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On Kronecker's limit formula in a totally imaginary quadratic field over a totally real algebraic number field

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Introduction

Let K be an algebraic number field of finite degree and $\zeta_K(s)$ be the Dedekind zeta-function of K. Then $\zeta_K(s)$ has an expansion of the form

$$\zeta_K(s)=A_{-1}/(s-1)+A_0+A_1(s-1)+\cdots.$$

Here A_{-1}, the residue of $\zeta_K(s)$ at $s=1$, was determined by Dirichlet and Dedekind for any algebraic number field K. However, little is known about the constant term A_0, in spite of its importance. As far as the author knows, A_0 has been investigated only in the cases where K is either a cyclotomic field or a quadratic field. The determination of A_0 for imaginary quadratic fields is known as "Kronecker's limit formula". The purpose of this paper is to consider this problem for any totally imaginary quadratic extension K of a totally real algebraic number field k. The main results are as follows. For any absolute ideal class $\mathfrak{K}$ in K, let $\zeta_K(s;\mathfrak{K})$ denote the zeta-function of the class $\mathfrak{K}$. Let n be the degree of k. We shall show that, in the expansion

$$\zeta_K(s;\mathfrak{K})=a_{-1}/(s-1)+a_0+a_1(s-1)+\cdots,$$

the constant a_0 can be expressed as a special value of $\log \Psi_k(z^{(1)},\cdots z^{(n)};\mathfrak{m},\mathfrak{n})$ with a certain analytic function Ψ_k defined on the product of n-copies of complex upper half-planes (Theorem 1.2). This function is a generalization of Dedekind's η-function. But this function cannot be a Hilbert's modular form. This fact was kindly mentioned to me by Professor Siegel. As an application we shall obtain a formula for the quotient of the class number of the absolute class field over K divided by the class number of K (Theorem 3).

Here the author wishes to express his hearty thanks to Professor C. L. Siegel.

Notation

We denote by $\boldsymbol{Z}$, $\boldsymbol{Q}$, $\boldsymbol{R}$ and $\boldsymbol{C}$, the ring of rational integers, the rational number field, the real number field, and the complex number field respectively. Let $\boldsymbol{C}^n$ be the product of n-copies of the complex number field. Its element will be denoted by $(Z^{(1)}, \cdots, Z^{(1)})$. For a totally real algebraic number field k we shall mean by $\lambda \gg 0$, $(\lambda \in k)$ that λ is totally positive.

§ 1. Reduction of the problem

Let k be a totally real algebraic number field of degree n and K be a totally imaginary quadratic extension of k. Let d_k (resp. d_K) be the absolute value of the discriminant of k (resp. of K), $\mathfrak{D}_{K/k}$ the relative different of the extension of K/k, and $\mathfrak{d}$ the different of the field k. Let $\mathfrak{o}$ and $\mathfrak{O}$ be the ring of algebraic integers in k and K respectively. We shall denote by $\mathfrak{a}, \mathfrak{b}, \mathfrak{m}, \mathfrak{n}, \cdots$, the ideals in k and by $\mathfrak{A}, \mathfrak{B}, \cdots$ the ideals in K.

Let $\mathfrak{K}$ be an absolute ideal class in K and

$$\zeta_K(s\,;\,\mathfrak{K}) = \sum_{\mathfrak{A} \in \mathfrak{K}} 1/N_K(\mathfrak{A})^s\,,$$

be the zeta function of the ideal class $\mathfrak{K}$, where the summation extends over all integral ideals in $\mathfrak{K}$. It is well known that the function $\zeta_K(s\,;\,\mathfrak{K})$ has the following property:

(A) The function $\zeta_K(s\,;\,\mathfrak{K})$, as a function of s, can be continued holomorphically to the whole s-plane except for $s=1$. At $s=1$, $\zeta_K(s\,;\,\mathfrak{K})$ has a simple pole with the residue equal to $a_{-1} = \dfrac{(2\pi)^n R_K}{w_K \sqrt{d_K}}$, where R_K denotes the regulator of the field K and w_K the number of roots of unity contained in K.

Therefore the function $\zeta_K(s\,;\,\mathfrak{K})$ has the expansion of the form at $s=1$

$$\zeta_K(s\,;\,\mathfrak{K}) = a_{-1}/(s-1) + a_0 + a_1(s-1) + \cdots\,, \tag{2}$$

where the constant a_{-1} depends only on K and not on $\mathfrak{K}$. Our main purpose is to obtain the constant term a_0 in terms of $\mathfrak{K}$.

Let $\mathfrak{A}$ be an integral ideal in K. Then $\mathfrak{A}$ is torsion free and of rank 2 regarded as an $\mathfrak{o}$-module. Hence there exist two integral ideals $\mathfrak{m}$, $\mathfrak{n}$ in k and two numbers $\Omega_1 \in \mathfrak{m}^{-1}\mathfrak{O}$ and $\Omega_2 \in \mathfrak{n}^{-1}\mathfrak{O}$ such that $\mathfrak{A} = \mathfrak{m}\Omega_1 + \mathfrak{n}\Omega_2$ is the direct sum of two $\mathfrak{o}$-modules $\mathfrak{m}\Omega_1$ and $\mathfrak{n}\Omega_2$. In particular, if we can choose $\mathfrak{m} = \mathfrak{n} = \mathfrak{o}$, $\mathfrak{A}$ is said to have a relative basis. It follows immediately:

(B) Let $\mathfrak{A}$ and $\mathfrak{B}$ be the equivalent integral ideals in K. If $\mathfrak{A} = \mathfrak{m}\Omega_1 + \mathfrak{n}\Omega_2$ for some integral ideals $\mathfrak{m}$, $\mathfrak{n}$ in k and $\Omega_1 \in \mathfrak{m}^{-1}\mathfrak{O}$, $\Omega_2 \in \mathfrak{n}^{-1}\mathfrak{O}$, then there exist $\Omega_1' \in \mathfrak{m}^{-1}\mathfrak{O}$ and $\Omega_2' \in \mathfrak{n}^{-1}\mathfrak{O}$ such that $\mathfrak{B} = \mathfrak{m}\Omega_1' + \mathfrak{n}\Omega_2'$.

For two integral ideals $\mathfrak{m}$ and $\mathfrak{n}$ in k, we shall denote by $\tilde{\Gamma}(\mathfrak{m}, \mathfrak{n})$ the group consisting of all matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that i) a, b, c, d lie in k and $ad-bc$ is a totally positive unit in k, ii) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ fixes the lattice $(\mathfrak{n}, \mathfrak{m})$ i.e., $(\mathfrak{n}, \mathfrak{m}) \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (\mathfrak{n}, \mathfrak{m})$. Clearly we obtain:

PROPOSITION 1. *Let $\mathfrak{A}$ be an integral ideal in K such that $\mathfrak{A} = \mathfrak{m}\Omega_1 + \mathfrak{n}\Omega_2$ for $\Omega_1 \in \mathfrak{m}^{-1}\mathfrak{O}$ and $\Omega_2 \in \mathfrak{n}^{-1}\mathfrak{O}$ where $\Omega_1^{-1}\Omega_2$ has totally positive imaginary part. Then the ideal $\mathfrak{A}$ can be written as $\mathfrak{m}\Omega_1' + \mathfrak{n}\Omega_2'$, for $\Omega_1' \in \mathfrak{m}^{-1}\mathfrak{O}$ and $\Omega_2' \in \mathfrak{n}^{-1}\mathfrak{O}$ where $\Omega_1'^{-1}\Omega_2'$ has totally positive imaginary part, if and only if there exists a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\tilde{\Gamma}(\mathfrak{m}, \mathfrak{n})$ such that*

$$\begin{pmatrix} \Omega_2' \\ \Omega_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \Omega_2 \\ \Omega_1 \end{pmatrix}.$$

Let $\mathfrak{A} = \mathfrak{m}\Omega_1 + \mathfrak{n}\Omega_2$ be as above. Then, by multiplying a suitable unit in k, we may always assume that the number $\Omega_1^{-1}\Omega_2$ has totally positive imaginary part. The ideals $\mathfrak{m}$ and $\mathfrak{n}$ are not uniquely determined by $\mathfrak{A}$, but they depend on the choice of Ω_1, Ω_2. We shall now consider this problem. Throughout this paper, we consider the ideal classes in k in a wide sense (i.e., two ideals $\mathfrak{m}$ and $\mathfrak{n}$ are equivalent if and only if $\mathfrak{m}^{-1}\mathfrak{n}$ is principal). Let Δ be a number in k such that $K = k(\sqrt{\Delta})$ and let $(\sqrt{\Delta})$ be the principal ideal in K generated by $\sqrt{\Delta}$. Considering prime ideal factors of $\mathfrak{D}_{K/k}^{-1}(\sqrt{\Delta})$ we can conclude that the ideal $\mathfrak{D}_{K/k}^{-1}(\sqrt{\Delta})$ is of the form $\mathfrak{a}\mathfrak{O}$ for some ideal $\mathfrak{a}$ in k. The ideal $\mathfrak{a}$ depends on the choice of $\sqrt{\Delta}$ such that $K = k(\sqrt{\Delta})$, but the ideal class of k containing $\mathfrak{a}$ does not depend on the choice of $\sqrt{\Delta}$. We shall denote by $\mathfrak{k}_1$ the ideal class containing $\mathfrak{a}$.

PROPOSITION 2. *Let $\mathfrak{A}$ be an integral ideal in K. Then the relative norm $N_{K/k}(\mathfrak{A})$ lies in the ideal class of the form $\mathfrak{a}\mathfrak{k}_1$, $\mathfrak{a}$ being an integral ideal in k, if and only if there exist $\Omega_1 \in \mathfrak{a}^{-1}\mathfrak{O}$ and $\Omega_2 \in \mathfrak{O}$ such that $\mathfrak{A} = \mathfrak{a}\Omega_1 + \mathfrak{o}\Omega_2$. In particular, an integral ideal $\mathfrak{A}$ has a relative basis if and only if its relative norm lies in $\mathfrak{k}_1$.*

For a proof of this proposition see C. Chevalley [1].

Let $\mathfrak{a}_1, \cdots, \mathfrak{a}_l$ be a complete set of representatives of the ideal classes in k such that each ideal class $\mathfrak{a}_i\mathfrak{k}_1$ contains the relative norm of an ideal in K. We may assume that the ideals $\mathfrak{a}_1, \cdots, \mathfrak{a}_l$ be integral. Then each integral ideal $\mathfrak{A}$ in K can be written as $\mathfrak{a}_i\Omega_1 + \mathfrak{o}\Omega_2$ for some i, $\Omega_1 \in \mathfrak{a}_i^{-1}\mathfrak{O}$ and $\Omega_2 \in \mathfrak{O}$. For each absolute ideal class $\mathfrak{K}$ in K, we choose once for all a representative $\mathfrak{L}_\mathfrak{k}$ such that $\mathfrak{L}_\mathfrak{k}$ is integral and belongs to $\mathfrak{K}^{-1}$. Let e be the index of the unit group of k in the unit group of K. We call two pairs of integers $\{\mu, \nu\}$ and $\{\mu', \nu'\}$ in k are associated with respect to the unit group of k, if there exists a unit η in k such that $\mu' = \mu\eta$, $\nu' = \nu\eta$. Let $\mathfrak{L}_\mathfrak{k}$ be an ideal such that

$\mathfrak{L}_{\mathfrak{R}}=\mathfrak{a}_{\iota}\Omega_1+\mathfrak{d}\Omega$ for $\Omega_1\in\mathfrak{a}_{\iota}^{-1}\mathfrak{D}$, $\Omega_2\in\mathfrak{D}$ where $\Omega_1^{-1}\Omega_2$ has totally positive imaginary part. When $\mathfrak{A}$ runs over all integral ideals in $\mathfrak{R}$ the ideal $\mathfrak{A}\mathfrak{L}_{\mathfrak{R}}$ runs over all principal ideals (λ) such that $\lambda\equiv 0$ mod. $\mathfrak{L}_{\mathfrak{t}}$. Therefore we have

$$\begin{aligned}\zeta_K(s:\mathfrak{R})&=\frac{1}{e}N_K(\mathfrak{L}_{\mathfrak{R}})^s\sum_{\{\mu,\nu\}}{}'\frac{1}{N_K(\mu\Omega_1+\nu\Omega_2)^s}\\ &=\frac{1}{e}N_K(\mathfrak{L}_{\mathfrak{R}})^s\sum_{\substack{\{\mu,\nu\}\\ \mu\in\mathfrak{a}_{\iota},\nu\in\mathfrak{o}}}{}' N_k(A\mu^2+2B\mu\nu+C\nu^2)^{-s}.\end{aligned}\tag{3}$$

Here the summation extends over all pairs $\{\mu,\nu\}\neq\{0,0\}$ which are not associated one another with respect to the unit group of k; the numbers A, B and C are given by $A=\Omega_1\Omega_1^{\tau}$, $2B=\Omega_1\Omega_2^{\tau}+\Omega_1\Omega_2$ and $C=\Omega_2\Omega_1^{\tau}$ and the quadratic form $Q(u,v)=Au^2+2Buv+Cv^2$ is totally positive definite, where τ denotes the generator of the galois group of K/k.

§ 2. Limit formula

In this section we shall consider a generalization of the Kronecker's limit formula for the zeta-function of a totally positive definite quadratic form.

Let k be a totally real algebraic number field of degree n and let $\sigma_1(=1)$, $\cdots,\sigma_n$ be the n distinct isomorphisms of k into $\boldsymbol{R}$. Then such σ_ι can be extended to an isomorphism of $\boldsymbol{R}$ into $k\otimes_{Q}\boldsymbol{R}$ which we shall denote again by σ_ι. For each λ in $\boldsymbol{R}$ we put $\lambda^{(p)}=\lambda^{\sigma_p}$ $(1\leqq p\leqq n)$, $N(\lambda)=\prod_{p=1}^{n}\lambda^{(p)}$ and $S(\lambda)=\sum_{p=1}^{n}\lambda^{(p)}$. If λ lies in k, these notations coincide with usual ones and we shall write N_k, S_k instead of N and S. We call a quadratic form $Q(u,v)=Au^2+2Buv+Cv^2$ with coefficients in $\boldsymbol{R}$ totally positive definite if $Q^{(p)}(u,v)=A^{(p)}u^2+2B^{(p)}uv+C^{(p)}v^2$ are positive definite form for all $p=1,\cdots,n$.

Let $Q(u,v)=Au^2+2Buv+Cv^2$ be a totally positive definite quadratic form with coefficients in $\boldsymbol{R}$. For two integral ideals $\mathfrak{m}$ and $\mathfrak{n}$ in k we define

$$Z(s;\mathfrak{m},\mathfrak{n};Q)=\sum_{\{\mu,\nu\}}N(A\mu^2+2B\mu\nu+C\nu^2)^{-s}\quad\text{for}\quad \mathrm{Re}(s)>1,\tag{4}$$

where the summation extends over all non-associated pairs $\{\mu,\nu\}\neq\{0,0\}$ $(\mu\in\mathfrak{m},\ \nu\in\mathfrak{n})$ with respect to the unit group of k and $N(A\mu^2+2B\mu\nu+C\nu^2)$ $=\prod_{p=1}^{n}(A^{(p)}\mu^{(p)2}+2B^{(p)}\mu^{(p)}\nu^{(p)}+C^{(p)}\nu^{(p)2})$. For this function we have:

Proposition 3. *The series* $Z(s;\mathfrak{m},\mathfrak{n};Q)$ *converges absolutely for* $\mathrm{Re}(s)>1$, *uniformly for* $\mathrm{Re}(s)\geqq 1+\delta(\delta>0)$ *and hence* $Z(s;\mathfrak{m},\mathfrak{n};Q)$ *is a holomorphic function of* s *in* $\mathrm{Re}(s)>1$.

Proof. Since $Q(u,v)$ is totally positive definite, there exist positive real numbers $\lambda_1,\cdots,\lambda_n$ such that $Q^{(p)}(u,v)=\lambda_p(u^2+u^2)$ $(1\leqq p\leqq n)$. Put $s=\sigma+$ it.

The series (4) is majorized by $c_1 \sum'_{\{\mu,\nu\}} N_k(\mu^2+\nu^2)^{-\sigma} = c_1 \sum'_{(\mu)} \sum_{\nu\in\mathfrak{n}} N_k(\mu^2+\nu^2)^{-\sigma}$ where c_1 depends only on Q and the summation $\sum'_{(\mu)}$ extends over all principal ideals $(\mu)\neq 0$ with $\mu\in\mathfrak{m}$. For every μ in $\mathfrak{m}$, we have

$$\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\frac{dt^{(1)}\cdots dt^{(n)}}{(\mu^{(1)2}+t^{(1)})^{\sigma}\cdots(\mu^{(n)2}+t^{(n)2})^{\sigma}}$$

$$=\frac{1}{|N_k(\mu)|^{2\sigma-1}}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\cdots\frac{dt^{(1)}\cdots dt^{(n)}}{(t^{(1)2}+1)\cdots(t^{(n)2}+1)}\,.$$

Thus, for $\mu\neq 0$ and $\sigma>\frac{1}{2}$, the series $\sum_{\mu\in\mathfrak{n}} N_k(\mu^2+\nu^2)^{-\sigma}$ converges absolutely and (4) is majorized by

$$\text{const.}\sum_{\substack{(\mu)\\ \mu\in\mathfrak{m}}}'\frac{1}{|N_k(\mu)|^{2\sigma-1}}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\frac{dt^{(1)}\cdots dt^{(n)}}{(t^{(1)2}+1)^{\sigma}\cdots(t^{(n)2}+1)^{\sigma}}$$

for $\sigma>1$.
This completes the proof.

For our later use we need a few lemmas.

LEMMA 1. (*Poisson's summation formula*) *Let $f(x_1,\cdots,x_n)$ be a continuous function on $\boldsymbol{R}^n$ such that the series $\sum_{m_1,\cdots,m_n} f(x_1+m_1,\cdots,x_n+m_n)$ converges uniformly for $0\leqq x_1\leqq 1,\cdots,0\leqq x_n\leqq 1$. Then we have*

$$(5)\qquad \sum_{m_1,\cdots,m_n=-\infty}^{+\infty} f(x_1+m_1,\cdots,mx_n+m_n)$$

$$=\sum_{k_1,\cdots,k_n=-\infty}^{+\infty} e^{-2\pi i(k_1x_1+\cdots+k_nx_n)}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} f(t_1,\cdots,t_n)e^{2\pi i(k_1t_1+\cdots+k_nt_n)}dt_1\cdots dt_n\,.$$

A proof of this lemma for $n=1$ is in Siegel [5] and the general case is proved by induction.

The following lemma is well known.

LEMMA 2. *Let $\alpha_1,\cdots,\alpha_n$ be a $\boldsymbol{Z}$-basis of the ideal $\mathfrak{m}$ in k. If we put*

$$\begin{pmatrix}\alpha_1^{(1)},\cdots,\alpha_n^{(1)}\\ \alpha_1^{(n)},\cdots,\alpha_n^{(n)}\end{pmatrix}^{-1}=\begin{pmatrix}A_1^{(1)},\cdots,A_1^{(n)}\\ A_n^{(1)},\cdots,A_n^{(n)}\end{pmatrix}$$

then $(A_1^{(1)},\cdots,A_n^{(1)})$ is a $\boldsymbol{Z}$-basis for the ideal $\mathfrak{m}^{-1}\mathfrak{D}^{-1}$.

LEMMA 3. *Let $\omega\neq 0$ be a real number and C a compact set in the domain of complex s-plane $\operatorname{Re}s\geqq\delta$ $(\delta>0)$. Put $\varepsilon=\operatorname{sgn}\omega$ and let Γ^{ε} be the contour in the complex ζ-plane composed of the circle $\zeta=\varepsilon i+\frac{1}{2}e^{i\varphi}$ $(0\leqq\varphi\leqq 2\pi)$ and the half line $\zeta=\varepsilon i\eta$ $\left(\frac{3}{2}\leqq\eta\leqq\infty\right)$. Then for any s_1 and s_2 in C, we have*

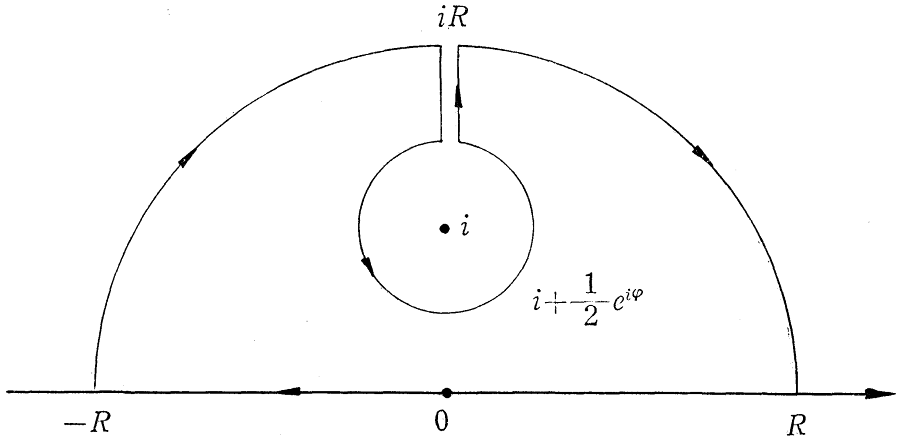
$$(6)\qquad \int_{-\infty}^{+\infty}\frac{e^{2\pi i\omega\xi}}{(\xi+i)^{s_1}\cdot(\xi-i)^{s_2}}d\xi=\int_{\Gamma^{\varepsilon}}\frac{e^{2\pi i\omega\zeta}}{(\zeta+i)^{s_1}\cdot(\zeta-i)^{s_2}}d\zeta\,,$$

and

$$(7)\qquad \left|\int_{-\infty}^{+\infty}\frac{e^{2\pi i\omega\xi}}{(\xi+i)^{s_1}\cdot(\xi-i)^{s_2}}d\xi\right|\leqq ce^{-\pi|\omega|}\int_0^{\infty}\left(\eta+\frac{1}{2}\right)^{-2\delta}e^{2\pi|\omega|\eta}d\eta<+\infty\,,$$

where the constant c depends only on C.

PROOF. When $\omega>0$, considering integrals on contour in the figure (1) and tending $R\to\infty$, we can show (6) and (7). When $\omega<0$, considering the reflection of the figure (1) with respect to the real axis, we can show them similarly.



(1)

Put $\mathscr{H}_n=\mathscr{H}\times\cdots\times\mathscr{H}=\{z=(z^{(1)},\cdots,z^{(n)})\,;\,z^{(p)}\in\boldsymbol{C},\,\mathrm{Im}\,z^{(p)}>0,\,1\leqq p\leqq n\}$. By our assumptions on $Q(u,v)$, A and $D_Q=AC-B^2$ are totally positive real numbers. Put $x^{(p)}=B^{(p)}/A^{(p)}$, $y^{(p)}=\sqrt{D^{(p)}}/A^{(p)}$ and $z^{(p)}=x^{(p)}+iy^{(p)}$. Then the point $z=(z^{(1)},\cdots,z^{(n)})$ lies in $\mathscr{H}_n$ and $A^{(p)}u^2+2B^{(p)}uv+C^{(p)}v^2=D^{(p)}y^{(p)-1}(u+vz^{(p)})(u+v\bar{z}^{(p)})$. Therefore we get

$$\begin{aligned}Z(s\,;\,\mathfrak{m},\mathfrak{n}\,;\,Q)&=\frac{N(y)^s}{\sqrt{N(D_Q)^s}}\sum_{\{\mu,\nu\}}{}'|N(\mu+\nu z)|^{-2s}\\&=\frac{N(y)^s}{\sqrt{N(D_Q)^s}}\sum_{\substack{(\mu)\\ \mu\in\mathfrak{m}}}\frac{1}{|N_k(\mu)|^{2s}}+\frac{N(y)^s}{\sqrt{N(D_Q)^s}}\times\sum_{(\mu)}\sum_{\substack{\nu\in\mathfrak{n}\\ \nu\neq 0}}\frac{1}{|N(\mu+\nu z)|^{2s}}\,.\end{aligned}$$

Thus

$$(8)\qquad \begin{aligned}Z(z\,;\,\mathfrak{m},\mathfrak{n}\,;\,Q)=&\frac{N(y)^s}{\sqrt{N(D_Q)^s}}N_k(\mathfrak{m})^{-2s}\zeta_k(s\,;\,\mathfrak{k}(\mathfrak{m}^{-1}))\\&+\frac{N(y)^s}{\sqrt{N(D_Q)^s}}\sum_{\substack{(\nu)\\ \nu\in\mathfrak{n}}}{}'\sum_{\mu\in\mathfrak{m}}\frac{1}{|N(\mu+\nu z)|^{2s}}\,,\end{aligned}$$

where the summation $\sum_{(\lambda)}$ (resp., $\sum_{\lambda}{}'$) extends over all principal ideal (λ) (resp., non-zero λ), and $\mathfrak{k}(\mathfrak{a})$ means the ideal class in k which contains $\mathfrak{a}$. In (8) the function $\zeta_k(2s;\mathfrak{k}(\mathfrak{a}^{-1}))$ can be continued analytically to the whole s-plane holomorphically in $\mathrm{Re}(s)>\frac{1}{2}$. Therefore to obtain the analytic continuation of $Z(s;\mathfrak{m},\mathfrak{n};Q)$, we have only to investigate the second term of (8).

Let $\alpha_1,\cdots,\alpha_n$ be a $\boldsymbol{Z}$-basis for the ideal $\mathfrak{m}$. Let $z=(z^{(1)},\cdots,z^{(n)})$ be a point in $\mathcal{H}_n$ and let $x^{(p)}=x_1\alpha_1^{(p)}+\cdots+x_n\alpha_n^{(p)}$, where $z^{(p)}=x^{(p)}+iy^{(p)}$ $(1\leqq p\leqq n)$. For every $Z=(z^{(1)},\cdots,z^{(n)})$ in $\mathcal{H}_n$, define

$$f(x_1,\cdots,x_n)=\prod_{p=1}|x_1\alpha_1^{(p)}+\cdots+x_n\alpha_n^{(p)}+iy^{(p)}|^{-2s}\qquad\text{for } \mathrm{Re}\,s>1\,.$$

As $y^{(p)}>0$ for $p=1,\cdots,n$, $f(x_1,\cdots,x_n)$ is continuous in $\boldsymbol{R}^n$. Let $\mu=m_1\alpha_1+\cdots+m_n\alpha_n$ $(m_\iota\in\boldsymbol{Z})$ be an integer in $\mathfrak{m}$. Then the series

$$\sum_{m_1,\cdots,m_n}f(x_1+m_1,\cdots,x_n+m_n)=\sum_{\mu\in\mathfrak{m}}|N_k(\mu+z)|^{-2s}\,,$$

converges absolutely for $\mathrm{Re}(s)>1$ and uniformly for $\mathrm{Re}(s)\geqq 1+\delta(\delta>0)$ when $0\leqq x_1\leqq 1,\cdots,0\leqq x_n\leqq 1$. Hence we can apply lemma 1 to the function $f(x_1,\cdots,x_n)$ and we get

$$\sum_{\mu\in\mathfrak{m}}|N(\mu+z)|^{-2s}=\sum_{k_1,\cdots,k_n}e^{-2\pi i(k_1x_1+\cdots+k_nx_n)} \tag{9}$$

$$\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\frac{e^{2\pi i(k_1t_1+\cdots+k_nt_n)}}{|N(t_1\alpha_1+\cdots+t_n\alpha_n+iy|^{2s}}dt_1\cdots dt_n\,,$$

whenever the right hand side converges. Here we denote

$$N(t_1\alpha_1+\cdots+t_n\alpha_n+iy)=\prod_{p=1}^{n}(t_1\alpha_1^{(p)}+\cdots+t_n\alpha_n^{(p)}+iy^{(p)})\,.$$

Now we shall prove the absolute convergence of the right hand side of (9) in $\mathrm{Re}(s)>\frac{1}{2}$. Let $A_1^{(1)},\cdots,A_n^{(1)}$ be a $\boldsymbol{Z}$-basis of the ideal $\mathfrak{m}^{-1}\mathfrak{D}^{-1}$ given by lemma 2. Consider the linear transformation $(t_1,\cdots,t_n)\to(\zeta^{(1)},\cdots,\zeta^{(n)})$ defined by

$$\begin{pmatrix}y^{(1)}\zeta^{(1)}\\ \vdots\\ y^{(n)}\zeta^{(n)}\end{pmatrix}=\begin{pmatrix}\alpha_1^{(1)},&\cdots,&\alpha_n^{(1)}\\ \vdots& &\vdots\\ \alpha_1^{(n)},&\cdots,&\alpha_n^{(n)}\end{pmatrix}\begin{pmatrix}t_1\\ \vdots\\ t_n\end{pmatrix}.$$

Then we have

$$\partial(\zeta^{(1)},\cdots,\zeta^{(n)})/\partial(t_1,\cdots,t_n)=N(y)^{-1}\sqrt{d_k}\,N_k(\mathfrak{m})\,,$$

and

$$\sum_{\iota=1}^{n}k_\iota t_\iota=\sum_{p=1}^{n}\beta^{(p)}y^{(p)}\zeta^{(p)}\qquad\text{for } \beta^{(p)}=k_1A_1^{(p)}+\cdots+k_nA_n^{(p)}\ (1\leqq p\leqq n)$$

$$\sum_{\iota=1}^{n}k_\iota x_\iota=\sum_{p=1}^{n}\beta^{(p)}x^{(p)}\qquad\text{for } x^{(p)}=x_1\alpha_1^{(p)}+\cdots+x_n\alpha_n^{(p)}\ (1\leqq p\leqq n)\,.$$

Therefore the right hand side of (9) is transformed as follows;

(10) $$N(y)^{1-2s}\frac{1}{\sqrt{d_k}N_k(\mathfrak{m})}\sum_{\beta\in\mathfrak{m}^{-1}\mathfrak{D}^{-1}}e^{-2\pi iS(\beta x)}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\frac{e^{2\pi iS(y\beta\zeta)}d\zeta^{(1)}\cdots d\zeta^{(n)}}{(\zeta^{(1)2}+1)^s\cdots(\zeta^{(n)2}+1)^s}.$$

Let C denote a compact set contained in the complex s-plane with $\mathrm{Re}(s)\geqq\delta$ $(\delta<0)$. Let $\beta\neq 0$ be a number in $\mathfrak{m}^{-1}\mathfrak{D}^{-1}$ and $\varepsilon=(\varepsilon^{(1)},\cdots,\varepsilon^{(n)})$ be the set of signatures of $(\beta^{(1)},\cdots,\beta^{(n)})$ defined by $\varepsilon^{(p)}=\mathrm{sgn}\,\beta^{(p)}$ for $p=1,\cdots,n$. Then, applying lemma 3 to the integral (10), we have, for any $\beta\neq 0$ in $\mathfrak{m}^{-1}\mathfrak{D}^{-1}$;

(11) $$\left|\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\frac{e^{2\pi iS(\beta y\zeta)}}{(\zeta^{(1)2}+1)^s\cdots(\zeta^{(n)2}+1)^s}d\zeta^{(1)}\cdots d\zeta^{(n)}\right|$$

$$\leqq c_1e^{-\pi(|\beta^{(1)}|y^{(1)}+\cdots+|\beta^{(n)}|y^{(n)})}\int_0^{+\infty}\cdots\int_0^{\infty}\left(\eta^{(1)}+\frac{1}{2}\right)^{-2\delta}\cdots\left(\eta^{(n)}+\frac{1}{2}\right)^{-2\delta}$$

$$\times e^{-2\pi(|\beta^{(1)}|y^{(1)}\eta^{(1)}+\cdots+|\beta^{(n)}|y^{(n)}\eta^{(n)})}d\eta^{(1)}\cdots d\eta^{(n)}$$

$$\leqq c_2e^{-\pi(|\beta^{(1)}|y^{(1)}+\cdots+|\beta^{(n)}|y^{(n)})}\frac{1}{N(y)^{1-2\delta}|N_k(\beta)|^{1-2\delta}}$$

$$\leqq c_3\frac{1}{N(y)^{1-2\delta}}e^{-\pi(|\beta^{(1)}|y^{(1)}+\cdots+|\beta^{(n)}|y^{(n)})},$$

where the constants C_1, C_2 and C_3 depend only on C. This tells us that each term of (10) with $\beta\neq 0$ is an entire function of s. On the other hand the series

$$\sum_{\beta}{}' e^{-\pi(|\beta^{(1)}|y^{(1)}+\cdots+|\beta^{(n)}|y^{(n)})},$$

converges absolutely, where the summation extends over all $\beta\neq 0$ in $\mathfrak{m}^{-1}\mathfrak{D}^{-1}$. Therefore the summation over all $\beta\neq 0$ in (10) converges and defines an entire function of s in the whole s-plane. Next, we consider the term $\beta=0$ in (10). We have

(12) $$\int_{-\infty}^{+\infty}\frac{d\zeta}{(\zeta^2+1)^s}=B\left(\frac{1}{2},s-\frac{1}{2}\right)=\frac{\pi^{\frac{1}{2}}\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)},$$

where $B(a,b)$ denotes the Bessel function. Therefore the series (10) converges absolutely for $\mathrm{Re}(s)>\frac{1}{2}$, uniformly for $\mathrm{Re}(s)\geqq\frac{1}{2}+\delta(\delta>0)$ and defines a holomorphic function of s in the domain $\mathrm{Re}\,s>\frac{1}{2}$. Thus, in view of (6), we get;

(13) $$\sum_{\mu\in\mathfrak{m}}|N(\mu+z)|^{-2s}=N(y)^{1-2s}\frac{1}{\sqrt{d_k}N_k(\mathfrak{m})}\frac{\pi^{\frac{n}{2}}\Gamma\left(s-\frac{1}{2}\right)^n}{\Gamma(s)^n}$$

$$+\frac{N(y)^{1-2s}}{\sqrt{d_k}N_k(\mathfrak{m})}\sum_{\substack{\beta\in\mathfrak{m}^{-1}\mathfrak{D}^{-1}\\ \beta\neq 0}}{}' e^{-2\pi iS(x\beta)}\int_{\Gamma\varepsilon^{(1)}}\cdots\int_{\Gamma\varepsilon^{(n)}}\frac{e^{2\pi iS(\beta y\zeta)}d\zeta^{(1)}\cdots d\zeta^{(n)}}{(\zeta^{(1)2}+1)^s\cdots(\zeta^{(n)2}+1)^s}$$

where $\varepsilon=(\varepsilon^{(1)},\cdots,\varepsilon^{(n)})$ denotes the set of signatures of $(\beta^{(1)},\cdots,\beta^{(n)})$. Substituting (13) in (8), we have

$$(14)\qquad Z(s;\mathfrak{m},\mathfrak{n};Q)=\frac{N(y)^{1-s}}{\sqrt{N(D_Q)^s}}N_k(\mathfrak{m})^{-2s}\zeta_k(2s;\mathfrak{k}(\mathfrak{m}^{-1}))$$

$$+\frac{N(y)^{1-s}}{\sqrt{N(D_Q)^s}\sqrt{d_k}N_k(\mathfrak{m})N_k(\mathfrak{n})^{2s-1}}\cdot\frac{\pi^{\frac{n}{2}}\Gamma\left(s-\frac{1}{2}\right)^n}{\Gamma(s)^n}\zeta_k(2s-1;\mathfrak{k}(\mathfrak{n}^{-1}))$$

$$+\frac{N(y)^{1-s}}{\sqrt{N(D_Q)^s}\sqrt{d_k}N_k(\mathfrak{m})}\sum_{\substack{(\nu)\\ \nu\gg0\\ \nu\in\mathfrak{n}}}\frac{1}{N_k(\nu)^{2s-1}}\sum_{\substack{\beta\in\mathfrak{m}^{-1}\mathfrak{D}^{-1}\\ \beta\neq0}}' e^{-2\pi iS(\nu\beta x)}$$

$$\times\int_{\Gamma\varepsilon^{(1)}}\cdots\int_{\Gamma\varepsilon^{(n)}}\frac{e^{2\pi iS(\nu\beta y\zeta)}d\zeta^{(1)}\cdots d\zeta^{(n)}}{(\zeta^{(1)2}+1)^s\cdots(\zeta^{(n)2}+1)^s},$$

where the summation $\sum_{(\nu)}'$ extends over all principal ideals (ν) such that ν is totally positive integers in $\mathfrak{n}$. Clearly the first term of (14) is continued analytically to the whole s-plane, holomorphically in $\mathrm{Re}\,s>\frac{1}{2}$. By a similar way as in (11), we can show that the third term of (14) is an entire function of s in the whole s-plane. Now the function $\zeta_k(s;\mathfrak{k}(\mathfrak{n}^{-1}))$ is continued analytically to the whole s-plane such that $\zeta_k(s;\mathfrak{k}(\mathfrak{n}^{-1}))-\kappa/(s-1)$ is holomorphic. Here we put $\kappa=2^{n-1}R_k/\sqrt{d_k}$ and R_k is the regulator of k. Therefore $\zeta_k(2s-1;\mathfrak{k}(\mathfrak{n}^{-1}))-\kappa/2(s-1)$ is holomorphic in $\mathrm{Re}\,s>\frac{1}{2}$. On the other hand $\pi^{\frac{n}{2}}\Gamma\left(s-\frac{1}{2}\right)^n\cdot\Gamma(s)^{-n}$ is continued analytically to the whole s-plane, holomorphically in $\mathrm{Re}\,s>\frac{1}{2}$ and non-zero at $s=1$. Consequently, the function $Z(s;\mathfrak{m};\mathfrak{n};Q)$ is continued analytically to the whole s-plane, holomorphically in $\mathrm{Re}\,s>\frac{1}{2}$ except for $s=1$. At $s=1$, $Z(s;\mathfrak{m},\mathfrak{n};Q)$ has a simple pole with the residue

$$(15)\qquad A_{-1}=\frac{\pi^n\kappa}{2\sqrt{N(D_Q)}\cdot d_k\cdot N_k(\mathfrak{m}\mathfrak{n})}\qquad \kappa=\frac{2^{n-1}R_k}{\sqrt{d_k}}.$$

From the above considerations, the function $Z(s;\mathfrak{m},\mathfrak{n};Q)-A_{-1}/(s-1)$ has the expansion of the form; $A_0+A_1(s-1)+\cdots$ at $s=1$. We consider the constant term A_0. The first and the third terms of the right hand side of (14) are holomorphic at $s=1$. By (14) and (15) we have;

$$(16)\quad A_0=\frac{N(y)}{\sqrt{N(D_Q)}}N_k(\mathfrak{m})^{-2}\zeta_k(2;\mathfrak{k}(\mathfrak{m}^{-1}))+\frac{1}{\sqrt{N(D_Q)}d_k N_k(\mathfrak{m}\mathfrak{n})}$$

$$\times\lim_{s\to1}\Big\{(\sqrt{N(D_Q)}\,N_k(\mathfrak{n})^2N(y))^{1-s}\frac{\pi^{\frac{n}{2}}\Gamma\left(s-\frac{1}{2}\right)^n}{\Gamma(s)^n}\zeta_k(2s-1\,;\,\mathfrak{k}(\mathfrak{n}^{-1}))$$

$$-\frac{\pi^n\kappa}{2(s-1)}\Big\}+\frac{1}{\sqrt{N(D_Q)}d_kN_k(\mathfrak{m})}\sum_{(\nu)}{}'\frac{1}{N_k(\nu)}\sum_{\substack{\beta\in\mathfrak{m}^{-1}\mathfrak{D}^{-1}\\ \beta\neq0}}{}' e^{-2\pi iS(\nu\beta x)}$$

$$\times\int_{\Gamma\varepsilon^{(1)}}\cdots\int_{\Gamma\varepsilon^{(n)}}\frac{e^{2\pi iS(\nu\beta y\zeta)}d\zeta^{(1)}\cdots d\zeta^{(1)}}{(\zeta^{(1)2}+1)\cdots(\zeta^{(n)2}+1)}.$$

As $\Gamma\left(s+\frac{1}{2}\right)\cdot\Gamma(s)=\sqrt{\pi}\cdot2^{1-2s}\cdot\Gamma(2s)$ and $\Gamma(s)$ has expansion of the form $1+a(s-1)+\cdots$; $\Gamma\left(s-\frac{1}{2}\right)^n\cdot\Gamma(s)^{-n}=\pi^{\frac{n}{2}}2^{2n(1-s)}\cdot\Gamma(2s-1)^n\cdot\Gamma(s)^{-2n}$

$$=\pi^{\frac{n}{2}}2^{2n(1-s)}(1+nb(s-1)^2+\cdots).$$

Since we have;

$$(\sqrt{N(D_Q)}N(y)N_k(\mathfrak{n})^2)^{1-s}\cdot\pi^{\frac{n}{2}}\Gamma\left(s-\frac{1}{2}\right)^n\Gamma(s)^{-n}\zeta_k(2s-1\,;\,\mathfrak{k}(\mathfrak{n}^{-1}))$$

$$=\pi^n\{2(\sqrt{N(D_Q)}N_k(\mathfrak{n})^2N(y))^{\frac{1}{2n}}\}^{2n(1-s)}\Gamma(2s-1)^n\Gamma(s)^{-2n}\zeta_k(2s-1\,;\,\mathfrak{k}(\mathfrak{n}^{-1}))$$

$$=\pi^n\{1-2n\log(2(\sqrt{N(D_Q)}N_k(\mathfrak{n})^2N(y)^{\frac{1}{2n}})(s-1)+\cdots\}\{1+nb(s-1)^2+\cdots\}$$

$$\times\Big\{\frac{\kappa}{2(s-1)}+\kappa_0(\mathfrak{n}^{-1})+\cdots\Big\}.$$

It follows immediately;

$$A_0=\frac{N(y)}{\sqrt{N(D_Q)}}N_k(\mathfrak{m})^{-2}\zeta_k(2:\mathfrak{k}(\mathfrak{m}^{-1}))\tag{17}$$

$$+\frac{\pi^n}{\sqrt{N(D_Q)}\cdot d_kN_k(\mathfrak{mn})}\Big\{(\kappa_0(\mathfrak{n}^{-1})-n\kappa\log2)-\kappa\log(\sqrt{N(D_Q)}N_k(\mathfrak{n})^2N(y))^{\frac{1}{2}}\Big\}$$

$$+\frac{1}{\sqrt{N(D_Q)}\cdot d_kN_k(\mathfrak{m})}\sum_{\substack{(\nu)\\ \nu\in\mathfrak{n}\\ \nu\gg0}}{}'\frac{1}{N_k(\nu)}\sum_{\substack{\beta\in\mathfrak{m}^{-1}\mathfrak{D}^{-1}\\ \beta\neq0}}{}' e^{-2\pi iS(\nu\beta x)}$$

$$\times\int_{\Gamma\varepsilon^{(1)}}\cdots\int_{\Gamma\varepsilon^{(n)}}\frac{e^{2\pi iS(\nu\beta y\zeta)}d\zeta^{(1)}\cdots d\zeta^{(n)}}{(\zeta^{(1)2}+1)\cdots(\zeta^{(n)2}+1)},$$

where we denote by $\varepsilon=(\varepsilon^{(1)},\cdots,\varepsilon^{(n)})$ the set of signatures of $(\beta^{(1)},\cdots,\beta^{(n)})$. By our assumptions $y^{(p)}>0$ and $\nu^{(p)}>0$ $(1\leqq p\leqq n)$, we have

$$\int_{\Gamma\varepsilon^{(p)}}\frac{e^{2\pi i\nu^{(p)}\beta^{(p)}y^{(p)}\zeta^{(p)}}}{(\zeta^{(p)2}+1)}d\zeta^{(p)}=\pi e^{-2\pi\nu^{(p)}|\beta^{(p)}|y^{(p)}}\quad(1\leqq p\leqq n).\tag{18}$$

For any set of signatures $\varepsilon=(\varepsilon^{(1)},\cdots,\varepsilon^{(n)})$ and $z=(z^{(1)},\cdots,z^{(n)})$ in $\mathscr{H}_n$ put

$$\varepsilon\cdot z=(\varepsilon^{(1)}z^{(1)},\cdots,\varepsilon^{(n)}z^{(n)}),\ \varepsilon^{(p)}z^{(p)}=\begin{cases} z^{(p)} & \text{if}\quad \varepsilon^{(p)}=1 \\ -\bar{z}^{(p)} & \text{if}\quad \varepsilon^{(p)}=-1 \end{cases}\quad (1\leqq p\leqq n).$$

Clearly, $z\rightarrow\varepsilon\cdot z$ defines a transformation of $\mathcal{H}_n$ onto itself and we have

(19) $$N(y)=y^{(1)}\cdots y^{(n)}=(2i)^{-n}\sum_{(\varepsilon)}\varepsilon^{(1)}z^{(1)}\cdots\varepsilon^{(n)}z^{(n)},$$

and

(20) $$\sum_{\substack{\beta\in\mathfrak{m}^{-1}\mathfrak{D}^{-1}\\ \beta\neq 0}}{}' e^{-2\pi iS(\nu\beta x)-2\pi S(\nu|\beta|y)}=\sum_{(\varepsilon)}\sum_{\substack{\beta\in\mathfrak{m}^{-1}\mathfrak{D}^{-1}\\ \beta\gg 0}}{}' e^{2\pi iS(\nu\beta\cdot\varepsilon z)},$$

for any $z=(z^{(1)},\cdots,z^{(n)})$ $(z^{(p)}=x^{(x)}+iy^{(p)})$ in $\mathcal{H}_n$. In the above, the summation $\sum_{(\varepsilon)}$ extends over the set of 2^n operators $\varepsilon=(\varepsilon^{(1)},\cdots,\varepsilon^{(n)})$ with $\varepsilon^{(p)}=\pm 1$ and we denote by $S(\nu\beta\cdot\varepsilon z)=\sum_{p=1}^{n}\nu^{(p)}\beta^{(p)}\varepsilon^{(p)}z^{(p)}$. Then, in view of (17), (18), (19), (20) we get

(21) $$A_0=\frac{\pi^n}{\sqrt{N(D_Q)\cdot d_k}N_k(\mathfrak{mn})}(\kappa_0(\mathfrak{n}^{-1})-n\kappa\log 2)$$
$$-\frac{\pi^n\kappa}{\sqrt{N(D_Q)\cdot d_k}N_k(\mathfrak{mn})}\log\{(\sqrt{N(D_Q)}N(\mathfrak{n})^2N(y))^{\frac{1}{2}}\prod_{(\varepsilon)}\Psi_k(\varepsilon^{(1)}z^{(1)},\cdots,\varepsilon^{(n)}z^{(n)};\mathfrak{m},\mathfrak{n})\},$$

where $\log\Psi_k(z;\mathfrak{m},\mathfrak{n})$ is defined formally by;

(22) $$-\log\Psi_k(z^{(1)},\cdots,z^{(n)};\mathfrak{m},\mathfrak{n})=\frac{\sqrt{d_k}N_k(\mathfrak{n})}{(2\pi i)^n\kappa N_k(\mathfrak{m})}\zeta_k(2;\mathfrak{k}(\mathfrak{n}^{-1}))z^{(1)}\cdots z^{(n)}$$
$$+\frac{N_k(\mathfrak{n})}{\kappa}\sum_{\substack{(\nu)\\ \nu\gg 0\\ \nu\in\mathfrak{n}}}{}'\frac{1}{N(\nu)}\sum_{\substack{\beta\in\mathfrak{m}^{-1}\mathfrak{D}^{-1}\\ \beta\gg 0}}{}' e^{2\pi iS(\nu\beta z)}.$$

Consequently we have obtained:

THEOREM 1. *Let k be a totally real algebraic number field of degree n and $\sigma_1,\cdots,\sigma_n$ be the n injections of $\boldsymbol{R}$ into $k\underset{Q}{\otimes}\boldsymbol{R}$ which are extensions of the isomorphisms of k into $\boldsymbol{R}$ $(\sigma_1=1)$. Let $Q(u,v)=Au^2+2Buv+Cv^2$ be a totally positive definite quadratic form in $\boldsymbol{R}$ with respect to the injections $\sigma_1,\cdots,\sigma_n$. We denote by $z=(z^{(1)},\cdots,z^{(1)})$ $(z^{(p)}=x^{(p)}+iy^{(p)})$ the point in $\mathcal{H}_n$ such that $Q^{(p)}(u,v)=\sqrt{D^{(p)}}y^{(p)-1}(u+uz^{(p)})(u+v\bar{z}^{(p)})$ for $D_Q=AC-B^2$ $(p=1,\cdots,n)$. For two integral ideals $\mathfrak{m}$ and $\mathfrak{n}$ in k, put*

(23) $$Z(s;\mathfrak{m},\mathfrak{n};Q)=\sum_{\substack{\{\mu,\nu\}\neq\{0,0\}\\ \mu\in\mathfrak{m},\nu\in\mathfrak{n}}}{}' N(A\mu^2+2B\mu\nu+C\nu^2)^{-s}.$$

Then the function $Z(s;\mathfrak{m},\mathfrak{n},Q)$ can be continued analytically to the whole s-plane, holomorphically in $\operatorname{Re}s>\frac{1}{2}$ except for $s=1$. At $s=1$, $Z(s;\mathfrak{m},\mathfrak{n};Q)$ has a *simple pole with the residue*

$$A_{-1}=\frac{\pi^{n}\kappa}{2\sqrt{N(D_{Q})d_{k}N_{k}(\mathfrak{mn})}}$$

where $\kappa=2^{n-1}R_{k}/\sqrt{d_{k}}$ *and* R_{k} *is the regulator of* k. *Moreover in the expansion* $Z(s;\mathfrak{m},\mathfrak{n};Q)=A_{-1}/(s-1)+A_{0}+A_{1}(s-1)+\cdots$, *the constant term* A_{0} *is given by*

$$A_{0}=\frac{\pi^{n}}{\sqrt{N(D_{Q})d_{k}N_{k}(\mathfrak{mn})}}(\kappa_{0}(\mathfrak{n}^{-1})-n\kappa\log 2)$$

$$-\frac{\pi^{n}\kappa}{\sqrt{N(D_{Q})\cdot d_{k}N_{k}(\mathfrak{mn})}}\log\{(\sqrt{N(D_{Q})}N_{k}(\mathfrak{n})^{2}N(y))^{\frac{1}{2}}\prod_{(\varepsilon)}\Psi_{k}(\varepsilon^{(1)}z^{(1)},\cdots,\varepsilon^{(n)}z^{(n)};\mathfrak{m},\mathfrak{n})\}$$

where the product $\prod_{(\varepsilon)}$ *extends over the set of* 2^{n} *operators* $\varepsilon=(\varepsilon^{(1)},\cdots,\varepsilon^{(n)})$, *and* $\log\Psi_{k}(z;\mathfrak{m},\mathfrak{n})$ *is given by*

$$-\log\Psi_{k}(z^{(1)},\cdots,z^{(n)};\mathfrak{m},\mathfrak{n})=\frac{\sqrt{d_{k}}N_{k}(\mathfrak{n})}{(2\pi i)^{n}\kappa N_{k}(\mathfrak{m})}\zeta_{k}(2;\mathfrak{k}(\mathfrak{m}^{-1}))z^{(1)}\cdots z^{(n)}$$

$$+\frac{N_{k}(\mathfrak{n})}{\kappa}\sum_{\substack{(\nu)\\ \nu\gg 0\\ \nu\in\mathfrak{n}}}'\frac{1}{N_{k}(\nu)}\sum_{\substack{\beta\in\mathfrak{m}^{-1}\mathfrak{D}^{-1}\\ \beta\gg 0}}'e^{2\pi iS(\nu\beta z)}.$$

Here we denote by $\mathfrak{k}(\mathfrak{a})$ *the ideal class in* k *containing* $\mathfrak{a}$ *and* $\kappa_{0}(\mathfrak{a})$ *is given by* $\zeta_{k}(s;\mathfrak{k}(\mathfrak{a}))=\kappa/(s-1)+\kappa_{0}(\mathfrak{a})+\kappa_{1}(s-1)+\cdots$.

In theorem 1, if we consider $\Psi_{k}(z^{(1)},\cdots,z^{(n)};\mathfrak{m},\mathfrak{n})$ as a function of $(z^{(1)},\cdots,z^{(n)})$ on $\mathcal{H}_{n}$, for the fixed integral ideals $\mathfrak{m}$ and $\mathfrak{n}$, Ψ_{k} is nothing but a generalization of Dedekind's η-function. But this function Ψ_{k} is not a Hilbert's modular form. For, by the definition of Ψ_{k} this function is everywhere $\neq 0$, so that Ψ_{k}^{-s} (s being positive real number) is everywhere regular. Now from Theorem (17) and its corollaries in page 280 of [5], Ψ_{k} cannot be a modular form.

Let K be a totally imaginary quadratic extension of a totally real algebraic number field of degree n. We shall use the same notations as in § 1. Let $\mathfrak{K}$ be an absolute ideal class in K and $\mathfrak{L}_{k}=\mathfrak{a}_{i}\Omega_{1}+\mathfrak{o}\Omega_{2}$ be the integral ideal in $\mathfrak{K}^{-1}$. If $\lambda=\mu\Omega_{1}+\nu\Omega_{2}$ ($\mu\in\mathfrak{a}_{i}$, $\nu\in\mathfrak{o}$) lie in $\mathfrak{L}_{k}$ and $N_{K/k}(\lambda)=A\mu^{2}+2B\mu\nu+C\nu^{2}$, then from the fact

$$4D=4Ac-4B^{2}=\begin{vmatrix}\Omega_{1} & \Omega_{1}^{\tau}\\ \Omega_{2} & \Omega_{2}^{\tau}\end{vmatrix}^{2} \text{ we have } \sqrt{N_{k}(D)}=(2^{n}N_{k}(\mathfrak{a}_{i})d_{k})^{-1}\sqrt{d_{K}}N_{K}(\mathfrak{L}_{\mathfrak{K}}).$$

Therefore, in view of (3) and theorem 1, in the expansion;

$$\zeta_{K}(s;\mathfrak{K})=\frac{1}{e}N_{k}(\mathfrak{L}_{\mathfrak{K}})\cdot N_{K}(\mathfrak{L}_{\mathfrak{K}})^{s-1}\sum_{\{\mu,\nu\}}'N_{k}(A\mu^{2}+2B\mu\nu+C\nu^{2})^{-s}$$

$$=\frac{1}{e}N_{K}(\mathfrak{L}_{\mathfrak{K}})\{1+\log N_{K}(\mathfrak{L}_{\mathfrak{K}})\cdot(s-1)+\cdots\}\{A_{-1}/(s-1)+A_{0}+A_{1}(s-1)+\cdots\}$$

$$=\frac{1}{e}N_K(\mathfrak{L}_{\mathfrak{K}})\Big\{\frac{A_{-1}}{s-1}+(A_0+A_{-1}\log N_K(\mathfrak{L}_{\mathfrak{K}}))+\cdots\Big\}$$

we have

$$A_{-1}=(2\pi)^n\kappa\sqrt{d_k}(2\sqrt{d_K}\,N_K(\mathfrak{L}_{\mathfrak{K}}))^{-1},$$

$$A_0=\frac{(2\pi)^n\sqrt{d_k}}{\sqrt{d_K}N_K(\mathfrak{L}_{\mathfrak{K}})}(\kappa_0(\mathfrak{o})-n\kappa\log 2)$$

$$-\frac{(2\pi)^n\kappa\sqrt{d_k}}{\sqrt{d_K}N_K(\mathfrak{L}_{\mathfrak{K}})}\log\Big\{\Big(\frac{\sqrt{d_K}N_K(\mathfrak{L}_{\mathfrak{K}})}{2^nN_k(\mathfrak{a}_i)d_k}N(y)^{\frac{1}{2}}\prod_{(\varepsilon)}\Psi_k(\varepsilon z\,;\,\mathfrak{a}_i,\mathfrak{o})\Big)\Big\},$$

for $z=x+iy=\Omega_1^{-1}\Omega_2$. Consequently we obtain:

THEOREM 2. *Let K be a totally imaginary quadratic extension of a totally real algebraic number field k of degree n. Let $\mathfrak{K}$ be an absolute ideal class in K and $\mathfrak{L}_{\mathfrak{K}}=\mathfrak{a}_i\Omega_1+\mathfrak{o}\Omega_2$ be the integral ideal in $\mathfrak{K}^{-1}$ (as defined in* § 1). *Put $z=x+iy=\Omega_1^{-1}\Omega_2$. Then we have:*

$$\zeta_K(s\,;\,\mathfrak{K})=\frac{(2\pi)^n\sqrt{d_k}}{e\sqrt{d_K}}\Big\{\frac{\kappa}{2(s-1)}+\kappa_0(\mathfrak{o})-\kappa\log\frac{\sqrt{d_K}}{d_k} \tag{24}$$

$$+\kappa\log\Big(\frac{N(y)}{N_k(\mathfrak{a}_i)}\Big)^{\frac{1}{2}}\prod_{(\varepsilon)}\Psi_k(\varepsilon^{(1)}z^{(1)},\cdots,\varepsilon^{(n)}z^{(n)}\,;\,\mathfrak{a}_i,\mathfrak{o})\Big\}$$

$$+\Big(\textit{higher terms in }(s-1)\Big).$$

REMARK. In Theorem 2, the absolute value $\Big|\Big(\frac{N(y)}{N_k(\mathfrak{a}_i)}\Big)^{\frac{1}{2}}\prod_{(\varepsilon)}\Psi_k(\varepsilon z\,;\,\mathfrak{a}_i,\mathfrak{o})\Big|$ is in fact a class invariant of $\mathfrak{K}$.

§ 4. An application of Kronecker's limit formula

As an application of theorem 2, we consider in this section the relative class number formula of the absolute class field F over K.

The notations being the same as in the previous sections. Let F be the absolute class field over K, i. e. the maximal unramified abelian extension of K, and h be the absolute class number of K. It follows from class field theory that;

$$\zeta_F(s)=\zeta_K(s)\prod_{\chi\neq 1}L(s\,;\,\chi), \tag{25}$$

where the product $\prod_{\chi\neq 1}$ extends over $h-1$ non-principal ideal class characters of K. The absolute value of the discrimiant of F is d_K^h and F is a totally imaginary extension of degree h over K. Thus, comparing the residues at $s=1$ on both sides of (25), we have

$$\frac{(2\pi)^{nh}R_F}{w_F \cdot \sqrt{d_K}^h} h_F = \frac{(2\pi)^n R_K h}{w_K \sqrt{d_K}} \prod_{\chi \neq 1} L(1\,;\chi) \tag{26}$$

where h_F, R_F and w_F denote, the class number of F, regulator of F and the number of roots of unity in F respectively. On the other hand we have

$$L(s\,;\chi) = \sum_{\mathfrak{K}} \chi(\mathfrak{K})\zeta_K(s\,;\mathfrak{K}) \tag{27}$$

Put $\mathfrak{L}_{\mathfrak{K}} = \mathfrak{a}_i\Omega_1 + \mathfrak{o}\Omega_2$ and $z = x+iy = \Omega_1^{-1}\Omega_2$. As the absolute value

$$|(N_k(\mathfrak{a}_i)^{-1}N(y))^{\frac{1}{2}} \prod \Psi_k(\varepsilon z\,;\mathfrak{a}_i, \mathfrak{o})|$$

is a class invariant of $\mathfrak{K}$, we shall denote it by $J(\mathfrak{K})$. Then, by (24) and (27) we get;

$$L(1\,;\chi) = \frac{(2\pi)^n \kappa \sqrt{d_k}}{e\sqrt{d_K}} \sum_{\mathfrak{K}} (-\chi(\mathfrak{K}) \log J(\mathfrak{K})) \tag{28}$$

Thus we obtained finally:

THEOREM 4. *Let K be a totally imaginary quadratic extension of a totally real algebraic number field k of degree n, F the absolute class field over K. Denote h_F, R_F and w_F the absolute class number of F, regulator of F and the number of roots of unity in F. Then the quotient $h_F h^{-1}$ is given by*

$$\frac{h_F}{h} = \frac{w_F \kappa^{h-1} \sqrt{d_k}^{h-1} R_K}{w_K e^{h-1} R_F} \prod_{\chi \neq 1} (-\sum_{\mathfrak{K}} \chi(\mathfrak{K}) \log J(\mathfrak{K}))\,. \tag{29}$$

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