into C; this system of np equations in np unknowns has a unique solution, since $\det(\omega_{i\alpha}^{\rho_j})\neq 0$, and we may write $\bar{a}_{\alpha j}=\bar{a}_{\alpha}^{\sigma_j}$ for some $a_\alpha\in k$; the vector a is denoted by $\delta(S)$. Then $a=(a_1,\cdots,a_p)\in L_1'$. Then for any $m\in L_1$, $m=\sum b_i\omega_i$, we have

$$\frac{1}{2}\sum_j S^{\rho_j}[m^{\rho_j}]\equiv\frac{1}{2}\sum_i b_i^2\sum_j S^{\rho_j}[\omega^{\rho_j}]$$
$$\equiv\frac{1}{2}\sum_i b_i\sum_j {}^t\bar{a}^{\sigma_j}\omega_i^{\sigma_j}=\frac{1}{2}tr({}^t\bar{a}m), \pmod 1.$$

Hence,

$$\varepsilon\Big(\frac{1}{2}Z[m+g]+{}^t(\overline{m+g})(\zeta+h)\Big)$$
$$=\varepsilon\Big(\frac{1}{2}Z'[m+g]+{}^t(\overline{m+g})\Big(\zeta+h-Sg-\frac{1}{2}a\Big)\Big)\varepsilon\Big(\frac{1}{2}S[g]-\frac{1}{2}{}^t\bar{g}a\Big),$$

where $Z'=Z+S$ and $\zeta'=\zeta$. Hence,

$$\theta[g, h](\zeta, Z)=\varepsilon\Big(\frac{1}{2}(S[g]-{}^t\bar{g}a)\Big)\theta[g', h'](\zeta', Z'),$$

where $g'=g$ and $h'=h-Sg-\frac{1}{2}a$. We observe that $a\in L_1'$. Finally, let T_q be defined as before with $0\leqq q\leqq p$. We shall treat this case by obtaining a particular version of the Poisson summation formula. Let ψ be an entire function of $z_1,\cdots,z_n$, $z_i\in C^q$, such that $\sum_{m^*\in L_1^*}\psi(z+m^*)$ converges absolutely and uniformly on every compact subset of C^{nq}, where

$$L_1^*=\left\{\begin{pmatrix}\lambda^{\rho_1}\\ \vdots\\ \lambda^{\rho_n}\end{pmatrix}\middle|\,\lambda=\begin{pmatrix}\lambda_1\\ \vdots\\ \lambda_q\end{pmatrix},\quad \binom{\lambda}{0}\in L_1\right\}.$$

Then $f(z)=\sum_{m\in L_1^*}\psi(z+m)$ is an entire function such that $f(z+m)=f(z)$ for all $m\in L_1^*$. Hence

$$f(z)=\sum_{m_1'\in L_1^{*\prime}} a_{m'}\varepsilon({}^t\overline{m}'z),$$

where $L_1^{*\prime}$ is the complementary lattice to L_1^* and

$$v(\square)\cdot a_{m'}=\int_{\square} f(z)\varepsilon(-{}^t\overline{m}'z)dx,$$

dx denoting an nq-dimensional (real) volume element and where $\square$ is a period parallelogram for the lattice $L_1^{*\prime}$ in the real vector space which it spans in C^{nq} ($v(\square)$ is the volume of $\square$). Hence

$$\begin{aligned} v(\square)f(z)&=\sum_{m'\in L_1^{*\prime}} v(\square)a_{m'}\varepsilon({}^t\overline{m}'z)\\ &=\sum_{m'\in L_1^{*\prime}}\sum_{m\in L_1^*}\int_{\square}\psi(x+m)\varepsilon({}^t\overline{m}'(z-x))dx\\ &=\sum_{m'\in L_1^{*\prime}}\int_{R^{nq}}\psi(x)\varepsilon({}^t\overline{m}'(z-x))dx. \end{aligned}$$

Putting $z=0$, we obtain

$$v(\square)\sum_{m\in L_1^{*\prime}}\psi(m)=v(\square)f(0)=\sum_{m'\in L_1^{*\prime}}\int_{R^{nq}}\psi(x)\varepsilon(-{}^t\overline{m}'x)dx,$$

where we identify R^{nq} with the real vector space spanned by $L_1^{*\prime}$ (which is the same as that spanned by L_1^*, since $(L_1^*, L_1^{*\prime})\subset Z\subset R$). Let

$$L_1^{**}=\left\{\kappa=\begin{pmatrix}\kappa_1\\ \vdots\\ \kappa_{p-q}\end{pmatrix}\middle|\begin{pmatrix}0\\ \kappa\end{pmatrix}\in L_1\right\}.$$

Put

$$\psi(x)=\sum_{\kappa\in L_1^{**}}\varepsilon\left(\frac{1}{2}Z\begin{bmatrix}x+g_1\\ \kappa+g_2\end{bmatrix}+{}^t\begin{pmatrix}x+g_1\\ \kappa+g_2\end{pmatrix}\left(\zeta+\begin{pmatrix}h_1\\ h_2\end{pmatrix}\right)\right),$$

where $g_1, h_1\in C_q$, $g_2, h_2\in C^{p-q}$. Then

$$\sum_{m\in L_1^*}\psi(m)=\theta[g,h](\zeta,Z).$$

Hence, since ψ evidently satisfies the necessary hypotheses, we have

$$\theta[g,h](\zeta,Z)=v(\square)^{-1}\cdot A,$$

where

$$\begin{aligned} A&=\sum_{m'\in L_1^*}\int_{R^{nq}}\psi(x)\varepsilon(-{}^t\overline{m}'x)dx\\ &=\sum_{m'\in L_1^{*\prime}}\sum_{\kappa\in L_1^{**}}\int_{R^{nq}}\varepsilon\left(\frac{1}{2}Z\begin{bmatrix}x+g_1\\ \kappa+g_2\end{bmatrix}+{}^t\begin{pmatrix}x_1+g_1\\ \kappa\ +g_2\end{pmatrix}\begin{pmatrix}\zeta_1+h_1\\ \zeta_2+h_2\end{pmatrix}-{}^t\overline{m}'x\right)dx, \end{aligned}$$

where $\zeta=\begin{pmatrix}\zeta_1\\ \zeta_2\end{pmatrix}$, ζ_1 being composed of the first q components of ζ, and ζ_2, of the last $p-q$. Put $Z=\begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix}$, and introduce the following notation:

$$Z^*=\begin{pmatrix} 0 & Z_{11} & 0 & Z_{12} \\ {}^tZ_{11} & 0 & {}^tZ_{21} & 0 \\ 0 & Z_{21} & 0 & Z_{22} \\ {}^tZ_{12} & 0 & {}^tZ_{22} & 0 \end{pmatrix}=\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$$

$$x_1=x+g_1\,,\quad x_2=\kappa+g_2\,,\quad \eta_1=\zeta_1+h_1-m_1'\,,\quad \eta_1'=\zeta_1^\tau+\bar{h}_1-\bar{m}_1'$$

$$\eta=\zeta_2+h_2\,,\quad \eta_2'=\zeta_2^\tau+\bar{h}_2\,,\quad \eta_1''=\begin{pmatrix}\eta_1\\ \eta_1'\end{pmatrix},\quad \eta_2''=\begin{pmatrix}\eta_2\\ \eta_2'\end{pmatrix},\quad \xi_1=\begin{pmatrix}\bar{x}_1\\ x_1\end{pmatrix},\quad \xi_2=\begin{pmatrix}\bar{x}_2\\ x_2\end{pmatrix}.$$

Then Z_{11} and therefore X_{11} are non-singular. Clearly ${}^tZ^*=Z^*$. So

$$\frac{1}{2}Z\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\frac{1}{2}{}^tZ\begin{bmatrix}\bar{x}_1\\ \bar{x}_2\end{bmatrix}+{}^t\begin{pmatrix}\bar{x}_1\\ \bar{x}_2\end{pmatrix}\begin{pmatrix}\zeta_1+h_1\\ \zeta_2+h_2\end{pmatrix}$$

$$+{}^t\begin{pmatrix}x_1\\ x_2\end{pmatrix}\begin{pmatrix}\zeta_1^\tau+\bar{h}_1\\ \zeta_2^\tau+\bar{h}_2\end{pmatrix}-{}^t\bar{m}_1'x-{}^tm_1'\bar{x}$$

$$=\frac{1}{2}Z^*\begin{Bmatrix}\xi_1\\ \xi_2\end{Bmatrix}+{}^t\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}\begin{pmatrix}\eta_1''\\ \eta_2''\end{pmatrix}+{}^t\bar{m}_1'g_1+{}^tm_1'\bar{g}_1=B\,,$$

where $X\{a\}={}^taXa$. (We observe that the vector space R^{nq} is spanned by vectors of the form $\begin{pmatrix}a\\ \bar{a}\end{pmatrix}$.) Then

$$B=\frac{1}{2}X_{11}\{\xi_1+X_{11}^{-1}X_{12}\xi_2+X_{11}^{-1}\eta_1''\}-\frac{1}{2}X_{11}\{X_{11}(X_{12}\xi_2+\eta_1'')\}$$

$$+\frac{1}{2}X_{22}\{\xi_2\}+{}^t\xi_2\eta_2''+{}^t\bar{m}_1'g_1+{}^tm_1'\bar{g}_1$$

$$=\frac{1}{2}X_{11}\{\xi_1+l\}-{}^t\zeta H\zeta-{}^t\zeta^\tau H^\tau\zeta+\frac{1}{2}Z'\begin{bmatrix}m_1'-h_1\\ \kappa+g_2\end{bmatrix}$$

$$+\frac{1}{2}{}^tZ'\begin{bmatrix}\overline{m_1'-h}\\ \overline{\kappa+g_2}\end{bmatrix}+{}^t\begin{pmatrix}\overline{m_1'-h_1}\\ \overline{\kappa+g_2}\end{pmatrix}\left(\zeta'+\begin{pmatrix}g_1\\ h_2\end{pmatrix}\right)$$

$$+{}^t\begin{pmatrix}m_1'-h_1\\ \kappa+g_2\end{pmatrix}\left(\zeta'^\tau+\begin{pmatrix}\bar{g}_1\\ \bar{h}_2\end{pmatrix}\right)+{}^tg_1\bar{h}_1+{}^t\bar{g}_1h_1\,,$$

where $\begin{pmatrix}A & B\\ C & D\end{pmatrix}=T_q$, $H=(CZ+D)^{-1}C$, $Z'=T_qZ$, $\zeta'={}^t(CZ+D)^{-1}\zeta$, and l is a complex vector independent of ξ_1. Finally, X_{11} is a symmetric form on $R^{nq}\times R^{nq}$, with $\det X_{11}=(\det Z_{11})^2$, having positive definite imaginary part. Hence,

$$\int_{R^{n}}\varepsilon\left(\frac{1}{2}X_1\{\xi_1+l\}\right)dx=\pm N(\det Z_{11})^{\frac{1}{2}}\,,$$

where N means "norm" in the sense of taking the product over all formal conjugates. Thus, if $\begin{pmatrix} A & B \\ C & D \end{pmatrix} = T_q$, we obtain

$$\theta[g, h](\zeta, Z)$$
$$= \pm v(\varXi)^{-1} N(\det Z_{11})^{-\frac{1}{2}} \varepsilon\Big(\frac{1}{2} Z_{11}^{-1}[\zeta_1]\Big)\varepsilon({}^t\bar{g}_1 h_1) \sum_{\substack{m' \in L_1^{*\prime} \\ m_2 \in L_1^{**}}} \varepsilon\Big(\frac{1}{2} Z'\begin{bmatrix} m' \\ m_2+g' \end{bmatrix}$$
$$+ {}^t\begin{pmatrix} m' \\ m_2+g' \end{pmatrix}(\zeta'+h')\Big),$$

where $g' = \begin{pmatrix} -h_1 \\ g_2 \end{pmatrix}$, $h' = \begin{pmatrix} g_1 \\ h_2 \end{pmatrix}$. Finally, we remark that the sum on the right above can be written as a sum of θ-functions attached to the lattice L_1. This follows from the observation that if Λ_1 and Λ_2 are lattices, then any of the θ-functions $\theta_{\Lambda_1}[g, h]$ can be expressed as a linear combination of the θ-functions $\theta_{\Lambda_2}[g', h']$, for suitable g' and h'. To prove this, it is sufficient to consider the cases (a) $\Lambda_1 \supset \Lambda_2$ and (b) $\Lambda_2 \supset \Lambda_1$. Case (a) is obvious since then $\Lambda_1 = \bigcup(\Lambda_2+c)$ for a finite number of c. For case (b), let $\chi_1, \cdots, \chi_N$ be the characters of the group Λ_2/Λ_1. Each χ_i can be written in the form $\chi_i(a) = \varepsilon({}^t\bar{\lambda}_i' a)$, for some $\lambda_i' \in \Lambda_1'$ and for all $a \in \Lambda_2$. Then if $m = (m_1, \cdots, m_p) \in \Lambda_2$,

$$\sum_{i=1}^{n} \chi_i(m) = \begin{cases} 0 & \text{if } m \notin \Lambda_1, \\ N & \text{if } m \in \Lambda_1. \end{cases}$$

Therefore

$$\theta_{\Lambda_1}[g, h](\zeta, Z) = \sum_{m \in L_1} \varepsilon\Big(\frac{1}{2} Z[m+g] + {}^t(\overline{m+g})(\zeta+h)\Big)$$
$$= N^{-1} \sum_{m \in \Lambda_2} \sum_i \chi_i(m) \varepsilon\Big(\frac{1}{2} Z[m+g] + {}^t(\overline{m+g})(\zeta+h)\Big)$$
$$= \sum_i \Big(\sum_{m \in \Lambda_2} \varepsilon\Big(\frac{1}{2} Z[m+g] + {}^t(\overline{m+g})(\zeta+h+\lambda_i')\Big)\varepsilon(-{}^t g\lambda_i')\Big),$$

where $\lambda' \in \Lambda_1'$. Hence

$$\theta_{\Lambda_1}[g, h] = \sum_i \theta_{\Lambda_2}[g, h+\lambda_i']\varepsilon(-{}^t\bar{g}\lambda_i') \cdot N^{-1}.$$

Combining the above results, we obtain, finally, for any $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(p, k)$, that

$$\theta_{L_1}[g, h](\zeta, Z) = \gamma(\zeta, Z) \sum_{g', h'} c_{g', h'} \theta_{L_1}[g', h'](\zeta', Z'),$$

where $\gamma(\zeta, Z) = (N\det(CZ+D))^{-\frac{1}{2}}\varepsilon(-{}^t\zeta' C\zeta)$, $\zeta' = {}^t(CZ+D)^{-1}\zeta$, $Z' = (AZ+B)(CZ+D)^{-1}$, and $c_{g', h'}$ are constants; the sum on the right side is finite, and if $g, h \in k$, then $g', h' \in k$, and $c_{g', h'}$ belong to a cyclotomic number field. We are particularly interested in the case for which ${}^t\overline{\begin{pmatrix} A & B \\ C & D \end{pmatrix}}L = L$. Such a matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is called an L-unit. The group of L-units is denoted by Γ_L.

If L_1 and L_2 are complimentary lattices in k^p, then the C-module of θ-functions of given characteristic g, h with respect to L and of 1st order is one-dimensional (*vide infra*). Assuming this we shall show in case $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_L$ that the above formula takes the form:

$$\theta_{L_1}[g, h](\zeta, Z) = c \cdot (N \det(CZ+D)^{-\frac{1}{2}}) \varepsilon\Big(-\frac{1}{2}{}^t\zeta' C\zeta\Big) \cdot \theta_L[g', h'](\zeta', Z'),$$

where $|c|=1$ and

$$g' = Dg - Ch + \frac{1}{2}\delta(C{}^t\bar{D})$$

$$h' = -Bg + Ah + \frac{1}{2}\delta({}^t\bar{B}A),$$

where $\delta(S)$ has the previously defined meaning for Hermitian S such that $\bar{S}L_1 \subset L_1'$, (we recall that $L_1 = \mathfrak{o}^p$) or $\bar{S}L_1' \subset L_1$. In fact, for this choice of (g', h'), put $\varphi(\zeta, Z) = \varepsilon\Big(-\frac{1}{2}{}^t\zeta' C\zeta\Big)\theta_{L_1}[g', h']$. Then for $\lambda_1 \in L_1$, $\lambda_2 \in L_1'$, we have

$$\begin{aligned}\varphi(\zeta + Z\lambda_1 + \lambda_2) &= \theta_{L_1}[g', h'](\zeta' + Z'(D\lambda_1 - C\lambda_2) + (-B\lambda_1 + A\lambda_2), Z') \cdot \\ &\quad \cdot \varepsilon\Big(-\frac{1}{2}{}^t(\zeta' + Z'(D\lambda_1 - C\lambda_2) + (-B\lambda_1 + A\lambda_2))C(\zeta + Z\lambda_1 + \lambda_2)\Big) \\ &= \theta_L[g', h'](\zeta', Z')\varepsilon\Big(-\frac{1}{2}{}^t\zeta' C\zeta\Big) \cdot \varepsilon(\sharp),\end{aligned}$$

where $\sharp = tr\Big\{-{}^t(\overline{D\lambda_1 - C\lambda_2})\Big(\frac{1}{2}Z'(D\lambda_1 - C\lambda_2) + \zeta'\Big) + \cdots\Big\}$. Direct computation shows the latter to be $\equiv tr\Big\{-\frac{1}{2}Z[\lambda_1] - {}^t\bar{\lambda}_1\zeta + {}^t\bar{\lambda}_2 g - {}^t\bar{\lambda}_1 h\Big\} \bmod 1$. Hence

$$\theta_{L_1}[g, h](\zeta, Z) = c \cdot \theta_{L_1}[g', h'](\zeta', Z')\varepsilon\Big(-\frac{1}{2}{}^t\zeta' C\zeta\Big),$$

where c depends only on Z. We know in any case from our preceding formulas that $c = (N\det(CZ+D)^{-\frac{1}{2}}) \cdot c_1$, where c_1 is a constant. For the purpose of computing c, it is easily seen to be sufficient to consider the case for which $\det C \neq 0$; for we can always write

$$M = \begin{pmatrix} E & 0 \\ \nu E & E \end{pmatrix}\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}$$

where $\nu \neq 0$ and $\det C_1 \neq 0$ for suitable choice of ν. In the following, we say a rational integer l is "sufficiently divisible" if l is divisible by $a!$ for a sufficiently large rational integer a. When $\det C \neq 0$, we may use the computations on 173-181 of [**9**] to compute c, and we find that

$$c = \varepsilon\Big(\frac{1}{2}\psi(g, h)\Big) N\det(CZ+D)^{-1/2}\chi(M),$$

where $\chi(M)$ is a complex number depending only on M. Using an argument similar to that of lines 7-12, p. 347 of [**2**], we see that there is an integer d such that $\chi(M)^d$ is a character on Γ_L, i.e., a homomorphism of Γ_L into the multiplicative group of non-zero complex numbers. As for the more precise nature of $\chi(M)$, we assume that

$$M\in\Gamma_L(l)=\{M_1\,|\,{}^t(\overline{M_1-E})L\subset lL\}$$

for sufficiently divisible l. Let $\varnothing$ denote a "period parallelogram" for L_1. If Ω is the matrix of the conjugates of a set of basis vectors of L_1 over the rational integers, numbered and arranged in suitable order, then $|\det\Omega|=v(\varnothing)$. We write the rational number $v(\varnothing)^2$ as the quotient $\frac{a}{b}$ of rational integers such that $a\Omega^{-1}$ and $b\Omega$ are algebraic integral matrices and assume $4ab\,|\,l$. Then using the argument of lines 12-17 of page 347 of [**2**], we can easily prove $\chi(M)$ is a root of unity, provided we can prove $|\chi(M)|=1$. To prove $|\chi(M)|=1$, we observe that the calculations 173-181 of [**9**] may be carried over to our case with certain minor, but essential, modifications which we now describe. First, Krazer's matrix T [p. 131 loc. cit.] is $\begin{pmatrix} D & -C \\ -B & A\end{pmatrix}$ in our notation. Secondly, Krazer's period matrix α and vector variable u are just πiZ and $\pi i\zeta$ in our notation. In the calculations 173-181 of [**9**], in formula (163) p. 173, the integral indicated there should be $v(\varnothing)^{-1}$, when not zero, instead of 1 (we note that integration must be carried out over a "period parallelogram" of L_1'). Moreover, the new indices of summation n and ρ introduced on 174-175 should be restricted by requiring that $n\in L_1'$ and $\rho\in L_1$. Then the number of "normal solutions" of (173), p. 175 is the number of solutions ρ of $C^{-1}\rho\in L_1'$, for $\rho\in L_1$, incongruent modulo $|\det\Omega^{-2}\cdot N\det C|\,L_1$. If $P=C$ or D, we put

$$P^*=\begin{pmatrix} P^{\sigma_1} & & & & & 0\\ & \ddots & & & & \\ & & P^{\sigma_{n_0}} & & & \\ & & & P^{\sigma_1\tau} & & \\ & & & & \ddots & \\ 0 & & & & & P^{\sigma_{n_0}\tau}\end{pmatrix}$$

and put $C_0=\Omega^{-1}C^{*t}\bar{\Omega}^{-1}$, $D_0=\Omega^{-1}D^*\Omega$. Then the number of normal solutions referred to above is just $|\det C_0|^{p-1}$. Moreover, if (C,D) is a primitive Hermitian pair and if, as we have supposed, $M\in\Gamma_L(l)$, then (C_0,D_0) is a primitive symmetric pair of rational integral matrices with $C_0\equiv 0$, $D_0\equiv E$ mod 4. In (XXXI) p. 181 of [**9**], $\left|\frac{(-\pi)^p}{\Delta_A}\right|^{\frac{1}{2}}=|N\det(CZ+D)^{-\frac{1}{2}}|$, $\Delta_{II}=N\det C$, $V_{II}^{1-p}=|\det C_0|^{1-p}$, and we see that G becomes the modified Gaussian sum associated with (C_0, D_0) in the classical transformation theory of θ-functions, so that $|G|=|\det C_0|^{p-\frac{1}{2}}$. Hence, in the formula (XXXI) we have

$$\begin{aligned}|v(\square)^{-1}c| &= |N\det(CZ+D)^{-\frac{1}{2}}|\cdot|\det C_0|^{\frac{1}{2}}|N\det C|^{-\frac{1}{2}} \\ &= |N\det(CZ+D)^{-\frac{1}{2}}||\det \Omega|^{-1}\end{aligned}$$

or since $v(\square)=|\det\Omega|$, we have $|N\det(CZ+D)^{-\frac{1}{2}}|=|c|$, so that $|\chi(M)|=1$. Since $\chi(M)$ is a product of a Gaussian sum by a root of unity and by a real number, it follows that $\chi(M)$ is a root of unity. Clearly its order is bounded since $\Gamma_L(l)$ is finitely generated. Hence, for any l_0, $l|l_0$, we have $\chi(M)^d=1$ for $M\in\Gamma_L(l_0)$, for suitable d.

Let $L=L_1\oplus L_2$ be as at the beginning of § 2. We assume that $\mathfrak{r}(L_1)=\mathfrak{r}(L_1')=\mathfrak{r}(L_2)=\mathfrak{r}(L_2')=\mathfrak{r}$, so in particular $\mathfrak{r}=\bar{\mathfrak{r}}$. For fixed characteristic (g, h), for fixed Z, and for a positive rational integer m, we wish to find a basis for the m^{th} order θ-functions Θ of characteristic (g, h), i.e., entire functions such that

$$\Theta(\zeta+Z\lambda_1+\lambda_2)=\varepsilon\Big(-m{}^t\bar{\lambda}_1\Big(\frac{1}{2}Z\lambda_1+\zeta\Big)+{}^t\bar{\lambda}_2-{}^t\bar{\lambda}_1h\Big)\Theta(\zeta).$$

We assume $mL_2\subset L_1'$, so that $mL_1\subset L_2'$. By multiplication of L by a suitable scalar, we may assume without loss of generality that $L_2\supset L_1'$. Θ may be expanded in a Fourier series:

$$\Theta(\zeta)=\varepsilon({}^t\bar{g}\zeta)\sum_{\mu\in L_2'}C_\mu\varepsilon({}^t\bar{\mu}\zeta),$$

and it is easy to show that for $\lambda_1\in L_1$ we have

$$C_{\mu_0+m\lambda_1}=C_{\mu_0}\varepsilon\Big(\frac{1}{2}mZ[\lambda_1]+{}^t\bar{\mu}_0Z\lambda_1+{}^t\bar{\lambda}_1h+{}^t\bar{g}Z\lambda_1\Big),$$

so that if $\mu_0\in L_2'$ runs over a complete set of representatives modulo mL_1, we have

$$\Theta(\zeta)=\varepsilon({}^t\bar{g}\zeta)\sum_{\mu_0}C_{\mu_0}\sum_{\lambda_1\in L_1}\varepsilon\Big(\frac{1}{2}mZ[\lambda_1]+{}^t\bar{\mu}_0Z\lambda_1+{}^t\bar{g}Z\lambda_1+{}^t(\bar{\mu}_0+m\bar{\lambda}_1)\zeta+{}^t\bar{\lambda}_1h\Big),$$

or

$$\Theta(\zeta)=\varepsilon({}^t\bar{g}\zeta)\cdot\Big(\sum_{\mu_0}C_{\mu_0}\varepsilon\Big(-\frac{1}{2m}Z[\mu_0+g]\Big)\theta_{L_1}\Big[\frac{\mu_0+g}{m}, h\Big](m\zeta, mZ)\Big).$$

Hence, the dimension of the module of such θ-functions is $[L_2':mL_1]$. Denote this module by $\mathcal{L}(m, L, g, h)$. Let m be such that in fact $mL_2\subset 3L_1'$ (or $mL_2\subset pL_1'$, for any $p\geqq 3$). Let $\Theta_0, \cdots, \Theta_N$ be a basis over C (for fixed Z) of $\mathcal{L}(m, L, g, h)$. Let L_Z be defined as in § 2, and let A_Z be the complex torus C^{np}/L_Z. We define a mapping

$$\theta : A_Z \to CP^N ,$$

where CP^N is the N-dimensional complex projective space, by $\theta(\zeta) = [\Theta_0(\zeta) : \cdots : \Theta_N(\zeta)]$. Then by a proof which is mutatis mutandis identical with the classical proof in [**5**], we may show that θ is a biregular immersion of A_Z as an Abelian variety in CP^N.

We now refer to [**10**], p. 32, formulas (Π_0') and $(\bar{\Pi}_0)$. Precisely the same formulas can be shown to hold in our case with n replaced by some positive integer m and with n^p replaced by m^{np}. The calculations are just as in [**10**] with appropriate change of notation, and we shall not dwell on them here except to note that the indices β on p. 19 (loc. cit.) should be taken as elements of L_1'. At any rate, the result which these formulas imply is that if $L = L_1 \oplus L_1'$ (i.e., if $L_2 = L_1'$, viewing each as a lattice in k^p), then the module of θ-functions of m^{th} order and characteristic (g, h) is spanned by the products

$$\prod_{i=1}^{m} \theta\left[\frac{g+\lambda_i}{m}, \frac{h+\kappa_i}{m}\right](\zeta, Z)$$

for which $\sum \lambda_i \in mL_1$, $\sum \kappa_i \in mL_1'$. The argument for proving this is just as on p. 344 of [**2**].

It is also easy to prove that

$$\theta[g+g', h+h'](\zeta, Z) = \varepsilon\Big(\frac{1}{2} Z[g'] + {}^t g'(\zeta+h+h')\Big)\theta[g, h](\zeta+Zg'+h', Z)$$

and in particular if $\kappa \in L_1$, $\lambda \in L_1'$, then

$$\theta[g+\kappa, h+\lambda](\zeta, Z) = \varepsilon({}^t\bar{\lambda} g)\theta[g, h](\zeta, Z).$$

Hence, if $\sum_{i=1}^{m} h_i^{(1)} = \sum_{i=1}^{m} g_i^{(1)} = 0 = \sum_{i=1}^{m} h_i^{(2)} = \sum_{i=1}^{m} g_i^{(2)}$, then

$$\frac{\prod_i \theta[g_i^{(1)}, h_i^{(1)}](Zg'+h', Z)}{\prod_i \theta[g_i^{(2)}, h_i^{(2)}](Zg'+h', Z)} = \frac{\prod_i \theta[g_i^{(1)}+g', h_i^{(1)}+h'](0, Z)}{\prod_i \theta[g_i^{(2)}+g', h_i^{(2)}+h'](0, Z)}.$$

Denote the right side by $f[g_1^{(1)}, \cdots, h_m^{(2)}](Z)$. Let $Z_1, Z_2 \in H_p^{n_0}$ and suppose for all $m, g_1^{(1)}, \cdots, h_m^{(2)}$, we have $f[g_1^{(1)}, \cdots, h_m^{(2)}](Z_1) = f[g_1^{(1)}, \cdots, h_m^{(2)}](Z_2)$. The equality of the expressions on the left side of the above equation for Z_1 and Z_2 says that if $L = L_1 \oplus L_1'$ and if $A_{Z_i} = C^{np}/L_{Z_i}$, $i = 1, 2$, then we have an isomorphism φ of A_{Z_1} onto A_{Z_2} which carries all the points of any fixed order m on A_{Z_1} onto the points of order m on A_{Z_2}. Hence φ is induced by a mapping $\psi : \zeta \to a\zeta + b$ of C^{np} onto itself, and we evidently have $a(Z_1 g' + h') + b = Z_2 g' + h'$ for all g', h' sufficiently small. Hence $b = 0$ and $a = E$, the identity matrix. Therefore $Z_1 = Z_2$.

Just as in [**2**], 344-345, we may also write down an "addition formula" for θ-functions similar in form to (9) of [**2**]; the essential details of the proof are those in [**10**], sections, 3, 6, 7 with notation adapted to our present situation and with minor modifications in the summations performed there. The essential point is that the final formula must be the same in form (compare p. 58, loc. cit.; replace r by m, α by πimZ, and u by $\pi im\zeta$).

Finally, let $\rho \in \mathfrak{r}(L)$, $\rho \neq 0$. Then $\theta(\rho\zeta, Z)$ is a first order θ-function with respect to the lattice $\rho^{-1}(L_1 \oplus L_1')$. Since ρ is an endomorphism of $L_1 \oplus L_1'$, and since the latter is commensurable with $L_1 \oplus L_2$, it is clear that if $\theta_0(\zeta, Z)$, $\cdots$, $\theta_{np}(\zeta, Z)$ are algebraically independent θ-functions of the type we have been considering, then $\theta(\rho\zeta, Z)$ and these must satisfy an algebraic relation with coefficients which are holomorphic on $H_p^{n_0}$ and which therefore (see [**2**] p. 364) must be automorphic forms with respect to some congruence group contained in Γ_L. It is a priori evident that the numerical coefficients of the Fourier series involved must lie in some cyclotomic field, and that the given algebraic equation is carried into another valid algebraic equation (involving different θ-functions) by application of an automorphism of the Galois group of that cyclotomic field to these. We are now in a position to apply the methods of [**2**], section 5.1; we shall do this in the next section.

§ 3. The moduli of Abelian varieties.

Let $L = L_1 \oplus L_2$, k, etc., have the same meaning as in the previous section (we shall not assume $L_2 = L_1'$ unless so stated). Let H_p be defined as before (see introduction). Let R_0 be the subring of the ring of holomorphic functions on $H_p^{n_0} \times C^{np}$ generated by the functions $\theta_{L_1}[g, h](\zeta, Z)$ for $g \in k^p$ and $h \in k^p$. As we have seen, given $Z_1, Z_2 \in H_p^{n_0}$, there exist $g, h, g', h' \in k^p$ such that $\theta_{L_1}[g', h'](0, Z_1)$, $\theta_{L_1}[g', h'](0, Z_2) \neq 0$ and such that if $f(Z) = \theta_{L_1}[g, h](0, Z)/\theta_{L_1}[g', h'](0, Z)$, then $f(Z_1) \neq f(Z_2)$. Define subsets of R_0 as follows:

1) For each positive rational integer m, let

$$R_{m0} = \{\text{ring generated by 1 and by all } \theta_{L_1}[g, h] \text{ such that } mg \in L_1, mh \in L_2\}.$$

2) If l is a non-negative rational integer, let $R_0^{(l)}$ denote the module of homogeneous polynomials of degree l with rational integral coefficients in the functions $\theta_{L_1}[g, h](\zeta, Z)$, $g, h \in k^p$.

3) Put $R_{m0}^{(l)} = R_{m0} \cap R_0^{(l)}$.

Clearly, $R_{m0} = \oplus_l R_{m0}^{(l)}$, $\oplus$ denoting restricted direct sum (this follows from the functional equation for θ-functions). If r is any of the sets defined above, let $r^* = \{g \mid g(Z) = f(0, Z),\ Z \in H_p^{n_0},\ f \in r\}$. Let $K(r)$ and $K(r^*)$ denote the quotient fields of the rings generated by r and r^* respectively. If m is a positive rational integer, let Q_m denote the cyclotomic field $Q\left(\varepsilon\left(\frac{1}{m}\right)\right)$. Put

$A = \cup_m Q_m$; A is the maximal Abelian extension of the rationals Q. Let $G(A/Q)$ denote the Galois group of A. It is easy to see that $Q_{m^2}K(R_{m0}^{(l)})$ and $Q_{m^2}K(R_{m0}^{(l)*})$ are regular extensions of Q_{m^2}. If $\sigma \in G(A/Q)$, σ may be made to act naturally on $R_0 (= R_{m0}^{(l)}$ or $R_{m0}^{(l)*})$, and hence on $K(R_0)$, by allowing σ to act on the Fourier coefficients of all $\theta_{L_1}[g, h]$, which belong to A as soon as $g, h \in k^p$. If R_0 is any of the above rings or Z-modules (Z=rational integers), let $R = R_0 \otimes_Z A$. We have seen that $Sp(p, k)$ acts naturally on $R^{(l)}$; namely, if $P \in R^{(l)}$, and if $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(p, k)$, define

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} P(\zeta, Z) = \varepsilon\left(-\frac{l}{2}{}^t\zeta' C\zeta\right) N \det(CZ+D)^{-\frac{l}{2}} P(\zeta', Z').$$

If Γ is a subgroup of $Sp(p, k)$, denote by $R_\Gamma^{(l)}$ the subring or submodule of $R^{(l)}$ consisting of the elements invariant under Γ.

It follows from our imbedding theorem for Abelian varieties that for $m \geqq 3$, the elements of $R_m^{(1)}$ have no common zeros on $H_p^{n_0} \times C^{np}$, and so the elements of $R_m^{(1)*}$, and therefore the elements of $R_m^{(l)*}$ have no common zeros on $H_p^{n_0}$. We define

$$\Gamma_L(m) = \{g \in \Gamma_L | ({}^t\bar{g} - E)L \subset mL\},$$

E denoting as always the identity matrix of appropriate dimension. Then for fixed m and for sufficiently divisible l and d, $R_{m,\Gamma_L(d)}^{(l)} = R_m^{(l)}$ and $R_{m,\Gamma_L(d)}^{(l)*} = R_m^{(l)*}$. These statements are easy consequences of the transformation formula. Let $\theta_0^*, \cdots, \theta_N^*$ be a basis of $R_m^{(l)*}$ and define $\theta^*: H_p^{n_0} \to CP^N$ by $\theta^*(Z) = [\theta_0^*(Z) : \cdots : \theta_N^*(Z)]$. Clearly, θ^* induces a well-defined mapping θ_I^* of the orbit space $H_p^n/\Gamma_L(d) = V(\Gamma_L(d))$ into CP^N. $V(\Gamma_L(d))$ has a natural compactification [1][1), $V(\Gamma_L(d))^*$, obtained by adjoining to $H_p^{n_0}$ all rational boundary components (in the sense of Satake), supplying the union $(H_p^{n_0})^\#$ of these with a suitable topology, and providing the quotient space $(H_p^{n_0})^\#/\Gamma_L(d)$ with the richest natural analytic structure; the resulting complex space $V(\Gamma_L(d))^*$ may be imbedded as an algebraic variety in some complex projective space by means of a mapping whose coordinates are automorphic forms of suitably high weight with respect to $\Gamma_L(d)$. Each rational boundary component in $H_p^{n_0}$ may be identified with $H_r^{n_0}$ for some $r, 0 \leqq r < p$, and there is a mapping Φ which maps the graded module of automorphic forms with respect to any group commensurable with Γ_L into a graded module of automorphic forms on $H_r^{n_0}$ with respect to some discontinuous group. It is easy to see that Φ maps $R_m^{(l)*}$ onto the similarly defined module of holomorphic functions on H_r^n. Hence, as it is easy to see, θ_I^* may be extended to a well-defined mapping also denoted by θ_I^* (whose coordinates have no common zeros) of $V(\Gamma_L(d))^*$ into CP^N. As

1) See end of § 1.

in [2], 353-354, it is easy to prove that for each $x \in CP^N$, $\theta_I^{*-1}(x)$ is at most a finite set of points. Let $y \in H_p^{n_0}$ be such that $\theta_I^{*-1}(\theta^*(y)) \subset V(\Gamma_L(d))$ and such that the number of points in $\theta_I^{*-1}(\theta^*(y))$ is minimal. Let $\theta_I^{*-1}(\theta^*(y)) = \{y_1, \cdots, y_s\}$ and let y_i be the canonical image of $Z_i \in H_p^{n_0}$, $i=1, \cdots, s$. We know there exists a sufficiently divisible m', $d|m'$, such that if we are given i, j with $1 \leqq i < j \leqq s$, then there exists an $f \in K(R_{m'}^{(l)*})$ with $f(Z_i) \neq f(Z_j)$. (For this, we should observe again the fact that any two lattices in k^p are commensurable.) Let $\phi_0, \cdots, \phi_{N'}$ be a basis of $R_{m', \Gamma_L(d)}^{(l')*}$ for suitably divisible l', $l|l'$. Clearly, $R_{m', \Gamma_L(d)}^{(l')*}$ contains all homogeneous polynomials of degree l'/l in $\theta_0, \cdots, \theta_N$ having coefficients in A. Define $\phi: H_p^{n_0} \to CP^{N'}$ by $\phi(Z) = [\phi_0(Z): \cdots : \phi_{N'}(Z)]$. Then ϕ induces an analytic mapping ϕ_I: $V(\Gamma_L(d))^* \to CP^{N'}$. By our choice of m', $\phi_I(y_1), \cdots, \phi_I(y_s)$ are all distinct. Hence $\phi_I^{-1}(x)$ is finite (if not empty) for all $x \in CP^N$, and if x is a generic point of $\phi_I(V_{\Gamma_L}(d)^*)$ over A, then $\phi_I^{-1}(x)$ is a single point. For any positive integer m_1, let $G(m_1, L)$ denote the subgroup of $Sp(p, k)$ leaving $K(R_{m_1})$ pointwise fixed. What we have just proved clearly implies that $G(m', L) \subset \Gamma_L(d) \subset G(m, L)$. Let $V(m', L) = V(G(m', L))$. Denote by $\mathcal{J}$ the homogeneous elements of degree l'', for sufficiently divisible, fixed l'', in the integral closure of the graded ring whose generating elements (of degree 1) are the elements of $R_{m'}^{(l')}$ for some fixed l'; let $\alpha_0, \cdots, \alpha_{N''}$ be a basis of $\mathcal{J}$; then the mapping

$$\alpha: V(m', L)^* \to CP^{N''}$$

with these as coordinates is a projective imbedding of $V(m', L)^*$ as a projective normal variety defined over A for suitably large l''. Clearly, for sufficiently divisible m', $G(m', L)$ is a normal subgroup of Γ_L, as we see from the transformation formulae for θ-functions. In fact, using the notation of the preceding section, if $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_L$, then $\theta_{L_1}[0, 0](\zeta, Z) = \lambda(M) \sum_{g_1, h_1} \varepsilon_{g_1 h_1} \theta[g_1, h_1](\zeta^*, Z)$, where $\lambda(M) = \chi(M) N \det(CZ+D)^{-\frac{1}{2}}$ and the sum on the right side is finite; replacing ζ by $\zeta + Zg + h$, we obtain the transformation formula for $\theta_{L_1}[g, h]$; then, finally, since Γ_L is finitely generated, we can find a positive integer l_0 such that all $l_0 g_1, l_0 h_1 \in L_i \cap L_i'$, $i=1, 2$ for all g_1 and h_1 which appear as M runs over a set of generators of Γ_L; therefore, if $l_0 | m'$, Γ_L is a transformation group of $K(R_{m'})$ and $G(m', L)$ is the kernel of this representation of Γ_L. Then if $\gamma \in \Gamma_L / G(m', L)$, γ is represented by a projective transformation defined over A. Hence $\Gamma_L / G(m', L) = G_{m'}$ is a finite group of projective transformations of $V(m', L)^*$, and the quotient variety is easily seen to be $V(\Gamma_L)^*$ defined over A. On the other hand, if $\sigma \in G(A/Q)$, we see that σ carries $R_m^{(l)*}$ onto itself for any l, m_1. Let T_σ: $V(m', L)^* \to V(m', L)^{*\sigma}$ be the projective transformation induced by this linear mapping of $R_{m'}^{(l)*}$ onto itself (see [2], p. 356, Lemma 2). If $\gamma \in G_{m'}$, then $\gamma^\sigma = T_\sigma \gamma T_\sigma^{-1}$. In fact, let (a, b) be a generic point of γ over

A, so that a and b are generic points of $V(m', L)^*$; then $a^\sigma = T_\sigma(a)$ and $b^\sigma = T_\sigma(b)$ are generic points of $V(m', L)^{*\sigma}$ and we have: $\gamma^\sigma(a^\sigma) = b^\sigma = T_\sigma(b) = T_\sigma(\gamma a) = T_\sigma \gamma T_\sigma^{-1}(a^\sigma)$ (as for the equations $a^\sigma = T_\sigma(a)$, etc., T_σ may be thought of as prescribing for any generic point a of $V(m', L)^*$ a fixed extension of $\sigma : \mathrm{A} \rightarrow \mathrm{A}$ to an isomorphism $T_\sigma : \mathrm{A}(a) \rightarrow \mathrm{A}(a^\sigma)$). Let $G^\sigma_{m'} = \{\gamma^\sigma \mid \gamma \in G_{m'}\}$, and define T_σ on $V(\Gamma_L)^*$ by $T_\sigma = \pi^\sigma T_\sigma \pi^{-1}$, where π is the canonical map of $V(m', L)^*$ onto $V(\Gamma_L)^*$. Then it is easy to show that T_σ is a well defined biregular map of $V(\Gamma_L)^*$ onto $V(\Gamma_L)^{*\sigma} = V(m', L)^{*\sigma}/G'^\sigma_m$. So we have the diagram:

$$\begin{tikzcd} V(m', L)^* \arrow[r, "T_\sigma"] \arrow[d, "\pi"] & V(m', L)^{*\sigma} \arrow[d, "\pi^\sigma"] \\ V(\Gamma_L)^* \arrow[r, "T_\sigma"] & V(\Gamma_L)^{*\sigma}. \end{tikzcd}$$

Evidently $T_\sigma^\tau \circ T_\tau = T_{\sigma\tau}$. Therefore, we may apply the criterion of André Weil [**15**] and find a biregular map f of $V(\Gamma_L)^*$ onto a variety V defined over the rational numbers such that $f = f_e = f^\sigma \circ T_\sigma$. Hence we have diagram:

$$\begin{tikzcd} V(m', L)^* \arrow[r, "T_\sigma"] \arrow[d, "\pi"'] & V(m', L)^{*\sigma} \arrow[d, "\pi^\sigma"] \\ V(\Gamma_L)^* \arrow[r, "T_\sigma"] \arrow[dr, "f"'] & V(\Gamma_L)^{*\sigma} \arrow[d, "f^\sigma"] \\ & V/Q \end{tikzcd}$$

Let α: $V(m', L)^* \rightarrow CP^{N''}$ be the projective imbedding defined previously. Let $\theta_0, \cdots, \theta_N$ be the basis $\theta_{L_1}\left[\frac{\eta}{m}, 0\right](m\zeta, mZ)$, $\eta \in L_2'$ mod mL_1, of the module of m^{th} order θ-functions (for each fixed Z) with respect to L_Z of characteristic (0, 0). Consider the map $\theta : H_p^{n_0} \times C^{np} \rightarrow CP^N$ defined by $\theta(\zeta, Z) = [\theta_0(\zeta, Z) : \cdots : \theta_N(\zeta, Z)]$; define $\mathrm{T} = \bar{\alpha} \times \theta : H_p^{n_0} \times C^{np} \rightarrow CP^M \times CP^N$ by $\mathrm{T}(Z, \zeta) = \bar{\alpha}(Z) \times \theta(\zeta, Z)$, $\bar{\alpha}$ being the map induced by α. For fixed $Z \in H_p^{n_0}$, $\mathrm{T}(Z \times C^{np}) = A_Z$ is an Abelian variety. For sufficiently divisible m', $G(m', L) \subset \Gamma_L(d)$ for some sufficiently divisible d, and therefore we may assume for such m' that $G(m', L)$ has no (non-trivial) fixed points in $H_p^{n_0}$, so that $\mathrm{T}(H_p^{n_0} \times C^{np})$ is a fiber system of Abelian varieties over $\alpha(H_p^n)$, and $\Gamma_L/G(m', L) = G_{m'}$ acts in a natural way as a finite group of projective transformations of $\mathrm{T}(H_p^{n_0} \times C^{np})$. Clearly $G_{m'}(V(m', L)^* - V(m', L)) = V(m', L)^* - V(m', L)$. Moreover, if $\sigma \in G(\mathrm{A}/Q)$, then $T_\sigma(V(m', L)^* - V(m', L)) = (V(m', L)^* - V(m', L))^\sigma$; this is true because application of σ to the Fourier coefficients of a modular form commutes with the Φ-operator (q. v., supra).

We now show that $\mathrm{T}(H_p^{n_0} \times C^{np})^*$ is an algebraic variety on which

$\mathrm{T}(H_p^{no}\times C^{np})$ is an A-open set. In fact, it is clear that the dimension of $\mathrm{T}(H_p^{no}\times C^{np})$ as a complex space is $n\Big(p+\frac{p(p+1)}{2}\Big)$. Moreover, if $\mathcal{A}$ is the smallest algebraic variety containing $\mathrm{T}(H_p^{no}\times C^{np})$, then $\dim\mathcal{A}=\dim_{\tilde{\alpha}}\mathrm{T}+\dim_Q\tilde{\alpha}$ $=n\Big(\frac{p(p+1)}{2}\Big)+\dim\mathrm{T}$. One may then show, using methods of a previous paper [**2**], that any $np+2$ θ-functions are algebraically dependent over the field of automorphic functions on H_p^{no} with respect to Γ_L, and hence $\dim_{\tilde{\alpha}}\mathrm{T}\leqq np$. It follows that $\dim\mathcal{A}\leqq\dim(\mathrm{T}(H_p^{no}\times C^{np}))$, and our assertions about $\mathrm{T}(H_p^{no}\times C^{np})$ are immediate. Put $\mathfrak{A}=\mathrm{T}(H_p^{no}\times C^{np})^*$, $\mathfrak{B}=\alpha(H_p^{no})^*$, $\lambda:\mathfrak{A}\to\mathfrak{B}$, the restriction of $pr_1: CP^M\times CP^N\to CP^M$. It is easy to see how to define T_σ on $\mathfrak{A}$ and that $T_\sigma(\mathfrak{A})=\mathfrak{A}^\sigma$, $T_\sigma[\lambda]=T_\sigma\lambda T_\sigma^{-1}=\lambda^\sigma$, etc. By our remarks at the end of the last section it is easy to see how to make the Galois group act on the graphs of the group law and of the elements of our endomorphism ring $\mathfrak{r}$ by projective transformations which we also denote by T_σ. Then T_σ is an isomorphism of $A_{\beta(Z)}=A_{\beta(\gamma Z)}$. If $\eta\in\mathfrak{r}$, then the graph $[\eta]$ of the endomorphism η is an algebraic subvariety of $\mathfrak{A}^{(2)}$ (see end of §2 and [**2**], p. 370), and $T_\sigma[\eta]=[\eta]^\sigma$, $\gamma[\eta]=[\eta]$. Thus we have the following diagram:

$$\begin{tikzcd} \mathfrak{A} \arrow[r, "T_\sigma"] \arrow[d, "\lambda"] & \mathfrak{A}^\sigma \arrow[d, "\lambda^\sigma"] \\ \mathfrak{B} \arrow[r, "T_\sigma"] \arrow[dr, "f_1"'] & \mathfrak{B}^\sigma \arrow[d, "f_1^\sigma"] \\ & V/Q \end{tikzcd} \quad , \quad f_1=f\circ\pi .$$

Let $Z\in H_p^{no}$, $\beta(Z)=x\in\mathfrak{B}$, $f_1(x)\in V=V(\Gamma_L)^*$. We want to prove that $Q(f_1(x))$ is the field k_0^x of moduli of (A_Z,θ^x,ι), ι being the "natural" injection of $\mathfrak{r}$ in the endomorphism ring of A_Z and θ^x being the polarization attached to the projective imbedding θ. First A_Z, θ^x, and $\iota(\eta)$ are all defined ($\eta\in\mathfrak{r}$) over a finitely generated subfield $k(x)$ of $\mathrm{A}(x)$. Let x be such that x and $f_1(x)$ are generic on $\mathfrak{B}$ and V respectively over A. If $\gamma\in\Gamma_L$, $A_Z\cong A_{\gamma Z}$, and if $\sigma\in G(\mathrm{A}/Q)$, then $A_Z\cong T_\sigma A_Z=A_Z^\sigma$. Hence k_0^x is contained in the fixed point field k_1 of all γ and σ in $k(x)$. The fixed point field of all γ is $k(\pi(x))$ (since π is the canonical map of $V(m',L)^*$ onto $V(\Gamma_L)^*=V(m',L)^*/G_{m'}$), and by the construction of [**15**, p. 511], the fixed point field in $k(\pi(x))$ of all σ is $Q(f(\pi(x)))=Q(f_1(x))$. Hence $k_0^x\subset Q(f_1(x))$. Then with minor changes of notation, we may apply the argument of [**2**; p. 364, l. 20–p. 365, l. 4] to show that for any specialization $\bar{x}$ of x over A, we have $k_0^{\bar{x}}=Q(f_1(\bar{x}))$. Thus we have:

THEOREM. *There exists a projective imbedding φ of the space $V(\Gamma_L)^*$ $=(H_p^n/\Gamma_L)^*$ defined over the rational numbers Q. The points of the orbit space*

$V(\Gamma_L)$ are in one-to-one correspondence with the isomorphism classes of polarized Abelian varieties of given dimension and polarization, given type [**12, 14**] *and with given injection of* $\mathfrak{o} \subset k$ *into their endomorphism rings satisfying* (1)-(3) *of* § 1. *If* $y \in V(\Gamma_L)$, *then* $Q(\varphi(y))$ *is the field of moduli of the corresponding isomorphism class.*

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