J. Math. Soc. Japan
Vol. 17, No. 2, 1965

Recursive progression of intuitionistic number theories

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(Received Jan. 6, 1964)

Introduction

In the research about intuitionism from the standpoint of the formalism, there are at least two fundamental subjects. The first is; how we should define the concept of the intuitionistic truth for formulas.

Many concepts around this have been proposed; for example,

"general recursive truth in the case of prenex formulas,"

"recursive realizability" (Kleene [**2**]),

"Gödel's interpretation" (Gödel [**5**]),

"no-counter example-interpretation" (Kreisel [**17**]),

"Gödel's interpretation by means of effective operations" (Kreisel [**6**]).

The second is; by what means we can recognize various formulas to be intuitionistically true, especially in what formal systems the formulas are provable.

In this paper, restricting our attention to the number theory, we shall answer partially to the second problem. This problem leads necessarily to establishing the formal systems or series of systems which deduce intuitionistically true formulas as many as possible.

The intuitionistic number theory Z^I, which will be explained in § 1, deduce only intuitionistically true formulas in any sense.

But Gödel's ω-incompleteness theorem asserts that in Z^I as well as more extended systems S with some conditions there is a formula $\forall x A(x)$, where $A(x)$ is a PR-formula 3), which is true in any sense, and therefore $\vdash_s A(\bar{x})$ for any numerals $\bar{x}$, but not $\vdash_s \forall x A(x)$. So, for example, Z^I is incomplete for intuitionistically true formulas.

The ω-rule is considered by many people in order to obtain such formulas as above $\forall x A(x)$ in formal systems. The ω-rule is defined as a rule of inference such that if $(x) \vdash_s A(\bar{x})$ we can infer $\vdash_s \forall x A(x)$.

But in this case the proofs are infinitely long extensive and "a is a proof of b" is no more recursive for a, b. However, when we know that $(x) \vdash_s A(\bar{x})$ holds, we can obtain that the formula expressing $(x) \vdash_s A(\bar{x})$ is provable in S or in other systems. Therefore, in these cases, in order to obtain such formulas the ω-rule is not always indispensable. It suffices to suppose the following rule or axiom;

$$(1)\qquad \frac{\text{formula expressing } (x)\vdash_s \mathrm{A}(\bar{x})}{\forall \mathrm{x}\mathrm{A}(\mathrm{x})}$$

$$(2)\qquad \{\text{formula expressing } (x)\vdash_s \mathrm{A}(\bar{x})\}\supset \forall \mathrm{x}\mathrm{A}(\mathrm{x})\,.$$

(1) is considered for the first time by Rosser in [8], and named *K-rule.* We call *Axiom K for S* for (2).

The system S added by (1) is equivalent to the system S added by (2) (Proposition 1.1).

The system $S^{(K)}$ which is the system S added by Axiom K is a proper extension of S, because there exist formulas probable in $S^{(K)}$ but not in S.

Adding successively K-rule starting from S, we obtain a hierarchy $S, S^{(K)}$, $S^{(K)^{(K)}}, \cdots, S^{(K)^{\cdot^{\cdot^{(K)}}}}, \cdots$. The most perfect theory of such hierarchy is Feferman's theory of transfinite recursive progression [1].

There, Feferman constructs the progression S_d adding Axiom K successively up to arbitrary constructive ordinal d starting from S such that S_d is recursively enumerable if S is so, and S_1 is S, S_{2^d} is S_d added by Axiom K for S_d, and $S_{3.5^e}$ where $3.5^e\in O$ is the union of $S_{\{e\}(n)}$. In § 1, we give the definition of K-rule, Axiom K and Axiom K' equivalent to K. Moreover transfinite recursive progressions of intuitionistic or classical number theories Z^I_d, Z^C_d, are given. In § 2, we give an example which is obtained by using K-rule. Firstly, the transfinite induction up to ε_0 is provable in $Z^{(K)}$. Secondly, a normal truth definition can be obtained in a predicative extension of systems considered by using K-rule. Z^I_d keeps still intuitionistic character as Z^I. This is formulated as follows in § 3. The provable formulas in Z^I_d are recursively realizable (Theorem 1). In § 4, we obtain a formal consistency proof of classical systems Z^C_d in Z^I_{2d} which is intuitionistic as above mentioned. (Corollary to Theorem 2.) By the way of this proof, we show that another formulation of consistency, which is not as usual but expresses substantially the consistency of Z^C_d, can be proved in Z^C_d itself, even in Z^I. It means that the 2nd incompleteness theorem of Gödel may not hold if we adopt the special method of Gödel's numbering. And this fact was known by Feferman and Nishimura by another way. In case $d=1$, that the consistency of Z^C is provable in $Z^{I(K)}$, can be expected from the fact that the transfinite induction up to ε_0 is provable in $Z^{I(K)}$ and from Gentzen's famous result.

One of the most distinguished results of Feferman [1] is the completeness property of Z^C_d, that is, the number theoretic formula A is classically true if and only if $\vdash_{Z^C_d}$A for some $d\in O$. Does analogous theorem hold for Z^I_d and for intuitionistically true formulas instead of Z^C_d and of classically true formulas?

But unfortunately, in order to show that the general recursively true formula is provable in Z^I_d for some $d \in O$, we need the following formulas

$$(\hat{I}) \quad \neg\forall x A(x) \supset \exists x \neg A(x).$$

These formulas can not be provable in Z^I in general, but are considered as quasi-intuitionistic when A(x) are universal prenex formulas. And these formulas are used in Gödel's interpretation. We define $\hat{Z}^I$ as Z^I added by Axiom $\hat{I}$ and $\hat{Z}^I_d$ as the recursive progression of $\hat{Z}^I$.

The formula provable in $\hat{Z}^I_d$ may not recursively realizable in general. However if the formula is of the prenex form and provable in $\hat{Z}^I_d$, then it is general recursively true; for, the formula provable in $\hat{Z}^I_d$ is true under the Gödel's interpretation by means of effective operations and the truth of it coincides with the general recursive truth in case of prenex formula.

In § 5 we show that the provable formula of $\hat{Z}^I_d$ is true under Gödel's interpretation by means of effective operations.

In § 6, we shall show that the prenex formulas are general recursively true if and only if these formulas are provable in $\hat{Z}^I_d$ for some d (Theorem 3).

We do not know what is the recursive progression such that the formulas (not necessary prenex) which are true under Gödel's interpretation by means of effective operations are provable.

Of course, it is impossible to be $\hat{Z}^I_d$; in fact there is a true formula under above interpretation which is not provable in Z^I_d for any $d \in O$. We notice that the Gödel's interpretations by means of effective operations can be expressed by the formula of Z. In appendix we shall show that the formulas provable in Z^C_d are not only true under no-counterexample-interpretation, but the functionals used there are provably effective in Z^I_d; i.e. it is provable in Z^I_d that these are effective operations.

It is a generalization of Kreisel's result in [**17**]. This generalization would answer better to Kreisel's idea, if we can get the following theorem: if A is provable in Z^C_d, then not only no-counterexample-interpretation of A is true, but also functionals used there can be defined by the transfinite induction on some well ordering $<_d$ depending on d.

Thus, we are led to the following problem: is Z^C_d (resp. Z^I_d) equivalent to the system Z^C (resp. Z^I) added by the transfinite induction on some well ordering $<_d$ depending on d?

And for $d=2$, i.e. for $Z^{(K)}$, the well ordering $<_d$ ought to mean the order in ordinals less than ε_0. In this paper, we often formalize in Z or in other systems metamathematical statements which can be proved in ordinary way, by means of Gödel numbering. To avoid the complication, we could not show this process precisely enough, especially in appendix. But readers who are familiar with formal number theory will see in our informal proof that this process is possible.

§ 1. Preliminaries, Notations

1.1. K-rule, Axiom K

1.1.1. Formal axiom systems S, Z^C, Z^I, $\hat{Z}^I$ and Z_0.

a) In the following, let S be an axiom system within the intuitionistic or classical 1st order predicate calculus with equality. Sometimes we shall write S^C and S^I to distinguish classical and intuitionistic logics.

b) S is recursively enumerable.

c) S is wide enough to include number-theory Z.

d) We shall also distinguish classical and intuitionistic number theory. Z^C will mean the number theory within the classical 1st order predicate calculus, i.e. the system of Kleene [**2**], p. 81. Z^I will mean the system obtained by replacing the logical axiom 8 by 8^I in Kleene [**2**], p. 101. Z_0 will mean R. Robinson's system (c.f. [**2**], p. 244, Lemma 18b).

e) $\hat{Z}^I$ will denote the number-theory obtained from Z^I by adding the following axiom scheme.

Axiom $[\hat{I}]$.

$$\neg\forall \mathrm{x}\mathrm{A}(\mathrm{x}) \supset \exists \mathrm{x} \neg \mathrm{A}(\mathrm{x})$$

Here, A(x) is a universal prenex PR-formula, i.e. a formula of the form $\forall \mathrm{y}_1 \forall \mathrm{y}_2 \cdots \forall \mathrm{y}_n \mathrm{B}(\mathrm{y}_1, \mathrm{y}_2 \cdots \mathrm{y}_n \mathrm{x})$, B being a PR-formula[3)].

This axiom scheme is, as well as that of higher-types, used in Gödel's interpretation ([**5**] [**6**], [**3**], [**4**]), and M. Yasugi ([**3**], [**4**]) considered it as $\exists\neg$-inference.

1.1.2. a) Following to Kleene [**2**], we shall use Roman letters for formal expressions, while italic letters for informal objects.

b) If x is a natural number, $\bar{x}$ will denote the numeral corresponding to x.

c) If A is a formal expression (of formulas, terms, etc.), $\ulcorner \mathrm{A} \urcorner$ will denote its Gödel number.

d) $N(x)$ is the primitive recursive function which assigns the Gödel number of the numeral $\bar{x}$ to a natural number x.

e) $Sb\ a(^b_c)$ or briefly $a(^b_c)$ is (the number of) the result by substitution of c for (the number of variable) b at all free occurrences of b in (the number of expression) a. $a(^b_c)$ is a primitive recursive function of a, b, c.

f) If M is a set of formal expressions, $\ulcorner M \urcorner$ will denote the set of Gödel numbers of the elements of M.

1.1.3. a) As S is recursively enumerable, there exists a recursively enumerable predicate $Pr[S](a)$ for which we have $Pr[S](\ulcorner \mathrm{A} \urcorner) \Leftrightarrow\ \vdash_S \mathrm{A}$ for a formula A.

b) By Craig's theorem [7], S is primitive recursively axiomatizable, i. e. there exists a primitive recursive subset S^+ of S such that $Pr[S](a) \Leftrightarrow Pr[S^+](a)$. Every formula of S^+ is of the form $\mathrm{A} \& \cdots \& \mathrm{A}$ where $\mathrm{A} \in S$.

c) Moreover, the assertion b) can be formalized in Z^I, i. e. if $\mathrm{Pr}[S](\mathrm{a})$ (resp. $\mathrm{Pr}[S^+](\mathrm{a})$) is the formula of Z expressing $Pr[S](a)$ (resp. $Pr[S^+](a)$), we have

$$\vdash_{Z^I} \mathrm{Pr}[S](\mathrm{a}) \sim \mathrm{Pr}[S^+](\mathrm{a}) .$$

1.1.4. Axiom K and K-rule (with respect to S)

Consider a formula in S of the form $\forall \mathrm{x} \mathrm{A}(\mathrm{x})$ where x is a variable for natural numbers and let $a(x)$ be

$$Sb \ulcorner \mathrm{A}(\mathrm{x}) \urcorner \binom{\ulcorner \mathrm{x} \urcorner}{N(x)} .$$

$$\text{Ax. K}: \quad \forall \mathrm{x} \, \mathrm{Pr}\,[S](\mathrm{a}(\mathrm{x})) \supset \forall \mathrm{x} \mathrm{A}(\mathrm{x})^{1)}$$

$$\text{Ax. K}': \quad \forall \mathrm{x}[\mathrm{Pr}\,[S](\mathrm{a}(\mathrm{x})) \supset \mathrm{A}(\mathrm{x})]$$

$$K\text{-rule}^{2)} \quad \frac{\forall \mathrm{x} \, \mathrm{Pr}\,[S](\mathrm{a}(\mathrm{x}))}{\forall \mathrm{x} \mathrm{A}(\mathrm{x})}$$

Let $S^{(K)}$ (resp. $S^{(K')}$ denote the system obtained from S by adding AxK (resp. Ax. K$'$). If a formula A is provable from S with K-rule, we shall write $\vdash_S^{(K)} A$. $S^{(K)}$, $S^{(K')}$ or $\vdash_S^{(K)}$ corresponds to $S^{(II)}$, $S^{(II')}$ or $S^{(III)}$ in Feferman [1] respectively.

PROPOSITION 1.1 (Feferman [1] p. 28). Ax. K, Ax. K$'$ *and K-rule are equivalent. That is, for a formula* A *of S,*

$$\vdash_{S^{(K')}} \mathrm{A} \Leftrightarrow \vdash_{S^{(K)}} \mathrm{A} \Leftrightarrow \vdash_S^{(K)} \mathrm{A} .$$

PROOF. It is clear that Ax. K$'$ implies Ax. K and that Ax. K with modus ponens implies K-rule. It suffices therefore to show that K-rule implies Ax.K$'$.

a) For any natural number k and m,

$$\vdash_S \mathrm{Prf}\,[S^+](\overline{\ulcorner \mathrm{A}(\bar{k}) \urcorner}, \bar{m}) \supset \mathrm{A}(\bar{k}) ,$$

where $\mathrm{Prf}\,[S^+](\mathrm{a}, \mathrm{b})$ is the formula expressing "b is the proof of a in S^+"

1) We consider Axiom K as a formula in Z. But, strictly speaking, the formal expression $\mathrm{a}(\mathrm{x})$ of $a(x)$, as well as $\mathrm{b}(\mathrm{x}, \mathrm{y})$ which will appear later, is not a term in Z, but a usable term in a primitive recursive extension Z' of Z with primitive recursive function symbols and their defining equations. Nevertheless, for example the expressing formula $\mathrm{a}(\mathrm{x}, \mathrm{y})$ which is equivalent to $\mathrm{a}(\mathrm{x}) = \mathrm{y}$ can be constructed in Z. In the following, if a formal primitive recursive function symbol appears in a formula of Z, we shall consider it as a formula in Z obtained by using its expressing formula. For example, $\mathrm{A}(\mathrm{a}(\mathrm{x}))$ is considered as $\exists \mathrm{y}(\mathrm{A}(\mathrm{y}) \ \& \ \mathrm{a}(\mathrm{x}, \mathrm{y}))$. Strictly speaking we must point out clearly that these formulas are obtained by the translation (P') or (M) as in Feferman [1] [10].

2) K-rule was named by Rosser [8], who studied this rule by Kleene's suggestion.

which is a primitive recursive predicate.

If $Prf\,[S^{+}](\ulcorner A(\bar{k})\urcorner, m)$ holds, 1.1.3. a) and b) implies $\vdash_S A(\bar{k})$. Hence we have a) in the case.

If $Prf\,[S^{+}](\ulcorner A(\bar{k})\urcorner, m)$ does not hold, we have

$$\vdash_S \neg \mathrm{Prf}\,[S^{+}](\overline{\ulcorner A(\bar{k})\urcorner}, \bar{m})$$

by virtue of the "numeralwise representable" property. Hence we also have a) in the case.

b) The informal proof of a) can be formalized in Z, therefore in S too.

Let $a(x)$ be $\ulcorner A(x)\urcorner\binom{\ulcorner x\urcorner}{N(x)}$ and let $b(x, y)$ be $\ulcorner \mathrm{Prf}\,[S^{+}](a(x), y) \supset A(x)\urcorner\binom{\ulcorner x\urcorner,\ \ulcorner y\urcorner}{N(x),\ N(y)}$. Then, by the formalization of the informal proof of a), we have

$$\vdash_Z \forall x\forall y \mathrm{Pr}[S](b(x, y)),$$

Therefore, we have

$$\vdash_S \forall x\forall y \mathrm{Pr}[S](b(x, y)).$$

By K-rule,

$$\forall x\forall y(\mathrm{Prf}\,[S^{+}](a(x), y) \supset A(x)),$$

so we have

$$(*) \qquad \forall x(\exists y\, \mathrm{Prf}\,[S^{+}](a(x), y) \supset A(x)).$$

By 1.1.3. c),

$$\exists y\, \mathrm{Prf}\,[S^{+}](a(x), y) \sim \mathrm{Pr}\,[S^{+}](a(x)) \sim \mathrm{Pr}\,[S](a(x)).$$

Hence the formula (*) means Ax. K′.

PROPOSITION 1.2. *For any formula* A *of* S,

$$\vdash_{S^{(K)}} \mathrm{Pr}\,[S](\overline{\ulcorner A\urcorner}) \supset A$$

PROOF. Let $b(x)$ be $\ulcorner A \;\&\; a = a\urcorner\binom{\ulcorner a\urcorner}{N(x)}$. Then, we have $\vdash_S A \Rightarrow \vdash_S A \;\&\; \bar{k} = \bar{k}$ for all k. So we have

$$\vdash_Z \mathrm{Pr}\,[S](\overline{\ulcorner A\urcorner}) \supset \forall x\, \mathrm{Pr}[S](b(x)).$$

Ax. K implies $\forall x\, \mathrm{Pr}\,[S](b(x)) \supset \forall x(A \;\&\; x = x)$. Therefore, $\vdash_S \forall x(A \;\&\; x = x) \supset A$ yield Proposition 1.2.

1.2. Recursive progression

Let O be the set of all constructive ordinals. We make an axiom system S_d correspond to every $d \in O$. In general S_d is the result of an infinite times of extensions by adding Ax. K′ corresponding to the construction of d, starting from $1 \in O$. The following proposition was given by Feferman [**1**].

PROPOSITION 1.3. *There exists a recursive progression $S_d (d \in O)$ of S satisfying the following properties:*

1. *for each $d \in O$, S_d is a recursively enumerable set of axioms,*
2. *S_1 is S,*
3. *$d \in O \Rightarrow S_{2^d}$ is $S_d^{(K')}$,*
4. $3.5^d \in O \Rightarrow S_{3.5^d} = \bigcup_{n \in \omega} S_{\{d\}(n_0)} = \bigcup_{b <_O 3.5^d} S_b$.

Roughly speaking, S_d is defined as follows: there exists an RE-formula[3] S(a, b) such that we have $n \in \ulcorner S_d \urcorner \Leftrightarrow \vdash_{Z_0} \mathrm{S}(\bar{d}, \bar{n})$ for each $d \in O$.

$S(a, b)$ is constructed by the following idea:

$S(1, a) \sim S(a)$ (where $n \in \ulcorner S \urcorner \Leftrightarrow \vdash_{Z_0} \mathrm{S}(n)$),

$S(2^d, a) \sim S(d, a) \vee a = \ulcorner \mathrm{Ax.\ K'}$ for $S_d \urcorner$,

$S(3.5^d, a) \sim \exists n \{$formula expressing "$\vdash_{Z_0} \mathrm{S}(\{\bar{d}\}(\bar{n}_0), \bar{a})$"$\}$.

§ 2. Applications of *K*-rule

2.1. Transfinite induction up to the first ε-number.

2.1.1. Hilbert and Bernays formalized in Z the transfinite induction up to ε_0 in [**9**], p. 360. This is as follows.

a) Well-ordering $\underset{n}{<}$.

If $n = 0$, $a \underset{n}{<} b$ means $a < b$ in usual sense. There exists an isomorphism between the set of all natural numbers with $\underset{n}{<}$ and the set of all ordinals smaller than ω_{n+1} with $<$ (order of ordinals), where $\omega_0 = \omega$ and $\omega_{n+1} = \omega^{\omega_n}$. $a \underset{n}{<} b$ is primitive recursive with respect to a, b and n.

b) Transfinite induction is formalized as follows: for any formula A(a),

$$\mathrm{Ind_x(A(x), n)}; \quad \forall \mathrm{x}(\forall \mathrm{y}(\mathrm{y} \underset{\mathrm{n}}{<} \mathrm{x} \supset \mathrm{A(y)}) \supset \mathrm{A(x)}) \supset \forall \mathrm{x A(x)}.$$

Here we used the same symbol $\mathrm{a} \underset{\mathrm{n}}{<} \mathrm{b}$ for the formal expression of $a \underset{n}{<} b$.

c) Consider the following predicate

$$Cn(m, l); \quad \forall x(x < m \ \&\ (m)_x \neq 0 \ \&\ x \neq l \supset l \underset{n}{<} x) \ \&\ m \neq 0.$$

Let $\mathrm{C_n(a, b)}$ be the formal expression of the predicate of $Cn(a, b)$. Then B(l) is the abbreviation of the formula. $\forall \mathrm{x}(\mathrm{C_n(x, l)} \ \&\ \forall \mathrm{y}(\mathrm{y} \underset{\mathrm{n+1}}{<} \mathrm{x} \supset \mathrm{A(y)}) \supset \forall \mathrm{y}(\mathrm{y} \underset{\mathrm{n+1}}{<} \mathrm{x} \cdot \mathrm{P}_l \supset \mathrm{A(y)}))$. $Cn(m, l)$ is a primitive recursive predicate and means that P_l is the prime factor immediately before $m \cdot P_l$ with respect to $\underset{n+1}{<}$.

We shall write $\mathrm{B_x(c, A(x), n)}$ for B(c), as it depends on n and A(a).

3) After Feferman [**1**], we define PR-formula and RE-formula as follows: PR-formula is a formula representing numeralwise a primitive recursive predicate in usual manner. RE-formula is a PR-formula prefixed by existential quantifiers.

d) Hilbert-Bernays showed that

$$\vdash_{Z^I} \mathrm{Ind}_x(B_y(x, A(y), \bar{n}), \bar{n}) \supset \mathrm{Ind}_x(A(x), \overline{n+1}).$$

e) Hilbert-Bernays proved $\vdash_{Z^I} \mathrm{Ind}_x(A(x), \bar{n})$ by induction on n for any formula A(a) and any natural number n. In fact, $\mathrm{Ind}_x(A(x), 0)$ is the usual mathematical induction and belongs to the axiom scheme of Z^I. Thus, if e) holds for some n, then e) holds for B(a) and therefore for $n+1$ by d).

f) The formula $\forall n\, \mathrm{Ind}_x(A(x), n)$ is equivalent to the transfinite induction up to ε_0 and cannot be proved in Z as is well known [**11**]. Hilbert-Bernays showed that it can be proved in an extension Z' of Z with predicate variable. In fact, introduce predicate variable with one argument U(a) and replace the usual mathematical induction by $\forall U\, \mathrm{Ind}_x(U(x), 0)$. As in d), we have

$$\vdash_{Z'} \forall U[\mathrm{Ind}_x(B_y(x, U(y), n), n) \supset \mathrm{Ind}_x(U(x), n+1)],$$

$$\vdash_{Z'} \forall U\, \mathrm{Ind}_x(U(x), n) \supset \mathrm{Ind}_x(B(x), n),$$

hence

$$\vdash_{Z'} \forall n \forall U\, \mathrm{Ind}_x(U(x), n).$$

2.1.2. The formula $\forall n\, \mathrm{Ind}_x(A(x), n)$ can be proved by adding K-rule to Z^I.

PROPOSITION 2.1. $\vdash_{Z^{I(K)}} \forall n\, \mathrm{Ind}_x(A(x), n)$.

PROOF. There exist primitive recursive functions $ind(a, n)$ and $b(a, n)$ such that

$$\ulcorner \forall x A(x) \urcorner \text{ is } a \Rightarrow \ulcorner \mathrm{Ind}_x(A(x), \bar{n}) \urcorner \text{ is } ind(a, n),$$

$$\ulcorner B_y(x, A(y), n) \urcorner \text{ is } b(a, n).$$

There exists a primitive recursive predicate $Fm'(a)$ expressing " a is the Gödel number of the formula of the form $\forall x A(x)$".

By virtue of $\vdash_{Z^I} A \supset B \Rightarrow (\vdash_{Z^I} A \Rightarrow \vdash_{Z^I} B)$, d) means

$$Fm'(a) \Rightarrow \{Pr[Z^I](ind(b(a, n), n)) \Rightarrow Pr[Z^I](ind(a, n'))\},$$

and

$$Fm'(a) \Rightarrow Pr[Z^I](ind(a, 0)).$$

This fact can be formalized in Z^I. The outline is as follows.

$$\vdash_{Z^I} \mathrm{Fm}'(a) \supset \mathrm{Pr}[Z^I](\mathrm{ind}(b(a, n), n)) \supset \mathrm{Pr}[Z^I](\mathrm{ind}(a, n')),$$

$$\vdash_{Z^I} \mathrm{Fm}'(a) \supset \mathrm{Pr}[Z^I](\mathrm{ind}(a, 0)).$$

As $\vdash_{Z^I} \mathrm{Fm}'(a) \supset \mathrm{Fm}'(b(a, \bar{n}))$, it is provable in Z^I that $\forall z\{\mathrm{Fm}'(z) \supset \mathrm{Pr}[Z^I](\mathrm{ind}(z, \bar{n}))\}$.

Put $\alpha = \ulcorner \forall x A(x) \urcorner$ for any $\forall x A(x)$. As $\vdash_{Z^I} \mathrm{Fm}'(\bar{\alpha})$ is clear, we have $\vdash_{Z^I} \forall n \mathrm{Pr}[Z^I](\mathrm{ind}(\bar{\alpha}, n))$. $\mathrm{ind}(\bar{\alpha}, \bar{n})$ being nothing but the formal expression of $Sb \ulcorner \mathrm{Ind}_x(A(x), a) \urcorner \binom{\ulcorner a \urcorner}{N(n)}$, we get by K-rule, $\vdash_{Z^{I(K)}} \forall n\, \mathrm{Ind}_x(A(x), n)$.

2.2. Normal truth definition.

2.2.1. A. Mostowski and Hao Wang gave Tarski's truth definition for sentences of Zermelo-Fraenkel set theory in the following form: we can construct a formula Tr(a) (a is a free variable of natural numbers) of a theory S' with class (large) variable, such that, for any closed formula A of Zermelo-Fraenkel set theory, we have

a) $$\vdash_{S'} \mathrm{A} \sim \mathrm{Tr}(\ulcorner \mathrm{A} \urcorner)$$

(cf. [**15**] Theorem 1, [**16**] Theorem 2).

Mostowski proved a) by taking Gödel's set theory as S' and Wang proved a) by taking a weaker predicative theory S_2 as S'.

Tr(a) is called normal truth definition with respect to S, if it satisfies the following condition:

b) $$\vdash_{S''} \forall \mathrm{x}(\mathrm{CF}(\mathrm{x}) \ \& \ \mathrm{Pr}[S](\mathrm{x}) \supset \mathrm{Tr}(\mathrm{x})),$$

where CF(x) expresses "x is a closed formula" and S'' is a some extension of S.

Wang showed that b) holds if one takes S_3 (cf. [**16**]) as S and an impredicative extension S_1 of S as S''. Mostowski showed that b) holds if one takes Zermelo-Fraenkel theory as S and an impredicative extension of S as S''. Mostowski showed also that b) does not hold if S is consistent and if one takes as S'' a predicative extension of S, for example, Gödel's set theory. This is because b) implies $\vdash_{S''} \mathrm{Con}[S]$, and so implies $\vdash_{S''} \mathrm{Con}[S'']$.

c) We assert the following proposition.

PROPOSITION 2.2. *Let S be a set theory without class variable* (e.g. Zermelo-Fraenkel theory), *S' be a predicative extension of S* (e.g. Bernays-Gödel set theory) *and $S'^{(K)}$ be the system S' added by* Ax. K. *Then, the normal truth defidition of S is possible in $S'^{(K)}$, i.e.*

$$\vdash_{S'^{(K)}} \forall \mathrm{x}(\mathrm{CF}(\mathrm{x}) \ \& \ \mathrm{Pr}[S](\mathrm{x}) \supset \mathrm{Tr}(\mathrm{x})).$$

PROOF. If m is the Gödel number of a closed formula, then for any n and k, we have

d) $$\vdash_{S'} \mathrm{Prf}\,[S](\bar{m}, \bar{n}) \ \& \ \lambda(\bar{n}) \leqq \bar{k}) \supset \mathrm{Tr}(\bar{m})$$

where $\lambda(n)$ is the number of lines in the proof n (cf. Wang [**16**], 4.6 p. 262). A similar assertion can be found in Σ_{11} of Mostowski [**15**].

We can obtain Proposition 2.2. by formalizing d) and by using K-rule, but this is very complicated. Here we give the following essential remark on the proof. In [**15**] and [**16**], it is pointed out that the following comprehension axiom e) is indispensable for the normal truth definition:

e) For a formula $\exists \mathrm{X}\mathrm{C}(\mathrm{X}, \mathrm{x}_1)$ such that $\mathrm{C}(\mathrm{X}, \mathrm{x}_1)$ does not contain bound class variables, we have

$$\exists \mathrm{Y} \forall \mathrm{x}_1 [\mathrm{x}_1 \in \mathrm{Y} \sim \exists \mathrm{X}\mathrm{C}(\mathrm{X}, \mathrm{x}_1)].$$

But, what is indispensable for the normal truth definition is not the comprehension axiom e) itself, but is the fact that the mathematical induction holds for a formula of the form $\exists XC(X, x_1)$. That is to say, if $A(x_1)$ is $\exists XC(X, x_1)$, we have

f) $$\vdash_{S''} A(0) \ \& \ \forall n(A(n) \supset A(n+1)) \supset \forall n A(n)\,,$$

where n is a variable of natural numbers.

In the proofs of 2.33, 2.37, 4.7 and 4.12 in [**16**], f) does not hold. However, $\vdash_{S'} A(0)$ and $\vdash_{S'} \forall n(A(n) \supset A(n+1))$ hold there. And we need only $\forall n A(n)$. Let $a(x)$ be $\ulcorner A(n) \urcorner \binom{\ulcorner n \urcorner}{N(x)}$. Then, it is not so difficult to see that the formulas $\vdash_Z \Pr[S'](a(0))$ and $\vdash_Z \forall n\{\Pr[S'](a(n) \supset \Pr[S'](a(n+1)))\}$ hold in each case of 2.33, 2.37, 4.7 and 4.12. Therefore, we have $\vdash_Z \forall n \Pr[S'](a(n))$ and so $\vdash_{S'} \forall n \Pr[S'](a(n))$. Therefore, we can obtain $\forall n A(n)$ in $S'^{(K)}$ in every case.

§ 3. Intuitionistic character of Z^I_d.

a) If provable formulas in S are intutionistically true in some sense, then provable formulas in $S^{(K)}$ are intuitionistically true in the same sense on account of the form of Ax. K or K-rule. Therefore, if Z^I deduces only intuitionistically true sentences, then it will be expected that Z^I_d does too.

It is very problematic how we should interpret the concept of the intuitionistic truth. As all considered formulas are in Z, we shall consider here Kleene's recursive realizability [2] § 82.

b) Kleene showed the following proposition in [2] § 82 Theorem 62^N: if $\vdash_{Z^I} A$, and if $\tilde{A}$ is the closure of A, then $\tilde{A}$ is recursively realizable. Moreover, observing his proof in details, we see that the number which realizes $\tilde{A}$ is determined effectively from the proof of A in Z^I.

c) Kleene showed, moreover, that if A is a prenex formula, A is realizable if and only if it is general recursively true ([2], p. 465, p. 516), and that recursive functions which make A recursively true are determined effectively from the numbers realizing A.

This fact holds also when A is of the form $Q_1x_1Q_2x_2 \cdots Q_jx_jB(x_1 \cdots x_j)$ where Q_ix_i is $\exists x_i$ or $\forall x_i$ and B is PR- formula.

For, in this case A is equivalent to a prenex formula in a primitive recursive extension Z'^I of Z^I, and the above assertion holds for formulas in Z'^I.

d) Theorem 1. *If* $\vdash_{Z^I_d} A$ *and if* $\tilde{A}$ *is the closure of* A, *then* $\tilde{A}$ *is recursively realizable. Moreover the number realizing* $\tilde{A}$ *is effectively determined from the proof of* A *in* Z^I_d.

Proof. We prove this by the transfinite induction on d according to the well-ordering $<_o$. In case $d=1$, Theorem 1 is obvious from b). Take $d \in O$, and assume that Theorem 1 holds for every $d'(d'<_o d)$. In case $d=3.5^{d_1}$, if $\vdash_{Z_d^I}$ A, there exist $d'(<_o d)$ such that $\vdash_{Z_{d'}^I}$A, so Theorem 1 holds. In case $d=2^{d_0}$, Theorem 1 holds for d_0 by our assumption of the induction. Suppose $\vdash_{Z_d^I}$A which is equivalent to $\vdash^{(K)}_{Z_d^I}$A by proposition 1.1.

If A is an axiom of $Z_{d_0}^I$, Theorem 1 holds from our assumption. It suffices to show that when A is the immediate consequence of some inference, Theorem 1 holds for A under the assumption that Theorem 1 holds for premise of this inferende. Moreover, it suffices to show this in the case that A is the conclusion of K-rule, because, the other cases are easy.

Suppose A is of the form $\forall$xB(x) and premise is $\forall$xPr$[Z_{d_0}^I]$(b(x)) where $b(x)$ is $\ulcorner$B(x)$\urcorner$ $\binom{\ulcorner x \urcorner}{N(x)}$. Let C denote $\forall$xPr$[Z_{d_0}^I]$(b(x)). Then there exists e effectively determined from the proof of C in $Z_{d_0}^I$ such that e realizes C.

Seeing the form of C, C is general recursively true from the last part of c). That is, there exists a general recursive function $\phi(x)$ effectively determined from e such that $(x)Prf\,[Z_{d_0}^I](b(x),\phi(x))$. Moreover from the assumption of Theorem 1 for $Z_{d_0}^I$, there exists a partial recursive function $\psi_{d_0}(a)$ defined on the numbers of proof in $Z_{d_0}^I$, such that $\psi_{d_0}(\phi(x))$ realizes B($\bar{x}$).

As $\psi_{d_0}(\phi(x))$ is a general recursive, there exists e' such that $\psi_{d_0}(\phi(x)) =\{e'\}(x)$, for every x. Hence for all x $\{e'\}(x)$ realize B($\bar{x}$). From the definition of realizability, it means that e' realizes $\forall$xB(x).

As e is effectively determined from the proof of C, and as $\phi(x)$ is effectively determined from e, so e' is effectively determined from the proof of A.

e) Axiom $\hat{I}$, i.e. $\neg\forall$xA(x)$\supset\exists$x$\neg$A(x), where A(x) is a universal prenex PR-formula which is also recursively realizable. But we notice that the number realizing Axiom $\hat{I}$ depends on x satisfying $\neg$A($\bar{x}$). Therefore we cannot say that numbers realizing provable formulas are determined effectively from the number of its formal proof. So by the above method, we can not conclude that provable formulas in $\hat{Z}_d^I$ are recursively realizable.

Intuitionistic character of $\hat{Z}_d^I$ will be assured in § 5 by another method, i.e. under Gödel interpretation by means of effective operations.

§ 4. Proof of Con $[Z_d^C]$ in $Z_{2^d}^I$.

4.1. Since the transfinite induction up to ε_0 holds in $Z^{I(K)}$, by formalizing Gentzen's famous proof, it is expected that Con $[Z^C]$ will be provable in $Z^{I(K)}$, which is the formal expression of the statement "Z^C is consistent".

4.2. The construction of a formal consistency proof of S in $S^{(K)}$ can be seen in Rosser [8].

Let f be $\ulcorner\bar{0}=\bar{1}\urcorner$. Then

$$\mathrm{Con}\,[S] \sim \forall x \neg \mathrm{Prf}\,[S^+](\bar{f}, x)\,.$$

Let $r(x)$ be $\ulcorner \neg \mathrm{Prf}\,[S^+](\bar{f}, \mathrm{a})\urcorner \binom{\ulcorner \mathrm{a} \urcorner}{N(x)}$. Then we have

$$\overline{Prf\,[S^+](f, x)} \Rightarrow \vdash_Z \neg \mathrm{Prf}\,[S^+](f, \bar{x})\,.$$

By formalizing the above assertion, we have

$$\vdash_Z \neg \mathrm{Prf}\,[S^+](\bar{f}, \mathrm{a}) \supset \mathrm{Pr}\,[Z](\mathrm{r}(\mathrm{a}))\,,$$

so we have on one hand

a) $$\vdash_Z \mathrm{Con}\,[S] \supset \forall x \mathrm{Pr}\,[S](\mathrm{r}(x))\,.$$

On the other hand,

$$\overline{Con\,[S]} \Rightarrow Pr\,[S](r(x))\,.$$

Formalizing this in Z, we also have

b) $$\vdash_Z \neg \mathrm{Con}\,[S] \supset \forall x \mathrm{Pr}\,[S](\mathrm{r}(x))\,.$$

From a) and b), we have

$$\vdash_S \forall x \mathrm{Pr}\,[S](\mathrm{r}(x))\,.$$

By K-rule, $\vdash_{S(K)} \forall x \neg \mathrm{Prf}\,[S](f, x)$ holds, so we have $\vdash_{S(K)} \mathrm{Con}\,[S]$.

However, in above proof, we use $\vdash_S \mathrm{Con}\,[S] \vee \neg \mathrm{Con}\,[S]$ which is law of excluded middle. So, it is not a formal proof corresponding a finitary consistency proof. As we have already known in § 3 that Z^I_d is an intuitionistic system in some sense, it is desirable, for example, that $\mathrm{Con}\,[Z^C]$ is provable in $Z^{I(K)}$.

In the following we can show in general that consistency of Z^C_d is provable in Z^I_{2d}. That is, in some sense, a generalization of the assertion in 4.1.

4.3. Consistency proof of Z^C_d in Z^I_{2d}.

4.3.1. Translation "$\circ$".

For each formula A in Z, we make correspond A°, i.e. $\circ$-translation of A ([**12**] or [2] § 81) in which we replace for $\vee$ (resp. $\exists$) in A by & and $\neg$ (resp. $\forall$ and $\neg$). Then the following proposition holds.

PROPOSITION 4.1.

(i) $$\vdash_{Z^C} \mathrm{A} \Rightarrow \vdash_{Z^I} \mathrm{A}^\circ\,.$$

More strictly, $\vdash_{Z^I} \mathrm{Fm}(\mathrm{a}) \supset (\mathrm{Pr}\,[Z^C](\mathrm{a}) \supset \mathrm{Pr}\,[Z^I](\mathrm{a}^\circ))$, where Fm(a) represents that a is a formula of Z, and a° is the Gödel number of $\circ$-translation of the formula with the Gödel number a.

(ii) If A is a PR-formula, then $\vdash_{Z^I} \mathrm{A} \sim \mathrm{A}^\circ$.

(i) can be seen in [2] § 81.

(ii) can be shown as follows.

Consider some primitive recursive extension $Z^{I'}$ of Z^I in which we have a term ε of $Z^{I'}$ (primitive recursive function symbol) such that $\mathrm{A} \sim \varepsilon = 0$ is provable in $Z^{I'}$. Then we have $\vdash_{Z^{I'}} A \sim \varepsilon = 0 \sim (\varepsilon = 0)^\circ \sim A^\circ$. Moreover there is a translation * from formulas of $Z^{I'}$ to those of Z^I such that $\vdash_{Z^{I'}} A \Rightarrow \vdash_{Z^I} A^*$ and $\vdash_{Z^I} A^* \sim A$ for formulas A of Z^I. From this. $\vdash_{Z^I} A \sim A^\circ$ is obvious.

THEOREM 2. $\vdash_{Z_d^C} A \Rightarrow \vdash_{Z_d^I} A^\circ$.

More strictly,

$$\vdash_{Z^I} \mathrm{Fm}(a) \supset (\Pr[Z_d^C](a) \supset \Pr[Z_d^I](a^\circ)\,.$$

PROOF. We prove this by the transfinite induction on d.

In case $d=1$, Theorem 2 means prop. 4.1. Assume that Theorem 2 holds for $d'(<_O d)$. In case $d = 3.5^e \in O$, Theorem 2 is obvious from our assumption. We shall show, under the assumption that Theorem 2 holds for d_0, that Theorem 2 also holds for $d = 2^{d_0}$. It suffices to prove that the $\circ$-translation of Ax. K′ of Z_d^C can be provable in Z_d^I.

Ax. K′ of Z_d^C is of the form $\forall x(\Pr[Z_{d_0}^C](a(x)) \supset A(x))$ which is equivalent in Z^I, by 1.1.3 c), to

$$\forall x \forall y(\mathrm{Prf}[Z_{d_0}^{+C}](a(x), y) \supset A(x))\,.$$

The $\circ$-translation of the above formula is equivalent in Z^I to

$$\forall x \forall y(\mathrm{Prf}[Z_{d_0}^{+C}](a(x), y) \supset A^\circ(x))\,.$$

It suffices to show that this formula can be provable in $Z_{2^{d_0}}^I$ (i.e. Z_d^I).

Proof is analogous to that of prop. 1.1.

a) For all k, m,

$$\vdash_{Z_{d_0}^C} \mathrm{Prf}[Z_{d_0}^{C+}](a(\bar{k}), \bar{m}) \supset A(\bar{k})\,,$$

where $a(k)$ is $\ulcorner A(\bar{k})\urcorner$.

Let $\ulcorner \mathrm{Prf}[Z_{d_0}^{C+}](a(\bar{k}), \bar{m}) \supset A(\bar{k})\urcorner$ be $b(k, m)$. The $\circ$-translation of this formula is, by prop. 1.4. (ii),

$$\mathrm{Prf}[Z_{d_0}^{C+}](a(\bar{k}), \bar{m}) \supset A^\circ(\bar{k})\,.$$

Let $b^\circ(m, k)$ be the Gödel number of this formula. By formalizing a) in Z^I, we get

$$\vdash_{Z^I} \forall x \forall y \Pr[Z_{d_0}^C](b(x, y))\,.$$

b) From the assumption of Theorem 2 for d_0,

$$\forall x \forall y \Pr[Z_{d_0}^C](b(x, y)) \supset \forall x \forall y \Pr[Z_{d'}^I](b^\circ(x, y))\,.$$

So, we have

$$\vdash_{Z_{d_0}^I} \forall x \forall y \Pr[Z_{d'}^I](b^\circ(x, y))\,.$$

Using K-rule for $Z_{d_0}^I$, we have

$$\vdash_{Z_{2^{d_0}}^I} \forall x \forall y(\mathrm{Prf}[Z_{d_0}^{C+}](a(x), y) \supset A^\circ(x))\,.$$

Second part of Theorem 2 for d can be obtained by formalizing the above informal proof in Z^I.

COROLLARY. $\vdash_{Z^I_{2d}} \mathrm{Con}[Z^C_d]$.

As $\mathrm{Con}[Z^C_d]$ is equivalent to $\forall y \neg \mathrm{Prf}[Z^{C+}_d](\bar{f}, y)$ in Z^I where $\mathrm{Prf}[Z^{C+}_d](a, b)$ is PR-formula, we have

$$\vdash_{Z^I} \mathrm{Con}[Z^C_d] \sim (\mathrm{Con}[Z^C_d])^\circ$$

by prop. 4.1. (ii).

Since $\vdash_{Z^C_{2d}} \mathrm{Con}[Z^C_d]$ by Rosser's proof in 4.2, we obtain

$$\vdash_{Z^I_{2d}} \mathrm{Con}[Z^C_d].$$

4.3.2. We point out the following fact related to Nishimura [**13**].

For all n, $\vdash_{Z_0} \neg \mathrm{Prf}[Z^{C+}_d](\bar{f}, \bar{n})$.

For, $\vdash_{Z^I_{2d}} \forall y \neg \mathrm{Prf}[Z^{C+}_d](\bar{f}, y)$ and this formula is recursively realizable from Theorem 1 in § 3 and so, recursively true. Therefore, $(n)\overline{Prf}[Z^{C+}_d](f, n)$, so that $\vdash_{Z_0} \neg \mathrm{Prf}[Z^{C+}_d](\bar{f}, \bar{n})$ by the property of numeralwise representability of primitive recursive predicate in Z_0.

4.3.3. Another formulation of Consistency.

PROPOSITION 4.2.

There exists a formula $[\neg \mathrm{Prf}]^*(a, b)$ which numerate the predicate $\overline{Prf}[Z^{C+}_d]\ (a, b)$

$$(\text{i. e. } \overline{Prf}[Z^{C+}_d](x, y) \Leftrightarrow \vdash_{Z_0} [\neg \mathrm{Prf}]^*(\bar{x}, \bar{y})).$$

If we denote $\forall y [\neg \mathrm{Prf}]^*(\bar{f}, y)$ by $\mathrm{Con}^*[Z^C_d]$, then $\vdash_{Z^C_d} \mathrm{Con}^*[Z^C_d]$ moreover, $\vdash_{Z^I} \mathrm{Con}^*[Z^C_d]$.

Theorems of this type are known by Nishimura, and Feferman. Nishimura showed the following result. Let BG^* be the Bernays Gödel set theory considered within G^1LC as logics. Fm(a) (a is formula), Prf(a, b) (b is proof of a) etc. are defined by another way as usual, by using class variable of type 1, by which we can define the formula $\mathrm{Con}^*[BG^*]$ which expresses substantially the consistency of BG^*. Then we have

$$\vdash_{BG^*} \mathrm{Con}^*[BG^*] \quad [\mathbf{14}].$$

Feferman showed the following.

Let S be the axiom system satisfying the following conditions,

1. S has infinitely many axioms including Z
2. S is consistent
3. the formula $\alpha(x)$ which expresses that x is axiom of S is PR-formula
4. S is reflexive, i. e. consistency of any finite number of axioms of S can be provable in S.

Construct $\mathrm{Con}(\alpha)$ from α as usual.

Let $\alpha^*(x)$ be $\alpha(x)\ \&\ \forall z(z \leqq x \supset \mathrm{Con}\, \alpha \restriction z)$ where $\alpha \restriction z$ denote the axiom x such that $\alpha(x)\ \&\ x \leqq z$. Then $\alpha^*(x)$ also numeralwise represent that x is axiom of S, and we have $\vdash_S \mathrm{Con}(\alpha^*)$ [**10**].

In other words, whether Gödel's 2nd incompleteness theorem may hold or not, depend on how we express the primitive recursive predicate in formal systems.

Suppose $P(x, y)$ is a primitive recursive predicate and $\mathrm{P(x, y)}$ is a formula expressing $P(x, y)$ in usual way.

As Z_d^C include Z_0, it holds that

$$P(x, y) \Rightarrow \vdash_{Z_d^C} \mathrm{P}(\bar{x}, \bar{y}) .$$

Suppose that Z_d^C is consistent, (this may be allowable because consistency of Z_d^C can be proved in Z_{2d}^I which is recursively realizable system).

Then $P(x, y) \Leftrightarrow \vdash_{Z_d^C} \mathrm{P}(\bar{x}, \bar{y})$.

Let $p(x, y)$ be $\ulcorner \mathrm{P(x, y)} \urcorner \binom{\ulcorner \mathrm{x} \urcorner,\ \ \ulcorner \mathrm{y} \urcorner}{N(x),\ \ N(y)}$. Then

$$P(x, y) \Leftrightarrow Pr[Z_d^C](p(x, y)) .$$

As $Pr[Z_d^C](p(x, y))$ is recursively enumerable, we have

$$Pr[Z_d^C](p(x, y)) \Leftrightarrow \vdash_{Z_0} \mathrm{Pr}[Z_d^C](\mathrm{p}(\bar{x}, \bar{y}))$$

from ω-consistency in weak form of Z_0.

Therefore, if we put $\mathrm{p}^{(d)}(\mathrm{a, b})$ for $\mathrm{Pr}\,[Z_d^C](\mathrm{p(a, b)})$ then

$$P(x, y) \Leftrightarrow \vdash_{Z_0} \mathrm{P}^{(d)}(\bar{x}, \bar{y}) :$$

that is, $\mathrm{P}^{(d)}$ numerates P in Z_0. Let $(\neg \mathrm{Prf}\,[Z_d^{C+}])^{(d)}$ be the formula which numerates $\overline{Prf}\,[Z_d^{C+}]$ (primitive recursive predicate) by the above mentioned method.

Let $(\mathrm{Prf}\,[Z_d^{C+}]) * (\mathrm{a, b})$ be $(\mathrm{Prf}\,[Z_d^{C+}])^{(d)}(\mathrm{a, b})$ and let $\mathrm{Con}^*[Z_d^C]$ be $\forall \mathrm{x}(\neg \mathrm{Prf}\,[Z_d^{C+}]) * (\bar{f}, \mathrm{x})$. Then,

$$\vdash_{Z^I} \mathrm{Con}^*[Z_d^C] ,$$

because; for all m,

$$\vdash_{Z_d^C} \mathrm{Prf}\,[Z_d^{C+}](\bar{f}, \bar{m}) \supset 1 = 0 ,$$

i. e.

$$\vdash_{Z_d^C} \neg \mathrm{Prf}\,[Z_d^{C+}](\bar{f}, \bar{m}) .$$

By formalizing the above assertion in Z^I,

$$\vdash_{Z^I} \forall \mathrm{x}(\neg \mathrm{Prf}\,[Z_d^{C+}])^*(\bar{f}, \mathrm{x}) ,$$

Therefore

$$\vdash_{Z^I} \mathrm{Con}^*[Z_d^C] .$$

§ 5. Intuitionistic character of $\hat{Z}_d^I$.

5.1. Gödel interpretation by means of effective operations.

To make sure an intuitionistic character of $\hat{Z}^I_a$, we consider the Gödel's interpretation by means of effective operations in stead of realizability.

5.1.1. N^τ

a) As types we use only those given by the following i) ii):

i) (0) is a type,

ii) if $\tau_0, \tau_1 \cdots \tau_n$ are types, then $(\tau_0, \tau_1 \cdots \tau_n)$ is a type.

b) we define a set of natural numbers N^τ recursively for every type τ by the following

(i) $N^{(0)} = N$ (= the set of all natural numbers)

(ii) If $N^{\tau_0}, N^{\tau_1} \cdots N^{\tau_n}$ are already defined for types $\tau_0, \tau_1 \cdots \tau_n$, then $N^{(\tau_0, \tau_1, \cdots, \tau_n)}$ is defined by the following condition:

$a \in N^{(\tau_0, \tau_1, \cdots \tau_n)} \Leftrightarrow \{a\}_n(x_1, \cdots, x_n)$ is defined for all $x_1 \in N_1^{\tau_1}, \cdots, x_n \in N^{\tau_n}$ and belongs to N^{τ_0}.

(Here, $\{a\}_n$ in a partial recursive function with Gödel number a with n arguments. We shall occasionally write $\{a\}$ for $\{a\}_n$.)

c) E-term, E-formula and verifiability. First of all, we prepare, for each type τ, symbols for free or bound variables

$$a_1^\tau, a_2^\tau, a_3^\tau \cdots$$

or

$$x_1^\tau, x_2^\tau, x_3^\tau \cdots$$

respectively.

E-terms are defined recursively as follows:

i) if $n \in N^\tau$, $\bar{n}$ is an E-term of type τ;

ii) a free variable of type τ is an E-term of type τ;

iii) if t is an E-term of type $(\tau_0, \tau_1, \cdots, \tau_n)$ and $\mathrm{s}_1, \mathrm{s}_2, \cdots, \mathrm{s}_n$ are E-terms of $\tau_1, \cdots, \tau_n$ respectively then $\{\mathrm{t}\}_n(\mathrm{s}_1, \cdots, \mathrm{s}_n)$ is E-terms of type τ_0 (here, $\{\ \}_n$ is considered as a formal symbol);

iv) the only E-terms are those given by i)–iii).

Our prime E-formulas are of the form $\mathrm{s} = \mathrm{t}$ where s and t are arbitrary E-terms of the same type. The E-formulas are constructed, as usual, by the propositional connectives &, $\vee$, $\supset$, $\neg$, and by quantifiers $\forall \mathrm{x}^\tau$, $\exists \mathrm{x}^\tau$ for every type, starting from the prime formulas.

A quantifier-free E-formula $\mathrm{A}(\mathrm{a}_1^{\tau_1}, \cdots, \mathrm{a}_n^{\tau_n})$ is said verifiable if and only if for all $x_1 \in N_1^{\tau_1} \cdots x_n \in N_n^{\tau_n}$ $\mathrm{A}(\bar{x}_1, \cdots, \bar{x}_n)$ is defined and holds under the interpretation of $\{\ \}$ as symbol for partial recursive function as usual and of = as weak sense [cf. [**2**], p. 328].

5.1.2. The numbers corresponding to Gödel's primitive recursive functionals of finite types.

To each Gödel's primitive recursive functional of finite type, we make a number correspond as follows.

a) If f is defined by the defining equation f $(X^{\tau}, X_1^{\sigma_1}, \cdots, X_n^{\sigma_n}) = X^{\tau}$ then we make e correspond to f which satisfies the following conditions:

$$\{e\}(x, t_1, \cdots, t_n) \simeq x.$$

b) To f satisfying the defining equation f $(X, T_1, \cdots, T_n) = 0$, we make e correspond such that

$$\{e\}(x, t_1, \cdots, t_n) \simeq 0$$

c) To f such that $f(a\mathfrak{T}) = a+1$ where $\mathfrak{T}$ is an ordered set $(T_1, \cdots, T_n)$ of free functional variables, we make e correspond such that

$$\{e\}(a, t_1, \cdots, t_n) \simeq a+1 .$$

d) If f is defined by $f(X\mathfrak{T}) = g(h(X, \mathfrak{T})\mathfrak{T})$ and if e_1 corresponds to h and e_2 corresponds to g, then we make e correspond to f such that

$$\{e\}(x, t_1, \cdots, t_n) \simeq \{e_2\}(\{e_1\}(x, t_1, \cdots, t_n), t_1, \cdots, t_n)$$

e) If f is defined by $f(\mathfrak{T}) = g(\mathfrak{T}_1)$ where g is previously defined primitive recursive functional and $\mathfrak{T}_1$ is a permutation of $\mathfrak{T}$, let $(t_1', \cdots, t_n')$ be the same permutation of $(t_1, \cdots, t_n)$ as above. If e_1 corresponds to g, then we make e correspond to f such that

$$\{e\}(t_1, \cdots, t_n) \simeq \{e_1\}(t_1', \cdots, t_n')$$

f) If f_0 is defined by the equation $f(X, X_1, \cdots, X_m\mathfrak{T}) = X(X_1, \cdots, X_n)$, then to f_0 we make e correspond such that

$$\{e\}(x, x_1, \cdots, x_m, t_1, \cdots, t_n) \simeq \{x\}(x_1, \cdots, x_m)$$

g) If f is defined by $f(0, \mathfrak{T}) = g(\mathfrak{T})$ and $f(a+1, \mathfrak{T}) = h(a, f(a, \mathfrak{T}), \mathfrak{T})$ where g, h are previously defined functionals and e_1 corresponds to g, and e_2 to h, then to f we make e correspond such that

$$\{e\}(0, t_1, \cdots, t_n) \simeq \{e_1\}(t_1, \cdots, t_n)$$
$$\{e\}(a+1, t_1, \cdots, t_n) \simeq \{e_2\}(a, \{e\}(a, t, \cdots, t_n), t_1, \cdots, t_n)$$

The corresponding number e by a)-g) exists; this is clear from the partial recursive founction Theory (c. f. [2] Chapter XII).

PROPOSITION 5.1.

If e corresponds to f^{τ} by one of the above procedure, then $e \in N^{\tau}$. This will be proved in the proof of prop. 5.2.

5.1.3. E-formula corresponding to formulas of analysis of finite type. We make an E-formula A^E correspond to a formula A of *SJ* in Yasugi [3], [4] as follows, which we call the E-translation of A.

a) Previously, we make an E-free (resp. E-bound) variable of type τ correspond in one-to-one way to each free (resp. bound) variable of type τ of *SJ*.

b) To a primitive recursive functional symbol of *SJ*, we make the numeral $\bar{e}$ in 5.1.2 correspond.

c) Let T be a term of type $(\tau_0, \tau_1, \cdots, \tau_n)$ of *SJ* and $S_1, \cdots, S_n$ be terms of type $\tau_1, \cdots, \tau_n$ respectively. If E-terms $t, s_1, \cdots, s_n$ correspond to $T, S_1, \cdots, S_n$, we make $\{t\}(s_1, \cdots, s_n)$ correspond to $T(S_1, \cdots, S_n)$.

d) To a formula $S = T$, we make $S^E = T^E$ correspond where S^E and T^E are E-terms corresponding respectively to S and T in a), b) and c).

e)
$$(A \,\&\, B)^E \text{ is } A^E \,\&\, B^E,$$
$$(A \vee B)^E \text{ is } A^E \vee B^E,$$
$$(A \supset B)^E \text{ is } A^E \supset B^E,$$
$$(\neg A)^E \text{ is } \neg A^E.$$

f) If $A(f^\tau)^E$ is $A^E(a^\tau)$, then $(\forall\phi^\tau A(\phi^\tau))^E$ is $\forall x^\tau A^E(x^\tau)$ and $(\exists\phi^\tau A(\phi^\tau))^E$ is $\mathrm{E}x^\tau A^E(x^\tau)$, where x^τ is the bound variable corresponding to ϕ^τ in a).

5.1.4. PROPOSITION 5.2. *Let* $A_1, \cdots, A_n, B_1, \cdots, B_m$ *be formulas in QF* (cf. [3], [4]). *If we have*

$$\vdash_{QF} A_1, \cdots, A_n \rightarrow B_1, \cdots, B_m,$$

then, $A_1^E, \cdots, A_n^E \rightarrow B_1^E, \cdots, B_m^E$ *is verifiable.*

PROOF. a) The proposition is obvious for the beginning sequence of *QF* except for defining equations of primitive recursive functionals. It suffices therefore, to prove the proposition for recursive schemes. Proposition 5.1 will also be proved here.

Consider for instance, the case

$$f(0, \mathfrak{T}) = g(\mathfrak{T}),\ f(a+1, \mathfrak{T}) = h(a, f(a, \mathfrak{T})\mathfrak{T}).$$

Let $\mathfrak{T}$ be $(T_1^{\tau_1}, \cdots, T_n^{\tau_n})$ and $T_i^{\tau_i}$ be free variables. Let g be of type $(\sigma, \tau_1, \cdots, \tau_n)$, h be of type $(\sigma, (0), \sigma, \tau_1, \cdots, \tau_n)$ and f be of type $(\sigma, (0), \tau_1, \cdots, \tau_n)$. Let e_1, e_2 and e be the numbers corresponding to g, h and f respectively. The E-formula corresponding to this formula is

i) $$\{\bar{e}\}(0, t_1^{\tau_1}, \cdots, t_n^{\tau_n}) = \{\bar{e}_1\}(t_1^{\tau_1}, \cdots, t_n^{\tau_n})$$

ii) $$\{\bar{e}\}(a+1, t_1^{\tau_1}, \cdots, t_n^{\tau_n}) = \{\bar{e}_2\}(a, \{\bar{e}\}(a, t_1^{\tau_1}, \cdots, t_n^{\tau_n}), t_1^{\tau_1}, \cdots, t_n^{\tau_n})$$

where $t_1, \cdots, t_n$ are E-free variables corresponding to $T_1, \cdots, T_n$ respectively.

Assume that Proposition 5.1 holds for g and h; i.e., $e_1 \in N^{(\sigma,\tau_1,\cdots,\tau_1)}$ and $e_2 \in N^{(\sigma,(0),\sigma,\tau_1,\cdots,\tau_n)}$. $\{e_1\}(x_1, \cdots, x_n)$ is defined and belongs to N^σ for any $x_1 \in N^\tau, \cdots, x_n \in N^{\tau_n}$. By the definition of e in 5.1.2. g), $\{e\}(0, x_1, \cdots, x_n)$ is defined and belongs to N^σ. And if $\{e\}(a, x_1, \cdots, x_n)$ is defined and belongs to N^σ, then, by our assumption on e_2,

$$\{e_2\}(a, \{e\}(a, x_1, \cdots, x_n), x, \cdots, x_n)$$

is defined and belongs to N^σ. Therefore by the definition of e in 5.1.2. g),

$\{e\}(a+1, x_1, \cdots, x_n)$ is defined and belongs to N^σ. Therefore, $\{e\}(a, x_1, \cdots, x_n)$ is defined and belongs to N^σ for any $a, x_1 \in N^{\tau_1}, \cdots, x_n \in N^{\tau_n}$. Hence we have $e \in N^{(\sigma,(0),\tau_1,\cdots,\tau_n)}$.

i) and ii) are clearly verifiable by the definition of e and Prop. 5.1. If the beginning sequence is

$$\mathrm{f}(\mathrm{X}, \mathrm{X}_1^{\sigma_1}, \cdots, \mathrm{X}_m^{\sigma_m}, \mathrm{T}_1^{\tau_1}, \cdots, \mathrm{T}_n^{\tau_n}) = \mathrm{X}(\mathrm{X}_1^{\sigma_1}, \cdots, \mathrm{X}_m^{\sigma_m})$$

and if X is of type $(\tau, \sigma_1, \cdots, \sigma_m)$, then f is of type $(\tau, (\tau, \sigma_1, \cdots, \sigma_m), \sigma_1, \cdots, \sigma_m$ $\tau_1, \cdots, \tau_n)$. If e is the number corresponding to f, the E-formula corresponding to the above formula is

$$\text{iii)} \qquad \{\bar{e}\}(\mathrm{a}^{(\tau,\sigma_1,\cdots,\sigma_m)}, \mathrm{b}_1^{\sigma_1}, \cdots, \mathrm{b}_m^{\sigma_m}, \mathrm{c}_1^{\tau_1}, \cdots, \mathrm{c}_n^{\tau_n}) = \{\mathrm{a}^{(\tau,\sigma_1,\cdots,\sigma_m)}\}(\mathrm{b}_1^{\sigma_1}, \cdots, \mathrm{b}_m^{\sigma_m}),$$

where a, b_i and c_j are free E-variables corresponding to X, X_i and T_j respectively.

Now, $\{x\}(x_1, \cdots, x_m)$ is defined and belongs to N^τ for any

$$x \in N^{(\tau,\sigma_1,\cdots,\sigma_m)}, x_1 \in N^{\sigma_1}, \cdots, x_m \in N^{\sigma_m}, y_1 \in N^{\tau_1}, \cdots, y_n \in N^{\tau_n}.$$

Since e is defined by $\{e\}(x, x_1, \cdots, x_m, y_1, \cdots, y_n) \simeq \{x\}(x_1, \cdots, x_m)$, so $\{e\}(x, x_1, \cdots, x_m, y_1, \cdots, y_n)$ is defined and belongs to N^τ; i.e., $e \in N^{(\tau,(\tau,\sigma_1,\cdots,\sigma_m),\sigma_1,\cdots\sigma_m,\tau_1,\cdots,\tau_n)}$ and (iii) is verifiable.

For the other defining equations, the proofs are analogous to the above case.

b) We shall show, under the application of inference rule of QF, that if the E-translation of upper-sequences is verifiable, then the E-translation of lower-sequences is also verifiable.

We shall show this only for the weak-induction, since other cases are trivial.

$$WJ\frac{\Gamma \to \Delta, \mathrm{F}(0, \mathrm{B}^\tau).\ \Gamma, \mathrm{F}(\mathrm{a}, \alpha_0^\sigma(\mathrm{a}, \mathrm{B}^\tau)) \to \Delta, \mathrm{F}(\mathrm{a}+1, \mathrm{B}^\tau)}{\Gamma \to \Delta, \mathrm{F}(\mathrm{t}, \mathrm{s})}$$

where a and B are free variables, α_0 is a primitive recursive functional of type $\sigma = (\tau, \cdots, (0), \cdots, \tau, \cdots)$ and t and s are arbitrary terms of type (0) and τ respectively. From the assumption,

$$\Gamma^{\mathrm{E}} \to \Delta^{\mathrm{E}}, \mathrm{F}^{\mathrm{E}}(0, \mathrm{b}^\tau)$$

$$\Gamma^{\mathrm{E}}, \mathrm{F}^{\mathrm{E}}(\mathrm{a}, \{\bar{e}_0\}(\mathrm{a}, \mathrm{b}^\tau)) \to \Delta^{\mathrm{E}}, \mathrm{F}^{\mathrm{E}}(\mathrm{a}+1, \mathrm{b}^\tau)$$

are verifiable, where e_0 is the number corresponding to α_0.

We shall show that $\Gamma^{\mathrm{E}} \to \Delta^{\mathrm{E}}, \mathrm{F}^{\mathrm{E}}(\bar{n}, \mathrm{b}^\tau)$ is verifiable for each n. This is obvious for $n=0$. If we assume this for some n, then, from the type of e_0, $\Gamma^{\mathrm{E}} \to \Delta^{\mathrm{E}}, \mathrm{F}^{\mathrm{E}}(\bar{n}, \{\bar{e}_0\}(\bar{n}, \mathrm{b}^\tau))$ is also verifiable. By the assumption, $\Gamma^{\mathrm{E}} \to \Delta^{\mathrm{E}}$, $\mathrm{F}^{\mathrm{E}}(\overline{n+1}, \mathrm{b}^\tau)$ is verifiable and so $\Gamma^{\mathrm{E}} \to \Delta^{\mathrm{E}}, \mathrm{F}^{\mathrm{E}}(\mathrm{a}, \mathrm{b}^\tau)$ is also verifiable.

5.1.5. Gödel's interpretation by means of effective operations.

For a formula A of *SJ*, let A^G denote the formal Gödel's-interpretation of A. A^G is in general of the form

$$\exists X_1^{\tau_1}, \cdots, \exists X_m^{\tau_m} \forall Y_1^{\sigma_1}, \cdots, \forall Y_n^{\sigma_n} A'(X_1^{\tau_1}, \cdots, X_m^{\tau_m}, Y_1^{\sigma_1}, \cdots, Y_n^{\sigma_n}),$$

and therefore A^{GE} is in general of the form

$$\exists x_1^{\tau_1}, \cdots, x_m^{\tau_m}, y_1^{\sigma_1}, \cdots, y_n^{\sigma_n}, A'^E(x_1^{\tau_1}, \cdots, x_m^{\tau_m}, y_1^{\sigma_1}, \cdots, y_n^{\sigma_n}).$$

PROPOSITION 5.3. *Suppose* $\vdash_{SJ} A_1, \cdots, A_n \to \Delta, B$ *and let* A_i^G *be* $\exists X_i^{\tau_i}, \forall Y^{\sigma_i}, A_i'(X_i^{\tau_i}, Y_i^{\sigma_i})$ *and* B^G *be* $\exists X^{\tau'} \forall Y^{\sigma'} B'(X^{\tau'}, Y^{\sigma'})$. *Then, there exist* n_i *in* $N^{(\sigma_i, \cdots, \tau_1, \cdots, \tau_n, \cdots, \sigma', \cdots,)}$ *and* m *in* $N^{(\tau', \cdots \tau_1, \cdots \tau_n, \cdots)}$ *such that the formula*

$$\{A_i'^E(a_1^{\tau_1}\{\bar{n}_i\}(a_1^{\tau_1}, \cdots, a_n^{\tau_n}, b^{\sigma'}))\}_{i=1}^n \to \Delta^E B'^E(\{\bar{m}\}(a_1^{\tau_1}, \cdots, a_n^{\tau_n}), b^{\sigma'})$$

is verifiable. Here n_i *and* m *are determined effectively from the proof in SJ of* $A_1, \cdots, A_n \to \Delta, B$.

PROOF. We get the proposition from [**3**], [**4**], and our Proposition 5.2.

COLLARY. *Let* A *be a formula in Z.*

Suppose $\vdash_{\hat{Z}^I} A$ *and let* A^{GE} *be*

$$\exists x_1^{\tau_1} \cdots \exists x_m^{\tau_m} \quad \forall y_1^{\sigma_1} \cdots y_n^{\sigma_n} \quad B(x_1^{\tau_1}, \cdots, x_m^{\tau_m}, y_1^{\sigma_1}, \cdots, y_n^{\sigma_n}).$$

Then, there exist $z_1 \in N^{\tau_1}, \cdots, z_m \in N^{\tau_m}$, *depending effectively on the proof of* A *in* $\hat{Z}^I$, *such that*

$$B(\bar{z}_1, \cdots, \bar{z}_m, b_1^{\sigma_1}, \cdots, b_n^{\sigma_m})$$

is verifiable.

PROOF. Since all primitive recursive functions of number theory are also primitive recursive functionals of *SJ*, there exists, for each PR-formula A a primitive recursive function f_0 such that $\vdash_{SJ} f_0 = 0 \sim A$. Therefore Axiom [$\hat{1}$] is provable in *SJ* by using $\exists 7$ of *SJ*. So, Corollary is obtained from Proposition 5.3.

5.2. Truth of GE-interpretation of provable formulas in $\hat{Z}_d^I$.

5.2.1. A formula A in Z is said to be true under GE-interpretation, if there exists a number $z \in N^\tau$ such that $B(\bar{z}, b^\sigma)$ is verifiable, where A^{GE} is $\exists x^\tau \forall y^\sigma B(x^\tau, y^\sigma)$.

In the case that A contains many variables, we define "GE-interpretation true" analogously.

5.2.2. PROPOSITION 5.4. *If a prenex formula* A *in Z is true under* GE-*interpretation, then* A *is general recursively true.*

PROOF. Suppose A is, for instance, of the form $\forall x_1 \exists y_1 \forall x_2 \exists y_2 B(x_1, x_2, y_1, y_2)$. If A is true under GE-interpretation, then, by the definition of GE, there exist numbers z_1 and z_2 such that

$$B(a_1, a_2, \{\bar{z}_1\}(a_1), \{\{\bar{z}_2\}(a_1)\}(a_2))$$

is verifiable, where $z_1 \in N^{((0),(0))}$, $z_2 \in N^{(((0),(0)),(0))}$. Define z_3 by $\{\{z_2\}(a_1)\}(a_2)$

$\simeq \{z_3\}(a_1, a_2)$. Since z_2 belongs to $N^{(((0),(0)),(0))}$, $\{z_3\}(a_1, a_2)$ is defined for all a_1 and a_2, and so is a general recursive function. As $B(a_1, a_2, \{\bar{z}_1\}(a_1), \{\bar{z}_3\}(a_1, a_2))$ turns to be verifiable, A is general recursively ture.

The proof is similar for any prenex PR-formula, because for PR-formula $B(a \cdots)$, $B(a \cdots)^{GE}$ is equivalent to $B(a \cdots)$ intuitionistically.

5.2.3. PROPOSITION 5.5. *If* $\vdash_{\hat{Z}^I} A$, *then* A *is true under GE-interpretation. More precisely, if* A^{GE} *is* $\exists x^\tau \forall y^\sigma A'(x^\tau, y^\sigma)$, *there exists a number* $z \in N^\tau$ *depending effectively on the proof of* A *in* $\hat{Z}^I_d$ *such that* $A'(\bar{z}, b^\sigma)$ *is verifiable.*

PROOF. The proof is analogous to that of Theorem 1 in § 3.

The proposition is obvious for $d=1$ from Corollary to Proposition 5.3. We shall prove the Proposition for d, assuming that the Proposition holds for $d' <_0 d$. If $d=3.5^e \in 0$, it is obvious. Let $d=2^{d'}$. We have $\vdash_{\hat{Z}^I_d} A \Leftrightarrow \vdash_{\hat{Z}^{I(K)}_{d'}} A$ by Proposition 1.1. If A is axiom in $Z^I_{d'}$, Proposition holds. It suffices to prove Proposition for an immediate cosequence A of inference rule, assuming the Proposition for a premise. Moreover, it suffices to prove Proposition for a conclusion A of *K-rule* of the form $\forall x B(x)$, assuming Proposition for the premise $\forall x \mathrm{Pr}[\hat{Z}^I_d](b(x))$ (where $\ulcorner B(x) \urcorner \binom{\ulcorner x \urcorner}{N(x)}$ is b(x)), since other cases are easy.

$$\forall x \mathrm{Pr}[\hat{Z}^I_{d'}](b(x)) \quad \text{is} \quad \forall x \exists y \, \mathrm{Prf}[\hat{Z}^I_{d'}](b(x), y) .$$

Since $\mathrm{Prf}[\hat{Z}^I_{d'}](a, b)$ is an RE-formula, there exists, from the last part of 5.2.2., a number $e \in N^{((0),(0))}$ depending effectively on the proof of this formula such that $(x)\, Prf[Z^I_{d'}](b(x), \{e\}(x))$.

Let $B(a)^{GE}$ be $\exists x^\tau \forall y^\sigma C(x^\tau, y^\sigma, a)$. Then, by our assumption on d', there exists a partial recursive function $\psi_{d'}(y)$ defined for Gödel numbers of proofs in $\hat{Z}^I_{d'}$ such that $\psi_{d'}(\{e\}(x))$ is determined, for any x, effectively from the proof $\{e\}(x)$ of $B(\bar{x})$ and that $C(\overline{\psi_{d'}(\{e\}(x))}, b^\sigma, a)$ is verifiable. Here, $\psi_{d'}(\{e\}(x)) \in N^\tau$, and, if $\psi_{d'}(\{e\}(x)) \simeq \{e'\}(x)$, e' is determined effectively from $\psi_{d'}$ and $\{e\}$. Since $\{e'\}(x)$ is defined for all x and since $\{e'\}(x) \in N^\tau$, we have $e' \in N^{(\tau,(0))}$ and $C(\{\bar{e}'\}(a), b^\sigma, a)$ is verifiable. The GE-interpretation of A, i.e. of $\forall x B(x)$, is

$$\exists z^{(\tau,(0))} \forall x \forall y^\sigma C(\{z^{(\tau,(0))}\}(x), y^\sigma, x) .$$

It is clear that e' depends effectively on the proof of A in $\hat{Z}^I_d$. Proposition 5.5 has therefore, been proved.

§ 6. Completeness theorem of Feferman's type related to intuitionistic number theory.

Feferman showed the following completeness property of Z^C_d.

PROPOSITION 6.1. ([1] 5.13 Theorem (II))

For every classically true formula A of Z, there exists $d \in O$ such that

$|d| < \omega^{\omega^{\omega}}$ and $\vdash_{Z_d^C} \mathrm{A}$.

Analogous theorem holds for the general recursively true formula instead of classically true one. But to prove this theorem, we must mention about the following two theorems which also appeared in [1].

PROPOSITION 6.2. ([1] 4.1 Theorem)

There is a binary primitive recursive function E_0 such that for any $b \in O$ and any PR-formula with one free variable a B(a), *if* $\forall x B(x)$ *is true, then*

(i) $E_0(b, \ulcorner B \urcorner) \in O$

(ii) $b <_O E_0(b, \ulcorner B \urcorner)$ *and* $|E_0(b, \ulcorner B \urcorner)| = |b| + \omega + 1$

(iii) $\vdash_{Z^C_{E_0(b, \ulcorner B \urcorner)}} \forall x B(x)$.

PROPOSITION 6.3. ([1] 5.2 Theorem)

There exists a binary primitive recursive function E_1, such that for $b \in O$ and any PR-formula with two free variables a, b B(a, b), *if* $\forall x \exists y B(x, y)$ *is true, then*

(i) $E_1(b \ulcorner B \urcorner \in O$

(ii) $b <_O E_1(b, \ulcorner B \urcorner)$ *and* $|E_1(b, \ulcorner B \urcorner)| = |b| + \omega^2 + \omega + 1$

(iii) $\vdash_{Z^C_{E_1(b, \ulcorner B \urcorner)}} \forall x \exists y B(x, y)$

THEOREM 3. *Let* A *be a formula of Z of the form* $Q_1x_1Q_2x_2 \cdots Q_nx_nB(x_1, x_2, \cdots, x_n)$ *where* Q_ix_i *is* $\exists x_i$ *or* $\forall x_i$ *and* B *is a PR-formula. Then,* A *is general recursively true if and only if there is $d \in O$ such that $|d| = \omega^2 + \omega + 1$ and* $\vdash_{\hat{Z}_d^I} \mathrm{A}$.

PROOF. If $\vdash_{\hat{Z}_d^I} \mathrm{A}$, then A is true under GE-interpretation, and therefore is general recursively true from Propositions 5.4 and 5.5.

So it suffices to show the inverse. Now, for instance, let A be of the form $\forall x_1 \exists y_1 \forall x_2 \exists y_2 B(x_1, x_2, y_1, y_2)$. We can treat general case similarly. If A is general recursively true, there exist e_1, e_2 such that $B(a_1, a_2, \{e_1\}(a_1), \{e_2\}(a_1a_2))$ is verifiable. That is

$$T_1(e_1, a, y_1) \ \& \ T_2(e_2, a_1, a_2, y_2) \Rightarrow B(a_1, a_2, U(y_1), U(y_2)).$$

If U(y, z) is a formal expression of "$U(y) = z$",

a) $\forall x_1 \forall x_2 \forall y_1 \forall y_2 \forall z_1 \forall z_2$

$$\{T_1(\bar{e}_1, x_1, y_1) \ \& \ T_2(\bar{e}_2, x_1, x_2, y_2) \ \& \ U(y_1, z_1) \ \& \ U(y_2, z_2) \supset B(x_1, x_2, z_1, z_2)\}$$

is true, and since the formula in { } is a PR-formula, from Prop. 6.2, there is a $d_0 \in O$ such that $|d_0| = \omega + 1$ (setting $b = 0$) and $\vdash_{Z^C_{d_0}}$ formula a).

e_1, e_2 are Gödel numbers of general recursive functions, so that

$$\forall x_1 \exists y_1 \exists z_1 (T_1(\bar{e}_1, x_1, y_1,) \ \& \ U(y_1, z_1))$$

$$\& \ \forall x_1 \forall x_2 \exists y_2 \exists z_2 (T_2(\bar{e}_2, x_1, x_2, y_2) \ \& \ U(y_2, z_2))$$

are true. The conjunction of the above two formulas is of the form $\forall x \exists y C(x, y)$ where $C(x, y)$ is a PR-formula. Therefore from Prop. 6.3, if we set $d = E_1(d_0, \ulcorner C \urcorner)$, then $d_0 <_O d$, $|d| = |d_0| + \omega^2 + \omega + 1 = \omega^2 + \omega + 1$ and

$$\vdash_{Z_d^C} \forall x \exists y C(x, y).$$

Since $d_0 <_O d$, formula a) is also provbale in Z_d^C, i. e. $\vdash_{Z_d^C}$ formula a).

Considering $\circ$-translation of the formula a), we have

$$\vdash_{Z_d^I} \text{formula a), from Theorem 2.}$$

On the other hand, $\circ$-translation of $\forall x \exists y C(x, y)$ is $\forall x \neg \forall y \neg C(x, y)$; but from Axiom $\hat{I}$,

$$\vdash_{\hat{Z}^I} \forall x \neg \forall y \neg C(x, y) \supset \forall x \exists y \neg\neg C(x, y).$$

Hence we have

$$\vdash_{\hat{Z}_d^I} \forall x \exists y C(x, y).$$

Formula a) is equivalent to the following formula in $\hat{Z}_d^I$, and we have

$$\vdash_{\hat{Z}_d^I} \forall x_1 \exists y_1 \exists z_1 T_1(\bar{e}_1, x_1, y_1) \ \& \ U(y_1, z_1)$$

$$\& \ \forall x_1 \forall x_2 \exists y_2 \exists z_2 T_2(\bar{e}_2, x_1, x_2, y_2) \ \& \ U(y_2, z_2)$$

$$\supset \forall x_1 \exists y_1 \forall x_2 \exists y_2 B(x_1, x_2, y_1, y_2).$$

Since the antecedent formula of $\supset$ is $\forall x \exists y C(x, y)$, we have at last

$$\vdash_{\hat{Z}_d^I} \forall x_1 \exists y_1 \forall x_2 \exists y_2 B(x_1, x_2, y_1, y_2).$$

Appendix. No-counter example-interpretation.

A.1. Formal version of Proposition 5.5.

If $\exists x^\tau \forall y^\sigma A'(x^\tau, y^\sigma)$ is the GE-interpretation of A, and if e_d is a Gödel number of the partial recursive function ψ_d appeared in the proof of Proposition 5.5 and if A is provable by a proof in $\hat{Z}_d^I$ with Gödel number p then $A'(\{\bar{e}_d\}(\bar{p}), b^\sigma)$ is verifiable. This is the assertion of Proposition 5.5.

In this section we try to obtain some formal version of the proposition which will play a role in the so called no-counter example-interpretation.

A.1.1. Formula $N^\tau(a)$.

$$N^{(0)}(a) \text{ is } a = a.$$

If for types $\tau_0, \tau_1, \cdots, \tau_n$, formulas $N^{\tau_0}(a), N^{\tau_1}(a), \cdots, N^{\tau_n}(a)$ are already defined, then $N^{(\tau_0, \tau_1, \cdots, \tau_n)}(a)$ is the following formula:

$$\{\forall x_1, \forall x_2, \cdots, \forall x_n(N^{\tau_1}(x_1) \;\&\; \cdots \;\&\; N^{\tau_n}(x_n)$$
$$\supset \exists y T_n(a, x_1, \cdots, x_n y)\} \;\&$$
$$\forall x_1 \forall x_2, \cdots, \forall x_n \forall y \forall z\{N^{\tau_1}(x_1) \;\&\; \cdots \;\&\; N^{\tau_n}(x_n) \;\&$$
$$T_n(a, x_1, \cdots, x_n y) \;\&\; U(y, z) \supset N^{\tau_0}(z)\}\,.$$

A.1.2. In formalism,

(a) $\{a\}(b_1, \cdots, b_n) = c$ means $\exists x(T_n(a, b_1, b_2, \cdots, b_n, x) \;\&\; U(x, c))$,

(b) $\forall x^\tau A(x^\tau)$ means $\forall x(N^\tau(x) \supset A(x))$,

$\exists x^\tau A(x^\tau)$ means $\exists x(N^\tau(x) \;\&\; A(x))$.

By (a), (b) we can make a formula of Z correspond to each E-formula, for example $A(\{e\}(p))$ means $\exists x \exists y(A(x) \;\&\; T_1(e, p, y) \;\&\; U(y, x))$.

PROPOSITION 1. *If* $\exists x^\tau \forall y^\sigma A'(x^\tau y^\sigma)$ *is the* GE-*interpretation of the closure of* A, *and if* $A^*(x^\tau)$ *is the formula of Z corresponding to* $\forall y^\sigma A'(x^\tau y^\sigma)$ *by* (a), (b), *then the following* (c), (d) *hold.*

(c) $\vdash_{\hat{Z}_d^I} A$ (with proof p)

$$\Rightarrow \vdash_{\hat{Z}_d^I} A^*(\{\bar{e}_d\}(\bar{p})) \;\&\; N^\tau(\{\bar{e}_d\}(\bar{p}))\,.$$

In other words, if $\vdash_{\hat{Z}_d^I} A$, then GE-interpretation is also provable in $\hat{Z}_d^I$, and number z inProposition 5.5 is provably effective in $\hat{Z}_d^I$.

(d) The formula obtained by formalizing (c) is provable in Z^I. That is

$$\vdash_{\hat{Z}^I} \mathrm{Prf}\,[\hat{Z}_d^I](a, p) \supset \mathrm{Pr}\,[Z_d^I](\cdots)\,,$$

where $(\cdots)$ is Gödel-number of the following formula of Z :

$$A^*(\{\bar{e}_d\}(p)) \;\&\; N^\tau(\{\bar{e}_d\}(p)\,.$$

The outline of the proof.

In case $d=1$, we have the proposition by formalizing the proof of Proposition 5.2.

Firstly, the closure of GE-interpretation of the defining equation (i), (ii), etc. is provable in Z^I. To show this, we need the formal theory of partial recursive function of [2] Chapter XII. Secondly, *WJ*-inference for E-formula holds in Z^I.

That is, if $A \supset \forall x(N^\tau(x) \supset B(0, x))$, and $\forall x \forall y\{N^\tau(x) \supset A \;\&\; B(y, \{\bar{e}_0\}(y, x))$ $\supset B(y+1, x)\}$ are provable in Z^I, then $\forall x \forall y(N^\tau(x) \supset (A \supset B(y, x))$ is provable in Z^I: this is obvious.

(d) is obtained by formalizing the above. This is provable in Z^I. As in Proposition 5.5, suppose that $\vdash_{\hat{Z}_{d'}^I} A$ and that A is $\forall x B(x)$ and is conclusion of *K-rule* whose premise is $\forall x \mathrm{Pr}\,[\hat{Z}_{d'}^I](b(x))$. And suppose that Prop. 1 (c), (d) hold for d' and for the proof of this premise. By our assumption,

(e) $\qquad \vdash_{Z^I_{2d'}} \forall x\, \mathrm{Prf}\,[\hat{Z}^I_{d'}](b(x)\{\bar{e}\}(x))\ \&\ N^{((0)(0))}(\bar{e})$,

where e is determined effectively from the proof of

$$\forall x\, \mathrm{Pr}\,[\hat{Z}^I_{d'}](b(x)) \quad \text{in} \quad \hat{Z}^I_{2d'}.$$

And by our assumption of (c) for d',

(f) $\qquad Prf\,[\hat{Z}^I_{d'}](b(x), \{\bar{e}\}(x))$

$$\Rightarrow \vdash_{Z^I_{d'}} (B(\bar{x}))^*(\{\bar{e}_{d'}\}(\{\bar{e}\}(x)))$$

$$\&\ N^{\tau}(\{\bar{e}_{d'}\}(\{\bar{e}\}(x))).$$

If $\exists x^{\tau}\forall y^{\sigma}C(x^{\tau}, y^{\sigma}, a)$ is $B(a)^{GE}$, and if we put $\{e'\}(x) \simeq \{e_d\}(\{e\}(x))$, then

(g) $\qquad (x)\vdash_{\hat{Z}^I_{d'}} \forall y^{\sigma}C(\{\bar{e}'\}(\bar{x}), y, \bar{x})\ \&\ N^{\tau}(\{\bar{e}'\}(\bar{x}))$.

Since the formalization of (f) is provable in Z^I by (d) and since (e) holds, we have the formalization of (g) in $Z^I_{2d'}$.

Therefore, by K-rule, $\forall x \forall y^{\sigma}C(\{\bar{e}'\}(x), y^{\sigma}, x)$ and $\forall x N^{\tau}(\{\bar{e}'\}(x))$ are provable in $Z^I_{2d'}$. That is, $\vdash_{Z^I_{2d'}} A^*(\bar{e}')\ \&\ N^{(\tau,(0))}(\bar{e}')$.

(d) of Proposition A, 1 for $Z^I_{2d'}$ is very complicated but the above inference remains in the frame of Z^I.

Z^I_d, and even Z^G_d is not complete progression for true formula under GE-interpretation (not necessarily prenex formula). For, there is a formula C, of the form $\forall x \exists y \forall z[A(y, x) \vee \neg A(z, x)$ which is not general recursively true [**17**] Appendix 1, or [**6**] P. 123). Then, the GE-interpretation of $\neg C$ is true, but from the form of $\neg C$, $\neg C$ is not provable in $\hat{Z}^I_d$ (even in Z^G_d) for any $d \in O$. We do not know what is the recursive progression which is complete for any GE-interpretation true formula or for any recursively realizable formula.

A.2. No-counterexample-interpretation.

If A is a prenex formula of the form $\forall x_1 \exists y_1, \cdots, \forall x_n \exists y_n B(x_1, \cdots, x_n, y_1, \cdots, y_n)$ and if A^* is the prenex form of $\neg A$, the no-counterexample-interpretation is defined as Gödel interpretation of $\neg A^*$ [**6**] [**17**];

i. e. $\exists \varphi_1, \cdots, \exists \varphi_n \forall x \forall f_2, \cdots, \forall f_n B[x_1, f_2(\varphi_1), \cdots, f_n(\varphi_1, \cdots, \varphi_{n-1}), \varphi_1, \cdots, \varphi_n]$.

If A is classically true, then there exist recursive functionals $\varphi_1, \cdots, \varphi_n$ such that for every x and every recursive functions $f_2 f_3, \cdots, f_n$,

$$B[x_1 f_2(\varphi_1), \cdots, f_n(\varphi_1, \cdots, \varphi_{n-1}), \varphi_1, \cdots, \varphi_n]$$

is true [**6**] p. 124. Moreover, Kreisel pointed out that if A is provable in some system, φ_i's are not only recursive, but also belong to some subclass depending on the system considered. For example, as Kreisel showed in [**17**], if A is provable in Z^C, φ can be taken as ordinal recursive functionals of finite order. On the other hand every classically true formula A is provable in Z^G_d for some d, and the following proposition holds.

PROPOSITION A.2. *If a prenex formula* A *is provable in* Z_d^G, *then its no-counterexample-interpretation is true and the above* $\varphi_1, \cdots, \varphi_n$ *can be taken as effective operations and it is provable in* Z_d^I *that* $\varphi_1, \cdots, \varphi_n$ *are effective operations. That is, if the types of* $\varphi_1, \cdots, \varphi_n$ *are* $\tau_1, \cdots, \tau_n$ *respectively, and if* φ_i *is* $\{n_i\}$, *then* $\vdash_{Z_d^I} \mathrm{N}^{\tau_i}(\bar{n}_i)$.

PROOF. If $\forall x_1 \exists y_1, \cdots, \forall x_n \exists y_n B(x_1, \cdots, x_n, y_1, \cdots, y_n)$ is provable in Z_d^G, then $\circ$-translation

$$\forall x_1 \neg \forall y_1 \neg \cdots \neg \forall x_n \neg \forall y_n \neg B(x_1, \cdots, x_n, y_1, \cdots, y_n)$$

is provable in Z_d^I by Theorem 2. Also in Z_d^I, we have

$$\exists x_1 \forall y_1, \cdots, \exists x_n \forall y_n \neg B(x_1, \cdots, x_n, y_1, \cdots, y_n)$$
$$\supset \neg \forall x_1 \neg \forall y_1 \neg \cdots \neg \forall x_n \neg \forall y_n \neg B(x_1, \cdots, x_n, y_1, \cdots, y_n),$$

and so

$$\neg \exists x_1 \forall y_1, \cdots, \exists x_n \forall y_n \neg B(x_1, \cdots, x_n, y_1, \cdots, y_n)$$

is provable in Z_d^I. For instance, in case $n=2$,

$$\neg \exists x_1 \forall y_1 \exists x_2 \forall y_2 \neg B(x_1, x_2, y_1, y_2)$$

is provable in Z_d^I.

Its GE-interpretation is true by proposition 5.5. So there are numbers w_1, w_2, such that

$$B[x_1, \{z_2^\tau\}(\{\bar{w}\}(x, z_2^\tau)), \{\bar{w}_1\}(x_1 z_2^\tau), \{\bar{w}_2\}(x_1 z_2^\tau)]$$

where τ is ((0) (0)) is verifiable and $\vdash_{Z_d^I} \mathrm{N}^{\tau_1}(\bar{w}_1)$ and $\vdash_{Z_d^I} \mathrm{N}^{\tau_2}(\bar{w}_2)$ by proposition 1, where τ_1 is of the type ((0), (0), ((0) (0))) and τ_2 is of the same type as τ_1.

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