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On injective modules and flat modules

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§ 0. Introduction.

In this paper we will first establish a duality between injectivity and flatness for modules. That is: Let E be a faithfully injective module[1]. Then, for each module A, we have $W.\dim. A = Id\,\mathrm{Hom}(A, E)$ and further A is faithfully flat if and only if $\mathrm{Hom}(A, E)$ is faithfully injective. (Theorem 1.4). Moreover, when the ring is noetherian, we have $Id\,A = W.\dim\,\mathrm{Hom}(A, E)$ and A is faithfully injective if and only if $\mathrm{Hom}(A, E)$ is faithfully flat (Theorem 1.5). In this case we have also $Id\,A = Id\,A \otimes M$, where M is a faithfully flat module, and A is faithfully injective if and only if so is $A \otimes M$ (Theorem 1.3). In case of (semi-) local rings, from these results, we can prove that the self-injective dimension of the ring is equal to the weak dimension of the canonical injective module[1] of the ring.

Next, using above results, we will treat a problem: Is the tensor product of injective modules over a commutative ring again injective[2]? When the ring is an integral domain, we can easily say " Yes " (cf. [4]). If the ring is noetherian and every principal ideal is projective (or flat), the ring is a direct sum of a finite number of integral domains [5, Lemma 3], so in this case the answer is also affirmative. In case of noetherian rings, we will give a condition for an affirmative answer and its several equivalent conditions (Theorem 2.4).

§ 1. Duality.

Let R, S be two rings. We consider the situation $({}_RA, {}_RB_S, {}_SC)$, that is, A is a left R-module, B is a left R- right S- bimodule and C is a left S-module. Then we can define a homomorphism:

$$\tau : \mathrm{Hom}_R(A, B) \otimes_S C \rightarrow \mathrm{Hom}_R(A, B \otimes_S C)$$

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1) See [6] for the definition.
2) This problem was first proposed by N. Yoneda.

by setting $[\tau(\sum_i (f_i \otimes c_i))]a = \sum_i (f_i(a) \otimes c_i)$ for $f_i \in \mathrm{Hom}_R(A, B)$, $a \in A$ and $c_i \in C$.

LEMMA 1.1. *1) τ is an isomorphism, if A is R-projective and finitely generated. 2) τ is a monomorphism (resp. an isomorphism), if A is R-finitely generated (resp. R-finitely presented[3]) and C is S-flat.*

PROOF. If $A = R$, τ is obviously an isomorphism. Hence 1) follows from an easy direct sum argument.

Now, let A be finitely generated (resp. finitely presented). Then we have an R-free module F with a finite base (resp. R-free modules F, F' with finite bases) such that $F \to A \to 0$ (resp. $F' \to F \to A \to 0$) is exact. Being C S-flat, we have the following commuative diagram with exact rows:

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{Hom}_R(A, B) \otimes_S C & \longrightarrow & \mathrm{Hom}_R(F, B) \otimes_S C & \longrightarrow & \mathrm{Hom}_R(F', B) \otimes_S C \\ & \downarrow \tau_A & & \downarrow \tau_F & & \downarrow \tau_{F'} \\ 0 \longrightarrow & \mathrm{Hom}_R(A, B \otimes_S C) & \longrightarrow & \mathrm{Hom}_R(F, B \otimes_S C) & \longrightarrow & \mathrm{Hom}_R(F', B \otimes_S C) . \end{array}$$

Since τ_F is an isomorphism (resp. τ_F and $\tau_{F'}$ are isomorphisms) by 1), τ_A is a monomorphism (resp. an isomorphism) by Five Lemma.

COROLLARY 1.2. *If A has a projective resolution composed of finitely generated modules (e.g. if R is left noetherian and A is finitely generated) and C is S-flat, then we have*

$$\mathrm{Ext}_R^n(A, B) \otimes_S C \cong \mathrm{Ext}_R^n(A, B \otimes_S C) \qquad (n \geqq 0) .$$

PROOF. This follows immediately from Lemma 1.1 and [**2**, IV, Th. 7.2].

THEOREM 1.3. *Let R be left noetherian, M be a faithfully flat left S-module and A be a left R- right S- bimodule. Then we have*

$$Id_R\, A = Id_R(A \otimes_S M)$$

Further, A is faithfully injective if and only if so is $A \otimes_S M$.

PROOF. The first part follows directly from the above corollary and [**6**, Th. 2.1.].

For the second part, by Lemma 1.1 we have $\mathrm{Hom}_R(R/\mathfrak{m}, A) \otimes_S M \cong \mathrm{Hom}_R(R/\mathfrak{m}, A \otimes_S M)$ for each left maximal ideal $\mathfrak{m}$ of R. Therefore, M being faithfully flat, $\mathrm{Hom}_R(R/\mathfrak{m}, A) = 0$ if and only if $\mathrm{Hom}_R(R/\mathfrak{m}, A \otimes_S M) = 0$, which implies the assertion by [**6**, Th. 3.1].

THEOREM 1.4[4]. *Let E be a faithfully injective right S-module and A be a left R- right S-bimodule. Then we have*

(1) $$W.\dim_R A = (r)\, Id_R\, \mathrm{Hom}_S(A, E)$$

Moreover, A is faithfully flat if and only if $\mathrm{Hom}_S(A, E)$ *is faithfully*

3) An R-module A is called finitely presented if there exist R-free modules F, F' with finite bases such that $F' \to F \to A \to 0$ is exact.

4) Cf. [**6**, Prop. 3.6].

injective.

PROOF. By [2, VI, Prop. 5.1] we have $\mathrm{Ext}_R^n(X, \mathrm{Hom}_S(A, E)) \cong \mathrm{Hom}_S(\mathrm{Tor}_n^R(X, A), E)$ for each right R-module X. Thus $\mathrm{Ext}_R^n(X, \mathrm{Hom}_S(A, E))=0$ if and only if $\mathrm{Tor}_n^R(X, A)=0$ by [6, Th. 3.1]. This implies the first part of the theorem. The second part is obtained similarly as in the proof of Theorem 1.3, using the isomorphism [2, II, Prop. 5.2] $\mathrm{Hom}_R(R/\mathfrak{m}, \mathrm{Hom}_S(A, E)) \cong \mathrm{Hom}_S(R/\mathfrak{m}\otimes_R A, E)$ for each right maximal ideal $\mathfrak{m}$ of R.

Now, we will prove the dual of Theorem 1.4. That is:

THEOREM 1.5[5]. *Let R be left noetherian and E, A be same as in Theorem 1.4. Then we have*

$$(l)\, Id_R\, A=(r)\, W.\dim_R \mathrm{Hom}_S(A, E)$$

Moreover, A is faithfully injective if and only if $\mathrm{Hom}_S(A, E)$ *is faithfully flat.*

PROOF. The first part follows similarly as in Theorem 1.4, using the isomorphism $\mathrm{Tor}^R(\mathrm{Hom}_S(A, E), X) \cong \mathrm{Hom}_S(\mathrm{Ext}_R(X, A), E)$ [2, VI, Prop. 5.3], where X is any finitely generated left R-module. The second part follows also similarly by means of the following.

LEMMA 1.6. *Let $({}_RX, {}_RA_S, E_S)$ be the situation. If E is S-injective and X is R-finitely presented, then the homomorphism*

$$\sigma: \mathrm{Hom}_S(A, E)\otimes_R X \rightarrow \mathrm{Hom}_S(\mathrm{Hom}_R(X, A), E)$$

in [2, VI, §5] *is an isomorphism.*

PROOF. There exists an exact sequence $F' \rightarrow F \rightarrow X \rightarrow 0$ with F', F free R-modules with finite bases. Since E is injective, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\mathrm{Hom}_S(A, E)\otimes_R F' & \rightarrow & \mathrm{Hom}_S(A, E)\otimes_R F & \rightarrow & \mathrm{Hom}_S(A, E)\otimes_R X & \rightarrow & 0 \\
\downarrow{\scriptstyle \sigma_{F'}} & & \downarrow{\scriptstyle \sigma_F} & & \downarrow{\scriptstyle \sigma_X} & & \\
\mathrm{Hom}_S(\mathrm{Hom}_R(F', A), E) & \rightarrow & \mathrm{Hom}_S(\mathrm{Hom}_R(F, A), E) & \rightarrow & \mathrm{Hom}_S(\mathrm{Hom}_R(X, A), E) & \rightarrow & 0\,.
\end{array}$$

Thus, σ_X is an isomorphism, since σ_F and $\sigma_{F'}$ are isomorphisms by [2, VI, Prop. 5.2].

Now we will apply above Theorem 1.5 to the case of commutative noetherian (semi-) local rings.

THEOREM 1.7. *Let R be a commutative noetherian (semi-) local ring, $\mathfrak{m}$ be its maximal ideal (Jacobson radical) and $\tilde{R}$ be the completion of R. Then we have*

(1) *$\tilde{R}$ is faithfully flat as an R-module.*

5) After I obtained Theorem 1.4 and 1.5, I accepted an information from H. Yanagihara that he obtained our Theorem 1.5 in case $S=Z$ (the ring of integers) and further he informed to me that our Theorem 1.4 was obtained in case $S=Z$ by J. Lambek in [Canad. Math. Bull. 7 (1964)].

(2) $Id_R(R/\mathfrak{m})=gl.\dim R$.

(3) $Id_R R=W.\dim_R E_R(R/\mathfrak{m})$[6] $=Id_{\tilde{R}}\tilde{R}=Id_R\tilde{R}$.

PROOF. (1) This is a well known result. Here we give a new proof. By [7, Th. 3.7], $\tilde{R}$ is isomorphic to $\mathrm{Hom}_R(E_R(R/\mathfrak{m}), E_R(R/\mathfrak{m}))$ as an R-module. Hence, the result follows from Theorem 1.5 by putting $S=R$ and $A=E=E_R(R/\mathfrak{m})$, since $E_R(R/\mathfrak{m})$ is actually faithfully injective.

(2) By [7, Th. 3.4], $\mathrm{Hom}_R(R/\mathfrak{m}, E_R(R/\mathfrak{m}))\cong R/\mathfrak{m}$. Hence $Id_R R/\mathfrak{m}=W.\dim_R R/\mathfrak{m}$ by Theorem 1.5. $W.\dim_R R/\mathfrak{m}=hd_R R/\mathfrak{m}=\mathrm{gl.}\dim R$ is well known.

(3) Put $S=A=R$ and $E=E_R(R/\mathfrak{m})$ in Theorem 1.5 and we have $Id_R R=W.\dim_R E_R(R/\mathfrak{m})$. Also we have $Id_{\tilde{R}}\tilde{R}=W.\dim_{\tilde{R}} E_{\tilde{R}}(\tilde{R}/\tilde{\mathfrak{m}})$ by replacing R with $\tilde{R}$. Further, put $S=A=\tilde{R}$ and $E=E_{\tilde{R}}(\tilde{R}/\tilde{\mathfrak{m}})$ and we have $Id_R\tilde{R}=W.\dim_R E_{\tilde{R}}(\tilde{R}/\tilde{\mathfrak{m}})$. On the other hand we have $E_R(R/\mathfrak{m})=E_{\tilde{R}}(\tilde{R}/\tilde{\mathfrak{m}})$ by [7, Th. 3.6]. Therefore we have only to prove $W.\dim_R E_R(R/\mathfrak{m})=W.\dim_{\tilde{R}} E_R(R/\mathfrak{m})$. Since $\tilde{R}$ is faithfully flat, we can easily see that the sequence $0\to X_n\to\cdots\to X_0\to E_R(R/\mathfrak{m})\to 0$ of R-modules is exact if and only if the sequence $0\to X_n\otimes_R\tilde{R}\to\cdots\to X_0\otimes_R\tilde{R}\to E_R(R/\mathfrak{m})\otimes_R\tilde{R}\to 0$ of $\tilde{R}$-modules is exact, and that X is R-flat if and only if $X\otimes_R\tilde{R}$ is $\tilde{R}$-flat. Therefore we have $W.\dim_R E(R/\mathfrak{m})=W.\dim_{\tilde{R}}(E_R(R/\mathfrak{m})\otimes_R\tilde{R})$. On the other hand, it is easily seen that $E_R(R/\mathfrak{m})\otimes_R\tilde{R}$ is isomorphic to $E_R(R/\mathfrak{m})$ by the facts that $E_R(R/\mathfrak{m})$ is an $\mathfrak{m}$-primary module and $\tilde{R}$ is R-faithfully flat.

§ 2. Tensor product of injective modules.

E. Matlis proved in [7] that any injective module over a commutative noetherian ring R is a direct sum of indecomposable injective modules and that each indecomposable injective module is isomorphic to $E_R(R/\mathfrak{p})$ for some prime ideal $\mathfrak{p}$ of R. So, in commutative noetherian case, to examine the injectivity of tensor product of injective modules, we may restrict our attention to the modules $E_R(R/\mathfrak{p})\otimes_R E_R(R/\mathfrak{p}')$, where $\mathfrak{p}$ and $\mathfrak{p}'$ are prime ideals of R, since the tensor product commutes with direct sums [2, V, Prop. 9.2] and a direct sum of any family of injective modules over a noetherian ring is again injective [cf. 2, I, Ex. 8].

From now on, R will always denote a commutative noetherian ring. We begin with easy lemmas.

LEMMA 2.1. *Let $\mathfrak{p}$ and $\mathfrak{p}'$ be two prime ideals of R. Then $E_R(R/\mathfrak{p})\otimes_R E_R(R/\mathfrak{p}')=0$ if $\mathfrak{p}\neq\mathfrak{p}'$ or $\mathfrak{p}=\mathfrak{p}'$ contains a non zero divisor.*

PROOF. Assume that $\mathfrak{p}$ is not contained in $\mathfrak{p}'$. Let x and y be any non zero elements of $E_R(R/\mathfrak{p})$ and $E_R(R/\mathfrak{p}')$ respectively. Then, $\mathfrak{p}^n x=0$ for some

6) $E_R(A)$ denotes the injective envelope of A as an R-module. See [3] for the definition and [7] for fundamental properties.

positive integer n, since $0(x)$ (the order ideal of x) is a $\mathfrak{p}$-primary ideal [7, Lemma 3.2]. Since $\mathfrak{p}^n$ is not contained in $\mathfrak{p}'$, there exists an element r of $\mathfrak{p}^n$ which is not contained in $\mathfrak{p}'$. Then $0(r)$ is contained in $0(y)$, since $0(y)$ is $\mathfrak{p}'$-primary. Therefore we can define a homomorphism $f: Rr \to E_R(R/\mathfrak{p}')$ by setting $f(\lambda r) = \lambda y$ for $\lambda r \in Rr$. Since $E_R(R/\mathfrak{p}')$ is injective, there exists a homomorphism $g: R \to E_R(R/\mathfrak{p}')$ such that $f = gi$, where i is the inclusion map of Rr into R. Hence y is divisible by r in $E_R(R/\mathfrak{p}')$, that is, we have an element y' $(= g(1))$ of $E_R(R/\mathfrak{p}')$ such that $y = ry'$. Thus $x \otimes y = x \otimes ry' = xr \otimes y' = 0$, which implies the assertion.

In case that $\mathfrak{p} = \mathfrak{p}'$ contains a non zero divisor, the result follows quite similarly from the fact that an element of an injective module is divisible by any non zero divisor of R.

LEMMA 2.2. *Assume that the rank of any belonging prime ideal of zero is not greater than one. Then every tensor product of injective modules is again injective if and only if $E_R(R) \otimes_R E_R(R)$ is injective.*

PROOF. Let $E_R(R) \otimes_R E_R(R)$ be injective. By Lemma 2.1 we have only to prove that $E_R(R/\mathfrak{p}) \otimes_R E_R(R/\mathfrak{p})$ is injective for any prime ideal $\mathfrak{p}$ which contains no non zero divisor. But such a $\mathfrak{p}$ belongs to zero, since the rank of any belonging prime ideal of zero is not greater than one.

Hence $E_R(R/\mathfrak{p})$ is isomorphic to a direct summand of $E_R(R)$ by [7, Th. 2.3 and Prop. 3.1]. Therefore $E_R(R/\mathfrak{p}) \otimes E_R(R/\mathfrak{p})$ is isomorphic to a direct summand of $E_R(R) \otimes E_R(R)$ and hence injective.

LEMMA 2.3. *Let K be the full ring of quotients of R. Then we have $E_R(R) = E_R(K) = E_K(K)$.*

PROOF. Since K is an essential extension of R as an R-module, we have $E_R(R) = E_R(K)$. $E_R(K)$ is an essential extension of K as an R-module, and hence as a K-module. Further, any R-injective K-module is obviously K-injective. So we have $E_R(K) = E_K(K)$.

Now, we will proceed to prove a theorem which gives sufficient conditions for the injectivity of tensor product of injective modules.

THEOREM 2.4. *Let R be a commutative noetherian ring and K be its full ring of quotients. Then the following conditions are equivalent:*

(1) *$K = E_R(R)$ i.e. K is K-injective (or R-injective).*

(2) *$Id_K K < \infty$ (or $Id_R K < \infty$).*

(3) *0 is unmixed and its primary components are irreducible.*

(4) *$E_R(R)$ is R-flat (or $E_R(R) = E_K(K)$ is K-flat).*

(5) *$W.\dim_R E_R(R) < \infty$ (or $W.\dim_K E_K(K) < \infty$).*

(6) *$\mathrm{Hom}_R(E_{R'}(R), E_R(R))$ is R-injective.*

(7) *$\mathrm{Hom}_R(E_R(R), E_R(R))$ is isomorphic to $E_R(R)$.*

(8) *$E_R(R)$ is a finite K-module and $E_R(R) \otimes_R E_R(R)$ is isomorphic to $E_R(R)$.*

Moreover, in this case, a tensor product of any family of injective modules is again injective.

PROOF. The equivalence of (1), (2) and (3) is given in [**1**, Prop. 6.1]. The implications (1)$\Rightarrow$(4)$\Rightarrow$(5), (1)$\Rightarrow$(7)$\Rightarrow$(6) and (1)$\Rightarrow$(8) follow directly from the facts that K is R-flat, and $\mathrm{Hom}_R(K, K)$ and $K\otimes_R K$ are both naturally isomorphic to K. Now we will prove implications (5)$\Rightarrow$(2), (6)$\Rightarrow$(1) and (8)$\Rightarrow$(3), which complete the proof of the first part. The second part follows directly from (3) and (8) by Lemma 2.2.

(5)$\Rightarrow$(2): Let $W.\dim_R E_R(R)<\infty$. By Lemma 2.3 this is obviously equivalent to $W.\dim_K E_K(K)<\infty$. Since R is noetherian, K is semi-local and its maximal ideals are the prime ideals which are maximal in the set of the belonging prime ideals of zero in K. Therefore $E_K(K)$ contains the canonical injective K-module $E_K(K/\mathfrak{n})$, where $\mathfrak{n}$ is the Jacobson radical of K. Hence we have $W.\dim_K E_K(K/\mathfrak{n})<\infty$, which implies $Id_K K<\infty$ by Theorem 1.7, (3).

(6)$\Rightarrow$(1): Obviously the assumption is equivalent to the K-injectivity of $\mathrm{Hom}_K(E_K(K), E_K(K))$. Let $\mathfrak{m}$ be a maximal ideal of K. Then $\tilde{K}_{\mathfrak{m}}$, the completion of $K_{\mathfrak{m}}$, is isomorphic to $\mathrm{Hom}_K(E_K(K/\mathfrak{m}), E_K(K/\mathfrak{m}))$ [**7**, Th. 3.7] and hence is isomorphic to a direct summand of $\mathrm{Hom}_K(E_K(K), E_K(K))$. Therefore $\tilde{K}_{\mathfrak{m}}$ is K-injective for each maximal ideal $\mathfrak{m}$ of K, which implies that $\tilde{K}$ is K-injective. Thus K is self injective by Theorem 1.7, (3).

(8)$\Rightarrow$(3): Let $\mathfrak{p}$ be any belonging prime ideal of zero in R. Then $E_K(K/\mathfrak{p}K)$ is isomorphic to a direct summand of $E_K(K)=E_R(R)$. Therefore $E_K(K/\mathfrak{p}K)$ is K-finitely generated and hence $E_K(K/\mathfrak{p}K)_{\mathfrak{p}K}=E_{K_{\mathfrak{p}K}}(K_{\mathfrak{p}K}/\mathfrak{p}K_{\mathfrak{p}K})$ is $K_{\mathfrak{p}K}$-finitely generated. Thus $K_{\mathfrak{p}K}$ has the minimum condition by [**6**, Cor. 4.6], which implies that $\mathfrak{p}$ is minimal. The irreducibility of $\mathfrak{p}$-primary component follows from the comparison between the numbers of indecomposable components isomorphic to $E_R(R/\mathfrak{p})$ in both decompositions of $E_R(R)\otimes E_R(R)$ and $E_R(R)$.

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