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Vector-valued holomorphic functions on a complex space

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§ 1. Introduction.

1. For a (reduced) complex space X and a Fréchet space F, an F-holomorphic function on X is defined to be an F-valued continuous function f on X if, for each continuous linear functional u on F, uf is holomorphic on X. In this paper, we attempt to extend some results in the theory of holomorphic functions of several complex variables to the case of F-holomorphic functions on X.

In [2] Bishop gave an expansion theorem, which asserts every F-holomorphic function f on a complex manifold is represented as a sum of (essentially) scalar-valued holomorphic functions and enables us to reduce the study of f to that of a sequence of ordinary holomorphic functions. Firstly, we generalize his expansion theorem to the case of F-holomorphic functions on a complex space. And, using this, we show an F-holomorphic function on a complex space is locally equal to the restriction of an F-holomorphic function in the ambiant space. Moreover, we get some theorems on the continuations and approximations of F-holomorphic functions, which include the following results:

(1) Let X' be a complex subspace of a complex space X. If each holomorphic function on X' is the restriction of a holomorphic function on X, then each F-holomorphic function on X' is also the restriction of an F-holomorphic function on X.

(2) Let X' be a subdomain of a complex space X. If (X, X') is a Runge pair, that is, each holomorphic function on X is compactly approximated on X' by holomorphic functions on X, then each F-holomorphic function on X' is also compactly approximated by F-holomorphic functions on X (§ 2).

2. Bishop introduced the notion of the vectorization S_F of a coherent analytic sheaf S with respect to a Fréchet space F and gave some interesting properties of it ([2]). These are proved essentially by his expansion theorem. Using our generalized expansion theorem for F-holomorphic functions on a complex space, we can generalize almost all results of Bishop [2] to the case

of the vectorization of a coherent analytic sheaf of a complex space. Especially, his generalization of H. Cartan's Theorem B is extended as follows:

If a coherent analytic sheaf S on a complex space X satisfies $H^N(X, S)=0$ for some $N \geqq 1$, then $H^N(X, S_F)=0$ for each Fréchet space F.

An analytic homomorphism of a coherent analytic sheaf S into another S' induces canonically the analytic homomorphism of the vectorization S_F into S'_F. Also a continuous linear map of a Fréchet space F into another F' induces canonically the analytic homomorphism of S_F into $S_{F'}$. By using our generalizations of Bishop's results, we can show these functors are exact (§ 3).

3. For a σ-compact complex space X, the set $A(X, F)$ of all F-holomorphic functions on X, with the topology of compact convergence, is a Fréchet space. We consider $A(X, F)$-holomorphic functions on another complex space Y. In § 4, we prove an $A(X, F)$-holomorphic function on Y is nothing but an F-holomorphic function on $X \times Y$. This shows that the study of F-holomorphic functions is not only to generalize the results on ordinary holomorphic functions, but also contributes to the study of ordinary holomorphic functions on a product space. For examples, by considering $A(Y, F)$-holomorphic functions on a complex space X and its subspace X', we see

(1) If each holomorphic function on X' is the restriction of a holomorphic function on X, then each holomorphic function on $X' \times Y$ is also the restriction of a holomorphic function on $X \times Y$.

(2) If (X, X') is a Runge pair, then $(X \times Y, X' \times Y)$ is also a Runge pair.

Moreover, we can give an application to the theory of cohomology with coefficients in the sheaf of germs of F-holomorphic functions as follows:

For a complex space X and a Stein space Y

$$H^N(X, O_{A(Y,F)}) \cong H^N(X \times Y, O_F)$$

where O_F denotes the sheaf of germs of F-holomorphic functions.

For a σ-compact indefinitely differentiable manifold M, we obtain the analogous results on F-valued differentiable functions of class $C^{\omega,\infty}$ (see Definition 3) on $X \times M$ and hence continuation theorems on such functions etc..

§ 2. Fundamental properties of vector-valued holomorphic functions.

1. Let F be a locally convex topological vector space over the complex number space C and X be a complex space.

DEFINITION 1. An F-valued function f on X is called to be *F-holomorphic* on X if f is continuous and uf is holomorphic on X for each u in F^*, where F^* is the dual of F.

By $A(X, F)$ we denote the set of all F-holomorphic functions on X. With the compact convergence topology, $A(X, F)$ constitutes a topological vector space over C.

Moreover, we have

LEMMA 1. *For a Fréchet space F and a σ-compact complex space X, $A(X, F)$ is also a Fréchet space.*

PROOF. By definition, F admits a countable family $\{\| \ \|_k\}$ of continuous semi-norms such that the sets $\{a \in F; \|a\|_k < 1\}$ form a fundamental system of neighborhoods of 0 in F. For a countable family $\{K_n\}$ of compact sets exhausting X, we define semi-norms $\| \ \|_{k,n}$ by the equality $\|f\|_{k,n} = \sup \|f(K_n)\|_k$ for each f in $A(X, F)$. The sets $\{f \in A(X, F); \|f\|_{k,n} < 1\}$ form a fundamental system of neighborhoods of 0 in $A(X, F)$. This shows that $A(X, F)$ is locally convex and metrizable. To show the completeness of $A(X, F)$, we take a Cauchy sequence $\{f_n\}$ in $A(X, F)$, which converges to an F-valued continuous function f on X. Obviously, $\{uf_n\}$ converges compactly to uf on X for each u in F^*. Then, according to Grauert and Rememrt ([**6**] p. 290), uf is holomorphic on X. Therefore, f is by definition an F-holomorphic function on X. This completes the proof. q. e. d.

For the most part in this paper, we treat Fréchet spaces. In the following, a complex space will be always assumed to be σ-compact.

2. For a Fréchet space F, a series $\sum_n a^n$ in F is called to be absolutely convergent in F if $\sum_n \|a^n\|$ is convergent for each continuous semi-norm $\| \ \|$ on F. Thus, the series $\sum_n f_n$ in $A(X, F)$ is absolutely convergent if, for each compact subset K of X and each continuous semi-norm $\| \ \|$, $\sum_n \sup \|f_n(K)\|$ is convergent.

LEMMA 2. *Let M be a nowhere dense analytic subset of a complex space X. For each compact subset K of X, there exist a neighborhood U of M and a relatively compact open set X' such that*

1°. $K \subset X' \Subset X$ *and* $X' - U \neq \phi$.

2°. $\sup |f(K)| \leqq \sup |f(X' - U)|$ *for each holomorphic function f on X.*

This was shown by Grauert and Remmert in [**6**], Hilfssatz 4, p. 292.

LEMMA 3. *Let M be a nowhere dense analytic subset of a complex space X. The space $A(X, C)$ is isomorphic with a closed subspace of the Fréchet space $A(X-M, C)$.*

In particular, if a series $\sum_n f_n$ in $A(X, C)$ converges absolutely as a series in $A(X-M, C)$, then it converges absolutely in $A(X, C)$.

PROOF. Obviously, the canonical restriction map of $A(X, C)$ into $A(X-M, C)$ is an injective continuous linear transformation and hence $A(X, C)$ is isomorphic with a vector subspace of $A(X-M, C)$. Now, we take a sequence $\{f_n\}$ in $A(X, C)$ which converges to 0 as a sequence in $A(X-M, C)$. By Lemma 2, for each compact subset K of X, there exist a neighborhood U of M and a relatively compact open set X' with the properties 1° and 2°. Especially, $\sup|f_n(K)|\leqq\sup|f_n(X'-U)|$ for every n. By hypothesis, the right hand side converges to zero. Therefore the left hand side converges also to zero. This shows $A(X, C)$ is a topological subspace of $A(X-M, C)$. On the other hand, according to Lemma 1, it is a closed subspace. q. e. d.

3. Now, we generalize the Bishop's expansion theorem for F-holomorphic functions defined on complex manifolds ([**2**], Theorem 1, p. 1182) to the case of F-holomorphic functions defined on complex spaces.

THEOREM 1. *Let F be a Fréchet space, $\{X_i\}$ be a countable family of complex spaces and f_i be an F-holomorphic function on X_i for each i. Then there exist a sequence $\{b_n\}$ in F and a sequence $\{P_n\}$ of mutually annihilating continuous projections on F such that*

1°. *$\{b_n\}$ is bounded, namely, $\{\|b_n\|\}$ is bounded for any continuous seminorm $\|\ \|$ on F.*

2°. *$P_nb_n=b_n$ and the image of P_n is a 1-dimensional subspace of F generated by b_n for each n.*

3°. *The series $\sum_n f_i^n$, where $P_nf_i=f_i^nb_n$, converges absolutely in $A(X, C)$.*

4°. *$\sum_n P_nf_i$ converges absolutely to f_i in $A(X, F)$.*

PROOF. By $\mathring{X}_i$ we denote the set of all regular points of X_i. Then $\mathring{X}_i$ is considered as a complex manifold, and the set $X_i-\mathring{X}_i$ of all singular points of X_i is a nowhere dense analytic subset of X_i. The restriction $\tilde{f}_i$ of each f_i in $A(X_i, F)$ to $\mathring{X}_i$ is an F-holomorphic function on the complex manifold $\mathring{X}_i$. Applying Bishop's theorem to the functions $\tilde{f}_i$ and the complex manifolds $\mathring{X}_i$, we can take a sequence $\{b_n\}$ in F and a sequence $\{P_n\}$ of mutually annihilating continuous projections satisfying the conditions 1°~4° as above.

We shall show these $\{b_n\}$ and $\{P_n\}$ satisfy the conditions 1°~4° in our case. Evidently the conditions 1° and 2° are satisfied. To see the conditions 3° and 4°, we put $P_n(a)=u_n(a)b_n$ for each a in F. Obviously, $u_n\in F^*$ and $\sum_n u_n\tilde{f}_i$ is absolutely convergent in $A(\mathring{X}_i, C)$ for each i. On the other hand, by Definition 1, u_nf_i is holomorphic on X_i. Lemma 3 implies that $\sum_n u_nf_i$ is absolutely convergent in $A(X_i, C)$. This proves the condition 3°. Moreover, the condition 1° implies that $\sum_n u_nf_ib_n$ converges absolutely in $A(X_i, F)$ to an F-holomorphic function g_i on X_i. Since $\sum_n P_n\tilde{f}_i=\tilde{f}_i$ on $\mathring{X}_i$, g_i is equal to $\tilde{f}_i$ on $\mathring{X}_i$. By the continuity of g_i and f_i, we have $g_i=f_i$ on X_i. Thus the condition 3° is also satisfied. q. e. d.

4. Let τ be a holomorphic map of a complex space X' into another X. The map τ induces canonically a continuous linear map τ_F of $A(X, F)$ into

$A(X', F)$ for each Fréchet space F.

THEOREM 2. *If the map* τ_C *of* $A(X, C)$ *into* $A(X', C)$ *is surjective, then* τ_F *is also surjective for each* F.

For the proof, we use the following Lemma, which was shown by Bishop ([2], Lemma 5, p. 1188).

LEMMA 4. *Let* σ *be a continuous linear map of a Fréchet space* F *onto another* F'. *Then for each absolutely convergent series* $\sum_n b'_n$ *in* F' *there exists an absolutely convergent series* $\sum_n b_n$ *in* F *such that* $\sigma(b_n) = b'_n$.

PROOF OF THEOREM 2. Take an F-holomorphic function g on X'. By Theorem 1, g is expanded as $g = \sum_n g^n b_n$, where $\sum_n g^n$ is absolutely convergent in $A(X', C)$ and $\{b_n\}$ is a bounded sequence in F. Then, there exists an absolutely convergent series $\sum_n f^n$ in $A(X, C)$ with $\tau_C(f^n) = g^n$ by Lemma 4. Since the sequence $\{b_n\}$ is bounded $\sum_n f^n b_n$ converges absolutely to an F-holomorphic function f in $A(X, F)$. Evidently, g is the τ_F-image of f. This shows τ_F is surjective. q. e. d.

COROLLARY 1. *Let* X *be a complex space and* X' *be an open subset of it. If each holomorphic function on* X' *is holomorphically continuable to the whole* X, *then for an arbitrary Fréchet space* F *each* F*-holomorphic function on* X' *is holomorphically continuable to* X.

PROOF. Apply Theorem 2 to the injection map τ of X' into X. q. e. d.

COROLLARY 2. *Let* Y *be an analytic subset of a Stein space* X. *Then, each* F*-holomorphic function on* Y *is the restriction of an* F*-holomorphic function on* X.

5. A holomorphic function on a complex space is, roughly speaking, locally equal to the restriction of a holomorphic function in the ambient space. We can give another restricted definition of F-holomorphic functions on a complex space. In fact, we consider frequently the class of all F-holomorphic functions which are locally equal to the restriction of F-holomorphic functions in the ambient space. However, we do not need a new definition of F-holomorphic function by the following theorem.

THEOREM 3. *An* F*-valued function* f *on a complex space* X *is* F*-holomorphic on* X *if and only if for each* p *in* X *there exists a neighborhood* U *of* p *such that by some mapping* τ U *is mapped biholomorphically onto an analytic subset* M *of a domain* D *in* C^N *and the function* $f\tau^{-1}$ *on* M *is the restriction of some* F*-holomorphic function on* D.

PROOF. The sufficiency is obvious. To see the neccessity, take a neighborhood U for each point p in X which is mapped biholomorphically onto a closed analytic subset M of a domain of holomorphy D in C^N by a mapping

τ. Then the function $f\tau^{-1}$ on M is F-holomorphic on M, which is equal to the restriction of some F-holomorphic function on D in virtue of Corollary 2 of Theorem 2. q. e. d.

6. As another application of Theorem 2, we give the following approximation theorem.

THEOREM 4. 1° *Take a continuous linear map σ of a Fréchet space F into another F'. If the image of σ is dense in F', then for the canonically induced map $\sigma^*(X)$ of $A(X, F)$ into $A(X, F')$ the image of $\sigma^*(X)$ is dense in $A(X, F')$, where X is an arbitrary complex space.*

2° *Take a holomorphic map τ of a complex space X' into another X. If the image of τ_C is dense in $A(X', C)$, then the image of τ_F is also dense in $A(X', F)$ for each Fréchet space F.*

PROOF. 1° According to Theorem 1, an F'-holomorphic function f has an expansion $f=\sum_n f^n b'_n$ such that $\{b'_n\}$ is a bounded sequence in F' and $\sum_n f^n$ is absolutely convergent in $A(X, C)$. It is sufficient to show that for each compact set K and each continuous semi-norm $\| \|$ on F' there exists an F-holomorphic function g on X with the property $\|\sigma^*(X)g-f\|<1$ on K. To this end, we take a sufficiently large N with $\sum_{n>N}|f^n|\|b'_n\|<1/2$ on K and b_n in F $(1\leqq n\leqq N)$ with $|f^n|\|\sigma^* b_n-b'_n\|<1/2N$ on K. The F-holomorphic function $g=\sum_{0\leqq n\leqq N} f^n b_n$ is a desired one.

2° For an F-holomorphic function f on X' with a similar expansion $f=\sum_n f^n b_n$ as above we take a sufficiently large N with the analogous property. By the hypothesis, there exists a holomorphic function g^n on X with $|\tau_C g^n -f^n|\|b_n\|<1/2N$ for each n $(1\leqq n\leqq N)$. Putting $g=\sum_{1\leqq n\leqq N} g^n b_n$ we have $\|\tau_F g-f\|<1$ on K. q. e. d.

COROLLARY. *If (X, X') is a Runge pair i. e. each holomorphic function on an open subset X' of a complex space X can be approximated compactly on X' by holomorphic functions on X, then, for each Fréchet space F, each F-holomorphic function can be also approximated by F-holomorphic functions on X.*

§ 3. The vectorizations of coherent analytic sheaves.

1. For a complex space X with the structure sheaf O and a Fréchet space F, we consider the sheaf of germs of locally-defined F-holomorphic functions on X. We denote it by O_F. Clearly, O_F is an analytic sheaf on X.

DEFINITION 2. Take an analytic sheaf S on X. We call the analytic sheaf $S_F:=S\otimes_O O_F$ *the vectorization of S with respect to F.*

Bishop gave some interesting properties on the vectorization S_F of a coherent analytic sheaf S on a complex manifold, and extended to S_F H.

Cartan's Theorem B [3] on a Stein manifold. The proofs of these results are essentially due to Theorem 1 in his paper [2], which we generalized to the case of a complex space in the previous section (Theorem 1).

Now, we can generalize almost all results of Bishop [2] to the case of the vectorization of a coherent analytic sheaf on a complex space. In this section we summarize them to give some applications.

LEMMA 5. *Let S be a coherent analytic subsheaf of O^k on a complex space X and F be a Fréchet space. For an open subset U of X, take the set $S_F'(U) = \{f=(f_1, \cdots, f_k) \in O_F^k ; uf=(uf_1, \cdots, uf_k) \in S(U)$ for each u in $F^*\}$, where $S(U)$ denotes all sections of S on U. Then for each point p in U there exist a neighborhood V of p and $s_1, \cdots, s_l$ in $S(V)$ such that each $f \in S_F'(V)$ has the expansion*

$$f=\sum_{i=1}^{l} g_i s_i$$

on V for suitable $g_1, \cdots, g_l$ in $O_F(V)$.

PROOF. This is a generalization of Bishop [2], Theorem 2, p. 1184. For the convenience of readers, we sketch the outline of the proof. Since S is coherent, for each point p in U there exist a neighborhood V of p and $s_1, \cdots, s_l$ in $S(V)$ such that the $O(V)$-homomorphism s of $O(V)^l$ into $S(V)$ defined by $sh=s_1h_1+\cdots+s_lh_l$ for $h=(h_1, \cdots, h_l)$ in $O^l(V)$ is surjective. Take $f=(f_1, \cdots, f_k)$ in $S_F'(V)$. By Theorem 1, there exist a bounded sequence $\{b_n\}$ and continuous projections $\{P_n\}$ such that $P_n f_j = f_j^n b_n$, $f_j=\sum_n f_j^n b_n$ and $\sum_n f_j^n$ is absolutely convergent in $O(V)$ for each j $(1 \leqq j \leqq k)$. Since s is a continuous linear map of a Fréchet space $O^l(V)$ onto another Fréchet space $S(V)$, there exists by Lemma 4 an absolutely convergent series $\sum_n (g_i^n)$ in $O^l(V)$ such that $s(g_i^n)=f^n := (f_1^n, \cdots, f_k^n)$. Putting $g_i=\sum_n g_i^n b_n$, we have $f=\sum_{i=1} g_i s_i$ on V. This shows Lemma 5. q. e. d.

2. THEOREM 5. *Under the same notations and assumptions the sheaf S_F' defined by the presheaf $S_F'(U)$ is canonically isomorphic with S_F.*

For the proof see Bishop [2], Theorem 3, p. 1187.

COROLLARY. *Let Y be a closed analytic subset of X. The sheaf $I_F[Y]$ defined by the presheaf $I_F[Y](U)=\{f \in O_F(U): f=0$ on $U \cap Y\}$ is isomorphic with the vectorization $I[Y]_F$ of the sheaf $I[Y]$ defined by the presheaf $I[Y](U)=\{f \in O(U); f=0$ on $U \cap Y\}$.*

PROOF. An element $f \in O_F(U)$ is contained in $I_F[Y](U)$ if and only if uf is contained in $I[Y](U)$ for all u in F^*. This shows $I_F[Y]=I[Y]_F'$, which is isomorphic with $I[Y]_F$ by Theorem 5. q. e. d.

3. Take a continuous linear map σ of a Fréchet space F into another F'. For an arbitrary analytic sheaf S, σ induces the natural homomorphism σ_S

$=1_S \otimes \sigma^*$ of S_F into $S_{F'}$, where 1_S denotes the identity map of S. In particular, each $u \in F^*$ induces a homomorphism u_S of S_F onto S.

LEMMA 6. *Let S be a coherent analytic sheaf on a complex space and F be a Fréchet space. If an element f in $S_F(U)$ satisfies $u_S f=0$ for each u in F^*, we have $f=0$.*

For the proof, see Bishop [2], Lemma 4, p. 1186.

THEOREM 6. *If a coherent analytic sheaf S on a complex space X satisfies $H^N(X, S)=0$ for some $N \geqq 1$, then $H^N(X, S_F)=0$ for each Fréchet space F.*

PROOF. See the proof of Bishop [2], Theorem 4, p. 1189. We note in his proof $H^N(M, S_F)=0$ is deduced only from the condition $H^N(M, S)=0$ for a coherent analytic sheaf S on a complex manifold M. Theorem 6 is its generalization. We omit the proof. q. e. d.

COROLLARY 1. *For a coherent analytic sheaf S on a Stein space X and a Fréchet space F, $H^N(X, S_F)=0$ $(N \geqq 1)$.*

PROOF. This is an immediate consequence of Theorem 6 and H. Cartan's Theorem B [3]. q. e. d.

COROLLARY 2. *For the structure sheaf O on the projective space P^n, $H^N(P^n, O_F)=0$ $(N \geqq 1)$.*

PROOF. This is due to H. Cartan séminaire [4], p. 218. q. e. d.

4. THEOREM 7. *If a sequence of Fréchet spaces*

$$0 \longrightarrow F' \xrightarrow{\sigma} F \xrightarrow{\tau} F'' \longrightarrow 0$$

is exact, then the sequence of the analytic sheaves

$$0 \longrightarrow S_{F'} \xrightarrow{\sigma_S} S_F \xrightarrow{\tau_S} S_{F''} \longrightarrow 0$$

is also exact for each coherent analytic sheaf S on a complex space X.

Firstly, we give the following

LEMMA 7. *Under the same assumption as above, we have the exact sequence of analytic sheaves*

$$0 \longrightarrow O_{F'} \xrightarrow{\sigma^*} O_F \xrightarrow{\tau^*} O_{F''} \longrightarrow 0 .$$

PROOF. The set $\operatorname{Im} \sigma = \operatorname{Ker} \tau$ is a closed subspace of F by the continuity of τ. Hence $\sigma: F' \rightarrow \sigma(F')$ is an open map by Banach's theorem and F' is considered as a closed subspace of F. For an open set U, each function $f \in O_F(U)$ with $\tau^*(U)(f)=0$ is considered as an F-holomorphic function on U with values in F'. By Definition 1 and Hahn-Banach's theorem f is an F'-holomorphic function. Since the map $\sigma^*(U)$ of $O_{F'}(U)$ into $O_F(U)$ is obviously injective, we have the exact sequence

$$0 \longrightarrow O_{F'}(U) \xrightarrow{\sigma^*(U)} O_F(U) \xrightarrow{\tau^*(U)} O_{F''}(U) .$$

Now, we take an element $f'' \in O_{F''}(U)$. By Theorem 1 we can take a bounded sequence $\{b''_n\}$ and continuous projections $\{P_n\}$ such that $P_n f'' = f^n b''_n$, $f'' = \sum_n P_n f''$ and $\sum_n f^n$ is absolutely convergent in $O(U)(=A(U, C))$. For each point p in X we take a neighborhood V of p with $V \Subset U$ and put $L_n := \sup |f^n(V)|$. Then $\sum_n L_n b''_n$ is absolutely convergent in F''. According to Lemma 3, $\sum_n L_n b''_n$ is the image of an absolutely convergent series $\sum_n b_n$ in F. Obviously, $\Sigma'_n (f^n/L_n) \cdot b_n$ is absolutely convergent in $O_F(V)$, where Σ'_n denotes the sum of all terms with $L_n \neq 0$. For $f = \Sigma'_n (f^n/L_n) \cdot b_n$, $\tau^*(V) f = f''$ on V. Thus we get the exact sequence

$$0 \longrightarrow O_{F'} \xrightarrow{\sigma^*} O_F \xrightarrow{\tau^*} O_{F''} \longrightarrow 0 . \qquad \text{q. e. d.}$$

PROOF OF THEOREM 7. By the fundamental theorem on cohomology and Lemma 7 we obtain the exact sequence

$$0 \longrightarrow O_{F'}(U) \xrightarrow{\sigma^*(U)} O_F(U) \xrightarrow{\tau^*(U)} O_{F''}(U) \longrightarrow H^1(U, O_{F'})$$

for any open set U. Especially, if U is a Stein open set (i. e. holomorphically separable and holomorphically convex open set), $\tau^*(U)$ is surjective because $H^1(U, O_{F'}) = 0$ in virtue of Corollary 1 of Theorem 6. Then, we have also the exact sequence

$$O_{F'}(U) \otimes S(U) \longrightarrow O_F(U) \otimes S(U) \longrightarrow O_{F''}(U) \otimes S(U) \longrightarrow 0$$

by the right exactness of the functor $\otimes_{O(U)} S(U)$. Since each point has a fundamental system of Stein neighborhoods, we see easily the exact sequence of coherent analytic sheaves

$$S_{F'} \xrightarrow{\sigma_S} S_F \xrightarrow{\tau_S} S_{F''} \longrightarrow 0 .$$

To complete the proof of Theorem 7, it is sufficient to show the injectivity of σ_S. Take an element $f' = \sum_i f_i \otimes g_i$ in $S_{F'}(U)$ with $\sigma_S f' = \sum_i (\sigma^*(U) f_i) \otimes g_i = 0$ on an open set $V (V \subset U)$, where $f_i \in O_{F'}(U)$ and $g_i \in S(U)$. As in the proof of Lemma 7, F' is considered as a closed subspace of F. Each $u \in F'^*$ has an extension v to F by Hahn-Banach's theorem. Then we see

$$u_S f' = \sum_i (u^* f_i) \otimes g_i = \sum_i (v\sigma)^* f_i \otimes g_i = v'_S(\sigma_S f') = 0$$

on V. Lemma 5 implies $f' = 0$ on V. This shows σ_S is injective. q. e. d.

5. Let φ be an analytic homomorphism of an analytic sheaf S into another analytic sheaf S'. For each Fréchet space F φ induces canonically the analytic homomorphism $\varphi_F = \varphi \otimes 1_F$ of S_F into S'_F.

THEOREM 8. *If a sequence of coherent analytic sheaves*

$$0 \longrightarrow S' \xrightarrow{\varphi} S \xrightarrow{\psi} S'' \longrightarrow 0$$

is exact, then so is the sequence

$$0 \longrightarrow S'_F \xrightarrow{\varphi_F} S_F \xrightarrow{\psi_F} S''_F \longrightarrow 0 .$$

PROOF. By the properties of tensor products,

$$S'_F \xrightarrow{\varphi_F} S_F \xrightarrow{\psi_F} S''_F \longrightarrow 0$$

is obviously exact. It is sufficient to show the injectivity of φ_F. To this end, take an element $f = \sum_i f_i \otimes g_i \in S'_F(U)$ with $\varphi_F(U)f = \sum_i \varphi(U)f_i \otimes g_i = 0$, where $f_i \in S(U)$ and $g_i \in O_F(U)$. For each $u \in F^*$, we see

$$\varphi(U)(u_S f) = \varphi(U)(\sum_i f_i \otimes u^* g_i) = \sum_i (\varphi(U)f_i) \otimes u^* g_i$$

$$= u_S(\sum_i \varphi(U)f_i \otimes g_i) = u_S(\varphi_F(U)f) = 0 .$$

By the hypothesis, $\varphi(U)$ is injective and therefore $u_S f = 0$ on U. Then $f = 0$ on U by Lemma 6.

§ 4. *F*-holomorphic functions with values in some function spaces.

1. Let X be a complex space. For another complex space Y, we consider $A(Y, F)$-holomorphic functions on X.

THEOREM 9. *The space $A(X, A(Y, F))$ is canonically isomorphic with $A(X \times Y, F)$ as topological vector spaces.*

PROOF. 1°. The space $C(X \times Y, F)$ of all F-valued continuous functions on $X \times Y$ constitutes a Fréchet space with the topology of compact convergence. As is well known, $C(X \times Y, F)$ is canonically isomorphic with the space $C(X, C(Y, F))$ of all continuous functions on X with values in the space of all F-valued continuous functions on X. It is sufficient to show that an F-holomorphic function on $X \times Y$ induces an $A(Y, F)$-holomorphic function on X and vice versa.

2°. Take an element $f^*(p) \in A(X, A(Y, F))$. For a point p_0 in X, there exists a neighborhood U of p_0 which can be considered as an analytic subset of a polydisc $G = \{|z_i| < r_i\}_{i=1}^N$. By Corollary 2 of Theorem 2, $f^*(p)$ is the restriction of an $A(Y, F)$-holomorphic function $\tilde{f}^*(p)$ on G. Then, we have the equality of $A(Y, F)$-valued functions

$$\tilde{f}^*(z) = \frac{1}{(2\pi i)^n} \int_{|\zeta_i| = r_i'} \cdots \int \frac{\tilde{f}^*(\zeta_1, \cdots, \zeta_N)}{(\zeta_1 - z_1) \cdots (\zeta_N - z_N)} d\zeta_1 \cdots d\zeta_N$$

for each $z = (z_1, \cdots, z_N)$ in $G' = \{|z_i| < r_i'\}$, where $0 < r_i' < r$. By the definition of Riemann integral, the right hand side is approximated compactly on $G'\times Y$ by the linear combinations of F-holomorphic functions on Y with coefficients of ordinary holomorphic functions on G', which are contained in $A(G'\times Y, F)$. By the completeness of $A(G'\times Y, F)$, the function $\tilde{f}(p, q):=\tilde{f}^*(p)(q)$ is contained in $A(G'\times Y, F)$. Thus we see $f(p, q):=f^*(p)(q)\in A(X\times Y, F)$.

3°. To prove the converse, we may assume $F=C$. For, each F-holomorphic function $f(p, q)$ on $X\times Y$ has the expansion

$$f=\sum_n f^n(p, q)\cdot c_n$$

where $f^n(p, q)$ is in $A(X\times Y, C)$, $\sum_n f^n(p, q)$ is absolutely convergent in $A(X\times Y, C)$ and $\{c_n\}$ is a bounded sequence in F. Suppose each $f^n(p, q)$ induces an $A(Y, C)$-holomorphic function $f^{n*}(p)(q):=f^n(p, q)$, then the series $\sum_n f^{n*}(p)$ is absolutely convergent in $A(X, A(Y, C))$ and hence the series $\sum_n c_n\cdot f^{n*}(p)$ is absolutely convergent in $A(X, A(Y, F))$. Easily, we see $f^*(p)(q)(:=f(p, q)) =\sum_n c_n\cdot f^{n*}(p)(q)$, which is contained in $A(X, A(Y, F))$.

4°. If a complex space X can be proved to have the property that for an arbitrary complex manifold Y each holomorphic function on $X\times Y$ induces an $A(Y, C)$-holomorphic function on X, then $A(X\times Y, F)$ is isomorphic with $A(X, A(Y, F))$ for an arbitrary complex space Y. In fact, a holomorphic function $f(p, q)$ on $X\times Y$ is holomorphic on the subspace $X\times \mathring{Y}$, where $\mathring{Y}$ denotes the complex manifold consisting of all regular points of Y. By the assumption, $f(p, q)$ induces an $A(\mathring{Y}, C)$-holomorphic function $f^*(p)(q):=f(p, q)$ on X. By Lemma 3, $A(Y, C)$ is a closed subspace of $A(\mathring{Y}, C)$. It follows from Hahn-Banach's theorem that the $A(\mathring{Y}, C)$-holomorphic function $f^*(p)$ on X with values in the closed subspace $A(Y, C)$ is an $A(Y, C)$-holomorphic function. Thus $A(X\times Y, C)$ is isomorphic with $A(X, A(Y, C))$ by 1° and 2°, and hence $A(X\times Y, F)$ is isomorphic with $A(X, A(Y, F))$ by 3° for each Fréchet space F.

5°. For complex manifolds X and Y, Theorem 9 is easily proved ([**5**]). Moreover, according to 3°, Theorem 9 holds for a complex manifold X and an arbitrary complex space Y.

6°. Now, we shall prove Theorem 9 for arbitrary complex spaces X and Y. To this end, we may assume $F=C$ by 3° and Y to be a complex manifold by 4°. Take an element $f(p, q)$ in $A(X\times Y, C)$. For a point p_0 in X, there exists a neighborhood U which we can regard as an analytic subset of a polydisc $G=\{|z_i|<r_i\}_{i=1}^N$. In virtue of 4°, the holomorphic function $f(p, q)$ on $U\times Y$ induces an $A(U, C)$-holomorphic function $f_*(q)(p):=f(p, q)$ on Y. By Theorem 1, there exist a bounded sequence $\{b_n\}$ in $A(U, C)$ and an absolutely convergent series $\sum_n f^n$ in $A(Y, C)$ such that $f_*(q)=\sum_n f^n(q)\cdot b_n$. Since G is a

domain of holomorphy, each b_n has a holomorphic extension b'_n to G. Let $U' := U \cap G'$ be another neighborhood of p_0, where $G' = \{|z_i| < r'_i\}_{i=1}^N$ $(r'_i < r_i)$. Then, since the canonical restriction map of $A(G.C)$ onto $A(U, C)$ is open by Banach's open map theorem, there exists a positive number M such that each holomorphic function g on U has a holomorphic extension $\tilde{g}$ to G with $|\tilde{g}| \leqq M \sup |g(K)|$ on G' for some compact subset K of U. Therefore, $\{b'_n\}$ can be chosen so as to be bounded in $A(U', C)$. Thus we obtain a holomorphic function $f = \sum_n f^n b'_n$ on $G' \times Y$, which is equal to $f(p, q)$ on $U' \times Y$. Since Y and G' are both complex manifolds, $\tilde{f}^*(p)(q) := \tilde{f}(p, q)$ is an $A(Y, C)$-holomorphic function on G'. Obviously, $f^*(p)(q) := f(p, q)$ is the restriction of an $A(Y, C)$-holomorphic function $\tilde{f}^*$ on G' to U'. This shows $f^*(p) \in A(X, A(Y, C))$.

q. e. d.

2. By Theorem 9, we can generalize the results of n° 4 and n° 6 in § 2.

THEOREM 10. *If for a holomorphic map τ of a complex space X' into another X the induced map τ_C of $A(X, C)$ into $A(X', C)$ is surjective, then the canonically induced map $(\tau \times 1_Y)_F$ of $A(X \times Y, F)$ into $A(X' \times Y, F)$ is surjective for each complex space Y and Fréchet space F.*

PROOF. By Theorem 9 identifying $A(X, A(Y, F))$ and $A(X', A(Y, F))$ with $A(X \times Y, F)$ and $A(X' \times Y, F)$ respectively, we can regard the map $(\tau \times 1_Y)_F$ of $A(X \times Y, F)$ into $A(X' \times Y, F)$ as the map $\tau_{A(Y,F)}$ of $A(X, A(Y, F))$ into $A(X', A(Y, F))$, which is surjective in virtue of Theorem 2. q. e. d.

COROLLARY 1. *Let X be a complex space and X' be an open subset of it. If each holomorphic function on X' is holomorphically continuable to the whole X, then for an arbitrary complex space Y and a Fréchet space F each F-holomorphic function on $X' \times Y$ is holomorphically continuable to $X \times Y$.*

COROLLARY 2. *Let X be an analytic subset of a Stein space X. Then, for an arbitrary complex space Y each F-holomorphic function on $X' \times Y$ is the restriction of an F-holomorphic function on $X \times Y$.*

THEOREM 11. *Take a holomorphic map τ of a complex space X' into another complex space X. If the image of τ_C is dense in $A(X', C)$, then for an arbitrary complex space Y the image of $(\tau \times 1_Y)_F$ is also dense in $A(X' \times Y, F)$.*

PROOF. Apply Theorem 4 to the Fréchet space $A(Y, F)$. q. e. d.

COROLLARY. *If (X, X') is a Runge pair, then for an arbitrary complex space Y, $(X \times Y, X' \times Y)$ is also a Runge pair.*

3. We give another application of Theorem 9 to the theory of cohomology with coefficients in the sheaf of F-holomorphic functions.

THEOREM 12. *Let X be a complex space and Y be a Stein space. Then, for an arbitrary Fréchet space F, we have*

$$H^N(X, O_{A(Y,F)}) \cong H^N(X \times Y, O_F).$$

PROOF. In case of $N=0$, this is a special case of Theorem 9. To see the case $N\geqq 1$, we take a Stein covering $\mathfrak{U}=\{U_j\}_{j\in J}$ i.e. an open covering such that for each finite subset $i_1, \cdots, i_s$ of J $U_{i_1}\cap\cdots\cap U_{i_s}$ is a Stein space. Then the covering $\mathfrak{U}\times Y=\{U_j\times Y\}_{j\in J}$ is also a Stein covering of $X\times Y$. According to Theorem 6, these coverings $\mathfrak{U}$ and $\mathfrak{U}\times Y$ are Leray coverings with respect to the sheaf $O_{A(Y,F)}$ on X and O_F on $X\times Y$, respectively. Thus we have

$$H^N(X, O_{A(Y,F)})\cong H^N(\mathfrak{U}, O_{A(Y,F)}) \tag{1}$$

and

$$H^N(X\times Y, O_F)\cong H^N(\mathfrak{U}\times Y, O_F)\,. \tag{2}$$

On the other hand, by Theorem 9 we have the isomorphism of the cochain groups

$$\begin{aligned} C^N(\mathfrak{U}, O_{A(Y,F)}) &:= \prod_{j_0,\cdots,j_N} H^0(U_{j_0}\cap\cdots\cap U_{j_N}, O_{A(Y,F)}) \\ &\cong C^N(\mathfrak{U}\times Y, O_F) := \prod_{j_0,\cdots,j_N} H^0((U_{j_0}\cap\cdots\cap U_{j_N})\times Y, O_F)\,. \end{aligned}$$

Since the coboundary operator δ commutes with this isomorphism, this shows

$$H^N(\mathfrak{U}, O_{A(Y,F)})\cong H^N(\mathfrak{U}\times Y, O_F)\,. \tag{3}$$

By (1), (2) and (3) $H^N(X, O_{A(Y,F)})$ is isomorphic with $H^N(X\times Y, O_F)$.

COROLLARY 1. *Under the same assumptions as above, if* $H^N(X, O)=0$, *then* $H^N(X\times Y, O_F)=0$.

PROOF. By Theorem 6 and Theorem 12, $H^N(X\times Y, O_F)\cong H^N(X, O_{A(Y,F)})=0$.

q. e. d.

COROLLARY 2. $H^N(X\times P^n, O_F)=H^N(X, O_F)$, *where* P^n *denotes the n-dimensional projective space.*

PROOF. As is well known, $H^N(P^n, O)=0$ for $N\geqq 1$ and therefore $H^N(P^n\times X, O_F)=0$ for $N\geqq 1$ for each Stein space X by the above corollary. Thus, for a Stein covering $\mathfrak{U}=\{U_j\}_{j\in J}$ of X the covering $\mathfrak{U}\times P^n=\{U_j\times P^n\}_{j\in J}$ is a Leray covering of $X\times P^n$ with respect to the analytic sheaf O_F. As in the proof of Theorem 12, we have $H^N(X\times P^n, O_F)\cong H^N(X, O_{A(P^n,F)})$ $(N\geqq 1)$. On the other hand, each F-holomorphic function f on a compact complex space must be constant. In fact, otherwise, we take two points p, q with $f(p)\neq f(q)$. By Hahn-Banach's Theorem, there exists a continuous linear functional u on F such that $uf(p)\neq uf(q)$. The holomorphic function uf is non-constant on the compact complex space, which contradicts the maximum principle. This shows $A(P^n, F)$ is isomorphic with F. Therefore, $H^N(X\times P^n, O_F)\cong H^N(X, O_F)$.

q. e. d.

4. Let M be a σ-compact differentiable manifold of class C^k $(0\leqq k\leqq\infty)$. For a Fréchet space F, we can define naturally F-valued differentiable functions of class C^k on X. The function space $C^k(M, F)$ of all F-valued differentiable functions of class C^k constitutes also a Fréchet space with the topology of compact convergence of functions and their local derivatives.

DEFINITION 3. We call an F-valued function f on $X\times M$ a *k-times differentiable family of F-holomorphic functions* on X with parameters in M, or simply, *of class* $C^{\omega,k}$ if $f(p, q)$ is holomorphic on X for each fixed point q in M and has k-th derivatives with respect to each local coordinates in M which are continuous with respect to the product topology of $X\times M$. By $C^{\omega,k}(X\times M, F)$ we denote the set of all F-valued function of class $C^{\omega,k}$ on $X\times M$.

THEOREM 13. *An F-valued function $f(p, q)$ on $X\times M$ induces an $A(X, F)$-valued differentiable function $f_*(q)(p)=f(p, q)$ of class C^k if and only if $f(p, q)$ is of class $C^{\omega,k}$.*

PROOF. Take an $A(X, F)$-valued differentiable function $f_*(q)$ on M. Obviously, the function $f(p, q):=f_*(q)(p)$ on $X\times M$ is holomorphic on X for each fixed point q in M and has k-th derivatives referred to each local coordinates in M because the topology of $A(X, F)$ is stronger than the simple convergence topology. Moreover, since f_* has continuous derivatives with values in $A(X, F)$, they are continuous on $X\times M$. This shows $f(p, q)\in C^{\omega,k}(X\times M, F)$.

Conversely, take an F-valued function $f(p, q)$ of class $C^{\omega,k}$ on $X\times M$. To a point q in M we correspond the mapping $f_*(q)(p):=f(p, q)$ of M into the space of F-valued functions on X, which is contained in $A(X, F)$ by Definition 3. Since the derivatives of $f(p, q)$ are continuous on $X\times M$, they induce the continuous derivatives of $A(X, F)$-valued function $f_*(q)$. This shows $f_*(q)\in C^k(M, A(X, F))$. q. e. d.

5. LEMMA 8. *Let X' be an analytic subset of a Stein space X and M be a σ-compact differentiable manifold of class C^∞. Then each F-valued function of class $C^{\omega,\infty}$ on $X'\times M$ is the restriction of an F-valued function of class $C^{\omega,\infty}$ on $X\times M$.*

PROOF. By Corollary 2 of Theorem 2, the restriction map τ of F-holomorphic functions on X to X' is surjective. On the other hand, according to Andreotti-Grauert ([1], Theorem 1, p. 205) the functor $C^\infty(M, F)$ is an exact covariant functor on the category of Fréchet spaces. From these facts, we conclude the sequence

$$C^\infty(M, A(X, F)) \xrightarrow{\tau'} C^\infty(M, A(X', F)) \longrightarrow 0$$

is exact.

Now, take an F-valued function $f(p, q)$ of class $C^{\omega,\infty}$ on $X'\times M$. By Theorem 13 $f(p, q)$ induces an $A(X', F)$-valued indefinitely differentiable function $f_*(q)(p):=f(p, q)$ on M. By the above argument, there exists an $A(X, F)$-valued differentiable function $\tilde{f}_*(q)$ such that $\tau'\tilde{f}_*=f_*$. Applying Theorem 13 again, the F-valued function $\tilde{f}(p, q):=\tilde{f}_*(q)(p)$ is of class $C^{\omega,\infty}$ on $X\times M$. Obviously, $f(p, q)$ is the restriction of $\tilde{f}(p, q)$ to $X'\times M$. q. e. d.

THEOREM 14. *For a complex space X and a σ-compact differentiable manifold M of class C^{∞}. An F-valued function $f(p, q)$ on $X\times M$ induces a $C^{\infty}(M, F)$-holomorphic function $f^{*}(p)(q)=f(p, q)$ on X if and only if $f(p, q)$ is of class $C^{\omega,\infty}$.*

PROOF. Take a $C^{\infty}(M, F)$-holomorphic function $f^{*}(p)$ on X. For a point p in X, there exists a neighborhood U which can be imbedded in a polydisc $G=\{|z_i|<r_i\}_{i=1}^{N}$ by a one-to-one proper regular holomorphic map ϕ. By Corollary 2, there exists a $C^{\infty}(M, F)$-holomorphic function $\tilde{f}^{*}$ on G such that $f^{*}=\tilde{f}^{*}$ on U. We can see easily the F-valued function $\tilde{f}(p, q)=\tilde{f}^{*}(p)(q)$ on $G\times M$ is of class $C^{\omega,\infty}$ (c.f. [**5**]). Therefore, $f(p, q)=\tilde{f}(p, q)$ is of class $C^{\omega,\infty}$ on $U\times M$. This shows $f(p, q)\in C^{\omega,\infty}(X\times M, F)$.

Conversely, take an F-valued function $f(p, q)$ of class $C^{\omega,\infty}$ on $X\times M$. For a point p_0, there exists a neighborhood U as above, which is considered as an analytic subset of a polydisc $G=\{|z_i|<r_i\}$. By Lemma 8, $f(p, q)$ is the restriction of an F-valued function $\tilde{f}(p, q)$ of class $C^{\omega,\infty}$ on $G\times M$. Easily we see $\tilde{f}(p, q)$ induces a $C^{\infty}(M, F)$-valued holomorphic function $\tilde{f}^{*}(p)(q):=\tilde{f}(p, q)$ on G (c.f. [**4**]). The $C^{\infty}(M, F)$-valued function $f^{*}(p)(q):=f(p, q)$ on U is the restriction of a $C^{\infty}(M, F)$-holomorphic function $\tilde{f}^{*}(p)$ on G to U. This shows $f^{*}(p)\in A(X, C^{\infty}(M, F))$. q. e. d.

COROLLARY. *Let X be a complex space and X' be an open subset of it. If each holomorphic function on X' is holomorphically continuable to the whole X, then for an arbitrary differentiable manifold M, the continuations of an indefinitely differentiable family of F-holomorphic functions with parameters in M constitute also an indefinitely differentiable family of F-holomorphic functions.*

Added in proof: Recently, we found the paper by L. Bungart. *Holomorphic functions with values in locally convex space and applications to integral formulas*, Trans. Amer. Math. Soc., **3** (1964), 317-344. Some of the results in this paper seems to be special cases of Bungart's, though his methods are different from ours.

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