J. Math. Soc. Japan
Vol. 16, No. 4, 1964

Martin boundary for linear elliptic differential operators of second order in a manifold

By Seizô ITô

(Received June 27, 1964)

§1. **Introduction.** The generalized boundary value problem of Dirichlet type for the Laplace operator in arbitrary region D of Euclidean space R^N was studied by R. S. Martin [8]. It is shown in [8] that any non-negative harmonic function on D is represented in the form of an integral over the set of so-called minimal positive harmonic functions; the set of minimal functions corresponds to 'ideal boundary' of D which we call Martin boundary. On this subject, we may find in [2] work of a more general nature.

Recently the theory of Martin boundary has been connected with the theory of Markov processes; Martin boundaries for Markov chains have been constructed by many authors, especially by J. L. Doob [3], G. A. Hunt [4] and T. Watanabe [**10**]. M. G. Šur [**9**] has constructed the Martin boundary for the linear elliptic operator of second order in an arbitrary region $D \subset R^N$, which corresponds to a diffusion process in D. His method is achieved along the contents of Martin's paper [8], but mostly due to the probabilistic treatment.

In the present paper, we shall construct the Martin boundary for a linear elliptic differential operator of second order in a subdomain D of a manifold $\boldsymbol{M}$ (D may coincide with $\boldsymbol{M}$) by means of purely analytical treatment, as a direct extension of Martin's method, and show that, if a part S of the boundary ∂D of the domain D considered in $\boldsymbol{M}$ is 'smooth' and the elliptic operator is regular on $D+S$, then S is homeomorphically imbedded into the Martin boundary; in general, we do not assume any regularity of ∂D and any restriction on the behavior of the elliptic operator near ∂D. Our method is essentially same as Martin's method in [8] except the result on the correspondence between the smooth part of ∂D and a subset of the Martin boundary. However we use some properties of fundamental solutions of diffusion equations shown in the author's previous papers [**5**], [**6**], instead of some classical results in potential theory which are well known in the case of usual Laplace operator but whose extension to the case of general elliptic operators is not necessarily evident.

The contents of the present paper are as follows. In §2, we state some properties of fundamental solutions of diffusion equations and some properties of solutions of Dirichlet problem. In § 3, we construct the Martin boundary for a linear elliptic operator A of second order, and we prove, in § 4, the integral representation formula of positive A-harmonic functions. In § 5, *extremal* (=*minimal* in Martin's paper [8]) functions are characterized and it is shown that every positive A-harmonic function admits of exactly one canonical representation which involves only extremal functions. The arguments in §§ 3 and 4 are not entirely same as the corresponding parts in [8]. However, once the results in §§ 3 and 4 are established, the argument in § 5 can be achieved in the same way as in [8]. So, in § 5, we state only the outline of the procedure. In § 6, we establish some theorems on imbedding the smooth part of the boundary of the domain into the Martin boundary. Contents of § 6 is not contained in works by R. S. Martin [8] and M. G. Šur [9]. In Appendix, we give the proofs of preliminary lemmas stated in § 2.

§ 2. **Preliminaries.** Let D be a subdomain of an orientable N-dimensional C^∞-manifold $\boldsymbol{M}$ ($N \geqq 2$), and A be a second order elliptic differential operator defined in D as follows:

$$Au(x) = \frac{1}{\sqrt{a(x)}} \frac{\partial}{\partial x^i} \left[\sqrt{a(x)}\; a^{ij}(x) \frac{\partial u(x)}{\partial x^j} \right] + b^i(x) \frac{\partial u(x)}{\partial x^i} \;{}^{1)},$$

where $\| a^{ij}(x) \|$ and $\| b^i(x) \|$ are contravariant tensors of class C^2 in D and $a(x) = \det \| a^{ij}(x) \|^{-1}$. We require neither regularity of the boundary of D, nor restriction on the behavior of $\| a^{ij}(x) \|$ and $\| b^i(x) \|$ near the boundary of D. We only assume that there exists at least one non-constant and non-negative valued function $u(x)$ satisfying $Au(x) = 0$ in D[2].

A subdomain Ω of $\boldsymbol{M}$ is called a *domain with property* (S) if the boundary of Ω consists of a finite number of $(N-1)$-dimensional simple closed hypersurfaces of class C^3.

The adjoint operator A^* of the differential operator A is defined by

$$A^*u(x) = \frac{1}{\sqrt{a(x)}} \frac{\partial}{\partial x^i} \sqrt{a(x)} \left[a^{ij}(x) \frac{\partial u(x)}{\partial x^j} - b^i(x) u(x) \right],$$

and we have the following Green's formula (2.1).

1°) If Ω is a subdomain of D with property (S) and with compact closure $\bar{\Omega} \subset D$[3], and if $u(x)$ and $v(x)$ are functions of class C^1 on $\bar{\Omega}$ and of class C^2 in Ω, then

1) We omit the summation sign $\sum$ according to the usual rule of tensor calculus.
2) This assumption implies that D itself is not a compact manifold.
3) $\bar{\Omega}$ denotes the closure of Ω as a subset of $\boldsymbol{M}$.

$$(2.1)\qquad \begin{aligned}&\int_{\Omega}\{Au(x)\cdot v(x)-u(x)\cdot A^{*}v(x)\}dx\\ &=\int_{\partial\Omega}\{\frac{\partial u(x)}{\partial \boldsymbol{n}}v(x)+\beta(x)u(x)v(x)-u(x)\frac{\partial v(x)}{\partial \boldsymbol{n}}\}dS_{x},\end{aligned}$$

where $\frac{\partial u(x)}{\partial \boldsymbol{n}}$ and $\beta(x)$ respectively denote the outer normal derivative of $u(x)$ and outer normal component of $\| b^{i}(x)\|$ on the boundary $\partial\Omega$ of the domain Ω, and $dx(=\sqrt{a(x)}\,dx^{1}\cdots dx^{N})$ and dS_{x} respectively denote the volume element in D and the hypersurface element on $\partial\Omega$ with respect to the Riemannian metric defined by the tensor $\| a_{ij}(x)\|(=\| a^{ij}(x)\|^{-1})$.

The following facts 2°), 3°) and 4°) are implied by the results of the author's previous paper [**6**][4].

2°) For any domain Ω as stated in 1°), there exists one and only one fundamental solution $U_{\Omega}(t, x, y)$ of the initial-boundary value problem for the parabolic equation:

$$(2.2)\qquad \frac{\partial u}{\partial t}=Au \quad \text{in} \quad (0, \infty)\times\Omega,\ u|_{t=0}=u_{0},\ u|_{x\in\partial\Omega}=\varphi;$$

the function $U_{\Omega}(t, x, y)$ is also the fundamental solution of the initial-boundary value problem for the adjoint parabolic equation:

$$(2.2^{*})\qquad \frac{\partial u}{\partial t}=A^{*}u \quad \text{in} \quad (0, \infty)\times\Omega,\ u|_{t=0}=u_{0},\ u|_{x\in\partial\Omega}=\varphi.$$

$U_{\Omega}(t, x, y)$ satisfies that

$$(2.3)\qquad \begin{cases} U_{\Omega}(t, x, y)\geqq 0 \text{ for any } \langle t, x, y\rangle\in(0, \infty)\times\bar{\Omega}\times\bar{\Omega}\text{; the equality}\\ \text{holds if and only if at least one of } x \text{ and } y \text{ belongs to } \partial\Omega,\end{cases}$$

and also that

$$(2.4)\qquad \begin{cases}\int_{\Omega}U_{\Omega}(t, x, z)U_{\Omega}(s, z, y)dz=U_{\Omega}(t+s, x, y) \text{ and } \int_{\Omega}U_{\Omega}(t, x, z)dz\leqq 1\\ \text{for any } t, s>0 \text{ and any } x, y\in\bar{\Omega}.\end{cases}$$

Furthermore

$$(2.5)\qquad G_{\Omega}(x, y)=\int_{0}^{\infty}U_{\Omega}(t, x, y)dt$$

is well-defined whenever $x, y\in\bar{\Omega}$ and $x\neq y$, and is the Green function of the boundary value problem for the elliptic equation:

$$(2.6)\qquad Au=f \quad \text{in } \Omega,\quad u|_{\partial\Omega}=\varphi,$$

that is, the unique solution of (2.6) is given by

4) Differential operators A and A^{*} in the present paper correspond to A^{*} and A in [**6**] respectively.

$$(2.7) \qquad u(x) = -\int_{\Omega} G_{\Omega}(x, y) f(y) dy - \int_{\partial\Omega} \frac{\partial G_{\Omega}(x, y)}{\partial \boldsymbol{n}_y} \varphi(y) dS_y$$

where $f(x)$ and $\varphi(x)$ are assumed to be Hölder-continuous on $\bar{\Omega}$ and on $\partial\Omega$ respectively. $G_{\Omega}(x, y)$ is also the Green function of the boundary value problem for the adjoint elliptic equation:

$$(2.6^*) \qquad A^* u = f \quad \text{in } \Omega, \quad u|_{\partial\Omega} = \varphi;$$

the solution of (2.6*) is given by a formula similar to (2.7). $G_{\Omega}(x, y)$ satisfies that

$$(2.8) \qquad A_x G_{\Omega} = 0 \quad \text{in} \quad \Omega - \{y\} \quad \text{and} \quad G_{\Omega}|_{\partial\Omega - \{y\}} = 0$$

as a function of x for any fixed $y \in \bar{\Omega}$, and

$$(2.8^*) \qquad A_y^* G_{\Omega} = 0 \quad \text{in} \quad \Omega - \{x\} \quad \text{and} \quad G_{\Omega}|_{\partial\Omega - \{x\}} = 0$$

as a function of y for any fixed $x \in \bar{\Omega}$.

3°) Let $\{D_n; n = 0, 1, 2, \cdots\}$ be a sequence of domains with property (S) such that $\bar{D}_n$ is compact and $\bar{D}_n \subset D_{n+1} \subset D$ for any $n \geqq 0$ and that $\lim_{n\to\infty} D_n = D$, and put

$$(2.9) \qquad U_n(t, x, y) = U_{D_n}(t, x, y) \quad \text{and} \quad G_n(x, y) = G_{D_n}(x, y) \quad (n = 1, 2, \cdots)$$

(see the above article 2°)). Then

$$(2.10) \qquad U_n(t, x, y) \leqq U_{n+1}(t, x, y) \text{ for any } \langle t, x, y \rangle \in (0, \infty) \times \bar{D}_n \times \bar{D}_n \ (n = 1, 2, \cdots),$$

and

$$(2.11) \qquad U(t, x, y) = \lim_{n\to\infty} U_n(t, x, y)$$

is well-defined on $(0, \infty) \times D \times D$ and is independent of the choice of sequence $\{D_n\}$, $U(t, x, y)$ is a fundamental solution of the initial-boundary value problem of the form (2.2) and also that of the form (2.2*) considered in $(0, \infty) \times D$. (Uniqueness of solutions of these initial-boundary value problems does not always hold unless $\bar{D}$ is compact.)

4°) If a part of the boundary ∂D of D consists of a simple hypersurface S of class C^3 and if $\|a^{ij}(x)\|$ and $\|b^i(x)\|$ are of class C^2 on $D+S$, then we can choose the sequence $\{D_n\}$ stated in 3°) in such a way that $\partial D_n \cap S$ contains a relatively open subregion of S and $\bar{D}_n \cap D \subset D_{n+1}$ (instead of: $\bar{D}_n \subset D_{n+1}$ in 3°)) for any $n \geqq 1$ and that $\lim_{n\to\infty} \partial D_n \cap S = S$. In this case,

$$(2.12) \qquad G(x, y) = \int_0^{\infty} U(t, x, y) dt$$

is well-defined whenever $x, y \in D+S$ and $x \neq y$, and is independent of the choice of $\{D_n\}$. $G(x, y)$ satisfies that

$$(2.13) \qquad A_x G = 0 \quad \text{in} \quad D - \{y\} \quad \text{and} \quad G|_{S - \{y\}} = 0$$

as a function of x for any fixed $y \in D+S$, and

(2.13*) $$A_y^* G = 0 \quad \text{in} \quad D-\{x\} \quad \text{and} \quad G|_{S-\{x\}} = 0$$

as a function of y for any fixed $x \in D+S$. Furthermore, we may easily show the following facts (2.14)-(2.17):

(2.14) $$0 \leqq G_n(x, y) \leqq G_{n+1}(x, y) \ (x, y \in \bar{D}_n\,;\ x \neq y),\ n=1, 2, \cdots,$$

(2.15) $$\lim_{n\to\infty} G_n(x, y) = G(x, y) \ (x, y \in D+S\,;\ x \neq y),$$

(2.16) $$\left.\begin{aligned} 0 > \frac{\partial U_n(t, x, y)}{\partial \boldsymbol{n}_y} \geqq \frac{\partial U_{n+1}(t, x, y)}{\partial \boldsymbol{n}_y} \\ 0 > \frac{\partial G_n(x, y)}{\partial \boldsymbol{n}_y} \geqq \frac{\partial G_{n+1}(x, y)}{\partial \boldsymbol{n}_y} \end{aligned}\right\} (x \in D_n,\ y \in \partial D_n \cap S)\ n=1, 2, \cdots,$$

and

(2.17) $$\left.\begin{aligned} \lim_{n\to\infty} \frac{\partial U_n(t, x, y)}{\partial \boldsymbol{n}_y} = \frac{\partial U(t, x, y)}{\partial \boldsymbol{n}_y} > -\infty \\ \lim_{n\to\infty} \frac{\partial G_n(x, y)}{\partial \boldsymbol{n}_y} = \frac{\partial G_n(x, y)}{\partial \boldsymbol{n}_y} > -\infty \end{aligned}\right\} (x \in D,\ y \in S).$$

If Ω is a subdomain of D such that $\Omega = \Omega_1 \cap \Omega_2$ where each Ω_ν is a subdomain of D with property (S) and that $\bar{\Omega}$ is a compact subset of D, then the above stated results may be applied to the domain Ω; here we choose the sequence $\{D_n\}$ in such a way that $\lim_{n\to\infty} D_n = \Omega$ and $\lim_{n\to\infty} \partial D_n \cap S = S$ where

$$S = \left\{ x \in \partial\Omega\,;\ \begin{array}{l} \text{for a suitable neighborhood } V_x \text{ of } x,\ V_x \cap \partial\Omega \text{ is contained} \\ \text{in only one of } \partial\Omega_1 - \partial\Omega_2,\ \partial\Omega_2 - \partial\Omega_1 \text{ and } \partial\Omega_1 \cap \partial\Omega_2 \end{array} \right\}.$$

Similar results to those in 2°) may be obtained in this case. For example, the unique solution of the boundary value problem:

(2.6′) $$Au = f \quad \text{in } \Omega, \quad u|_S = \varphi,$$

is given by the formula:

(2.7′) $$u(x) = -\int_\Omega G(x, y) f(y) dy - \int_S \frac{\partial G(x, y)}{\partial \boldsymbol{n}_y} \varphi(y) dS_y,$$

where $f(x)$ and $\varphi(x)$ are assumed to be Hölder-continuous on $\bar{\Omega}$ and on $\partial\Omega$ respectively—cf. (2.6) and (2.7).

5°) Let $\{D_n\}$ be as mentioned in 3°), and $\gamma(z)$ be a continuous and non-negative valued function on D with support contained in D_0 and satisfying $\int_D \gamma(z) dz = 1$. For any function $u(x)$ defined on D_m and any function $H(x, y)$ defined almost everywhere on $D_m \times D_n$ (for certain m and n), we put

(2.18) $$u(\gamma) = \int_{D_m} \gamma(x) u(x) dx, \quad H(\gamma\,; y) = \int_{D_m} \gamma(x) H(x, y) dx$$

and

$$(2.19)\qquad \begin{cases} |\nabla u(x)| = \left\{a^{ij}(x)\dfrac{\partial u(x)}{\partial x^i}\cdot\dfrac{\partial u(x)}{\partial x^j}\right\}^{\frac{1}{2}} \\ |\nabla_x H(x, y)| = \left\{a^{ij}(x)\dfrac{\partial H(x, y)}{\partial x^i}\cdot\dfrac{\partial H(x, y)}{\partial x^j}\right\}^{\frac{1}{2}} \end{cases}$$

whenever the right-hand side of each formula makes sense (for example, $|\nabla_x G_n(x, y)|$ is defined whenever $x \in D_n$, $y \in D_n$ and $x \neq y$).

Let Ω be a subdomain of D. A function $u(x)$ is said to be *A-harmonic* in Ω if it satisfies $Au(x) = 0$ in Ω.

Proof of the following lemmas will be given in Appendix.

LEMMA 2.1. *If $u(x)$ is A-harmonic in Ω, then $u(x)$ takes neither the maximum nor the minimum at any interior point of Ω.*

This fact implies the following

COROLLARY. *If $u(x)$ is continuous on $\bar{\Omega}$ and A-harmonic in Ω, and if $u(x) \geqq 0$ on $\partial\Omega$, then $u(x) > 0$ in Ω; here the smoothness of $\partial\Omega$ does not necessarily assumed.*

LEMMA 2.2. *Let $\{u_n(x)\}$ be a sequence of A-harmonic functions in Ω and assume that either* i) *$\{u_n(x)\}$ converges to a function $u(x)$ uniformly on any compact subset of Ω, or* ii) *it converges to a locally bounded function $u(x)$ monotonically in n. Then $u(x)$ is A-harmonic in Ω.*

LEMMA 2.3. *If $\{u_\lambda(x); \lambda \in \Lambda\}$ is a family of A-harmonic functions on Ω and is uniformly bounded on any compact subset of Ω, then $\{|\nabla u_\lambda(x)|; \lambda \in \Lambda\}$ is uniformly bounded on any compact subset of Ω.*

LEMMA 2.4. *Assume that Ω is a domain with property* (S) *and containing $\bar{D}_0$, and that $u(x)$ is a function non-negative and of class C^1 on $\overline{D_m \cap \Omega}$ (m being fixed and $\geqq 1$), A-harmonic in $D_m \cap \Omega$ and satisfying $u(x) = 0$ on $\partial(D_m \cap \Omega) - D_n$ for some $n \leqq m$. Then there exists a bounded Borel measure on $\partial\Omega \cap \bar{D}_n$ such that*

$$u(x) = \int_{\partial\Omega \cap \bar{D}_n} \frac{\partial G_m(x, y)}{\partial G_m(\gamma; y)} d\mu(y) \quad \textit{for any} \quad x \in D_m \cap \Omega$$

and $\mu(\partial\Omega \cap \bar{D}_n) = u(\gamma)$.

6°) It follows from Theorem 2 in [7] that, if there exists a non-constant positive A-harmonic function in D, then

$$(2.20)\qquad G(x, y) = \int_0^\infty U(t, x, y)dt \quad (x, y \in D; x \neq y)$$

is well-defined and is a Green function of the elliptic differential operator A. This fact plays a fundamental role throughout the present paper. In such case, we have the following

LEMMA 2.5. *Let $G_n(x, y)$ $(n = 1, 2, \cdots)$ be as mentioned in* 3°) *and $G(x, y)$*

be as defined just above. Then

(2.21) $$G(x, y)=\lim_{n\to\infty} G_n(x, y)$$

uniformly on $E\times F$ for any mutually disjoint compact subsets E and F of D, and

(2.22) $$\lim_{\substack{x\to z\\ y\to z}} G(x, y)=\infty \qquad \text{for any } z\in D.$$

LEMMA 2.6. *If E is a compact subset of D and F is a subset of $D-(E\cup\bar{D}_0)$ relatively closed in D, then*

(2.23) $$\sup_{x\in E, y\in F}\frac{G(x, y)}{G(\gamma\,;y)}<\infty.$$

Proofs of these two lemmas will be given in Appendix.

§3. **Construction of the ideal boundary.** By means of 6°) in §2, it is sufficient to consider the case where a Green function is well-defined by (2.20).

In the sequel, we assume the existence of the Green function $G(x, y)$ defined by (2.20), and we fix a sequence $\{D_n\}$ of subdomains of D as mentioned in 3°) of §2 and a function $\gamma(x)$ as mentioned in 5°) of §2. Let $G_n(x, y)$ $(n=1, 2, \cdots)$ be as defined in 3°) of §2, and put

(3.1) $$\begin{cases} K_n(x, y)=\dfrac{G_n(x, y)}{G_n(\gamma\,;y)} \quad (n=1, 2, \cdots) \text{ and} \\ K(x, y)=\dfrac{G(x, y)}{G(\gamma\,;y)} \end{cases}$$

(see (2.18)). Then, by means of 6°) in §2, we may see that

(3.2) $$\begin{cases} K(x, y) \text{ is positive and } A\text{-harmonic in } x\in D-\{y\} \text{ for any fixed} \\ y\in D, \text{ and is continuous in } y\in D-\{x\} \text{ for any fixed } x\in D, \end{cases}$$

(3.3) $$\begin{cases} \lim_{n\to\infty} K_n(x, y)=K(x, y) \text{ uniformly on } E\times F \text{ for any} \\ \text{mutally disjoint compact subsets } E \text{ and } F \text{ of } D \end{cases}$$

and

(3.4) $$\lim_{\substack{x\to z\\ y\to z}} K(x, y)=\infty \qquad \text{for any } z\in D.$$

It is also clear that

(3.5) $$K(\gamma\,;y)=1 \qquad \text{for any } y\in D \text{ (see (2.18))}.$$

LEMMA 3.1. *Let F_1 be a compact subset of D and F_2 be a subset of $D-F_1$ relatively closed in D. Then*

(3.6) $$\sup_{x\in F_1, y\in F_2} K(x, y)<\infty \quad\text{and}\quad \sup_{x\in F_1, y\in F_2} |\nabla_x K(x, y)|<\infty.$$

Proof. The first inequality in (3.6) is nothing else than the conclusion (2.23) of Lemma 2.6. Furthermore $\{K(\cdot, y); y \in F_2\}$ is a family of A-harmonic functions in the domain $D-F_2$, and is uniformly bounded on any compact subset E of $D-F_2$ by means of Lemma 2.6. Hence, by Lemma 2.3, $\{|\nabla K(\cdot, y)|; y \in F_2\}$ is uniformly bounded on the compact set F_1; this fact implies the second inequality of (3.6), q. e. d.

For any y and y' in D, we define

$$\rho(y, y') = \int_{D_0} \frac{|K(x, y)-K(x, y')|}{1+|K(x, y)-K(x, y')|} dx. \tag{3.7}$$

Then ;—

Lemma 3.2. *The function $\rho(y, y')$ is a metric in D, which defines the same topology as the original one in D.*

Proof. It is clear that $\rho(y, y')$ is finite, non-negative, symmetric and that it satisfies the triangular inequality and vanishes if $y=y'$. If $\rho(y, y')=0$, then $K(x, y)=K(x, y')$ for almost all $x \in D_0$ by (3.7). Since $K(x, y)$ is A-harmonic in $D-\{y\}$, $K(x, y)=K(x, y')$ for all $x \in D_0-\{y, y'\}$, and accordingly for all $x \in D-\{y, y'\}$ by the unique continuation theorem of Aronszajn [1]. Hence we get $y=y'$ by virtue of (3.2) and (3.4). Thus we see that $\rho(y, y')$ is a metric in D. To prove the remaining part, it is sufficient to show that the topology defined by ρ is equivalent to the original one in D_n for any n. Since $\bar{D}_n$ is compact with respect to the original topology, we have only to prove that, if a sequence $\{y_\nu\}$ of points in $\bar{D}_n$ converges to a point y in $\bar{D}_n$ with respect to the original topology, then $\lim_{\nu\to\infty} \rho(y_\nu, y)=0$ holds. For any such sequence $\{y_\nu\}$, we have $\lim_{\nu\to\infty} K(x, y_\nu)=K(x, y)$ for almost all $x \in D_0$ (in fact, this convergence holds for any $x \in D-\{y, y_1, y_2, \cdots, y_\nu, \cdots\}$ by virtue of (3.2)), and hence $\lim_{\nu\to\infty} \rho(y_\nu, y)=0$ by (3.7) and by Lebesgue's convergence theorem.

Lemma 3.3. *D is totally bounded with respect to the metric ρ.*

Proof. It suffices to prove that any sequence $\{y_n\} \subset D$ contains a subsequence $\{y_{n_\nu}\}$ satisfying $\lim_{\nu,\nu'\to\infty} \rho(y_{n_\nu}, y_{n_{\nu'}})=0$. If $y_n \in D_1$ for infinitely many n's, then such subsequence clearly exists by virtue of Lemma 3.2 since $\bar{D}_1$ is compact. If $\{y_n, n \geqq n_0\} \subset D-D_1$ for some n_0, then the family $\{K(\cdot, y_n); n \geqq n_0\}$ of functions considered on $\bar{D}_0$ is uniformly bounded and equi-continuous since

$$\sup_{x\in\bar{D}_0, y\in D-D_1} K(x, y) < \infty \qquad \text{and} \qquad \sup_{x\in\bar{D}_0, y\in D-D_1} |\nabla_x K(x, y)| < \infty$$

by Lemma 3.1. Hence, by the Ascoli-Arzelà theorem, $\{K(\cdot, y_{n_\nu}); \nu=1, 2, \cdots\}$ converges uniformly on $\bar{D}_0$ for a suitable subsequence $\{n_\nu\}$; this fact implies that $\lim_{\nu,\nu'\to\infty} \rho(y_{n_\nu}, y_{n_{\nu'}})=0$, q. e. d.

Let $\mathfrak{D}$ be the completion of D with respect to the metric ρ; the function

$\rho(y, y')$ naturally extended to $\mathfrak{D}\times\mathfrak{D}$ will be denoted by the same notation. Then

THEOREM 3.1. *With respect to the metric* ρ, $\mathfrak{D}$ *is complete and compact, and* $\mathfrak{D}-D$ *is closed subset of* $\mathfrak{D}$. *The relative topology in* D *arising from the metric is equivalent to the original topology in* D.

PROOF. We have only to prove that $\mathfrak{D}-D$ is closed in $\mathfrak{D}$ with respect to ρ, since the other assertions are evident from the argument just above. Suppose that a sequence $\{\xi_n\}$ in $\mathfrak{D}-D$ converges to a point $y\in D$. Then $y\in D_m$ for some m, while, for each n, there exists a sequence $\{x_{n\nu}\,;\nu=1,2,\cdots\}\subset D$ such that $\lim_{\nu\to\infty}\rho(x_{n\nu}, \xi_n)=0$. For any fixed n, only a finite number of $x_{n\nu}$'s may belong to $\overline{D_{m+1}}$ (since, otherwise, the sequence $\{x_{n\nu}\,;\nu=1,2,\cdots\}$ would have an accumulating point in the compact set $\overline{D_{m+1}}$ with respect to ρ by Lemma 3.2), and hence there exists ν_n such that $x_n=x_{n\nu_n}$ belongs to $D-\overline{D_{m+1}}$ and that $\rho(x_n, \xi_n)<1/n$. Hence we obtain that

$$\rho(x_n, y)\leqq\rho(x_n, \xi_n)+\rho(\xi_n, y)\to 0 \quad \text{as} \quad n\to\infty\,;$$

this contradicts to the fact: $x_n\notin D_{m+1}$, $y\in D_m$ and $\bar{D}_m\subset D_{m+1}$. Theorem 3.1 is thus proved.

It is clear that $\mathfrak{D}-D$ has no inner point, and the above theorem implies that D is an open subset of $\mathfrak{D}$ with respect to ρ. So we can state the following

DEFINITION. The set $\mathfrak{S}=\mathfrak{D}-D$ is called the *ideal boundary* or *Martin boundary* of D for the elliptic operator A.

THEOREM 3.2. *The function* $K(x, y)$ *is extended to a function continuous on* $D\times\mathfrak{D}-\{\langle z, z\rangle\,; z\in D\}$, *and the extended function* $K(x, y)$ *is* A*-harmonic in* $x\in D-\{y\}$ *for any fixed* $y\in\mathfrak{D}$.

PROOF. For any $\xi\in\mathfrak{S}$, there exists a sequence $\{y_n\}\subset D$ such that $\lim_{n\to\infty}\rho(y_n, \xi)=0$. Then, for any D_m, there exists n_m such that $y_n\in D-D_{m+1}$ for any $n>n_m$. The family of functions $\{K(\cdot, y_n)\,; n>n_m\}$ is uniformly bounded and equi-continuous on D_m by Lemma 3.1; here m is arbitrary. Hence, by the Ascoli-Alzelà theorem and by diagonal process, we may take a subsequence $\{n_\nu\}$ such that the sequence of functions $\{K(\cdot, y_{n_\nu})\,;\nu=1,2,\cdots\}$ converges to a function $v(x)$ uniformly on any compact subset of D, and $v(x)$ is A-harmonic on D by Lemma 2.2. If $\{z_m\}$ is a sequence in D such that $\lim_{m\to\infty}\rho(z_m, \xi)=0$, and if $\{K(\cdot, z_{m_\nu})\,;\nu=1,2,\cdots\}$ converges, for a subsequence $\{m_\nu\}$, to a certain A-harmonic function $w(x)$ on D uniformly on any compact subset of D, then

$$\int_{D_0}\frac{|K(x, y_{n_\nu})-K(x, z_{m_\nu})|}{1+|K(x, y_{n_\nu})-K(x, z_{m_\nu})|}dx=\rho(y_{n_\nu}, z_{m_\nu})\leqq\rho(y_{n_\nu}, \xi)+\rho(\xi, z_{m_\nu})$$

and hence, by Fatou's lemma, we get

$$\int_{D_0} |v(x)-w(x)|\,dx \leq \lim_{\nu\to\infty}\int_{D_0}\frac{|K(x, y_{n_\nu})-K(x, z_{m_\nu})|}{1+|K(x, y_{n_\nu})-K(x, z_{m_\nu})|}dx=0\,,$$

which implies $v(x)=w(x)$ on D_0, and accordingly on the whole domain D by the unique continuation theorem of Aronszajn [1]. From this argument, we may see that $\{K(\cdot, y_n); n=1, 2, \cdots\}$ for the original sequence $\{y_n\}$ converges to $v(x)$ uniformly on any compact subset of D and that $v(x)$ depends only on ξ ($\in \mathfrak{S}$) and is independent of the sequence $\{y_n\}$. Hence we can define $K(x, \xi)$ by

$$K(x, \xi)=\lim_{n\to\infty} K(x, y_n)$$

for any sequences $\{y_n\}\subset D$ such that $\lim_{n\to\infty}\rho(y_n, \xi)=0$. Thus $K(x, y)$ is defined on $D\times\mathfrak{D}-\{\langle z, z\rangle; z\in D\}$, and it may be seen from Lemma 3.1, Theorem 3.1 and the above definition of $K(x, \xi)$ for $\xi\in\mathfrak{S}$ that $K(x, y)$ is continuous in y on $\mathfrak{D}-\{x\}$ for any fixed $x\in D$ and A-harmonic in x on $D-\{y\}$ for any fixed $y\in\mathfrak{D}$ and that

$$\sup_{x\in E, y\in F} K(x, y)<\infty \qquad \text{and} \qquad \sup_{x\in E, y\in F} |\nabla_x K(x, y)|<\infty$$

for any compact subset E of D and any closed subset F of $\mathfrak{D}$ such that $E\cap F$ is empty. Hence $K(x, y)$ is continuous on $D\times\mathfrak{D}-\{\langle z, z\rangle; z\in D\}$.

COROLLARY 1. *For any compact subset E of D and closed subset F of $\mathfrak{D}$ such that $E\cap F$ is empty, the function $K(x, y)$ is uniformly continuous on $E\times F$ with respect to the metric ρ.*

This fact immediately follows from Theorem 3.2 since $\mathfrak{D}$ is, and accordingly $E\times F$ is, compact with respect to the metric ρ.

COROLLARY 2. *If $\xi, \eta\in\mathfrak{S}$ and $K(x, \xi)=K(x, \eta)$ for any $x\in D$, then $\xi=\eta$.*

PROOF. Let $\{x_n\}$ and $\{y_n\}$ be sequences in D such that $\lim_{n\to\infty}\rho(x_n, \xi)=0$ and $\lim_{n\to\infty}\rho(y_n, \eta)=0$. Then if follows from the assumption that

$$\rho(\xi, \eta)=\lim_{n\to\infty}\rho(x_n, y_n)=\lim_{n\to\infty}\int_{D_0}\frac{|K(x, x_n)-K(x, y_n)|}{1+|K(x, x_n)-K(x, y_n)|}dx$$
$$=\int_{D_0}\frac{|K(x, \xi)-K(x, \eta)|}{1+|K(x, \xi)-K(x, \eta)|}dx=0\,,$$

which implies that $\xi=\eta$.

§4. **The function $u_\Gamma(x)$ and the integral representation.** Hereafter we shall consider the *compact metric space* $\mathfrak{D}$ *with metric* ρ, and the terms: *open, closed, interior*, etc., will be understood in the sense of this metric considered in $\mathfrak{D}$. However, only the boundary notation ∂ and the closure notation $\overline{}$ will denote respectively the boundary operation and the closure operation considered in the original manifold $\boldsymbol{M}$.

Let $\mathfrak{F}$ be the totality of closed subsets F of $\mathfrak{D}$ such that $F^{\circ}\cap D$ is a subdomain of D with property (S) (see § 2), where F° denotes the interior of F. Let φ be a function of class C^2 on D such that $0\leqq\varphi(x)\leqq 1$ and that the support of φ, which we shall denote by spt(φ), is a compact subset of D, and let Ω be any subdomain of D with property (S) such that spt$(\varphi)\subset\Omega\subset\bar{\Omega}\subset D$ and $\bar{\Omega}$ is compact.

Let $F\in\mathfrak{F}$ be fixed, and let $u(x)$ be a non-negative A-harmonic function on $D-F^{\circ}$ [5]. For any φ and Ω with properties stated above, we define the function $u_F(x;\varphi,\Omega)$, by means of the formula (2.7′), satisfying that

$$(4.1)\qquad \begin{cases} Au_F(x;\varphi,\Omega)=0 & \text{in}\quad \Omega-F \quad\text{and} \\ u_F(x;\varphi,\Omega)=\varphi(x)u(x) & \text{on}\quad \partial(\Omega-F)\,. \end{cases}$$

Then $u_F(x;\varphi,\Omega)\geqq 0$ for any $x\in\Omega-F$. Further we put

$$(4.2)\qquad u_F(x;\varphi)=\sup_{\Omega} u_F(x;\varphi,\Omega)$$

and

$$(4.3)\qquad u_F(x)=\sup_{\varphi} u_F(x;\varphi)$$

where Ω in (4.2) ranges over all domains as stated above for any fixed φ, and φ in (4.3) ranges over all functions as stated above.

LEMMA 4.1. *If* spt$(\varphi)\subset\Omega\subset\Omega'\subset\bar{\Omega}'\subset D$, *then*

$$(4.4)\qquad u_F(x;\varphi,\Omega)\leqq u_F(x;\varphi,\Omega')\leqq u(x)\quad \textit{for any}\quad x\in\Omega-F\,.$$

PROOF. Both $u_F(x;\varphi,\Omega)$ and $u_F(x;\varphi,\Omega')$ are A-harmonic in $\Omega-F$. Hence, comparing the 'boundary values' on $\partial(\Omega-F)$ and using Corollary to Lemma 2.1, we obtain the first inequality of (4.4). The second inequality may be proved similarly.

LEMMA 4.2. *$u_F(x;\varphi,D_m)$ is defined for sufficiently large m, and is monotone increasing with respect to m. Furthermore,*

$$(4.5)\qquad \lim_{m\to\infty} u_F(x;\varphi,D_m)=u_F(x;\varphi)\leqq u(x)\quad \textit{for any}\quad x\in D-F$$

and $u_F(x;\varphi)$ is A-harmonic in $D-F$.

PROOF. Since spt(φ) is compact subset of $D=\lim_{m\to\infty} D_m$, we have spt$(\varphi)\subset D_m$ for sufficiently large m, and hence $u_F(x;\varphi,D_m)$ is well-defined for any such m and is monotone increasing with respect to m by Lemma 4.1. Proof of (4.5) may be achieved in the same way as that of Lemma 1 in [8, § 3, p. 151], and the A-harmonicity of $u_F(x;\varphi)$ may be seen by Lemma 2.2.

5) For any relatively closed set E in D, the statement '$u(x)$ is A-harmonic on E' means that $u(x)$ is A-harmonic in a domain containing E.

Lemma 4.3. *If* $0 \leqq \varphi \leqq \varphi'$, φ *and* φ' *being as stated above, then*

$$u_F(x;\varphi) \leqq u_F(x;\varphi') \leqq u(x) \quad \textit{for any} \quad x \in D-F. \tag{4.6}$$

Proof. For any m such that $\mathrm{spt}(\varphi') \subset D_m$, we may show that

$$u_F(x;\varphi, D_m) \leqq u_F(x;\varphi', D_m) \leqq u(x) \quad \text{for any} \quad x \in D_m - F$$

by comparing the 'boundary values' on $\partial(D_m - F)$ and using Corollary to Lemma 2.1. Hence, letting $m \to \infty$, we obtain (4.6) by virtue of (4.5).

Lemma 4.4. *Let* $\{\varphi_n\}$ *be a monotone increasing sequence of functions with properties stated above, and assume that* $\lim_{n\to\infty} \{x;\varphi_n(x)=1\}^\circ = D$. *Then* $u_F(x;\varphi_n)$ *is monotone increasing with respect to* n,

$$\lim_{n\to\infty} u_F(x;\varphi_n) = u_F(x) \leqq u(x) \quad \textit{for any} \quad x \in D-F \tag{4.7}$$

and $u_F(x)$ *is* A*-harmonic in* $D-F$.

Proof. The monotonicity of $u_F(x;\varphi_n)$ with respect to n is clear by Lemma 4.3, and the remaining part may be proved similarly to Lemma 4.2.

Let $\{\varphi_n\}$ be a sequence of functions of class C^2 on D such that $0 \leqq \varphi_n(x) \leqq 1$ and that

$$\varphi_n(x) = \begin{cases} 1 & \text{on} \quad \overline{D_{n-1}} \\ 0 & \text{on} \quad D - D_n, \end{cases}$$

and put

$$u^F_{nm}(x) = u_F(x;\varphi_n, D_m) \quad \text{and} \quad u^F_n(x) = u_F(x;\varphi_n)$$

for any $m > n > 0$. Then it follows from those lemmas stated above that

$$u^F_{nm}(x) \uparrow u^F_n(x) \quad \text{as} \quad m \uparrow \infty \text{ (for any fixed } n) \tag{4.8}$$

and

$$u^F_n(x) \uparrow u_F(x) \quad \text{as} \quad n \uparrow \infty \tag{4.9}$$

for any $x \in D-F$ and that

$$\begin{cases} \text{if } u(x) \text{ and } v(x) \text{ are non-negative and } A\text{-harmonic on } D-F^\circ \\ \text{and if } u(x) \geqq v(x) \text{ in } D-F, \text{ then } u_F(x) \geqq v_F(x) \text{ in } D-F. \end{cases} \tag{4.10}$$

By means of the formula (2.7′), we have

$$u^F_{nm}(x) = -\int_{\partial F \cap D_m} \frac{\partial G_{D_m-F}(x,y)}{\partial \boldsymbol{n}_y} \varphi_n(y) u(y) dS_y \quad (x \in D_m - F) \tag{4.11}$$

since $\varphi_n(y) = 0$ on ∂D_m. $-\dfrac{\partial G_{D_m-F}(x,y)}{\partial \boldsymbol{n}_y}$ is non-negative and tends to $-\dfrac{\partial G_{D-F}(x,y)}{\partial \boldsymbol{n}_y}$ monotonically as $m \to \infty$ for any $x \in D$ and $y \in \partial F$ (see (2.16) and (2.17)). Hence letting $m \to \infty$ and then $n \to \infty$ in (4.11), we obtain by (4.8) and (4.9) that

$$u_F(x) = -\int_{\partial F} \frac{\partial G_{D-F}(x, y)}{\partial \boldsymbol{n}_y} u(y) dS_y \qquad (x \in D-F). \tag{4.12}$$

LEMMA 4.5. *Let $u_\nu(x)$ $(\nu = 1, 2, \cdots)$ and $u(x)$ be non-negative A-harmonic functions on $D-F^\circ$. Then;—*

i) *If $\varliminf_{\nu\to\infty} u_\nu(x) \geqq u(x)$ on ∂F, then $\varliminf_{\nu\to\infty} [u_\nu]_F(x) \geqq u_F(x)$ in $D-F$.*

ii) *If $\lim_{\nu\to\infty} u_\nu(x) = u(x)$ on ∂F and if there exists a majorant $v(x)$ to all of $u_n(x)$'s on $D-F^\circ$ where $v(x)$ is A-harmonic on $D-F^\circ$, then $\lim_{\nu\to\infty} [u_\nu]_F(x) = u_F(x)$ in $D-F$.*

PROOF. The first assertion is clear by the formula (4.12) and Fatou's lemma since $-\dfrac{\partial G_{D-F}(x, y)}{\partial \boldsymbol{n}_y} \geqq 0$. The second assertion also may be proved by (4.12) and Lebesgue's convergence theorem since $\int_{\partial F} \left\{-\dfrac{\partial G_{D-F}(x, y)}{\partial \boldsymbol{n}_y}\right\} v(y) dS_y = v_F(x) < \infty$ for any fixed $x \in D-F$.

LEMMA 4.6. *If $F, F' \in \mathfrak{F}$ and $F \supset F'$, then $[u_{F'}]_F(x) = u_{F'}(x)$ in $D-F$ for any non-negative A-harmonic function $u(x)$ on $D-(F')^\circ$.*

PROOF. By means of the formula (2.7′), we have

$$u_{nm}^{F'}(x) = -\int_{\partial F \cap D_m} \frac{\partial G_{D_m-F}(x, y)}{\partial \boldsymbol{n}_y} u_{nm}^{F'}(y) dS_y \qquad (x \in D_m - F)$$

since $u_{nm}^{F'}(y) = 0$ on ∂D_m. Letting $m \to \infty$ and then $n \to \infty$, we get

$$u_{F'}(x) = -\int_{\partial F} \frac{\partial G_{D-F}(x, y)}{\partial \boldsymbol{n}_y} u_{F'}(y) dS_y \qquad (x \in D-F)$$

by the same argument as we have derived (4.12) from (4.11). On the other hand, if we replace $u(y)$ in (4.11) by $u_{F'}(y)$, we get

$$[u_{F'}]_F(x) = -\int_{\partial F} \frac{\partial G_{D-F}(x, y)}{\partial \boldsymbol{n}_y} u_{F'}(y) dS_y \qquad (x \in D-F).$$

Hence we obtain $[u_{F'}]_F(x) = u_{F'}(x)$ for any $x \in D-F$.

COROLLARY. *If $F, F' \in \mathfrak{F}$ and $F \supset F'$, then $u_F(x) \geqq u_{F'}(x)$ in $D-F$.*

This is clear from (4.7), (4.10) and Lemma 4.6.

LEMMA 4.7. *If $F, F', F'' \in \mathfrak{F}$ and $F \cup F' \supset F''$, then*

$$u_F(x) + u_{F'}(x) \geqq u_{F''}(x) \quad \text{in} \quad D-(F \cup F').$$

PROOF. If $m > n > m' > n'$, then

$$u_{nm}^F(x) + u_{nm}^{F'}(x) \geqq \begin{cases} u_{nm}^F(x) = u(x) \geqq u_{n'm'}^{F''}(x) & \text{on} \quad \partial F \cap (D_{m'} - F') \\ u_{nm}^{F'}(x) = u(x) \geqq u_{n'm'}^{F''}(x) & \text{on} \quad \partial F \cap (D_{m'} - F) \end{cases}$$

and

$$u_{nm}^F(x) + u_{nm}^{F'}(x) \geqq 0 = u_{n'm'}^{F''}(x) \quad \text{on} \quad \partial D_{m'} - (F \cup F'),$$

and both $u_{nm}^F(x) + u_{nm}^{F'}(x)$ and $u_{n'm'}^{F''}(x)$ are A-harmonic in $D_{m'} - (F \cup F')$. Hence, by means of Corollary to Lemma 2.1, we get

$$u_{nm}^{F}(x)+u_{nm}^{F'}(x) \geqq u_{n'm'}^{F''}(x) \quad \text{in} \quad D_{m'}-(F \cup F').$$

Letting $m \to \infty$, then $n \to \infty$, then $m' \to \infty$ and then $n' \to \infty$, we obtain the conclusion of this lemma by means of (4.8) and (4.9).

Now, for any closed subset Γ of $\mathfrak{S}$, we denote by $\mathfrak{F}_{\Gamma}$ the totality of the sets $F \in \mathfrak{F}$ such that $F^{\circ} \supset \Gamma$, and put

$$u_{\Gamma}(x) = \inf_{F \in \mathfrak{F}_{\Gamma}} u_F(x) \tag{4.13}$$

for any non-negative A-harmonic function $u(x)$ in D. Then ;—

LEMMA 4.8. *For any closed subset Γ of $\mathfrak{S}$, there exists a monotone decreasing sequence $\{F_n\} \subset \mathfrak{F}_{\Gamma}$ such that $\lim_{n\to\infty} F_n = \Gamma$. Furthermore*

$$\lim_{n\to\infty} u_{F_n}(x) = u_{\Gamma}(x) \leqq u(x) \tag{4.14}$$

for any such sequence $\{F_n\}$, and $u_{\Gamma}(x)$ is A-harmonic in D.

The existence of such sequence $\{F_n\}$ may easily be shown. Proof of (4.14) is just the same as that of Lemma 1 in [8 ; § 3, p. 151], and A-harmonicity of $u_{\Gamma}(x)$ may be seen by Lemma 2.2.

A number of fundamental properties of the function $u_{\Gamma}(x)$ will be derived in the following theorem, in which $u(x)$ and $v(x)$ will denote non-negative A-harmonic functions in D, and Γ, Γ', etc.—closed subsets of $\mathfrak{S}$.

THEOREM 4.1. *The function $u_{\Gamma}(x)$ is non-negative and A-hormonic in D, and has the following properties:*

(a) $u(x) \geqq u_{\Gamma}(x)$ *for any* $x \in D$.

(b) *If* $u(x) \geqq v(x)$ *for any* $x \in D$, *then* $u_{\Gamma}(x) \geqq v_{\Gamma}(x)$.

(c) $[u+v]_{\Gamma}(x) = u_{\Gamma}(x)+v_{\Gamma}(x)$.

(d) $[c \cdot u]_{\Gamma}(x) = c \cdot u_{\Gamma}(x)$ *for any non-negative constant* c.

(e) $u_{\mathfrak{S}}(x) = u(x)$.

(f) *If* $\Gamma \supset \Gamma'$, *then* $[u_{\Gamma'}]_{\Gamma}(x) = u_{\Gamma'}(x)$.

(g) *If* $\Gamma \supset \Gamma'$, *then* $u_{\Gamma}(x) \geqq u_{\Gamma'}(x)$. *If* $\Gamma_n \downarrow \Gamma$, *then* $u_{\Gamma_n}(x) \downarrow u_{\Gamma}(x)$.

(h) $u_{\Gamma \cup \Gamma'}(x) \leqq u_{\Gamma}(x)+u_{\Gamma'}(x)$.

PROOF. A-harmonicity of $u_{\Gamma}(x)$ and the statements (a), (b), (c) and (d) are immediate consequences of the definition of $u_{nm}^{F}(x)$, (4.8), (4.9) and Lemma 4.8.

To prove (e), we take a closed set $F \in \mathfrak{F}_{\mathfrak{S}}$. Then $D-F^{\circ}$ is a compact subset of D, and accordingly $D-F^{\circ} \subset D_{n_0}$ for a suitable n_0. Hence, if $m>n>n_0$, we have $u_{nm}^{F}(x) = u(x)$ on $\partial(D-F)$ and accordingly in $D-F$. Letting $m \to \infty$, and then $n \to \infty$, we obtain $u_F(x) = u(x)$ in $D-F$ by (4.8) and (4.9). This result implies the statement (e) by (4.13).

To prove (f), we take an arbitrary closed set $F \in \mathfrak{F}_{\Gamma}$. Then, since $F^{\circ} \supset \Gamma \supset \Gamma'$, there exists a monotone decreasing sequence $\{F_n\} \subset \mathfrak{F}_{\Gamma'}$ such that

$F_n \subset F$ for any n and that $\lim_{n\to\infty} F_n = \Gamma'$. Hence $[u_{F_n}]_F(x) = u_{F_n}(x)$ for any n and $\lim_{n\to\infty} u_{F_n}(x) = u_{\Gamma'}(x)$ in $D-F$ by Lemmas 4.6 and 4.8, and hence we get $[u_{\Gamma'}]_F(x) = u_{\Gamma'}(x)$ in $D-F$ by Lemma 4.5 (using $u(x)$ as a majorant), and accordingly $[u_{\Gamma'}]_\Gamma(x) = u_{\Gamma'}(x)$ in D by (4.13).

The first assertion of (g) is evident from (4.13) since $\Gamma \supset \Gamma'$ implies $\mathfrak{F}_\Gamma \subset \mathfrak{F}_{\Gamma'}$. For the second assertion, the sequence $\{u_{\Gamma_n}(x)\}$ is monotone decreasing with respect to n, and hence $v(x) = \lim_{n\to\infty} u_{\Gamma_n}(x)$ exists and $\geqq u_\Gamma(x)$ for any $x \in D$. On the other hand, for any $F \in \mathfrak{F}_\Gamma$, $\lim_{n\to\infty} \Gamma_n = \Gamma \subset F^\circ$ and each Γ_n is compact, and accordingly $\Gamma_n \subset F^\circ$ except for a finite number of n's. Hence we have $v(x) = \lim_{n\to\infty} u_{\Gamma_n}(x) \leqq u_F(x)$, which implies that $v(x) \leqq u_\Gamma(x)$ by (4.13). Therefore we get $u_\Gamma(x) = \lim_{n\to\infty} u_{\Gamma_n}(x)$.

Finally the assertion (h) is proved as follows. For any $F \in \mathfrak{F}_\Gamma$ and $F' \in \mathfrak{F}_{\Gamma'}$, we have $F^\circ \cup F'^\circ \supset \Gamma \cup \Gamma'$ and accordingly there exists $F'' \in \mathfrak{F}_{\Gamma\cup\Gamma'}$ such that $F \cup F' \supset F''$. Hence, by Lemma 4.7, $u_F(x) + u_{F'}(x) \geqq u_{F''}(x) \geqq u_{\Gamma\cup\Gamma'}(x)$ in $D-(F\cup F')$. Since F and F' respectively run over $\mathfrak{F}_\Gamma$ and $\mathfrak{F}_{\Gamma'}$ independently, we obtain (h) from the above inequality.

THEOREM 4.2. *If $u(x)$ is non-negative and A-harmonic in D and Γ is a closed subset of $\mathfrak{S}$, then there exists a bounded Borel measure μ_Γ on Γ such that*

$$(4.15)\qquad u_\Gamma(x) = \int_\Gamma K(x, \xi) d\mu_\Gamma(\xi) \qquad \text{in } D\,,$$

and $\mu_\Gamma(\Gamma) = u_\Gamma(\gamma)$.

The uniqueness of such μ_Γ does not always hold as shown in [8; §5].

PROOF OF THEOREM 4.2. For any $F \in \mathfrak{F}_\Gamma$ not intersecting with $\bar{D}_0$ and any m and n $(m > n)$, there exists a bounded Borel measure μ^F_{nm} on $\partial F \cap \bar{D}_n$ such that

$$(4.16)\qquad \begin{cases} u^F_{nm}(x) = \int_{\partial F\cap \bar{D}_n} K_m(x, y) d\mu^F_{nm}(y) \text{ for any } x \in D_n - F \text{ and} \\ \mu^F_{nm}(\partial F \cap \bar{D}_n) = u^F_{nm}(\gamma) \leqq u(\gamma) \end{cases}$$

by Lemma 2.4 (with $\Omega = D-F$) and (3.1). Since $\{\mu^F_{nm}\,;\, m = 1, 2, \cdots\}$ is a sequence of Borel measures on the compact set $\partial F \cap \bar{D}_n$ uniformly bounded by $u(\gamma)$, a suitable subsequence $\{\mu^F_{nm'}\}$ converges to a Borel measure μ^F_n on $\partial F \cap \bar{D}_n$ weakly as bounded linear functionals on $C(\partial F \cap \bar{D}_n)$. On the other hand, by means of (3.3), $\lim_{n\to\infty} K_n(x, y) = K(x, y)$ uniformly in y in the compact set $\partial F \cap \bar{D}_n$ for any fixed x. Hence, letting $m = m' \to \infty$ in (4.16), we obtain by (4.8) that

$$(4.17)\qquad \begin{cases} u^F_n(x) = \int_{\partial F\cap \bar{D}_n} K(x, y) d\mu^F_n(y) \text{ for any } x \in D_n - F \text{ and} \\ u^F_n(\partial F \cap \bar{D}_n) \leqq u^F_n(\gamma) \leqq u(\gamma)\,. \end{cases}$$

Similarly, since $\{\mu_n^F; n=1,2,\cdots\}$ may be considered as a sequence of Borel measures on the compact set ∂F uniformly bounded by $u(\gamma)$, we may show that

$$(4.18)\qquad \begin{cases} u_F(x)=\int_{\partial F} K(x,y)d\mu_F(y) \text{ for any } x\in D-F \text{ and} \\ \mu_F(\partial F)\leqq u_F(\gamma)\leqq u(\gamma) \end{cases}$$

for a suitable bounded Borel measure μ_F on ∂F. Now we take a monotone decreasing sequence $\{F_n\}\subset\mathfrak{F}_\Gamma$ such that $F_n\subset\mathfrak{D}-\bar{D}_0$ and $\lim_{n\to\infty} F_n=\Gamma$. Then $\{\mu_{F_n}\}$ may be considered to be a sequence of Borel measures on the compact set F_1 uniformly bounded by $u(\gamma)$. Hence a suitable subsequence $\{\mu_{F_{n'}}\}$ converges to a bounded Borel measure μ_Γ on F_1 weakly as bounded linear functionals on $C(F_1)$. Since $\{\mu_{F_n}; n\geqq m\}$ is a sequence of measures on F_m for any fixed m, μ_Γ is a measure on F_m; here m is arbitrary. Hence μ_Γ is a measure on $\Gamma\equiv\bigcap_{m=1}^{\infty} F_m$. Letting $F=F_{m'}$ and $m'\to\infty$ in (4.18), we obtain (4.15); accordingly we get $u_\Gamma(\gamma)=\mu_\Gamma(\Gamma)$ by (2.18) and (3.5).

Theorem 4.3 (Representation theorem). *If $u(x)$ is non-negative and A-harmonic in D, then there exists a bounded Borel measure on $\mathfrak{S}$ such that*

$$(4.19)\qquad u(x)=\int_{\mathfrak{S}} K(x,\xi)d\mu(\xi) \qquad in\ D$$

and $\mu(\mathfrak{S})=u(\gamma)$. Conversely, for any bounded Borel measure μ on $\mathfrak{S}$, the formula (4.19) *represents a non-negative A-harmonic function $u(x)$ in D.*

Proof. The first part of this theorem immediately follows from Theorem 4.2 (with $\Gamma=\mathfrak{S}$) and (e) in Theorem 4.1. The converse statement is proved as follows. For any compact subset E of D, $K(x,\xi)$ is uniformly continuous on $E\times\mathfrak{S}$ by Corollary 1 to Theorem 3.2. Hence the integral in (4.19) is approximated uniformly on E by means of 'Riemann sum,' that is, the function of the form

$$(4.20)\qquad \sum_{\nu=1}^{l} K(x,\xi_\nu)c_\nu \qquad \text{with} \qquad c_\nu\text{'s}>0\,.$$

Since any function of the form (4.20) is non-negative and A-harmonic in D, so is the function $u(x)$ defined by (4.19) by virtue of Lemma 2.2.

§5. The extremal functions and the uniqueness theorem. In this §, we shall give a characterization of the extremal A-harmonic functions and mention the existence of a unique canonical representation in terms of extremal A-harmonic functions.

The argument in the preceding sections are similar to, but not quite the same as, those in the corresponding sections in Martin's paper [8]. However, all properties of non-negative A-harmonic functions corresponding to those established in § 3 of [8] and also some properties corresponding to those stated in § 1 and quoted in § 4 of [8] are already shown in § 4 of the present paper. So we may achieve the essentially same arguments for A-harmonic functions as those for classical harmonic functions in § 4 of [8]—only some minor technical modifications may be necessary. Thence it seems not to be necessary to mention the arguments in detail. We shall state only the outline of the process to give a characterization of the extremal A-harmonic functions and to get a unique canonical representation.

By definition, a positive A-harmonic function $u(x)$ in D is said to be *extremal*[6] if every non-negative A-harmonic function in D not exceeding[7] $u(x)$ is a constant multiple of $u(x)$.

Since $K^{\xi}(x)\equiv K(x, \xi)$ is positive and A-harmonic in $x\in D$ for any fixed $\xi\in\mathfrak{S}$, $[K^{\xi}]_{\Gamma}(x)$ is defined for any closed subset Γ of $\mathfrak{S}$ as stated in the preceding section.

LEMMA 5.1. *Let $u(x)$ be positive and A-harmonic in D and extremal, let B be any Borel subset of $\mathfrak{S}$, and assume that*

(5.1) $$u(x)\geqq\int_{B}K(x, \xi)d\mu(\xi)>0 \quad \textit{for any} \quad x\in D\,.$$

Then $u(x)=u(\gamma)K(x, \xi)$ for some point $\xi\in B$. (Cf. Lemma 1 in § 4 of [8].)

From this lemma immediately follows that

COROLLARY 1. *Every extremal positive A-harmonic function in D is a positive multiple of $K(x, \xi)$ for some $\xi\in\mathfrak{S}$.*

COROLLARY 2. *If $K^{\xi}(x)$ is extremal, and Γ is a closed subset of $\mathfrak{S}$ such that $[K^{\xi}]_{\Gamma}(x)$ is positive, then ξ is in Γ.*

Now we put

$$\psi(\xi)=[K^{\xi}]_{\{\xi\}}(\gamma) \quad \text{for any} \quad \xi\in\mathfrak{S}$$

($\{\xi\}$ is the closed set which consists of the single point ξ). Then,

THEOREM 5.1. *The function $\psi(\xi)$ takes only two possible values* 1 *and* 0. *The function $K^{\xi}(x)$ is extremal if and only if $\psi(\xi)=1$.* (Cf. Theorem I in § 4 of [8].)

THEOREM 5.2. *The set $\mathfrak{S}_0=\{\xi\in\mathfrak{S}\,;\,\psi(\xi)=0\}$ is an F_{σ}-set (possibly closed or empty).* (Cf. Theorem II in § 4 of [8].)

In fact, we may show that $\mathfrak{S}_0$ is the sum of the monotone increasing sequence $\{\Gamma_n\}$ of closed (possibly empty) subset of $\mathfrak{S}$ defined as follows:

$$\Gamma_n=\left\{\ \xi\in\mathfrak{S}\,;\ \begin{array}{l}[K^{\xi}]_F(\gamma)\leqq 1/2 \text{ for any } F\in\mathfrak{F} \text{ such that } \xi\in F^{0}\\ \text{and that the } \rho\text{-diameter of } F \text{ is less than } 1/n\end{array}\right\}.$$

6) It is called *minimal* in Martin's paper [8].

7) '$v(x)$ does not exceed $u(x)$' means that $v(x)\leqq u(x)$ for any $x\in D$.

By virture of these two theorems, we may see that $\mathfrak{S}_0$ and

$$\mathfrak{S}_1 = \mathfrak{S} - \mathfrak{S}_0 \equiv \{\xi \in \mathfrak{S}\,;\, \psi(\xi) = 1\}$$

are Borel subsets of $\mathfrak{S}$ and we can state the following definition of the essential part of the Martin boundary and that of a canonical representation involving only those $K(x, \xi)$'s which are extremal.

DEFINITION 1. $\mathfrak{S}_1$ is called the *essential part* of the Martin boundary $\mathfrak{S}$.

DEFINITION 2. A bounded Borel measure μ on $\mathfrak{S}$ is called *canonical* if $\mu(\mathfrak{S}_0) = 0$. A representation of the form given in Theorem 4.3 is called a *canonical representation* if the measure μ occurring in it is canonical.

The following lemmas will give some steps to approach the theorem establishing the unique existence of a canonical representation. These may be proved by the essentially same arguments as proofs of corresponding lemmas in §4 of [8]; Lemma 4.5 in the present paper corresponds to part (e) of Theorem II in §1 of [8] which is used in the proof of Lemma 2 in §4 of [8].

LEMMA 5.2. *Let $\{\Gamma_n\}$ be as stated above. Then $u_{\Gamma_n}(x) = 0$ for any positive and A-harmonic function $u(x)$ in D and any n.*

LEMMA 5.3. *For any positive A-harmonic function $u(x)$ in D and any $\varepsilon > 0$, there exists a closed subset Γ of $\mathfrak{S}_1$ such that $u(\gamma) \leqq u_\Gamma(\gamma) + \varepsilon$.*

LEMMA 5.4. *Let Γ and Γ' be closed subsets of $\mathfrak{S}$ such that $\Gamma \cap \Gamma'$ is empty and $\Gamma' \subset \mathfrak{S}_1$, and let ε be an arbitrary positive number. Then there exists $F \in \mathfrak{F}_\Gamma$ such that $[K^\xi]_F(\gamma) < \varepsilon$ for any $\xi \in \Gamma'$.*

LEMMA 5.5. *Let Γ be a closed subset of $\mathfrak{S}$ and B be a Borel subset of $\mathfrak{S}_1$ not intersecting with Γ. Let $u(x)$ be a harmonic function of the form*

$$u(x) = \int_B K(x, \xi) d\mu(\xi)\,.$$

Then $u_\Gamma(x) \equiv 0$.

(Cf. Lemmas 2, 3, 4 and 5 in §4 of [8].)

Using these lemmas, we may prove the following

THEOREM 5.3. *Every non-negative A-harmonic function $u(x)$ in D admits of exactly one canonical representation, that is, $u(x)$ is represented in a unique manner in the form*

$$u(x) = \int_{\mathfrak{S}_1} K(x, \xi) d\mu_1(\xi) \qquad (x \in D) \tag{5.2}$$

where μ_1 is a bounded Borel measure on $\mathfrak{S}_1$. The canonical measure μ_1 representing $u(x)$ is characterized by the relation:

$$u_\Gamma(x) = \int_\Gamma K(x, \xi) d\mu_1(\xi) \qquad (x \in D) \tag{5.3}$$

for every closed subset Γ of $\mathfrak{S}$.

COROLLARY 1. *The function $u_\Gamma(x)$, defined for closed subsets Γ of $\mathfrak{S}$,* *admits of extension to a countably additive function of Borel sets in* $\mathfrak{S}$.

COROLLARY 2. *In order for the representation of Theorem* 4.3 *to be unique in general, it is necessary and sufficient that* $\mathfrak{S}_0$ *is empty.*

§ 6. Imbedding of the smooth boundary of the domain into the Martin boundary. In this §, we prove the following

THEOREM 6.1. *Assume that a part* S *of the boundary* ∂D *of the domain* D *considered in* $\boldsymbol{M}$ *consists of an* $(N-1)$*-dimensional simple hypersurface of class* C^3, *and that* $\|a^{ij}(x)\|$ *and* $\|b^i(x)\|$ *are of class* C^2 *on* $D+S$. *Then* S *is homeomorphically imbedded in the essential part* $\mathfrak{S}_1$ *of the Martin boundary* $\mathfrak{S}$[8]: *more precisely, for any point* $z \in S$, *there corresponds a point* $\xi_z \in \mathfrak{S}_1$ *in one-to-one way, and the mapping* Φ *defined by*

$$\Phi(x) = x \text{ for } x \in D \text{ and } \Phi(z) = \xi_z \text{ for } z \in S \tag{6.1}$$

gives a homeomorphism of $D+S$ *as a subspace of the original manifold* $\boldsymbol{M}$ *onto* $D+\{\xi_z\,;\,z \in S\}$ *as a subspace of the compact metric space* $\mathfrak{D}$.

THEOREM 6.2. *Under the assumption of the preceding theorem, we have*

$$K(x, \xi_z) = \frac{\partial G(x, z)}{\partial \boldsymbol{n}_z} \Big/ \frac{\partial G(\gamma\,;\,z)}{\partial \boldsymbol{n}_z} \quad \text{for any } x \in D \text{ and any } z \in S,$$

and accordingly $-\dfrac{\partial G(x, z)}{\partial \boldsymbol{n}_z}$ *is an extremal A-harmonic function of* $x \in D$ *for any fixed* $z \in S$.

Under the assumption of Theorem 6.1, we denote by $\mathrm{dis}(x, y)$ the Riemannian distance between the points x and y in $D+S$ defined by $\|a_{ij}(x)\|$.

LEMMA 6.1. *For any fixed* $x \in D$, $K(x, y)$ *is extended to a continuous function of* y *in* $D+S-\{x\}$ *by putting*

$$K(x, z) = \frac{\partial G(x, z)}{\partial \boldsymbol{n}_z} \Big/ \frac{\partial G(\gamma\,;\,z)}{\partial \boldsymbol{n}_z} \qquad \text{for } z \in S\,. \tag{6.2}$$

PROOF. Let z_0 be any fixed point in S. Then, as is shown in Lemma 2.1 in [5], there exists a neighborhood $U(z_0)$ of z_0 and a local coordinate system $(x^1, \cdots, x^N)$ defined in $U(z)$ with respect to which i) $S \cap U(z_0)$ is represented by the equation $x^1 = 0$, ii) $x^1 > 0$ in $D \cap U(z_0)$ and iii) $\dfrac{\partial f(z)}{\partial \boldsymbol{n}_z} = -\dfrac{\partial f(z)}{\partial z^1}$ for any $z \in S \cap U(z_0)$. We may take a domain Ω with property (S), with compact closure and such that

$$(U(z_0) \cap D) \cup \bar{D}_0 \subset \Omega \subset D$$

and that $\overline{(\partial\Omega - S)}$ does not intersect with $\overline{U(z_0)} \cup \bar{D}_0$. Then, by a similar argument to the proof of Lemma 2.6 (given in Appendix), we may obtain that

8) See Definition 1 in § 5.

$$G(x, y) = G_{\Omega}(x, y) - \int_{\partial\Omega - S} G(x, z) \frac{\partial G_{\Omega}(z, y)}{\partial \boldsymbol{n}_z} dS_z$$

for any x and $y \in \bar{\Omega}$.

On the other hand, $G_{\Omega}(x, y)$ is of class C^1 in $\langle x, y \rangle \in \bar{\Omega} \times \bar{\Omega} - \{\langle z, z \rangle ; z \in \bar{\Omega}\}$ and satisfies

$$\frac{\partial G_{\Omega}(x, y)}{\partial \boldsymbol{n}_y} < 0 \quad \text{and} \quad \frac{\partial^2 G_{\Omega}(z, y)}{\partial \boldsymbol{n}_z \partial \boldsymbol{n}_y} > 0$$

whenever $x \in \Omega$, y, $z \in \partial\Omega$ and $y \neq z$. Hence, for any fixed $x \in \Omega$, $G(x, y)$ is of class C^1 in $y \in \bar{\Omega} - \{x\}$ and satisfies $\partial G(x, y)/\partial \boldsymbol{n}_y < 0$ for any $y \in \partial\Omega \cap S$. Therefore, using the properties i), ii) and iii) of the local coordinate stated above, we may easily show that $K(x, y)$ is continuous in y on $D+S-\{x\}$, for any fixed $x \in D$, if $K(x, z)$ is defined by (6.2) for $z \in S$.

LEMMA 6.2. *For any $z \in S$, there corresponds one and only one point $\xi_z \in \mathfrak{S}$ such that $\lim_{\nu\to\infty} \rho(y_\nu, \xi_z) = 0$ holds for any sequence $\{y_\nu\} \subset D$ satisfying $\lim_{\nu\to\infty} \operatorname{dis}(y_\nu, z) = 0$.*

PROOF. For any given $z \in S$, we may take a sequence $\{z_n\} \subset D$ such that $\lim_{n\to\infty} \operatorname{dis}(z_n, z) = 0$. The sequence $\{z_n\}$ has no accumulating point in D with respect to ρ, while $\mathfrak{D}$ is (ρ-) compact. Hence there exists a subsequence $\{z_{n_\nu}\}$ of $\{z_n\}$ and a point $\xi \in \mathfrak{S}$ such that $\lim_{\nu\to\infty} \rho(z_{n_\nu}, \xi) = 0$. Let $\{y_\nu\}$ be an arbitrary sequence in D satisfying $\lim_{\nu\to\infty} \operatorname{dis}(y_\nu, z) = 0$. Then, by Lemma 6.1, we have

$$\lim_{\nu\to\infty} | K(x, y_\nu) - K(x, z_{n_\nu}) | = 0 \quad \text{for any fixed} \quad x \in D_0 .$$

Hence we obtain $\lim_{\nu\to\infty} \rho(y_\nu, z_{n_\nu}) = 0$ by means of the definition (3.7) of the metric ρ, and accordingly we get $\lim_{\nu\to\infty} \rho(y_\nu, \xi) = 0$. This result implies also that $\lim_{n\to\infty} \rho(z_n, \xi) = 0$ for the original sequence $\{z_n\}$ and consequently that the point $\xi \in \mathfrak{S}$ is uniquely determined by $z \in S$; so we may write $\xi = \xi_z$. Lemma 6.2 is thus proved.

COROLLARY.

$$K(x, \xi_z) = \frac{\partial G(x, z)}{\partial \boldsymbol{n}_z} \Big/ \frac{\partial G(\gamma ; z)}{\partial \boldsymbol{n}_z}$$

for any $x \in D$ and any $z \in S$,

This is a direct consequence of Theorem 3.2 and the preceding two lemmas.

LEMMA 6.3. i) *If E is a compact subset of $D+S$ and F is a subset of $D-(E \cup \bar{D}_0)$ relatively closed in D, then $K(x, y)$ is bounded on $E \times F$.*

ii) *For any $z, z' \in S$ and any sequence $\{x_n\} \subset D$ satisfying $\lim_{n\to\infty} \operatorname{dis}(x_n, z') = 0$, it holds that $\lim_{n\to\infty} K(x_n, \xi_z) = \infty$ or 0 according as $z' = z$ or $z' \neq z$.*

(Part i) is a generalization of Lemma 2.6.)

PROOF. i) By virtue of the assumption, we may take a domain Ω with property (S) and such that $E\cup\bar{D}_0\subset\bar{\Omega}\subset\bar{D}-F$, $(E\cup\bar{D}_0)\cap(\overline{\partial\Omega-S})$ is empty and $\bar{\Omega}$ is compact. Then, by the same argument as the proof of Lemma 2.6 (given in Appendix), we may obtain that

$$G(x, y)=G_{\Omega}(x, y)+\int_{\partial\Omega-S}\left\{-\frac{\partial G_{\Omega}(x, z)}{\partial \boldsymbol{n}_z}\right\}G(z, y)dS_z \tag{6.3}$$

for any $x\in\Omega$ and $y\in D$

where

$$G_{\Omega}(x, y)=0 \quad \text{for} \quad x\in\bar{\Omega} \quad \text{and} \quad y\in D-\Omega\,, \tag{6.4}$$

and that there exist constants C_1 and C_2 such that

$$0\leqq-\frac{\partial G_{\Omega}(x, z)}{\partial \boldsymbol{n}_z}\leqq C_1 \quad \text{for any } x\in E \text{ and } z\in\partial\Omega-S \tag{6.5}$$

and

$$0<C_2\leqq-\frac{\partial G_{\Omega}(\gamma\,;z)}{\partial \boldsymbol{n}_z}\leqq C_1 \quad \text{for any } z\in\partial\Omega-S\,. \tag{6.6}$$

Hence, combining (6.3) and (6.4), we get

$$K(x, y)=\frac{G(x, y)}{G(\gamma\,;y)}\leqq\frac{\int_{\partial\Omega-S}C_1G(z, y)dS_z}{\int_{\partial\Omega-S}C_2G(z, y)dS_z}=\frac{C_1}{C_2}<\infty$$

for any $x\in E$ and $y\in D-\Omega$, and accordingly $K(x, y)$ is bounded on $E\times F$.

ii) It follows from the assumption that there exists a subdomain Ω of D with property (S), with compact closure and such that

$$E=\{z, z', x_1, x_2, \cdots, x_n, \cdots\}^{9)} \quad \text{and} \quad F=D-\Omega$$

satisfy the assumption in i); accordingly we may use (6.3), (6.4), (6.5) and (6.6) stated above. Furthermore, since $-\dfrac{\partial G(y, z)}{\partial \boldsymbol{n}_z}$ is non-negative and continuous in $y\in\bar{D}-\{z\}$, there exists a constant C_3 such that

$$0\leqq-\frac{\partial G(y, z)}{\partial \boldsymbol{n}_z}\leqq C_3 \quad \text{for any } y\in\partial\Omega-S\,.$$

Hence, by Corollary to Lemma 6.2, we have

$$K(x_n, \xi_z)=\frac{-\dfrac{\partial G(x_n, z)}{\partial \boldsymbol{n}_z}}{-\dfrac{\partial G(\gamma\,;z)}{\partial \boldsymbol{n}_z}}\begin{cases}\geqq\dfrac{-\dfrac{\partial G_{\Omega}(x_n, z)}{\partial \boldsymbol{n}_z}}{-\dfrac{\partial G_{\Omega}(\gamma\,;z)}{\partial \boldsymbol{n}_z}-\int_{\partial\Omega-S}C_1C_3dS_y}>0\,,\\[2ex] \leqq\dfrac{-\dfrac{\partial G_{\Omega}(x_n, z)}{\partial \boldsymbol{n}_z}-\int_{\partial\Omega-S}\dfrac{\partial G_{\Omega}(x_n, y)}{\partial \boldsymbol{n}_y}C_3dS_y}{C_2}\,.\end{cases}$$

9) z' may coincide with z.

On the other hand, it may be seen from the construction of the fundamental solution $U_{\Omega}(t, x, y)$ and the Green function $G_{\Omega}(x, y)$ (stated in [6]) that

$$\lim_{n\to\infty} \frac{\partial G_{\Omega}(x_n, z')}{\partial \boldsymbol{n}_z} = \infty$$

and

$$\lim_{n\to\infty} \frac{\partial G_{\Omega}(x_n, y)}{\partial \boldsymbol{n}_y} = 0 \qquad \text{uniformily in } y \in B$$

for each compact subset B of $\partial\Omega$ not containing z'. Hence we obtain $\lim_{n\to\infty} K(x_n, \xi_z) = \infty$ or 0 according as $z' = z$ or $z' \neq z$.

LEMMA 6.4. *If* $\lim_{n\to\infty} \rho(y_n, \xi_z) = 0$ *where* $\{y_n\} \subset D$ *and* $z \in S$, *then* $\lim_{n\to\infty} \mathrm{dis}\,(y_n, z) = 0$.

PROOF. Suppose that $\lim_{n\to\infty} \mathrm{dis}\,(y_n, z) = 0$ does not hold. Then there exists a neighborhood $U(z)$ of z and a subsequence $\{y_{n_\nu}\}$ of the sequence $\{y_n\}$ such that $y_{n_\nu} \notin U(z)$ for any ν. Let $\{x_m\}$ be a sequence in $U(z) \cap D$ such that $\lim_{m\to\infty} \mathrm{dis}\,(x_m, z) = 0$. Then, by part i) of Lemma 6.3 (with $E = \{z, x_1, x_2, \cdots\}$ and $F = D - U(z)$), there exists a constant C such that

$$K(x_m, y_{n_\nu}) \leqq C \qquad \text{for any } m \text{ and } \nu\,. \tag{6.7}$$

Since $\lim_{\nu\to\infty} \rho(y_{n_\nu}, \xi_z) = 0$ and $K(x, y)$ is continuous in $y \in D$ with metric ρ for any fixed x, it follows from (6.7) that

$$K(x_m, \xi_z) \leqq C \qquad \text{for any } m\,,$$

which contradicts to part ii) of Lemma 6.3.

From this lemma, immediately follows that

COROLLARY. *$z \neq z'$ ($z, z' \in S$) implies $\xi_z \neq \xi_{z'}$, and accordingly, for any $\xi \in \mathfrak{S}$, there exists at most one point $z \in S$ such that $\xi = \xi_z$.*

LEMMA 6.5. *For any point $z \in S$, there corresponds a point $\xi_z \in \mathfrak{S}$ in one to one way, and the mapping Φ defined by*

$$\Phi(x) = x \text{ for } x \in D \text{ and } \Phi(z) = \xi_z \text{ for } z \in S \tag{6.1'}$$

gives a homeomorphism of $D+S$ as a subspace of $\boldsymbol{M}$ into $\mathfrak{D}$.

(This lemma would be nothing else than Theorem 6.1 if $\mathfrak{S}$ be restricted to $\mathfrak{S}_1$. We first prove this lemma, by virtue of which we can prove the following two lemmas which imply that $\Phi(\mathfrak{S}) \subset \mathfrak{S}_1$. Combining this result with the above lemma, we can finally obtain Theorem 6.1.)

PROOF. For any $z \in S$, there corresponds one and only one point $\xi_z \in \mathfrak{S}$ with the property stated in Lemma 6.2. From this fact and Corollary to Lemma 6.4, it follows that (6.1′) defines a one-to-one mapping of $D+S$ into $\mathfrak{D}(= D+\mathfrak{S})$. The bi-continuity of the mapping Φ at any point $x \in D$ is obvious. We shall prove the bi-continuity at any point $z \in S$.

For any sequence $\{x_n\} \subset D$ and any $z \in S$, $\lim_{n\to\infty} \operatorname{dis}(x_n, z) = 0$ implies, and is implied by, $\lim_{n\to\infty} \rho(\Phi(x_n), \Phi(z)) \equiv \lim_{n\to\infty} \rho(x_n, \xi_z) = 0$ by means of Lemma 6.2 and 6.4. Therefore, it is sufficient to prove, under the condition: $\{z, z_1, z_2, \cdots\} \subset S$, that $\lim_{n\to\infty} \rho(\xi_{z_n}, \xi_z) = 0$ if and only if $\lim_{n\to\infty} \operatorname{dis}(z_n, z) = 0$.

For each $z_n \in S$, we may take $x_n \in D$ such that both $\operatorname{dis}(x_n, z_n) < 1/n$ and $\rho(x_n, \xi_{z_n}) < 1/n$ hold (by Lemma 6.2), and consequently

$$\begin{cases} \operatorname{dis}(x_n, z) - \dfrac{1}{n} \leqq \operatorname{dis}(z_n, z) \leqq \operatorname{dis}(x_n, z) + \dfrac{1}{n} \quad \text{and} \\ \rho(x_n, \xi_z) - \dfrac{1}{n} \leqq \rho(\xi_{z_n}, \xi_z) \leqq \rho(x_n, \xi_z) + \dfrac{1}{n} . \end{cases}$$

Since $\lim_{n\to\infty} \operatorname{dis}(x_n, z) = 0$ implies and is implied by $\lim_{n\to\infty} \rho(x_n, \xi_z) = 0$ (by Lemmas 6.2 and 6.4), we may see from the above inequalities that $\lim_{n\to\infty} \operatorname{dis}(z_n, z) = 0$ is equivalent to $\lim_{n\to\infty} \rho(\xi_{z_n}, \xi_z) = 0$.

LEMMA 6.6. *Let Γ be a compact subset of S, and $u(x)$ be a non-negative A-harmonic function in D satisfying that $\lim_{\operatorname{dis}(x,z)\to 0} u(x) = 0$ for any $z \in \Gamma$. Then the canonical measure μ_1 representing $u(x)$ satisfies that $\mu_1(\Gamma) = 0$.*

PROOF. For any $\varepsilon > 0$, there exists a subdomain Ω of $\boldsymbol{M}$ with property (S) and such that $\Omega \supset \Gamma$ and $u(x) \leqq \varepsilon$ for any $x \in \bar{\Omega} \frown D$; here we may assume that $\bar{\Omega}$ does not intersect with $\bar{D}_0$ and that the set $F = \bar{\Omega} \frown \bar{D}$ is compact. In view point of the preceding lemma, we may consider that Γ is a compact subset of $\mathfrak{S}$ and accordingly that $F \in \mathfrak{F}_\Gamma$ (see § 4). Then the function $u_{nm}^F(x)$ (defined in § 4) satisfies $u_{nm}^F(x) \leqq \varepsilon$ on $\partial(D_m - F)$, and accordingly on $D_m - F$ by Lemma 2.1. Hence, by means of (4.8), (4.9), (4.13) and (2.18), we have

$$u_\Gamma(\gamma) \leqq u_F(\gamma) = \lim_{n\to\infty} u_n^F(\gamma) = \lim_{n\to\infty} \lim_{m\to\infty} u_{nm}^F(\gamma) \leqq \varepsilon \int_{D_0} \gamma(x) dx = \varepsilon .$$

On the other hand, the formula (5.3) in Theorem 5.3 implies that $u_\Gamma(\gamma) = \mu_1(\Gamma)$ since $K(\gamma\,;\xi) = 1$ by (3.5) and Theorem 3.2. Hence we get $\mu_1(\Gamma) \leqq \varepsilon$; here ε is arbitrary. So we may conclude that $\mu_1(\Gamma) = 0$.

LEMMA 6.7. *For any $z_0 \in S$, $K(x, \xi_{z_0})$ is an extremal positive harmonic function of x.*

PROOF. Let μ_1 be the canonical measure representing the function

$$u(x) = K(x, \xi_{z_0}) , \tag{6.8}$$

and suppose that $K(x, \xi_{z_0})$ is not extremal. Then

$$\mu_1(\{\xi_{z_0}\}) = 0 . \tag{6.9}$$

Let Ω be an open subset of $\boldsymbol{M}$ containing z_0 and with compact closure $\bar{\Omega}$. Then, for any compact set $\Gamma \subset S \frown \Omega - \{z_0\}$, the function $u(x)$ defined by (6.8) satisfies the assumption of Lemma 6.6 by virtue of part ii) of Lemma 6.3. Hence we have $\mu_1(\Gamma)=0$, and accordingly we may obtain $\mu_1(S\cap\Omega-\{z_0\})=0$. Combining this result with (6.9), we get $\mu_1(S\cap\Omega)=0$. Hence the canonical representation of $u(x)$ is reduced to the following form:

$$u(x)=\int_{\mathfrak{S}_1-\Omega}K(x,\xi)d\mu_1(\xi)\,. \tag{6.10}$$

Now let $\{x_n\}$ be a sequence in $D\cap\Omega$ such that $\lim_{n\to\infty}\operatorname{dis}(x_n,z_0)=0$. Then, by part i) of Lemma 6.3 and Theorem 3.2, we have

$$\sup_{\xi\in S-\Omega,n\geqq 1}K(x_n,\xi)\leqq C$$

for a suitable constant C. Hence we obtain from (6.10) that

$$\sup_{n\geqq 1}u(x_n)\leqq C\mu_1(\mathfrak{S}_1)=Cu(\gamma)<\infty\,.$$

On the other hand, it follows from (6.8) and part ii) of Lemma 6.3 that $\lim_{n\to\infty}u(x_n)=\infty$ contrary to the above result. Hence $K(x,\xi_{z_0})$ must be extremal.

Proof of Theorems 6.1. and 6.2. By Lemma 6.5, there corresponds $\xi_z\in\mathfrak{S}$ for any $z\in S$ in one-to-one way, and the mapping Φ defined by (6.1) gives a homeomorphism of $D+S$ as a subspace of $\boldsymbol{M}$ into $\mathfrak{D}=D+S$. Furthermore, for any $z\in S$, $K(x,\xi_z)$ is extremal by Lemma 6.7, and hence ξ_z belongs to $\mathfrak{S}_1$ by Theorem 5.1 and Definition 1 in §5. Therefore we get the conclusion of Theorem 6.1, and accordingly Theorem 6.2 follows immediately from Corollary to Lemma 6.2 and Lemma 6.7.

Appendix. Proofs of Lemmas stated in §2.

In the sequel, notations should be understood as stated in §2.

Proof of Lemma 2.1 may be easily obtained by Lemma 3.1 and Theorem 1 in the author's previous paper [7].

Proof of Lemma 2.2. Let Ω_1 be an arbitrary subdomain of Ω with compact closure $\bar{\Omega}_1\subset\Omega$ and with property (S). Then, by the formula (2.7), we have

$$u_n(x)=-\int_{\partial\Omega_1}\frac{\partial G_{\Omega_1}(x,y)}{\partial\boldsymbol{n}_y}u_n(y)dS_y\qquad(n=1,2,\cdots)\,.$$

Hence, if i) or ii) in Lemma 2.2 is assumed, we obtain by Lebesgue's convergence theorem that

$$u(x)=-\int_{\partial\Omega_1}\frac{\partial G_{\Omega_1}(x,y)}{\partial\boldsymbol{n}_y}u(y)dS_y\,.$$

Consequently $u(x)$ is A-harmonic in Ω_1; this shows Lemma 2.2 by arbitrariness of Ω_1.

PROOF OF LEMMA 2.3. For any compact subset E of Ω, there exists a domain Ω_1 with property (S) such that $E \subset \Omega_1 \subset \bar{\Omega}_1 \subset \Omega$ and that $\bar{\Omega}_1$ is compact. Then, by the formula (2.7), we have

$$u_\lambda(x) = -\int_{\partial\Omega_1} \frac{\partial G_{\Omega_1}(x, y)}{\partial \boldsymbol{n}_y} u_\lambda(y) dS_y \qquad \text{for any } x \in \Omega_1 .$$

On the other hand,

$$\sup_{x\in E, y\in\partial\Omega_1} \left| \nabla_x \frac{\partial G_{\Omega_1}(x, y)}{\partial \boldsymbol{n}_y} \right| < \infty$$

since E and $\partial\Omega_1$ are mutually disjoint compact sets, and

$$\sup_{y\in\Omega_1, \lambda\in\Lambda} |u_\lambda(y)| < \infty$$

by the assumption. Hence $\{|\nabla u_\lambda(x)| ; \lambda \in \Lambda\}$ is uniformly bounded on E.

PROOF OF LEMMA 2.4. For any $y \in \partial\Omega \cap D_m$, $\frac{\partial}{\partial \boldsymbol{n}_y}$ denotes the *outer* normal derivative at y as a boundary point of the domain $D_m \cap \Omega$ (accordingly it denotes the *inner* normal derivative at y as a boundary point of $D_m - \Omega$). Since $Au(y) = 0$ in $D_m \cap \Omega$ and $u(y) = 0$ on $\partial(D_m \cap \Omega) - D_n$, we have, by Green's formula (2.1),

$$-\frac{\partial}{\partial t}\int_{D_m\cap\Omega} U_m(t, x, y)u(y)dy = -\int_{D_m\cap\Omega} A_y^* U_m(t, x, y)u(y)dy$$
$$= \int_{\partial\Omega\cap D_n}\left\{U_m(t, x, y)\frac{\partial u(y)}{\partial \boldsymbol{n}_y} - \frac{\partial U_m(t, x, y)}{\partial \boldsymbol{n}_y}u(y)\right\}dS_y$$
$$\text{for any } x \in D_m \cap \Omega .$$

Integrating in t over $(0, \infty)$, we obtain

$$(1) \qquad u(x) = \int_{\partial\Omega\cap D_n}\left\{G_m(x, y)\frac{\partial u(y)}{\partial \boldsymbol{n}_y} - \frac{\partial G_m(x, y)}{\partial \boldsymbol{n}_y}u(y)\right\}dS_y$$

by means of (2.5) and by the fact; $\lim_{t\to\infty}\int_{D_m} U_m(t, x, y)dy = 0$, which follows from

$$(2) \qquad \int_0^\infty dt \int_{D_m} U_m(t, x, y)dy < \infty$$

proved in [6][10]. Next we put

$$(3) \qquad v(x) = \int_{\partial\Omega - D_n} \frac{\partial G_{D_m-\Omega}(x, y)}{\partial \boldsymbol{n}_y} u(y) dS_y{}^{11)} .$$

Then $v(x)$ is A-harmonic in $D_m - \Omega$ and

10) See Lemma 10. 1 in [6].

11) As for the equalities (3), (4) and (5), readers should remember the definition of $\frac{\partial}{\partial \boldsymbol{n}_y}$ mentioned at the beginning of this proof of Lemma 2.4.

$$v(x)=\begin{cases} u(x) & \text{for } x\in\partial\Omega\cap D_n \\ 0 & \text{for } x\in\partial(D_m-\Omega)-D_n\,. \end{cases}$$

On the other hand, for any fixed $x\in D_m\cap\Omega$, $G_m(x, y)$ satisfies $A_y^*G_m(x, y)=0$ in $D_m-\Omega$ as a function of y by (2.8*). Hence, by Green's formula (2.1), we have

(4) $$0=\int_{\partial\Omega\cap D_n}\Big\{G_m(x, y)\frac{\partial v(y)}{\partial \boldsymbol{n}_y}-\frac{\partial G_m(x, y)}{\partial \boldsymbol{n}_y}u(y)\Big\}dS_y \text{ for any } x\in D_m\cap\Omega\,.$$

Substracting (4) from (1) term by term, we get

(5) $$u(x)=\int_{\partial\Omega\cap D_n}G_m(x, y)\Big\{\frac{\partial u(y)}{\partial \boldsymbol{n}_y}-\frac{\partial v(y)}{\partial \boldsymbol{n}_y}\Big\}dS_y \text{ for any } x\in D_m\cap\Omega\,.$$

Here we prove that the function $\varphi(y)$ defined by

(6) $$\varphi(y)=\frac{\partial u(y)}{\partial \boldsymbol{n}_y}-\frac{\partial v(y)}{\partial \boldsymbol{n}_y}$$

is non-negative on $\partial\Omega\cap D_n$. Clearly $\varphi(y)$ is continuous on $\partial\Omega\cap D_n$. Hence, if $\varphi(y_0)<0$ at some point $y_0\in\partial\Omega\cap D_n$, then $\varphi(y)<0$ in $V(y_0)\cap\partial\Omega$ for a suitable neighborhood $V(y_0)$ of y_0. On the other hand, it may be seen from the construction of $U_m(t, x, y)$ (stated in [**6**]) that

(7) $$\lim_{x\to y_0, y\to y_0}G_m(x, y)=\lim_{x\to y_0, y\to y_0}\int_0^\infty U_m(t, x, y)dt=\infty\,.$$

Hence, by means of (5), we obtain $\varliminf_{x\to y_0} u(x)<0$ contrary to the assumption of this lemma. Thus we see that $\varphi(y)\geqq 0$ on $\partial\Omega\cap D_n$. Hence

$$d\mu(y)=G_m(\gamma\,;\,y)\varphi(y)dS_y$$

is a Borel measure on $\partial\Omega\cap\bar{D}_n$, and it follows from (5) and (6) that

$$u(x)=\int_{\partial\Omega\cap\bar{D}_n}\frac{G_m(x, y)}{G_m(\gamma\,;\,y)}d\mu(y) \text{ for any } x\in D_m\cap\Omega\,.$$

Multiplying both sides by $\gamma(x)$ and integrating in x over D_0, we obtain

$$u(\gamma)=\mu(\partial\Omega\cap\bar{D}_n)\,.$$

Lemma 2.4 is thus proved.

Proof of Lemma 2.5. We may see from (2.5), (2.10), (2.11) and (2.20) that

$$G(x, y)=\int_0^\infty U(t, x, y)dt=\lim_{n\to\infty}\int_0^\infty U_n(t, x, y)dt=\lim_{n\to\infty}G_n(x, y)$$

whenever $x\neq y$ and that $\{G_n(x, y)\}$ is monotone increasing with respect to n. On the other hand, $G(x, y)$ and $G_n(x, y)$ are continuous on the compact set $E\times F$ whenever $D_n\supset E\cup F$. Hence the convergence in (2.21) holds uniformly on $E\times F$. Furthermore, by the fact (7) mentioned in the proof of Lemma 2.4 just above, we have

$$\lim_{x\to z, y\to z} G(x, y) \geqq \lim_{x\to z, y\to z} G_m(x, y) = \infty$$

for any $z \in D_m$; here m may be chosen arbitrarily. Hence (2.22) holds for any $z \in D$.

PROOF OF LEMMA 2.6. By virtue of the assumption of Lemma 2.6, we may take a domain Ω with property (S) and such that $E \cup \bar{D}_0 \subset \Omega \subset \bar{\Omega} \subset D-F$ and $\bar{\Omega}$ is compact. Then we may see from Lemma 2.1 in [7] that, for any $x \in \bar{\Omega}$, $y \in D$ and $t > 0$,

$$U(t, x, y) = U_\Omega(t, x, y) - \int_0^t d\tau \int_{\partial\Omega} \frac{\partial U_\Omega(t-\tau, x, z)}{\partial \boldsymbol{n}_z} U(\tau, z, y) dS_z$$

where we define $U_\Omega(t, x, y) = 0$ for any $x \in \bar{\Omega}, y \in D-\bar{\Omega}$ and $t > 0$. Integrating both sides of the above equality in t over $(0, \infty)$, we obtain, by (2.5) and (2.20),

(8)
$$G(x, y) = G_\Omega(x, y) + \int_{\partial\Omega} \left\{ -\frac{\partial G_\Omega(x, z)}{\partial \boldsymbol{n}_z} \right\} G(z, y) dS_z$$

for $x \in \bar{\Omega}$ and $y \in D$

where we put

(9) $G_\Omega(x, y) = 0$ for $x \in \bar{\Omega}$ and $y \in D-\bar{\Omega}$.

Since $E \cup \bar{D}_0$ and $\partial\Omega$ are mutually disjoint compact sets, we have, for suitable constants C_1 and C_2,

(10) $0 \leqq -\dfrac{\partial G_\Omega(x ; z)}{\partial \boldsymbol{n}_z} \leqq C_1$ for any $x \in E$ and $z \in \partial\Omega$

and

(11) $0 < C_2 \leqq -\dfrac{\partial G_\Omega(\gamma ; z)}{\partial \boldsymbol{n}_z} \leqq C_1$ for any $z \in \partial\Omega$.

Hence, combining (8) with (9), we get

$$\frac{G(x, y)}{G(\gamma ; y)} \leqq \frac{\int_{\partial\Omega} C_1 \cdot G(z, y) dS_z}{\int_{\partial\Omega} C_2 \cdot G(z, y) dS_z} = \frac{C_1}{C_2} < \infty$$

for any $x \in E$ and $y \in D-\Omega$; this fact implies (2.23).

Department of Mathematics
University of Tokyo

References

[1] N. Aronszajn, A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of second order, Tech. Report **16**, Univ. of Kansas (1956).

[2] M. Brelot, Lectures on potential theory, Tata Inst. of Fundamental Research,

Bombay, 1960.
[3] J.L. Doob, Discrete potential theory and boundaries, J. Math. Mech., **8** (1959,) 433-458.
[4] G.A. Hunt, Markov chains and Martin boundaries, Ill, J. Math., **4** (1960), 313-340.
[5] S. Itô, A boundary value problem of partial differential equations of parabolic type, Duke Math. J., **24** (1957), 299-312.
[6] S. Itô, Fundamental solutions of parabolic differential equations and boundary value problems, Japan. J. Math., **27** (1957), 55-102.
[7] S. Itô, On existence of Green function and positive superharmonic functions for linear elliptic operators of second order, this volume, 299-306.
[8] R.S. Martin, Minimal positive harmonic functions, Trans. Amer. Math. Soc., **49** (1941), 137-172.
[9] M.G. Šur, Martin's boundary for linear elliptic operators of second order, Izv. Akad. Nauk, SSSR., **27** (1963), 45-60 (Russian).
[10] T. Watanabe, On the theory of Martin boundaries induced by countable Markov processes, Mem. Coll. Sci. Univ. Kyoto Ser. A, **33** (1960), 39-108.