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On some double coset decompositions of complex semi-simple Lie groups

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It is well known that a homogeneous space X of a complex semi-simple Lie group G^c by a Borel subgroup B is a compact complex Kähler manifold. A compact form G_u of G^c also operates transitively on X and so X can be considered as a homogeneous space of G_u. This manifold was studied profoundly by Borel [1], Borel-Hirzebruch [2], Bott [3], and by Kostant [15]. This manifold is intimately related to the theory of finite dimensional representations of G_u (or of G^c). On the other hand, Gelfand and Graev [11] paid attention to that since it is also very important for the theory of unitary representations of a non compact real form G of G^c which is a real semi-simple Lie group. They studied the orbits of G in X when G is of AI-type (or of AIII-type) and, in fact, they showed that one can construct some new irreducible unitary representations of G, so-called "discrete series" in their sense, which are realized on some suitable spaces of functions on these orbits [12].

In this paper we shall show that, in general, the space of the double cosets $B\backslash G^c/G$ is finite, i.e. the number of the G-orbits in X is finite. In several examples, we shall calculate the number of G-orbits. In particular, open orbits in X seem very interesting. These are complex manifolds on which G operates holomorphically, and can be considered as homogeneous spaces of G by its Cartan subgroups.

It seems to be interesting to determine whether they are Stein manifolds or not, and further to study the structure of their cohomologies, because these properties have a close connection with representations of G. The method which I have made use of is essentially the same as that of Sulʹdin in [12].

In § 1, we first prove that the subgroup of all elements are fixed by an anti-linear involutive automorphism of a connected, simply connected complex semi-simple group, which derives easily from the well known Cartan's theorem on symmetric spaces. The decomposition theorem of Bruhat-Chevalley deduces the theorem (Theorem 1):

Let G^c be a connected complex semi-simple group, $\tilde{\sigma}$ be an anti-linear involutive automorphism of G^c, Φ be the set of all elements g of G which satisfy

$\tilde{\sigma}(g)=g^{-1}$. *For an arbitrary Borel subgroup of* G^c, *there exists a finite subset* Ψ *of* Φ *such that any element* g *of* Φ *can be written* $g=b\cdot g_1\cdot\tilde{\sigma}(b)^{-1}$, *where* b *is in* B, g_1 *is an element of* Ψ.

This is a generalization of Suĺdin's result. Let Φ_0 be the set of all elements of the form $g\cdot\tilde{\sigma}(g)^{-1}$ where g is an arbitrary element in G^c. Obviously, Φ_0 is contained in Φ. But it is not obvious if Φ_0 coincides with Φ. In § 2 we show that, in general, Φ_0 does not coincides with Φ and this problem has a close relation with the classification of real semi-simple Lie groups. For the real form G which $\tilde{\sigma}$ determines, the double coset $B\backslash G^c/G$ is finite. This is the main result of § 3.

G operates on the flag manifold $B\backslash G^c$. We show that open G-orbits in this manifold are all isomorphic with $A\backslash G$ where A is a fundamental Cartan subgroup of G. It is known that $A\backslash G$ has a complex structure, and moreover it has a canonical holomorphic imbedding in this way. This is a generalization of Harish-Chandra's theory (see [9]). The double coset space $B\backslash G^c/G$ has a close connection with the conjugate classes of Cartan subgroups of A. This approach was obtained by H. Matsumoto (unpublished). In § 4, we show several examples when G^c is of simple classical type and shall count the number of double cosets

Professor N. Iwahori has kindly given me many useful advices. In particular Lemma 3′ which is a generalization of Chevalley's theorem was communicated to me by him. Owing to this lemma, the proof of Theorem 1 has become much simpler and I have been able to formulate it in more general situation than I have expected. I must be grateful to Mr. T. Yokonuma and Mr. M. Takeuchi who have helped me in many points.

§ 1.

Let $\mathfrak{g}$ be a non-compact real semi-simple Lie algebra, $\mathfrak{g}^c$ be its complexification, $\tilde{\sigma}$ be an involutive semi-linear automorphism of $\mathfrak{g}^c$ defined by

$$\tilde{\sigma}(X+\sqrt{-1}\,Y)=X-\sqrt{-1}\,Y,\qquad X,Y\in\mathfrak{g}.$$

There exists a compact real form $\mathfrak{g}_u$ of $\mathfrak{g}^c$ which is invariant with respect to $\tilde{\sigma}$. The restriction of $\tilde{\sigma}$ to $\mathfrak{g}_u$ is an involutive automorphism of $\mathfrak{g}_u$, denoted by σ:

$$\tilde{\sigma}|\mathfrak{g}_u=\sigma.$$

Let $\tilde{\theta}$ be an involutive semi-linear automorphism of $\mathfrak{g}^c$ defined by

$$\tilde{\theta}(X+\sqrt{-1}\,Y)=X-\sqrt{-1}\,Y,\qquad X,Y\in\mathfrak{g}_u.$$

Then $\tilde{\theta}$ transforms $\mathfrak{g}$ into itself. $\theta=\tilde{\theta}|\mathfrak{g}$ is an involutive automorphism of $\mathfrak{g}$. $\mathfrak{g}$ and $\mathfrak{g}_u$ are decomposed in the following way:

$$\mathfrak{g}=\mathfrak{k}+\mathfrak{p}\,,\qquad \mathfrak{g}_u=\mathfrak{k}+\sqrt{-1}\,\mathfrak{p}$$

where $\mathfrak{k}=\mathfrak{g}\cap\mathfrak{g}_u$, $\mathfrak{p}=\mathfrak{g}\cap\sqrt{-1}\,\mathfrak{g}_u$. It is a Cartan decomposition.

Let G^c be a connected, complex semi-simple Lie group with the Lie algebra $\mathfrak{g}^c$. We assume $\tilde{\sigma}$ can be extended to an automorphism of G^c. We define a subgroup G of G^c : $G=\{g\in G^c\,;\ \tilde{\sigma}(g)=g\}$. Obviously the Lie algebra of G is $\mathfrak{g}$. G may not be connected.

If G^c is simply connected, $\tilde{\sigma}$ can be extended uniquely to an automorphism of G^c.

NOTATIONS. Let $\mathfrak{g}$ be the Lie algebra of a Lie group G, and $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$, then there exists a unique analytic subgroup H of G corresponding to $\mathfrak{h}$. It is denoted by $H\curvearrowright\mathfrak{h}$ in G or simply by $H\curvearrowright\mathfrak{h}$. We shall frequently make use of this notation.

LEMMA 1. (Cartan) *Let G_u be a simply connected compact semi-simple group, $\mathfrak{g}_u$ be its Lie algebra. We consider an involution σ on $\mathfrak{g}_u$ corresponding to a non-compact real semi-simple algebra. Then $\mathfrak{g}_u$ is decomposed as follows:*

$$\mathfrak{g}_u=\mathfrak{k}+\mathfrak{p}$$

where

$$\mathfrak{k}=\{X\in\mathfrak{g}_u\,;\ \sigma(X)=X\}\,,\qquad \mathfrak{p}=\{X\in\mathfrak{g}_u\,;\ \sigma(X)=-X\}$$

σ is extended uniquely to an automorphism of G_u. Let $K\curvearrowright\mathfrak{k}$ in G. Then $K=\{x\in G_u\,;\ \sigma(x)=x\}$, that is, the subgroup of σ-fixed elements is connected.

For the proof see Helgason [13], p. 272 Theorem 7.2.

LEMMA 2. *Let G^c be a simply connected complex semi-simple Lie group, G_u be its maximal compact subgroup, which is simply connected, as well known. Let $G^c\curvearrowright\mathfrak{g}^c$, $G_u\curvearrowright\mathfrak{g}_u$, $K\curvearrowright\mathfrak{k}$ as in Lemma 1. Then the subgroup G of $\tilde{\sigma}$-fixed elements of G^c is connected.*

PROOF. It is well known that $G^c=G_u\cdot\exp(\sqrt{-1}\,\mathfrak{g}_u)$ (unique expression). The map $\exp:\sqrt{-1}\,\mathfrak{g}_u\to\exp\sqrt{-1}\,\mathfrak{g}_u$ is a diffeomorphism. Obviously $G_u\cap G\supset K$. Let $g=u\cdot\exp X$ ($u\in G_u$, $X\in\sqrt{-1}\,\mathfrak{g}_u$) be an element of G. Then $\tilde{\sigma}(g)=\tilde{\sigma}(u)\cdot\tilde{\sigma}(\exp X)=u\cdot\exp X$. On the other hand we have $\tilde{\sigma}(G_u)=G_u$, $\tilde{\sigma}(\exp\sqrt{-1}\,\mathfrak{g}_u)=\exp\sqrt{-1}\,\mathfrak{g}_u$. Therefore $\tilde{\sigma}(u)=u$, $\tilde{\sigma}(\exp X)=\exp X$. By Lemma 1, we have $u\in K$, $\tilde{\sigma}(X)=X$ and $X\in\mathfrak{g}\cap\sqrt{-1}\,\mathfrak{g}_u=\mathfrak{p}$. We have proved thus $G\subset K\cdot\exp\mathfrak{p}$. But $K\cdot\exp\mathfrak{p}$ is connected and is contained in G. Therefore we obtain finally $G=K\cdot\exp\mathfrak{p}$.

Let G^c be a connected complex semi-simple group with $G^c\curvearrowright\mathfrak{g}^c$, and G_u a maximal compact subgroup with $G_u\curvearrowright\mathfrak{g}_u$. Then G_u is connected. Let T be a maximal torus of G_u with $T\curvearrowright\mathfrak{t}$. Then $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}_u$. Its complexification $\mathfrak{t}^c$ is a Cartan subalgebra of $\mathfrak{g}^c$. Let $\mathfrak{t}^c\curvearrowright A^c$ and B be a Borel subgroup of G^c containing A^c. Let $\mathfrak{N}(T)$ be the normalizer of T in G_u. $W=\mathfrak{N}(T)/T$ is a finite group (Weyl group) and operates on $\mathfrak{t}$ as linear trans-

formations via the adjoint representation. For each $w \in W$, we choose once for all a representative element ω in $\mathcal{N}(T)$, and denote it by $\omega = \omega(w)$. Then one has the Bruhat decomposition:

LEMMA 3.

$$G^C = \bigcup_{w \in W} B \cdot \omega(w) \cdot B \qquad (\textit{disjoint sum}). \tag{1}$$

For the proof, see [4].

We need a more precise decomposition which is essentially the same as the Chevalley's [5]. Let $\sqrt{-1}\,\mathfrak{t} = \mathfrak{h}_R$. We denote its dual (over $\boldsymbol{R}$) by $\mathfrak{h}_R^*$, which is identified as the real part of the dual space $(\mathfrak{t}^C)^*$ of $\mathfrak{t}^C$ (over $\boldsymbol{C}$). We consid er two orderings $\mathcal{O}_1$ and $\mathcal{O}_2$ on $\mathfrak{h}_R^*$. Let Δ be roots of $\mathfrak{g}^C$ with respect to $\mathfrak{h}_R$, and $\boldsymbol{P}_1$, $\boldsymbol{P}_2$ be positive roots with respect to $\mathcal{O}_1$, $\mathcal{O}_2$, respectively. Then

$$\mathfrak{g}^C = \sum_{\alpha \in \boldsymbol{P}_1} \boldsymbol{C}E_\alpha + \sum_{\alpha \in \boldsymbol{P}_1} \boldsymbol{C}E_{-\alpha} + \mathfrak{t}^C \tag{2}$$

and

$$\mathfrak{g}^C = \sum_{\alpha \in \boldsymbol{P}_2} \boldsymbol{C}E_\alpha + \sum_{\alpha \in \boldsymbol{P}_2} \boldsymbol{C}E_{-\alpha} + \mathfrak{t}^C \tag{3}$$

where

$$[H, E_\alpha] = \alpha(H)E_\alpha, \qquad (H \in \mathfrak{t}^C,\ \alpha \in \Delta).$$

Put

$$\mathfrak{n}_1^C = \sum_{\alpha \in \boldsymbol{P}_1} \boldsymbol{C}E_\alpha, \qquad \mathfrak{n}_2^C = \sum_{\alpha \in \boldsymbol{P}_2} \boldsymbol{C}E_\alpha.$$

For $\lambda \in (\mathfrak{t}^C)^*$ and $w \in W$ we define $w(\lambda) \in (\mathfrak{t}^C)^*$ by

$$w(\lambda)(H) = \lambda(w^{-1}(H)), \qquad H \in \mathfrak{t}^C.$$

Then for any $\alpha \in \Delta$, one has $Ad\omega \cdot (\boldsymbol{C}E_\alpha) = \boldsymbol{C}E_{w(\alpha)}$ with $\omega = \omega(w)$. We put

$$\begin{cases} \boldsymbol{P}_2'(w) = \{\alpha \in \boldsymbol{P}_2,\ w(\alpha) \in \boldsymbol{P}_1\} \\ \boldsymbol{P}_2''(w) = \{\alpha \in \boldsymbol{P}_2,\ w(\alpha) \in -\boldsymbol{P}_1\} \\ \mathfrak{n}_2'(w) = \sum_{\alpha \in \boldsymbol{P}_2'(w)} \boldsymbol{C}E_\alpha, \qquad \mathfrak{n}_2''(w) = \sum_{\alpha \in \boldsymbol{P}_2''(w)} \boldsymbol{C}E_\alpha, \end{cases} \tag{4}$$

then $\mathfrak{n}_2^C = \mathfrak{n}_2'(w) + \mathfrak{n}_2''(w)$ (direct sum), and $\mathfrak{n}_2'(w)$, $\mathfrak{n}_2''(w)$ are subalgebras of $\mathfrak{n}_2^C$. Put

$$\mathfrak{b}_1 = \mathfrak{t}^C + \mathfrak{n}_1^C, \qquad \mathfrak{b}_2 = \mathfrak{t}^C + \mathfrak{n}_2^C.$$

Let $B_1 \curvearrowright \mathfrak{b}_1$, $B_2 \curvearrowright \mathfrak{b}_2$, $N_1^C \curvearrowright \mathfrak{n}_1^C$, $N_2^C \curvearrowright \mathfrak{n}_2^C$, $N_2'(w) \curvearrowright \mathfrak{n}_2'(w)$, $N_2''(w) \curvearrowright \mathfrak{n}_2''(w)$. Then $N_2^C = N_2'(w) \cdot N_2''(w)$ which is a unique factorization, i.e. $N_2'(w) \cap N_2''(w) = \{1\}$.

LEMMA 3′.

$$\begin{aligned} G^C &= \bigcup_{w \in W} B_1 \cdot \omega(w) B_2 \qquad (\textit{disjoint union}) \\ &= \bigcup_{w \in W} B_1 \cdot \omega(w) N_2''(w) \qquad (\textit{unique expression}) \end{aligned} \tag{5}$$

i.e. the map $B_1 \times N_2''(w) \to B_1\omega(w)N_2''(w)$ *defined by* $(b, n) \to b\omega(w)n$ *is bijective.*

For the proof see [5].

Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$, $\mathfrak{g}_u=\mathfrak{k}+\sqrt{-1}\mathfrak{p}$ and σ, θ, $\tilde{\sigma}$, $\tilde{\theta}$ be defined as before. Let $\mathfrak{h}$ be a Cartan subalgebra which is invariant by θ. Then $\mathfrak{h}=\mathfrak{h}_\mathfrak{k}+\mathfrak{h}_\mathfrak{p}$ (direct sum), $\mathfrak{h}_\mathfrak{k}=\mathfrak{h}\cap\mathfrak{k}$, $\mathfrak{h}_\mathfrak{p}=\mathfrak{h}\cap\mathfrak{p}$, $\mathfrak{h}_u=\mathfrak{h}_\mathfrak{k}+\sqrt{-1}\mathfrak{h}_\mathfrak{p}$ is a Cartan subalgebra of $\mathfrak{g}_u$, $\mathfrak{h}_R=\sqrt{-1}\mathfrak{h}_\mathfrak{k}+\mathfrak{h}_\mathfrak{p}$ and $\mathfrak{h}^C=\mathfrak{h}_u^C$ is a Cartan subalgebra of $\mathfrak{g}^C$. Let G^C be a connected complex semi-simple Lie group in which $\tilde{\sigma}$ is defined and G be the subgroup of G^C consisting of all $\tilde{\sigma}$-fixed elements. We remark that G need not be connected. We fix an order on $\mathfrak{h}_R^*$ and we denote the root system of $\mathfrak{g}^C$ with respect to $\mathfrak{h}^C$ by Δ, and positive roots by $\boldsymbol{P}$. Thus

$$\mathfrak{g}^C=\sum_{\alpha\in\boldsymbol{P}}\boldsymbol{C}E_\alpha+\sum_{\alpha\in\boldsymbol{P}}\boldsymbol{C}E_{-\alpha}+\mathfrak{h}^C\,.$$

Let $\mathfrak{n}^C=\sum_{\alpha\in\boldsymbol{P}}\boldsymbol{C}E_\alpha$, $\mathfrak{b}=\mathfrak{n}^C+\mathfrak{h}^C$, B (a Borel subgroup)$\curvearrowright\mathfrak{b}$, $N^C\curvearrowright\mathfrak{n}^C$, $A^C\curvearrowright\mathfrak{h}^C$, $\mathfrak{h}_u\curvearrowright T\subset G_u$. We put $\tilde{\sigma}(\mathfrak{n}^C)=\mathfrak{n}_1^C$, $\mathfrak{b}_1=\mathfrak{h}^C+\mathfrak{n}_1^C$. For any $\lambda\in(\mathfrak{h}^C)^*$ we define $\tilde{\sigma}(\lambda)$ by $\tilde{\sigma}(\lambda)(H)=\lambda(\tilde{\sigma}(H))$ for $H\in\mathfrak{h}_R$. Put $\boldsymbol{P}_1=\tilde{\sigma}(\boldsymbol{P})$, then $\mathfrak{n}_1^C=\sum_{\alpha\in\boldsymbol{P}_1}\boldsymbol{C}E_\alpha$. By Lemma 3′ we have:

PROPOSITION 1.

$$G^C=\bigcup_{w\in W}B\cdot\omega(w)\tilde{\sigma}(B)\qquad(\textit{disjoint union})\tag{6}$$

$$=\bigcup_{w\in W}B\cdot\omega(w)N_1''(w)\qquad(\textit{unique expression}),\tag{7}$$

where $N_1''(w)\curvearrowright\mathfrak{n}_1''(w)$, $\mathfrak{n}_1''(w)=\sum_{\alpha\in\boldsymbol{P}_1'(w)}\boldsymbol{C}E_\alpha$, $\boldsymbol{P}_1''(w)=\{\alpha\in\tilde{\sigma}(\boldsymbol{P});\ w\alpha\in-\boldsymbol{P}\}$.

PROPOSITION 2. *If* $g\in G^C$ *satisfies* $\tilde{\sigma}(g)=g^{-1}$, g *can be described as follows:*

$$g=z\cdot a'\omega\cdot\tilde{\sigma}(z)^{-1}=z\cdot\omega a\cdot\tilde{\sigma}(z)^{-1}\,,\tag{8}$$

where $z\in N^C$, $\omega\in\mathfrak{N}(T)$ (*normalizer of* T *in* G_u), a, $a'\in A^C$, $\omega=\omega(w)$, $w\in W$.

PROOF. Let $g=na\omega n_1$, $n\in N^C$, $a\in A^C$, $\omega=\omega(w)\in\mathfrak{N}(T)$, $n_1\in N_1''(w)$ in Proposition 1. The equality $\tilde{\sigma}(g)=g^{-1}$ is written as follows:

$$n\omega an_1=\tilde{\sigma}(n_1)^{-1}\tilde{\sigma}(a)^{-1}\tilde{\sigma}(\omega)^{-1}\tilde{\sigma}(n)^{-1}\,,\qquad\tilde{\sigma}(n)^{-1}\in\tilde{\sigma}(N^C)=N_1^C\,.$$

Put $\tilde{\sigma}(n)^{-1}=n_2'n_2''$, $n_2'\in N_1'(w)$, $n_2''\in N_1''(w)$. Because of the disjointness, we have $\tilde{\sigma}(\omega)^{-1}=\omega a_0$ where $a_0\in T$, and $n\omega an_1=\tilde{\sigma}(n_1)^{-1}\tilde{\sigma}(a)^{-1}\omega a_0n_2'n_2''=\tilde{\sigma}(n_1)^{-1}\omega_1n_2'n_2''$ $=\tilde{\sigma}(n_1)^{-1}\omega_1n_2'\omega_1^{-1}\omega_1n_2''$, where $\omega_1=\tilde{\sigma}(a)^{-1}\omega a_0$. $\tilde{\sigma}(n_1)^{-1}\omega_1n_2'\omega_1^{-1}$ is an element of N^C and n_2'' is an element of N_1^C. Therefore, by Proposition 1, we have

$$n=\tilde{\sigma}(n_1)^{-1}\omega_1n_2'\omega_1^{-1}\,,\qquad\omega a=\omega_1\,,\qquad n_1=n_2''$$

and

$$n=\tilde{\sigma}(n_2'')^{-1}\tilde{\sigma}(n_2')^{-1}=\tilde{\sigma}(n_1)^{-1}\omega_1n_2'\omega_1^{-1}\,.$$

Hence from $\tilde{\sigma}(n_2'')^{-1}=\tilde{\sigma}(n_1)^{-1}$ follows $\tilde{\sigma}(n_2')^{-1}=\omega_1n_2'\omega_1^{-1}$. n_2' is an element of $N_1'(w)$ and therefore $n_2'=n_3'^2$, for a suitable $n_3'\in N_1'(w)$. Substituting n_2' by $n_3'^2$,

we obtain $\tilde{\sigma}(n_3')^{-2}=\omega_1 n_3'^2\omega_1^{-1}=(\omega_1 n_3'\omega_1^{-1})^2$, where $\tilde{\sigma}(n_3')\in N^c$, $\omega_1 n_3'\omega_1^{-1}\in N^c$. Since the correspondence between $\mathfrak{n}^c$ and N_c is bijective, we have $\tilde{\sigma}(n_3')^{-1}=\omega_1 n_3'\omega_1^{-1}$. Thus $n=\tilde{\sigma}(n_1)^{-1}\omega_1 n_3'^2\omega_1^{-1}$, $g=\tilde{\sigma}(n_1)^{-1}\omega_1 n_3'^2\omega_1^{-1}\omega a n_1=\tilde{\sigma}(n_1)^{-1}\omega_1 n_3'\omega_1^{-1}\cdot\omega_1 n_3'\omega_1^{-1}\cdot\omega a n_1$ $=\tilde{\sigma}(n_1)^{-1}\tilde{\sigma}(n_3')^{-1}\omega_1 n_3' n_1'$, where $\tilde{\sigma}(n_3' n_1)^{-1}\in N^c$.

We put $z=\tilde{\sigma}(n_3' n_1)^{-1}$, then we have $g=z\cdot\omega a\tilde{\sigma}(z)^{-1}$, which we had to prove. ωa in (8) must satisfy

$$\tilde{\sigma}(\omega a)=(\omega a)^{-1}. \tag{9}$$

$A\curvearrowright\mathfrak{h}_R$ in G^c is an analytic isomorphism. $A^c=T\cdot A$ is the direct product where $\tilde{\sigma}(T)=T$, and $\tilde{\sigma}(A)=A$. Let $a=t\cdot\xi$, $t\in T$, $\xi\in A$. Then (8) can be written as: $\tilde{\sigma}(\omega)\tilde{\sigma}(t)\tilde{\sigma}(\xi)=t^{-1}\xi^{-1}\omega^{-1}$, $\tilde{\sigma}(\omega)^{-1}=\omega a_0$, $a_0^{-1}\omega^{-1}\tilde{\sigma}(t)\tilde{\sigma}(\xi)=t^{-1}\xi^{-1}\omega^{-1}$ and $a_0^{-1}\omega^{-1}\tilde{\sigma}(t)\omega\cdot\omega^{-1}\tilde{\sigma}(\xi)\omega=t^{-1}\xi^{-1}$. Therefore

$$a_0^{-1}\omega^{-1}\tilde{\sigma}(t)\omega=t^{-1},\qquad \omega^{-1}\tilde{\sigma}(\xi)\omega=\xi^{-1}. \tag{10}$$

Let $\xi=\xi_1^2$ and $\xi_1\in A$. Then $\omega^{-1}\tilde{\sigma}(\xi_1)^2\omega=\xi^{-2}$, $\omega^{-1}\tilde{\sigma}(\xi_1)\omega=\xi_1^{-1}$. Consequently

$$\omega a=\omega t\xi=\omega t\xi_1^2=\omega t\xi_1\xi_1=\omega\xi_1\omega^{-1}\omega t\xi_1=\tilde{\sigma}(\xi_1)^{-1}\omega t\xi_1 \tag{11}$$

$\tilde{\sigma}$ and $\mathrm{Ad}\,\omega$ operate on $\mathfrak{h}_u$ as linear transformations and $\mathrm{Ad}\,\omega|\mathfrak{h}_u=w$. For any $H\in\mathfrak{h}_u$ $\tilde{\sigma}\circ w(H)=\tilde{\sigma}(\mathrm{Ad}\,\omega(H))$ is an involution, because $(\tilde{\sigma}\circ w)^2(H)$ $=\tilde{\sigma}[\mathrm{Ad}\,\omega\{\tilde{\sigma}[\mathrm{Ad}\,\omega(H)]\}]=\mathrm{Ad}\,\tilde{\sigma}(\omega)\,\mathrm{Ad}\,\omega(H)=\mathrm{Ad}\,a_0^{-1}(H)=H$.

Let

$$\begin{cases}\mathfrak{h}_u^+=\{X\in\mathfrak{h}_u,\ \tilde{\sigma}\circ w(X)=X\}\\ \mathfrak{h}_u^-=\{X\in\mathfrak{h}_u,\ \tilde{\sigma}\circ w(X)=-X\}.\end{cases} \tag{12}$$

Then $\mathfrak{h}_u=\mathfrak{h}_u^++\mathfrak{h}_u^-$ is a direct sum. Let $\mathfrak{h}_u^+\curvearrowright T_w^+$, $\mathfrak{h}_u^-\curvearrowright T_w^-$. Then $T=T_w^+\cdot T_w^-$, where $T_w^+\cap T_w^-$ may be non-trivial.

We also define two subgroups ${}'T_w^+$ and ${}'T_w^-$ as follows:

$$\begin{cases}{}'T_w^+=\{t\in T;\ \tilde{\sigma}(t)=\omega t\omega^{-1}\}\\ {}'T_w^-=\{t\in T;\ \tilde{\sigma}(t)=\omega t^{-1}\omega^{-1}\}.\end{cases} \tag{13}$$

Obviously ${}'T_w^+\supset T_w^+$, ${}'T_w^-\supset T_w^-$ with the index $[{}'T_w^+:T_w^+]<\infty$, $[{}'T_w^-:T_w^-]<\infty$.

Let $t=\exp(H_++H_-)$, $H_+\in\mathfrak{h}_u^+$, $H_-\in\mathfrak{h}_u^-$. Then

$$\begin{aligned}\omega t&=\omega\exp(H_++H_-)=\omega\exp H_-\exp H_+\\ &=\omega\left(\exp\frac{1}{2}H_-\right)^2\exp H_+=\omega\exp\frac{1}{2}H_-\omega^{-1}\omega\exp\frac{1}{2}H_-\exp H_+\\ &=\tilde{\sigma}\left(\exp\frac{1}{2}H_-\right)^{-1}\omega\exp H_+\exp\frac{1}{2}H_-.\end{aligned} \tag{14}$$

Because of $\tilde{\sigma}(\omega t)=(\omega t)^{-1}$ we have $\tilde{\sigma}(\omega\exp H_+)=(\omega\exp H_+)^{-1}$, $\tilde{\sigma}(\omega)\tilde{\sigma}(\exp H_+)$ $=\exp(-H_+)\cdot\omega^{-1}$, $a_0^{-1}\omega^{-1}\tilde{\sigma}(\exp H_+)=\exp(-H_+)\cdot\omega^{-1}$ and $a^{-1}\exp(H_+)$ $=\exp(-H_+)$. Thus

(15) $$(\exp H_+)^2 = a_0\,, \qquad H_+ \in \mathfrak{h}_u^+\,.$$

The set of elements H_+ which satisfy (15) is finite which may be empty, $t = \exp H_+$ is contained in T_w^+ and satisfies $t^2 = a_0$. If t and t_1 (elements of T_w^+) satisfy $t^2 = t_1^2 = a_0$, then

(16) $$(t\,t_1^{-1})^2 = 1\,, \qquad t\,t_1^{-1} \in T_w^+ \quad \text{and} \quad t\,t_1^{-1} \in {}^{\backprime}T_w^-\,.$$

Conversely, suppose $t\,t_1^{-1} \in {}^{\backprime}T_w^-$, then $t^2 = t_1^2$.

If $t, t_1 \in T_w^+$ satisfy $\omega t = a\omega\tilde{\sigma}(a)^{-1}$ for a suitable element $a \in A^c$, then

(17) $$t \equiv t_1 \mod T_w^-\,.$$

More generally, suppose that

(18) $$\omega t = b\omega_1 t_1 \tilde{\sigma}(b)^{-1}\,,$$

for $b \in B$, $\omega = \omega(w)$, $\omega_1 = \omega(w_1)$, $t \in T_w^+$, $t_1 \in T_{w_1}^+$ which satisfy $t^2 = a_0$, $t_1^2 = a_1$ and $a_0^{-1} = \tilde{\sigma}(\omega)\omega$, $a_1^{-1} = \tilde{\sigma}(\omega_1)\omega_1$.

By (6), $w = w_1$ and so $\omega = \omega_1$. Therefore $T_w^+ = T_{w_1}^+$, $a_0 = a_1$. Let $b = na$, $n \in N^c$, $a \in A^c$, then (18) is rewritten as follows

$$\omega t = na\omega t_1 \tilde{\sigma}(a)^{-1}\tilde{\sigma}(n)^{-1}\,.$$

Let $\tilde{\sigma}(n)^{-1} = n_1' n_2''$, $n_1' \in N_1'(w)$, $n_2'' \in N_1''(w)$ as in Proposition 1. Then $\omega t = na\omega t_1 \tilde{\sigma}(a)^{-1} n_1' n_1'' = n\omega' n_1' \omega'^{-1} \cdot \omega' n_1''$ where $\omega' = a\omega t_1 \tilde{\sigma}(a)^{-1}$. Here $n\omega' n_1' \omega'^{-1}$ is contained in N^c and $n_1'' \in N_1''(w)$. By Proposition 1 $n\omega' n_1' \omega'^{-1} = 1$, $\omega t = \omega'$, $n_1'' = 1$ and $\omega t = a\omega t_1 \tilde{\sigma}(a)^{-1}$ $(a \in A^c)$. By (17) $t \equiv t_1 \mod T_w^-$.

We define $\boldsymbol{\Phi}$, $\boldsymbol{\Phi}_0$, $\boldsymbol{\Psi}$, $\boldsymbol{\Psi}_0$, $[\boldsymbol{\Psi}]$, $[\boldsymbol{\Psi}_0]$ in the following way:

(19)
$$\begin{cases} \boldsymbol{\Phi} = \{g \in G^c\,;\ \tilde{\sigma}(g) = g^{-1}\} \\ \boldsymbol{\Phi}_0 = \{g \in G^c\,;\ g = g_1\tilde{\sigma}(g_1)^{-1} \text{ for a certain element } g_1 \in G^c\} \\ \boldsymbol{\Psi} = \{\omega a\,;\ a^2 = a_0,\ a_0^{-1} = \tilde{\sigma}(\omega)\omega,\ a \in T_w^+,\ \omega = \omega(w),\ w \in W\} \\ \qquad \text{(The set of such elements } a \text{ is denoted by } \hat{T}_w^+) \\ \boldsymbol{\Psi}_0 = \boldsymbol{\Psi} \cap \boldsymbol{\Phi}_0\,. \end{cases}$$

Two elements of $\boldsymbol{\Psi}$ (or of $\boldsymbol{\Psi}_0$) ωa and $\omega_1 a_1$ are said to be equivalent if and only if $\omega = \omega_1$ and $a \equiv a_1 \mod (T_w^+ \cap T_w^-)$. The set of these equivalence classes of $\boldsymbol{\Psi}$ (or $\boldsymbol{\Psi}_0$) is denoted by $[\boldsymbol{\Psi}]$ (or $[\boldsymbol{\Psi}_0]$).

We shall also denote by $[\boldsymbol{\Psi}]$ (or $[\boldsymbol{\Psi}_0]$) the set of representatives in $\boldsymbol{\Psi}$ (or $\boldsymbol{\Psi}_0$) of all the equivalence classes $[\boldsymbol{\Psi}]$ (or $[\boldsymbol{\Psi}_0]$). We have arrived at:

THEOREM 1. *(The Borel subgroup) B in Proposition* 1 *operates on $\boldsymbol{\Phi}$, by taking*

(20) $$b\colon g \to b \cdot g \cdot \tilde{\sigma}(b)^{-1} \text{ for any } g \in \boldsymbol{\Phi} \qquad (b \in B)\,.$$

This operation is not generally transitive, and $\boldsymbol{\Phi}$ is decomposed into finite B-orbits. This orbit set is identified with $[\boldsymbol{\Psi}]$. More precisely, in any B-orbit

there exists an element ωa of Ψ; and two elements of Ψ contained in the same orbit are in the same equivalence class of $[\Psi]$.

PROPOSITION 4. *Let $\sigma\circ w$ and $\sigma\circ w_1$ are conjugate with respect to W, that is, $\sigma\circ w_1=w_0\,(\sigma\circ w)w_0^{-1}$ for a suitable w_0. Then we can take as $\omega(w_1)$ the element $\omega_1=\sigma(\omega_0)\omega\omega_0^{-1}$, where $\omega_0=\omega(w_0)$. Moreover if $\omega a\in\Psi$ (or $\omega a\in\Psi_0$), then $\omega_1 a_1\in\Psi$ (or Ψ_0 respectively) with $a_1=\omega_0 a\omega_0^{-1}$, and conversely. Consequently, the property whether ωa is an element of Ψ_0 or not depends only on the conjugate class of $\sigma\circ w$ with respect to W.*

PROOF. For any $t\in T$

$$\sigma(\omega_1 t\omega_1^{-1})=\sigma(\sigma(\omega_0)\omega\omega_0^{-1}t\omega_0\omega^{-1}\sigma(\omega_0)^{-1})=\omega_0\sigma(\omega\omega_0^{-1}t\omega_0\omega^{-1})\omega_0^{-1}=\sigma\cdot w_1(t)\,.$$

Therefore we can take $\omega_1=\omega(w_1)$ and $\sigma(\omega_0)\omega a\omega_0^{-1}=\sigma(\omega_0)\omega\omega_0^{-1}\omega_0 a\omega_0^{-1}=\omega_1 a_1$ where $\omega_1=\omega_0 a\omega_0^{-1}$. The decomposition $T=T_w^+T_w^-$ is isomorphic to the decomposition $T=T_{w_1}^+T_{w_1}^-$.

§ 2. The relation between Φ and Φ_0.

Obviously $\Phi\supset\Phi_0$. We consider the problem whether Φ_0 coincides with Φ or not?

Let $\mathfrak{g}$ be a real semi-simple Lie algebra, $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be a Cartan decomposition and θ be an involution automorphism determining this decomposition: $\theta(X+Y)=X-Y$ for $X\in\mathfrak{k}$, $Y\in\mathfrak{p}$. Let G be a connected real semi-simple Lie group with finite centre corresponding to $\mathfrak{g}$. Let $K\curvearrowright\mathfrak{k}$ in G. K is a maximal compact subgroup of G. Let $P=\exp\mathfrak{p}$, then $G=K\cdot P$ is a Mostow decomposition. Let $\mathfrak{h}$ be a θ-invariant Cartan subalgebra of $\mathfrak{g}$. Then $\mathfrak{h}=\mathfrak{h}_t+\mathfrak{h}_\mathfrak{p}$ ($\mathfrak{h}_t=\mathfrak{h}\cap\mathfrak{k}$, $\mathfrak{h}_\mathfrak{p}=\mathfrak{h}\cap\mathfrak{p}$) is a direct sum. We define $\mathfrak{h}_u=\mathfrak{h}_t+\sqrt{-1}\,\mathfrak{h}_\mathfrak{p}$, $\mathfrak{h}_{\boldsymbol{R}}=\sqrt{-1}\,\mathfrak{h}_u$ and $\mathfrak{h}^{\boldsymbol{C}}$ (the complexification of $\mathfrak{h}$). We also define the dual spaces $\mathfrak{h}_{\boldsymbol{R}}^*$ (over $\boldsymbol{R}$) and $(\mathfrak{h}^{\boldsymbol{C}})^*$ (over $\boldsymbol{C}$) of $\mathfrak{h}_{\boldsymbol{R}}$ and $\mathfrak{h}^{\boldsymbol{C}}$ respectively. $\mathfrak{h}_{\boldsymbol{R}}^*$ can be identified with $(\mathfrak{h}^{\boldsymbol{C}})^*$. We introduce the order of $\mathfrak{h}_{\boldsymbol{R}}^*$ as follows: We take the orders of $(\mathfrak{h}_\mathfrak{p})^*$ and $(\sqrt{-1}\,\mathfrak{h}_t)^*$ and define the order of $(\mathfrak{h}^{\boldsymbol{C}})^*$, compatible with those of $(\mathfrak{h}_\mathfrak{p})^*$ and $(\sqrt{-1}\,\mathfrak{h}_t)^*$, as follows,

$$\text{(21)}\qquad \mathfrak{h}_{\boldsymbol{R}}^*\ni\lambda>0\Leftrightarrow\lambda|\mathfrak{h}_\mathfrak{p}>0\quad\text{or}\quad\lambda|\mathfrak{h}_\mathfrak{p}=0\quad\text{and}\quad\lambda|\sqrt{-1}\,\mathfrak{h}_t>0\,.$$

Let Δ be a root-system of $\mathfrak{g}^{\boldsymbol{C}}$ with respect to $\mathfrak{h}^{\boldsymbol{C}}$, and Δ^+ be the set of positive root: Δ_1^+ (or Δ_0^+) be the set of positive roots α such that $\alpha|\mathfrak{h}_\mathfrak{p}>0$ (or $\alpha|\mathfrak{h}_\mathfrak{p}=0$ and $\alpha|\sqrt{-1}\,\mathfrak{h}_t>0$). Then $\Delta^+=\Delta_1^+\cup\Delta_0^+$. Let $\mathfrak{n}_+^{\boldsymbol{C}}=\sum_{\alpha\in\Delta_1^+}\boldsymbol{C}E_\alpha$, $\mathfrak{n}_+=\mathfrak{n}_+^{\boldsymbol{C}}\cap\mathfrak{g}$, $\mathfrak{m}^{\boldsymbol{C}}=\sum_{\alpha\in\Delta_0^+}(\boldsymbol{C}E_\alpha+\boldsymbol{C}E_{-\alpha})+\mathfrak{h}_t^{\boldsymbol{C}}$, where $\mathfrak{h}_t^{\boldsymbol{C}}$ is the complexification of $\mathfrak{h}_t$ and $\mathfrak{m}=\mathfrak{m}^{\boldsymbol{C}}\cap\mathfrak{g}$. $\mathfrak{m}$ is reductive and θ-invariant. Therefore $\mathfrak{m}$ is decomposed into

$$\mathfrak{m}=\mathfrak{m}_t+\mathfrak{m}_\mathfrak{p}\,,\qquad\mathfrak{m}_t=\mathfrak{k}\cap\mathfrak{m}\,,\qquad\mathfrak{m}_\mathfrak{p}=\mathfrak{m}\cap\mathfrak{p}\,.$$

Let $\mathfrak{n}_+ \curvearrowright N_+$, $\mathfrak{h}_\mathfrak{p} \curvearrowright A_\mathfrak{p}$, $\mathfrak{m} \curvearrowright M$, $\exp \mathfrak{m}_\mathfrak{p} = M_\mathfrak{p}$ and $M_K = M \cap K$. $M_\mathfrak{p}$ is contained in P. Then,

LEMMA 4. (Harish-Chandra)

i) *$M = M_K \cdot M_\mathfrak{p}$ and the map $M_K \times m_\mathfrak{p} \to M$, defined by*

$$(k, X) \to k \exp X \tag{22}$$

is an analytic diffeomorphism.

ii) *The map $K \times \mathfrak{m}_\mathfrak{p} \times A_\mathfrak{p} \times N_+ \to G$ defined by*

$$(k, X, a, n) \to k \exp X \cdot a \cdot n \tag{23}$$

is an analytic diffeomorphism.

For the proof see [**10**].

We apply this lemma to G^C itself in the following situation.

G^C: a connected complex semi-simple Lie group $\curvearrowright \mathfrak{g}^C$,

$\tilde{\sigma}$: (anti-linear) involutive automorphism of $\mathfrak{g}^C$ with respect to a non-compact real form $\mathfrak{g}$, such that this automorphism can be lifted to G^C,

$\mathfrak{g}_u$: σ-invariant compact real form of $\mathfrak{g}^C$ and $\mathfrak{g}_u \curvearrowright G_u$,

$\tilde{\theta}$: (anti-linear) involutive automorphism of $\mathfrak{g}^C$ with respect to $\mathfrak{g}_u$,

$$\sigma = \tilde{\sigma} | \mathfrak{g}_u, \qquad \theta = \tilde{\theta} | \mathfrak{g}.$$

Let $\mathfrak{h}$, $\mathfrak{h}_u$, $\mathfrak{h}^C$, Δ, Δ^+, Δ_1^+, Δ_0^+ be as in the beginning of § 2.

Let $\mathfrak{n}^C = \sum_{\alpha \in \Delta^+} \boldsymbol{C} E_\alpha$, $\mathfrak{n}_+^C = \sum_{\alpha \in \Delta_1^+} \boldsymbol{C} E_\alpha$, $\mathfrak{n}_0^C = \sum_{\alpha \in \Delta_0^+} \boldsymbol{C} E_\alpha$, $\mathfrak{m}^C = \sum_{\alpha \in \Delta_0^+} (\boldsymbol{C} E_\alpha + \boldsymbol{C} E_{-\alpha}) + \mathfrak{h}_t^C$.

Then $\mathfrak{m}^C = \tilde{\mathfrak{m}}_1 + \tilde{\mathfrak{m}}_2$ where $\tilde{\mathfrak{m}}_1 = \mathfrak{m}^C \cap \mathfrak{g}_u$, $\tilde{\mathfrak{m}}_2 = \mathfrak{m}^C \cap \sqrt{-1}\,\mathfrak{g}_u$, $\mathfrak{m}^C \curvearrowright \tilde{M}$, $\tilde{M}_1 = \tilde{M} \cap G_u$, $\tilde{M}_2 = \tilde{M} \cap \exp(\sqrt{-1}\,\mathfrak{g}_u)$, $\tilde{M} = \tilde{M}_1 \cdot \tilde{M}_2$, $\mathfrak{n}_+^C \curvearrowright N_+^C$, $\mathfrak{h}_\mathfrak{p} \curvearrowright A_\mathfrak{p}$, $\sqrt{-1}\,\mathfrak{h}_t \curvearrowright A_K$, $\mathfrak{h}_R \curvearrowright A$, and $\mathfrak{h}^C \curvearrowright A^C$.

We have

$$\begin{aligned} G^C &= G_u \cdot AN^C \text{ (Iwasawa decomposition)} \\ &= G_u A N_0^C N_+^C = G_u \cdot N_0^C A_K A_\mathfrak{p} N_+^C \\ &= G_u \tilde{M}_1 \tilde{M}_2 A_\mathfrak{p} N_+^C = G_u \tilde{M}_2 A_\mathfrak{p} N_+^C . \end{aligned}$$

By Lemma 4 the map $G_u \times \tilde{M}_2 \times A_\mathfrak{p} \times N_+^C \to G^C$ defined by

$$(u, m, a, n) \to uman \tag{24}$$

and $\exp : \tilde{\mathfrak{m}}_2 \to \tilde{M}_2$, $\exp : \mathfrak{n}_+^C \to N_+^C$ defined by $X \to \exp X$, $Y \to \exp Y$ respectively are analytic diffeomorphisms. Now, suppose $v \in G_u \cap \Phi_0$, $v = g \cdot \tilde{\sigma}(g)^{-1}$, $g \in G^C$. If we write $g = uman$, then

$$v = uman\, \tilde{\sigma}(uman)^{-1} = uman\, \tilde{\sigma}(n)^{-1} \tilde{\sigma}(a)^{-1} \tilde{\sigma}(m)^{-1} \tilde{\sigma}(u)^{-1}$$

and

$$u^{-1} v \cdot \tilde{\sigma}(u) = man\, \tilde{\sigma}(n)^{-1} \tilde{\sigma}(a)^{-1} \tilde{\sigma}(m)^{-1} \tag{*}$$

$\tilde{\sigma}(\tilde{\mathfrak{m}}_2) = \tilde{\mathfrak{m}}_2$, $\tilde{\sigma}(\mathfrak{n}_+^C) = \mathfrak{n}_+^C$ and therefore $\tilde{\sigma}(\tilde{M}_2) = \tilde{M}_2$, $\tilde{\sigma}(N_+^C) = N_+^C$, $[\mathfrak{m}_+^C, \mathfrak{n}_+^C] \subset \mathfrak{n}_+^C$, and

$\tilde{\sigma}(a)=a$ for $a\in A_{\mathfrak{p}}$. The right hand side of (*) is equal to

$$man\tilde{\sigma}(n)^{-1}a^{-1}\tilde{\sigma}(m)^{-1}=m\tilde{\sigma}(m)^{-1}a\tilde{\sigma}(m)n\tilde{\sigma}(n)^{-1}\tilde{\sigma}(m)^{-1}a^{-1}$$

because any element of $A_{\mathfrak{p}}$ commutes with any one of $\tilde{M}_2$. Now $m\tilde{\sigma}(m)^{-1}\in\tilde{M}_1\tilde{M}_2$, $a\tilde{\sigma}(m)n\tilde{\sigma}(n)^{-1}\tilde{\sigma}(m)^{-1}a^{-1}\in N_+^c$. Therefore, by (24) we have $u^{-1}v\cdot\tilde{\sigma}(u)=m\tilde{\sigma}(m)^{-1}$, $n\tilde{\sigma}(n)^{-1}=1$ and $m\cdot\tilde{\sigma}(m)^{-1}\in\exp(\sqrt{-1}\,\mathfrak{g}_u)$, obviously, $\tilde{\sigma}(G_u)=G_u$. Put $\exp(\sqrt{-1}\,\mathfrak{g}_u)=\tilde{P}$. Then $\tilde{P}\cdot\tilde{P}\cap G_u=\{1\}$. For, suppose $p, p_1\in\tilde{P}$ such that $pp_1\in G_u$, then $\tilde{\theta}(pp_1)=pp_1$, $\tilde{\theta}(p)\tilde{\theta}(p_1)=pp_1$, $p^{-1}p_1^{-1}=pp_1$, therefore $p^{-2}=p_1^2$ and $p^{-1}=p_1$ i.e. $pp_1=1$. Thus we have proved: $u^{-1}v\tilde{\sigma}(u)=1$, $m\tilde{\sigma}(m)^{-1}=1$, $n\tilde{\sigma}(n)^{-1}=1$ and

$$v=u\cdot\tilde{\sigma}(u)^{-1}\,. \tag{25}$$

We define subsets of G_u:

$$\begin{cases}\Theta=\{u\in G_u\,;\ \sigma(u)=u^{-1}\}\\ \Theta_0=\{u\in G_u\,;\ u=v\cdot\sigma(v)^{-1}\text{ for a suitable }v\in G_u\}\,.\end{cases} \tag{26}$$

It is well known that Θ_0 coincides with the totally geodesic submanifold $\exp\sqrt{-1}\,\mathfrak{p}$ and is a compact symmetric space [**14**]. We have proved:

PROPOSITION 5. *$\Theta\cap\Phi_0=\Theta_0$, that is, if any element u of G_u satisfies $u=g\cdot\tilde{\sigma}(g)^{-1}$ for a suitable g of G^c, then there exists v of G_u such that $u=v\cdot\sigma(v)^{-1}$ and therefore u can be written* $\exp Y$ *for a suitable $Y\in\sqrt{-1}\,\mathfrak{p}$.*

COROLLARY. *In order that $\Phi=\Phi_0$ may hold, it is necessary and sufficient that $\Theta=\Theta_0$.*

Proof follows from Theorem 1 and Proposition 5.

LEMMA 5. *Let G be a compact, connected Lie group.*

i) *Let S be a toral subgroup of G. Suppose a is an element of G which commutes with each member of S. Then there exists a torus T such that $T\ni a$ and $T\supset S$.*

ii) *Any maximal torus of G is conjugate with each other.*

For the proof see [**13**], p 212 and p 247.

Let $\mathfrak{g}$ be any compact semi-simple Lie algebra, G' be the automorphism group of $\mathfrak{g}$, G be the connected component of G' containing the identity. As is well known, G' is a compact group, G is a normal subgroup of G' and $[G':G]<\infty$.

Let T be a maximal torus of G, its Lie algebra be $\mathfrak{h}$. We put $\mathfrak{h}_R=\sqrt{-1}\,\mathfrak{h}$. Let N' (or N) be a normalizer of T in G' (or in G respectively). $W'=N'/T$ and $W=N/T$ are both finite groups which operate on $\mathfrak{h}_R$ as rotation groups with respect to the Killing form of $\mathfrak{g}_u$. W is a normal subgroup of W', and $[G':G]=[W':W]$. $W'=P\cdot W$ where P is the so-called particular rotation group. This expression is unique.

Let $P=\{\tau_0=1,\tau_1,\cdots,\tau_{s-1}\}\subset W'$. For any τ_i we take a representative

ω_i' in N' corresponding to τ_i.

Let $\mathfrak{h}_i^+ = \{H \in \mathfrak{h},\ \tau_i H = H\}$ $0 \leqq i \leqq s-1$, $\mathfrak{h}_0^+ = \mathfrak{h}$. We put $T_i^+ = \exp \mathfrak{h}_i^+$ $(0 \leqq i \leqq s-1)$, $T_0^+ = T$, $T_i^+ \subset T$.

G' is decomposed into connected components:

$$G' = G + \omega_1 G + \cdots + \omega_{s-1} G . \tag{27}$$

LEMMA 6. *For each i $0 \leqq i \leqq s-1$, any element g of $\omega_i G$ is conjugate to some element of $\omega_i T_i^+$ with respect to G, that is, there exists $g_1 \in G$ and $\xi \in \omega_i T_i^+$ such that*

$$g = g_1^{-1} \cdot \xi \cdot g_1 . \tag{28}$$

For the proof see Gantmacher [7].

LEMMA 7*. *If ξ, ξ_i are contained in $\omega_i T_i^+$, satisfy $\xi^2 = \xi_1^2 = 1$ and are conjugate to each other with respect to G, then there exists $n \in N$ such that*

$$\xi_1 = n \xi n^{-1} . \tag{29}$$

PROOF. Suppose we have two Cartan decompositions of a compact real semi-simple algebra $\mathfrak{g}$ with respect to two involutions ξ, ξ_1 respectively,

$$\begin{aligned} \mathfrak{g} &= \mathfrak{k} + \sqrt{-1}\,\mathfrak{p} \qquad \text{(with respect to } \xi), \\ &= \mathfrak{k}_1 + \sqrt{-1}\,\mathfrak{p}_1 \qquad \text{(with respect to } \xi_1). \end{aligned}$$

ξ and ξ_1 are contained in the same component $\omega_j G$, for example. Then $\mathfrak{k} \cap \mathfrak{h} = \mathfrak{k}_1 \cap \mathfrak{h} = \mathfrak{h}_j^+$. Let there exist some $g \in G$ such that $\xi_1 = g \xi g^{-1}$. Then g transforms $\mathfrak{k}$ into $\mathfrak{k}_1$, $\sqrt{-1}\,\mathfrak{p}$ into $\sqrt{-1}\,\mathfrak{p}_1$ respectively. As $\mathfrak{h}_j^+$ is a Cartan subalgebra of $\mathfrak{k}$ and of $\mathfrak{k}_1$ so is $g(\mathfrak{h}_j^+)$. Let $K_1 \curvearrowright \mathfrak{k}_1$ in G. In virtue of Lemma 5 $g(\mathfrak{h}_j^+)$ is conjugate to $\mathfrak{h}_j^+$ with respect to K_1, and there exists $k_1 \in K_1$ such that $k_1(\mathfrak{h}_j^+) = g(\mathfrak{h}_j^+)$ and $\mathfrak{h}_j^+ = k_1^{-1} g(\mathfrak{h}_j^+)$. Since a Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}_j^+$ is unique and coincides with the centralizer of $\mathfrak{h}_j^+$ in $\mathfrak{g}$, we have $k_1^{-1} g(\mathfrak{h}) = \mathfrak{h}$ and $k_1^{-1} g \in N$. $k_1^{-1} g$ transforms $\mathfrak{k}$ into $\mathfrak{k}_1$, and $\sqrt{-1}\,\mathfrak{p}$ into $\sqrt{-1}\,\mathfrak{p}_1$ respectively, so we get $\xi_1 = k_1^{-1} g \xi (k_1^{-1} g)^{-1}$.

Let G_u be a connected compact semi-simple group with Lie algebra $\mathfrak{g}_u$, and an involutive automorphism σ of G_u be defined: $\sigma : G_u \to G_u$ and $\sigma : \mathfrak{g}_u \to \mathfrak{g}_u$. $\mathfrak{g}_u$ has a Cartan decomposition with respect to σ: $\mathfrak{g}_u = \mathfrak{k} + \sqrt{-1}\,\mathfrak{p}$. Let $\mathfrak{h}_u$ be a Cartan subalgebra of $\mathfrak{g}_u$ such that $\mathfrak{h}_u \cap \mathfrak{k}$ be a Cartan subalgebra of $\mathfrak{k}$. $\mathfrak{h}_u$ is called the "fundamental Cartan subalgebra" by Harish-Chandra. $\mathfrak{h}_u$ is invariant for σ. Let Z be the centre of G_u. We define Θ and Θ_0 by (26). Let $\mathfrak{h}_u \curvearrowright T_u$, $\mathfrak{h}_u^+ = \mathfrak{h} \cap \mathfrak{k} \curvearrowright T_u^+$, $\mathfrak{h}_u^- = \mathfrak{h}^- \cap \sqrt{-1}\,\mathfrak{p} \curvearrowright T_u^-$. $T_u = T_u^+ T_u^-$, $T_u^+ \subset K$ where $\mathfrak{k} \curvearrowright K$.

PROPOSITION 6. *For any $u \in \Theta$, there exist $v \in G_u$, $\xi \in T_u^+$, $z \in Z$ such that*

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* This lemma was proved by S. Murakami in his lecture at Univ. of Tokyo in 1962. Also see his paper [17].

(30) $$u = v \cdot \xi z \sigma(v)^{-1} \quad \textit{and} \quad \xi^2 = z^{-1}\sigma(z)^{-1}.$$

Further, if the equality

(31) $$\xi_1 z_1 = v \cdot \xi z \sigma(v)^{-1}$$

holds for two elements ξz, $\xi_1 z_1 \in T_u^+ Z$ *satisfying* $\xi^2 = z^{-1}\sigma(z)^{-1}$, $\xi_1^2 = z_1^{-1}\sigma(z_1)^{-1}$ *and for some element* $v \in G_u$, *then there exist* $n \in N$ (*normalizer of* T_u) *and* $z_0 \in Z$ *such that*

(32) $$\xi_1 z_1 = n \cdot \xi z z_0 \sigma(n)^{-1} \quad \textit{and} \quad z_0 \in Z \cap \Theta.$$

PROOF. $\sigma \cdot \mathrm{Ad}\, u$ is an involutive automorphism of $\mathfrak{g}_u$. By Lemma 6 there exist $v_1 \in G_u$ and $\xi_1 \in T_u^+$ such that $\sigma \cdot \mathrm{Ad}\, u = \mathrm{Ad}\, v_1 \cdot \sigma \cdot \mathrm{Ad}\, \xi\, \mathrm{Ad}\, v_1^{-1}$ and $\mathrm{Ad}\, u = \mathrm{Ad}\, \sigma(v_1)\xi v_1^{-1}$. Therefore $u = \sigma(v_1)\xi z v_1^{-1}$ for some element z of Z. We get (30) by putting $v = \sigma(v_1)$. (32) follows easily from Lemma 7.

Let ι be the projection of N onto $W = N/T_u$. W operates on $\sqrt{-1}\,\mathfrak{h}_u = \mathfrak{h}_R$ as a group of linear transformations. σ also operates on $\mathfrak{h}_R$. Let W_σ be the subgroup of W consisting of elements which commute with σ, and $N_\sigma = \iota^{-1}(W_\sigma)$. We take a representative element ω in $\iota^{-1}(w)$ for each $w \in W$ and denote it by $\omega = \omega(w)$. Suppose $\xi z \in T_u^+ Z$ be contained in Θ_0, then by (32) $\xi z = n \cdot z_0 \sigma(n)^{-1}$ ($n \in N$, $z_0 \in Z \cap \Theta_0$). Therefore $\iota(n) = \iota(\sigma(n))$. We put $\iota(n) = w$, and $n = \omega a$, $\omega = \omega(w)$, $a \in T_u$. w lies in W_σ. We have $\xi z = \omega\sigma(a) z_0 a^{-1}\sigma(\omega)^{-1} = \omega\sigma(a)a^{-1} z_0 \sigma(\omega)^{-1} = \omega\sigma(\omega)^{-1} z_0 \sigma(\omega)\sigma(a)a^{-1}\sigma(\omega)^{-1}$. $\sigma(a)a^{-1} \in T_u^+$ and w makes $\mathfrak{h}_u^+$ and $\mathfrak{h}_u^-$ invariant respectively, because $w \in W_\sigma$. Therefore $\sigma(\omega)\sigma(a)a^{-1}\sigma(\omega)^{-1} \in T_u^-$. We define the set

(33) $$\Gamma = \{\omega \cdot \sigma(\omega)^{-1};\ \omega = \omega(w),\ w \in W_\sigma\} \subset T_u.$$

Then $\xi z \in \Gamma \cdot (Z \cap \Theta) \cdot T_u^-$. But $\Gamma \cdot (Z \cap \Theta_0) \cdot T_u^- \subset \Theta_0$ holds because any element $\gamma \cdot z \cdot \exp H_1$ of $\Gamma \cdot (Z \cap \Theta) \cdot T_u^-$, where $\gamma = \omega \cdot \sigma(\omega)^{-1}$, $z \in Z \cap \Theta_0$, $H_1 \in \mathfrak{h}_u^-$, can be written

$$\gamma \cdot z \cdot \exp H_1 = \exp \frac{1}{2} H_1 \cdot \omega\sigma(\omega)^{-1} z \exp \frac{1}{2} H_1$$

$$= \exp \frac{1}{2} H_1 \cdot \omega \cdot z \cdot \sigma(\omega)^{-1} \sigma\left(\exp \frac{1}{2} H_1\right)^{-1}$$

and therefore $\gamma z \exp H_1$ belongs to Θ_0. We have proved:

PROPOSITION 7.

(34) $$(T_u^+ \cdot Z) \cap \Theta_0 = \Gamma \cdot (Z \cap \Theta) T_u^- \cap (T_u^+ Z).$$

COROLLARY. *In case* G_u *is the adjoint group of* $\mathfrak{g}_u$, σ *is defined on* G_u *and* $Z = \{1\}$, *and therefore*

(35) $$T_u^+ \cap \Theta_0 = \Gamma \cdot (T_u^- \cap T_u^+).$$

PROPOSITION 8. *In order that any element* $u \in \Theta$ *is contained in* Θ_0, *it is*

necessary that $\sigma \cdot \mathrm{Ad}\, u$ *is conjugate to* σ *in the adjoint group and this condition is sufficient when* G_u *coincides with the adjoint group itself.*

We can now answer if Θ coincides with Θ_0 or not, by calculating (34) or (35).

REMARK. Let G_u be the adjoint group, $\mathfrak{g}_u$ be the Lie algebra of G_u and $B(X, Y)$ $(X, Y \in G_u)$ be the Killing form of $\mathfrak{g}_u$, then

$$B(\sigma X, Y) \tag{36}$$

is a symmetric bilinear form on $\mathfrak{g}_u$, where σ is any involutive automorphism. Suppose, further, $\mathfrak{g}_u$ be simple. If two involutions σ, τ define real simple algebras (non-compact) of the same-type, then the signature $B(\sigma X, Y)$ and $B(\tau X, Y)$ are equal. But the converse is not always true, though it was conjectured by E. Cartan (See [**13**]). The exceptional case can take place when $\mathfrak{g}_u$ is of type D_n. Therefore we can not determine whether $u \in \Theta$ belongs to Θ_0 or not, by calculating the signature, when $\mathfrak{g}$ is of type D_n. But in other cases, it is possible to determine whether $u \in \Theta_0$ or not by calculating the sign. We shall use this method later in an example.

§ 3. The finiteness of double cosets $B\backslash G^c/G$.

It follows easily $\Psi_0 = \Psi \cap \Theta_0$ where $\Theta_0 = \exp \sqrt{-1}\,\mathfrak{p}$. For any element $u \in \Theta_0$ there exists an element $\tilde{u} \in \Theta_0$ such that $u = \tilde{u}^2$. There may exist many such elements, but we shall fix one element of them and denote it by $\sqrt{u}$. We define

$$\sqrt{[\Psi_0]} = \{\sqrt{\omega a}\,;\ \omega a \in [\Psi_0]\}\,. \tag{37}$$

Suppose $g \in G^c$ (in the situation of Theorem 1), then $g \cdot \tilde{\sigma}(g)^{-1} \in \Phi_0$. By Theorem 1 there exist $b \in B$, $\omega a \in [\Psi_0]$ such that

$$g \cdot \tilde{\sigma}(g)^{-1} = b \cdot \omega a \cdot \tilde{\sigma}(b)^{-1}\,. \tag{38}$$

Let $\sqrt{\omega a} = \gamma$, then $\omega a = \gamma^2 = \gamma \cdot \tilde{\sigma}(\gamma)^{-1}$ and (38) becomes $g \cdot \tilde{\sigma}(g)^{-1} = b \cdot \gamma \cdot \tilde{\sigma}(b\gamma)^{-1}$ where $(b\gamma)^{-1}$ is $\tilde{\sigma}$-fixed. Therefore $(b\gamma)^{-1} g \in G$ and

$$g \in b\gamma G \subset B \cdot \gamma \cdot G\,. \tag{39}$$

Conversely, for $\gamma_1, \gamma_2 \in \sqrt{[\Psi_0]}$ such that $\gamma_1 \in B\gamma_2 G$ we have $\gamma_1^2 = b \cdot \gamma_2^2 \cdot \tilde{\sigma}(b)^{-1}$ for some $b \in B$. Consequently, $\gamma_1^2 = \gamma_2^2$ by Theorem 1 and $\gamma_1 = \gamma_2$. We now have:

THEOREM 2. (In the situation of Theorem 1.) $G^c = B \cdot \sqrt{[\Psi_0]} \cdot G$ *holds. The double coset* $B\backslash G^c/G$ *is finite and can be identified with* $\sqrt{[\Psi_0]}$ *canonically.*

It is well known that any Borel subgroup of G^c is conjugate with each other, and therefore the Borel subgroup B in Theorem 2 may be replaced by an arbitrary Borel subgroup of G^c. Thus

THEOREM 2′. *Let G^c be any connected complex semi-simple Lie group, B be any Borel subgroup of G^c, and G be any real form of G^c (connected or not), then the double coset $B\backslash G^c/G$ is finite.*

In Theorem 2 the right coset $B\backslash G^c$ is a compact complex algebraic manifold on which real semi-simple Lie group G operates. G does not operate transitively, in general. But we have found the number of orbits is finite by Theorem 2.

Next we shall calculate the dimension of each orbit. Let $\gamma=\sqrt{\omega a}$, $\omega a \in [\Psi_0]$. $B\gamma$ is a point of $B\backslash G^c$. The G-orbit of $B\gamma$ is $B\gamma G$ and its isotropy subgroup is the group S:

$$S=\{x\in G;\ \gamma x\gamma^{-1}\in B\}\ . \tag{40}$$

Let $\gamma x\gamma^{-1}=b$, $b\in B$, $x=\gamma^{-1}b\gamma$. $x\in G$ is equivalent to $\tilde{\sigma}(\gamma^{-1}b\gamma)=\gamma^{-1}b\gamma$, that is $\gamma\cdot\tilde{\sigma}(b)\gamma^{-1}=\gamma^{-1}b\gamma$, $\gamma^2\tilde{\sigma}(b)\gamma^{-2}=b$ and

$$\omega a\tilde{\sigma}(b)(\omega a)^{-1}=b\ . \tag{41}$$

We have $\mathfrak{h}^c\subset\mathfrak{b}$, $A^c\subset B$. We put $\tilde{\sigma}_1=\mathrm{Ad}(\omega a)\cdot\tilde{\sigma}$. $\tilde{\sigma}_1$ operates on $\mathfrak{h}^c$ as a conjugate isomorphism. Let $\mathfrak{h}_1$ be the set of $\tilde{\sigma}_1$-fixed elements of $\mathfrak{h}^c$. Then $\mathfrak{h}_1$ is a real form of $\mathfrak{h}^c$, and therefore $\mathrm{Ad}\,\gamma^{-1}(\mathfrak{h}_1)$ is a Cartan subalgebra of $\mathfrak{g}$. As $\mathfrak{h}_1$ contains a regular element of $\mathfrak{g}$, and A^c is a Cartan subgroup of G^c, $G\cap\mathrm{Ad}\,\gamma^{-1}(A^c)=A_1$ is a Catran subgroup of G in the sense of Harish-Chandra. Therefore S always contains a Cartan subgroup of G.

We define the subsets of positive roots:

$$\begin{aligned}\boldsymbol{P}_+^{\sigma_1}&=\{\alpha\in\boldsymbol{P},\ \sigma_1(\alpha)>0\}\ ,\\ \boldsymbol{P}_-^{\sigma_1}&=\{\alpha\in\boldsymbol{P},\ \sigma_1(\alpha)<0\}\ .\end{aligned} \tag{42}$$

The Lie algebra $\mathfrak{s}$ of S is

$$\mathfrak{s}=\mathrm{Ad}\,\gamma^{-1}(\mathfrak{n}_1+\mathfrak{h}_1) \tag{43}$$

where

$$\mathfrak{n}_1=\sum_{\alpha\in\boldsymbol{P}_+^{\sigma_1}}\boldsymbol{R}(E_\alpha+\sigma_1E_\alpha)+\sum_{\alpha\in\boldsymbol{P}_+^{\sigma_1}}\sqrt{-1}\,\boldsymbol{R}(E_\alpha-\sigma_1E_\alpha) \tag{44}$$

is a subalgebra of $\mathfrak{n}^c$. In particular, if $\boldsymbol{P}_+^{\sigma_1}=\phi$ then $S=A_1$ is an abelian group. If $\mathfrak{h}$ is a fundamental Cartan subalgebra of $\mathfrak{g}$,

$$\mathfrak{h}=\mathfrak{h}_t+\mathfrak{h}_p\ ,\qquad \mathfrak{h}_R=\sqrt{-1}\,\mathfrak{h}_t+\mathfrak{h}_p$$

and if we take the order of $(\mathfrak{h}_R)^*$ (the dual space of $\mathfrak{h}_R$) such that for $\lambda\in\mathfrak{h}_R^*$

$$\begin{aligned}&\lambda>0\Leftrightarrow\lambda|\sqrt{-1}\,\mathfrak{h}_t>0\quad\text{or}\\ &\qquad\lambda|\sqrt{-1}\,\mathfrak{h}_t=0\quad\text{and}\quad\lambda|\mathfrak{h}_p>0\ .\end{aligned} \tag{45}$$

For any $\alpha \in \Delta$, we have $\alpha|\sqrt{-1}\mathfrak{h}_\mathfrak{k} \neq 0$, as is easily proved. Seeing that $\sigma(\boldsymbol{P}) = -\boldsymbol{P}$ and $w(\boldsymbol{P}) \neq \boldsymbol{P}$, for any $w \neq 1 \in W$,

$$\sigma_1(\boldsymbol{P}) = -\boldsymbol{P} \tag{46}$$

is equivalent to $\omega = 1$. Let $\tilde{\mathfrak{n}}^c = \sum_{\alpha \in \boldsymbol{P}} \boldsymbol{C}E_{-\alpha}$, and $\tilde{\mathfrak{n}}^c \curvearrowright \tilde{N}^c$ in G^c, $\mathfrak{b} = \sum_{\alpha \in \boldsymbol{P}} \boldsymbol{C}E_\alpha + \mathfrak{h}^c$ and $\mathfrak{b} \curvearrowright B$ in G^c. When (46) holds, $\omega a = a$ $(a \in T^+)$, $a^2 = 1$, $\tilde{\sigma}_1(N^c) = \tilde{N}^c$ and $\mathfrak{s} = \mathfrak{h}_1$. $\mathfrak{s}$ has the minimal dimension which is equal to the dimension of $B\backslash G^c$, and $S\backslash G$ is an open orbit in $B\backslash G^c$. In other cases, the dimension of $S\backslash G$ is smaller than that of $B\backslash G^c$ and therefore is not an open orbit. We shall see later some examples of open orbits. The number of these orbits is equal to $[(T^+ \cap \Theta_0) : T^+ \cap T^-]$.

Let $n = \dim \mathfrak{g}$, $l = \text{rank of } \mathfrak{g}$, $m = \text{number of } \boldsymbol{P}$, $m_1 = \dim \mathfrak{n}_1$ $(l + 2m = n)$. Then $\dim \mathfrak{s} = l + m_1$ and

$$\dim S\backslash G = n - l - m_1 = 2m - m_1 . \tag{47}$$

LEMMA 8. *Let $\mathfrak{g}$ be a compact Lie algebra, G be the group of inner automorphisms on $\mathfrak{g}$. Suppose σ be the involution on $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{k} + \sqrt{-1}\mathfrak{p}$, $\mathfrak{k} = \{\sigma X = X\}$, $\sqrt{-1}\mathfrak{p} = \{\sigma X = -X\}$. Let $K \curvearrowright \mathfrak{k}$ in G. Then any two maximal abelian subalgebras in $\sqrt{-1}\mathfrak{p}$ are conjugate with each other with respect to K.*

PROOF. $\mathfrak{g}$ is a direct sum of its centre $\mathfrak{c}$ and its derived algebra $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$: $\mathfrak{g} = \mathfrak{g}' + \mathfrak{c}$, $\mathfrak{g}'$ and $\mathfrak{c}$ are both invariant with respect to σ. Therefore $\mathfrak{g}' = \mathfrak{g}'_+ + \mathfrak{g}'_-$, $\mathfrak{c} = \mathfrak{c}_+ + \mathfrak{c}_-$, where $\mathfrak{g}'_+ = \mathfrak{g}' \cap \mathfrak{k}$, $\mathfrak{g}'_- = \mathfrak{g}' \cap \sqrt{-1}\mathfrak{p}$, $\mathfrak{c}_+ = \mathfrak{c} \cap \mathfrak{k}$, $\mathfrak{c}_- = \mathfrak{c} \cap \sqrt{-1}\mathfrak{p}$, and $\sqrt{-1}\mathfrak{p} = \mathfrak{g}'_- + \mathfrak{c}_-$. Any maximal abelian subalgebra $\mathfrak{h}_-$ in $\sqrt{-1}\mathfrak{p}$ contains $\mathfrak{c}_-$, and therefore $\mathfrak{h}_-$ is written in the following form

$$\mathfrak{h}_- = \mathfrak{h}'_- + \mathfrak{c}_- , \qquad \mathfrak{h}'_- = \mathfrak{h}_- \cap \mathfrak{g}'$$

$\mathfrak{h}'_-$ is a maximal abelian subalgebra in $\mathfrak{g}'_-$. Thus we have reduced this problem into that of Cartan's well-known theorem about conjugacy of $\mathfrak{h}'_-$ [**14**]. q. e. d.

Let $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ with the largest non-compact part. We define an order of $\mathfrak{h}^*_R$ as follows:

$$\lambda \in \mathfrak{h}^*_R , \quad \lambda > 0 \Leftrightarrow \lambda|\mathfrak{h}_\mathfrak{p} > 0 \quad \text{or} \tag{48}$$
$$\lambda|\sqrt{-1}\mathfrak{h}_\mathfrak{k} > 0 , \quad \lambda|\mathfrak{h}_\mathfrak{p} = 0 .$$

We define Δ (root-system), $\boldsymbol{P}$ (positive roots), $\boldsymbol{P}_+ = \{\alpha \in \boldsymbol{P},\ \alpha|\mathfrak{h}_\mathfrak{p} > 0\}$, $\boldsymbol{P}_0 = \{\alpha \in \boldsymbol{P},\ \alpha|\sqrt{-1}\mathfrak{h}_\mathfrak{k} > 0,\ \alpha|\mathfrak{h}_\mathfrak{p} = 0\}$, $\boldsymbol{P} = \boldsymbol{P}_+ \cup \boldsymbol{P}_0$ and $\mathfrak{g} = \sum_{\alpha \in \boldsymbol{P}} (\boldsymbol{C}E_\alpha + \boldsymbol{C}E_{-\alpha}) + \mathfrak{h}^c$. $\mathfrak{b} = \sum_{\alpha \in \boldsymbol{P}} \boldsymbol{C}E_\alpha + \mathfrak{h}^c$ is a Lie algebra of a Borel subgroup B of G^c. Put $\mathfrak{n}^c = \sum_{\alpha \in \boldsymbol{P}} \boldsymbol{C}E_\alpha$, $\mathfrak{n}^c_+ = \sum_{\alpha \in \boldsymbol{P}_+} \boldsymbol{C}E_\alpha$ and $\mathfrak{n}^c_0 = \sum_{\alpha \in \boldsymbol{P}_0} \boldsymbol{C}E_\alpha$.

In case of $\omega = 1$, (44) becomes

$$\mathfrak{n}_1 = \mathfrak{n}^c_+ \cap \mathfrak{g} , \qquad \mathfrak{s} = \mathfrak{n}_1 + \mathfrak{h}_\mathfrak{p} , \tag{49}$$

where $\mathfrak{h}_\mathfrak{p}$ is a maximal abelian subalgebra of $\mathfrak{p}$. In this case, $\mathfrak{s}$ has a maximal dimension and such $\mathfrak{s}$ is a maximal solvable subalgebra of $\mathfrak{g}$ [**14**].

Moreover, it seems to me that this is the only case where $\mathfrak{s}$ has a maximal dimension, but I can not prove it in general.

Let $T=T^+T^-$, where $T^+\curvearrowleft\mathfrak{h}_\mathfrak{t}$, $T^-\curvearrowleft\sqrt{-1}\,\mathfrak{h}_\mathfrak{p}$. We claim:

PROPOSITION 9.

$$T^+\cap\Theta_0=T^+\cap T^-. \tag{50}$$

PROOF. Let $a\in T^+\cap\Theta_0$ then $a^2=1$. Let $\mathfrak{g}'_u$ be the centralizer of a in $\mathfrak{g}_u$ and $\mathfrak{g}'_u\curvearrowleft G'_u$ in G_u. a lies in the centre of G'_u. As $\mathfrak{g}'_u$ is invariant with respect to σ, so is G'_u. G'_u contains T. Since $a\in\Theta_0$, there exists $k\in K$ such that $a\in k\cdot T^-\cdot k^{-1}$. But any elements of $k\cdot T^-\cdot k^{-1}$ commute with a, so $kT^-k^{-1}\subset G'_u$, and $\mathrm{Ad}\,k(\sqrt{-1}\,\mathfrak{h}_\mathfrak{p})$ is a maximal abelian subalgebra in $\mathfrak{g}'_u\cap\sqrt{-1}\,\mathfrak{p}$. By Lemma 8 there exists $k_1\in G'_u\cap K$ such that $kT^-k^{-1}=k_1T^-k_1^{-1}$, because $\sqrt{-1}\,\mathfrak{h}_\mathfrak{p}\subset\mathfrak{g}'_u\cap\sqrt{-1}\,\mathfrak{p}$. Hence $a\in kT^-k^{-1}=k_1T^-k_1^{-1}$ and k_1 commutes with a, $a\in T^-$. q. e. d.

COROLLARY. *$T^+\cap[\Psi_0]$ consists of only one element.*

CONJECTURE. *The number of G-orbits in $B\backslash G^C$ which have the smallest dimension is one, and this orbit is isomorphic to the coset space G by the maximal solvable subgroup S defined by* (49).

Next we shall find the relation between orbit spaces and conjugate classes of Cartan subalgebras of $\mathfrak{g}$. The same result was obtained by H. Matsumoto independently.

We recall that $\Psi_0=\{u\sigma(u)^{-1}\in\mathcal{N}(T);\ u\in G_u\}$ where $\mathcal{N}(T)$ is a normalizer of T in G_u. Let $T\curvearrowleft\mathfrak{h}_u$, $\sigma(\mathfrak{h}_u)=\mathfrak{h}_u$. If $u\sigma(u)^{-1}\in\mathcal{N}(T)$, $\mathrm{Ad}\,u^{-1}(\mathfrak{h}_u)=\mathfrak{h}'_u$ is also σ-invariant. The converse is also true. $u\sigma(u)^{-1}\in\mathcal{N}(T)$ if and only if $\mathrm{Ad}\,u^{-1}(\mathfrak{h}_u)$ is σ-invariant.

LEMMA 9. *Let $\mathfrak{h}_u$ and $\mathfrak{h}'_u$ be two σ-invariant Cartan subalgebras. $\mathfrak{h}_u=\mathfrak{h}_\mathfrak{t}+\sqrt{-1}\,\mathfrak{h}_\mathfrak{p}$, $\mathfrak{h}_\mathfrak{t}=\mathfrak{h}_u\cap\mathfrak{k}$, $\sqrt{-1}\,\mathfrak{h}_\mathfrak{p}=\mathfrak{h}_u\cap\sqrt{-1}\,\mathfrak{p}$, $\mathfrak{h}'_u=\mathfrak{h}'_\mathfrak{t}+\sqrt{-1}\,\mathfrak{h}'_\mathfrak{p}$, $\mathfrak{h}'_\mathfrak{t}=\mathfrak{h}'_u\cap\mathfrak{k}$, $\sqrt{-1}\,\mathfrak{h}'_\mathfrak{p}=\mathfrak{h}'_u\cap\sqrt{-1}\,\mathfrak{p}$. Then $\mathfrak{h}=\mathfrak{h}_\mathfrak{t}+\mathfrak{h}_\mathfrak{p}$, $\mathfrak{h}'=\mathfrak{h}'_\mathfrak{t}+\mathfrak{h}'_\mathfrak{p}$ are both Cartan subalgebras of $\mathfrak{g}$. Moreover, in order that $\mathfrak{h}$ and $\mathfrak{h}'$ are conjugate with each other under the group of inner automorphisms $\mathrm{Ad}(G)$ of $\mathfrak{g}$, it is necessary and sufficient that $\mathfrak{h}$ and $\mathfrak{h}'$ are conjugate only under $\mathrm{Ad}\,K$. The last condition is equivalent to that $\mathfrak{h}_u$ and $\mathfrak{h}'_u$ are conjugate under $\mathrm{Ad}\,K$.*

PROOF. It follows immediately from Theorem 2 and Proposition 5 in [**19**] p. 383.

Let $\mathfrak{h}^i$ be the standard Cartan subalgebra of $\mathfrak{g}$ in each conjugate class ($i=1, 2, \cdots, r$) such that $\mathfrak{h}^i=(\mathfrak{h}^i\cap\mathfrak{k})+(\mathfrak{h}^i\cap\mathfrak{p})$. We define $\mathfrak{h}^i_u=\mathfrak{h}^i\cap\mathfrak{k}+\mathfrak{h}^i\cap\sqrt{-1}\,\mathfrak{p}$. Let $\mathfrak{h}^i_u\curvearrowleft T^i_u$ in G_u, $\mathcal{N}(T^i_u)=\mathcal{N}^i$ be the normalizer of T^i_u in G. Suppose $\omega a\in\Psi_0$ be written in the form $\omega a=u\sigma(u)^{-1}$, $u\in G$. $\mathrm{Ad}\,u^{-1}(\mathfrak{h}_u)$ is conjugate to some

$\mathfrak{h}_u^j$ under $\mathrm{Ad}\,K$:

$$\mathrm{Ad}\,u^{-1}(\mathfrak{h}_u)=\mathrm{Ad}\,k(\mathfrak{h}_u^j) \qquad k\in K.$$

We fix an element γ_j for each j such that

$$\mathrm{Ad}\,\gamma_j^{-1}(\mathfrak{h}_u)=\mathfrak{h}_u^j \qquad \gamma_j\in G_u. \tag{51}$$

Then $\gamma_j^{-1}uk\in\mathfrak{N}^j$ and

$$\omega a=u\sigma(u)^{-1}=\gamma_j n\sigma(n)^{-1}\sigma(\gamma_j)^{-1}, \tag{52}$$

$$\gamma_j^{-1}\mathfrak{N}\gamma_j=\mathfrak{N}^j. \tag{53}$$

If ωa and $\omega_1 a_1$ be in the same class of $[\Psi_0]$, and ωa is represented in the form (52), then $\omega_1 a_1$ can be written

$$\omega_1 a_1=\gamma_j(na')\sigma(na')^{-1}\sigma(\gamma_j)^{-1} \tag{54}$$

for suitable $a'\in T_u^j$ by Theorem 1, and conversely.

The set

$$\Psi^j=\{n\sigma(n)^{-1};\ n\in\mathfrak{N}^j\} \qquad 1\leqq j\leqq r \tag{56}$$

is canonically isomorphic to the coset space of $\mathfrak{N}^j$: $\mathfrak{N}^j/(\mathfrak{N}^j\cap K)$. In virtue of (54), $[\Psi_0]$ is canonically isomorphic to the set

$$\bigcup_{j=1}^{r}\mathfrak{N}^j/(\mathfrak{N}^j\cap K)\cdot T_u^j. \tag{57}$$

We have arrived at:

THEOREM 3. *Let $\mathfrak{h}^1,\mathfrak{h}^2,\cdots,\mathfrak{h}^r$ be standard Cartan subalgebras of $\mathfrak{g}$ and $\mathfrak{h}_u^1,\mathfrak{h}_u^2,\cdots,\mathfrak{h}_u^r$ be their compact form. Let T_u^j be a maximal torus corresponding to $\mathfrak{h}_u^j$ and $\mathfrak{N}^j$ be the normalizer of T_u^j in G_u for each j. Then*

$$[\Psi_0]\approx\bigcup_{1\leqq j\leqq r}\mathfrak{N}^j/(\mathfrak{N}^j\cap K)\cdot T_u^j=\bigcup_{1\leqq j\leqq r}W/W^j$$

where $W=\mathfrak{N}^j/T_u^j$ and $W^j=(\mathfrak{N}^j\cap K)\cdot T_u^j/T_u^j$ is a subgroup of W.

§ 4. Examples.

NOTATION. (We shall make use of Sugiura's notations in [19] from now on.) $\mathfrak{M}(m,n,\boldsymbol{C})$ = the set of complex matrices with m lines and n columns. $\mathfrak{M}(m,n,\boldsymbol{R})$ = the set of real matrices with m lines and n columns. $\mathfrak{gl}(n,\boldsymbol{C})=\mathfrak{M}(n,n,\boldsymbol{C})$. $\mathfrak{gl}(n,\boldsymbol{R})=\mathfrak{M}(n,n,\boldsymbol{R})$. $\mathfrak{u}(n)$ = the set of skew hermitian matrices. $\mathfrak{su}(n)$ = the set of skew hermitian matrices whose traces are zero. $\mathfrak{sl}(n,\boldsymbol{C})$ (or $\mathfrak{s}(n,\boldsymbol{R})$) = the set of complex (or real) square matrices whose traces are zero. $He(n)$ = the set of hermitian matrices. $Sy(n,\boldsymbol{C})$ (or $Sy(n,\boldsymbol{R})$) = the set of complex (or real) symmetric matrices. $o(n,\boldsymbol{C})$ (or $o(n)$) = the set of complex (or real) skew symmetric matrices. ${}^tX$ = transposed matrix of X. $\det X$ = determinant of X. $\bar{X}$ = complex conjugate of X. $Tr(X)$ = trace of X.

1. $G^c=SL(n,\boldsymbol{C})$, $G=SL(n,\boldsymbol{R})$, $G_u=SU(n)$, $K=SO(n)$, then $Z=\{$scalar

matrices with determinants 1}, $\mathfrak{k}=\{$skew symmetric, real$\}$, $\sqrt{-1}\mathfrak{p}=\{$symmetric, skew hermitian, trace $=0\}$. We put $n=2\nu$ or $2\nu+1$ according as n is even or odd.

We denote the diagonal matrix $\begin{pmatrix}\lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n\end{pmatrix}$ by Diag $[\lambda_1\lambda_2\cdots\lambda_n]$. Let $\mathfrak{h}'$ be a fundamental Cartan subalgebra, $\mathfrak{h}'=\mathfrak{h}'_{\mathfrak{k}}+\mathfrak{h}'_{\mathfrak{p}}$, $\mathfrak{h}'_{\mathfrak{k}}\curvearrowright T'_+$:

$$T'_+=\left\{\begin{pmatrix} \cos\theta_1 & \sin\theta_1 & & & & \\ -\sin\theta_1 & \cos\theta_1 & & & & \\ & & \cos\theta_2 & \sin\theta_2 & & \\ & & -\sin\theta_2 & \cos\theta_2 & & \\ & & & & \ddots & \\ & & & & & 1\end{pmatrix}=\xi\right\} \tag{58}$$

The lowest term 1 appears only in case of $n=2\nu+1$.

If $\xi^2=1$, $\xi=$ Diag $[\pm(1,1),\pm(1,1),\cdots,\pm(1,1),1]$. But these ξ are contained in Θ_0.

$$\sqrt{-1}\mathfrak{p}\ni X=\text{Diag}\,[\sqrt{-1}\varphi_1\ \sqrt{-1}\varphi_1\ \sqrt{-1}\varphi_2\ \sqrt{-1}\varphi_2,$$
$$\cdots\sqrt{-1}\varphi_\nu\ \sqrt{-1}\varphi_\nu\ \sqrt{-1}\varphi_{\nu+1}]$$
$$2\varphi_1+2\varphi_2+\cdots+2\varphi_\nu+\varphi_{\nu+1}=0 \qquad \varphi_i \text{ real.}$$

If we put $\varphi_i=0$ or π, $1\leqq i\leqq\nu$, $\varphi_{\nu+1}=0$ (mod $2\pi\boldsymbol{Z}$ integers) suitably, we obtain all ξ, $\xi^2=1$. On the other $z\sigma(z)=1$ for any $z\in Z$. And if $\xi z\in\Theta$, then $\xi^2=1$. $Z\subset\Theta_0$ and therefore $\xi z\in\Theta_0$. By Proposition 6, $\Theta=\Theta_0$ and $\Phi=\Phi_0$. This fact is well-known and can be proved by a direct method.

If G^c is replaced by the adjoint group of $SL(n,\boldsymbol{C})$, $\Theta\neq\Theta_0$ in general. This occurs for example in case $n=2\nu$ and

$$\mathfrak{g}=\begin{pmatrix} & \overset{\nu}{\curvearrowleft} & & \overset{\nu}{\curvearrowright} & \\ & & 1 & & \\ & & & 1 & \\ & & & & \ddots \\ & & & & 1 \\ -1 & & & & \\ & -1 & & & \\ & & \ddots & & \\ & & & -1 & \end{pmatrix}\in SL(n,\boldsymbol{C}) \tag{59}$$

$\tilde{\sigma}(g\cdot Z)=\bar{g}\cdot Z=g\cdot Z$, $\sigma(g\cdot Z)\cdot(g\cdot Z)=g^2\cdot Z=Z$, $\tilde{\sigma}(g\cdot Z)=g^{-1}\cdot Z$. But $g\cdot Z$ can not be written in the form $g_1\cdot\bar{g}_1^{-1}Z$ for any element of $SL(n, \boldsymbol{C})$. For, if we put $g_1^{-1}=\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22}\end{pmatrix}$, where g_{ij} $\nu\times\nu$ matrix, then $\lambda\begin{pmatrix} -g_{12} & g_{11} \\ -g_{22} & g_{21}\end{pmatrix}=\lambda\begin{pmatrix} \bar{g}_{11} & \bar{g}_{12} \\ \bar{g}_{21} & \bar{g}_{22}\end{pmatrix}$, $\lambda^{2\nu}=1$.

$$-g_{12}=\lambda\bar{g}_{11}\,,\quad g_{11}=\lambda\bar{g}_{12}\,,\quad -g_{22}=\lambda\bar{g}_{21}\,,\quad g_{21}=\lambda\bar{g}_{22}\,,$$
$$-g_{12}=|\lambda|^2g_{12}\,,\quad -g_{22}=|\lambda|^2g_{22}\qquad \text{for some}\quad \lambda\in\boldsymbol{C}\,.$$

But these equations are impossible except for $g_{12}=g_{22}=0$ and therefore except for $g_1^{-1}=0$, which is impossible. This fact is deduced also from Proposition 8 and the classification of real simple Lie algebras of A_n type.

Let $B=\{\text{upper triangular matrices}\}\cap SL(n, \boldsymbol{C})$,

$$\text{(60)}\qquad \mathfrak{h}=\{H=\mathrm{Diag}\,[t_1t_2\cdots t_n],\ \sum_{i=1}^{n}t_i=0,\ t_i\ \text{real}\}\,,\qquad \mathfrak{h}_u=\sqrt{-1}\,\mathfrak{h}\,.$$

For any $H\in\mathfrak{h}_u$, $\sigma(H)=-H$ therefore σ is commutable with each member of W (Weyl group with respect to $\mathfrak{h}_u$) and

$$\text{(61)}\qquad (\sigma\cdot w)^2=1\Leftrightarrow w^2=1\qquad \text{for}\quad w\in W\,,$$

where W is the symmetric group of degree n.

$$\text{(62)}\qquad w[\sqrt{-1}t_1\ \sqrt{-1}t_2\cdots\sqrt{-1}t_n]=[\sqrt{-1}t_{i_1}\ \sqrt{-1}t_{i_2}\cdots\sqrt{-1}t_{i_n}]$$

where $\begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n\end{pmatrix}$ is a permutation of $(1, 2, 3, \cdots, n)$. For any two elements w, w' of W,

$$\sigma\cdot w \text{ is conjugate with } \sigma\cdot w' \text{ under } W\,,$$
$$\Leftrightarrow w \text{ is conjugate with } w' \text{ under } W\,.$$

The conjugate classes of W are represented by cyclic permutations. The classes of elements of order 2 are:

$$1,\ (12),\ (12)(34),\ \cdots,\ (12)(34)\cdots(2\nu-1,\ 2\nu)\,.$$

We denote by $\#[w]$ the number of conjugate elements of w.

Case $w=1$, $\omega(w)=1$, $\sigma(\omega)\omega=1$, $\sigma\cdot w(H)=-H$, $H\in\mathfrak{h}_u$, $\mathfrak{h}_u^+=\{0\}$, $\mathfrak{h}_u^-=\mathfrak{h}_u$, $T_w^+=\{1\}$ (Notations are the same as in (12), (13)) $\hat{T}_w^+=\{1\}$. The total $=1$.

Case $w=(12)$, $\#[w]=\binom{n}{2}$

$$\omega=\begin{pmatrix} & 1 & & & \\ 1 & & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ & & & & & -1\end{pmatrix}\qquad \sigma(\omega)\omega=1$$

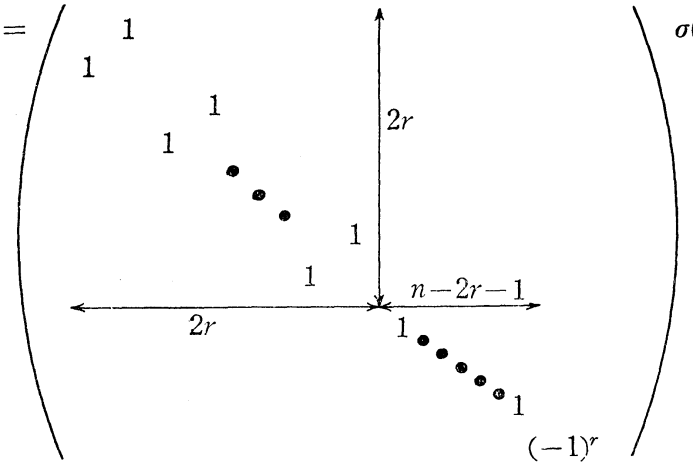
$\sigma \cdot w\,\mathrm{Diag}\,[\sqrt{-1}t_1\ \sqrt{-1}t_2 \cdots \sqrt{-1}t_n]$
$= \mathrm{Diag}\,[-\sqrt{-1}t_2, -\sqrt{-1}t_1, -\sqrt{-1}t_3, \cdots, -\sqrt{-1}t_n]$
$\mathfrak{h}_u^+ = \mathrm{Diag}\,[\sqrt{-1}t_1, -\sqrt{-1}t_1, 0, \cdots, 0]$
$\mathfrak{h}_u^- = \mathrm{Diag}\,[\sqrt{-1}t_1, \sqrt{-1}t_1, \sqrt{-1}t_3, \cdots, \sqrt{-1}t_n]$
$T_u^+ = \mathrm{Diag}\,[\lambda_1, \lambda_1^{-1}, 1, \cdots, 1]$
$T_u^- = \mathrm{Diag}\,[\lambda_1, \lambda_1, \lambda_3, \cdots, \lambda_n]$
$a \in T_w^+,\ a^2 = 1 \Rightarrow a = [\pm(1,1), 1, \cdots, 1]$.

These two elements are congruent with each other mod T_w^-. The total $= \binom{n}{2}$.

Case $w = (12)(34)\cdots(2r-1, 2r)$ $1 \leqq r \leqq \nu$ if $n = 2\nu+1$, $1 \leqq r \leqq \nu-1$ if $n = 2\nu$.

$$\#[w] = \frac{n(n-1)(n-2)\cdots(n-2r+1)}{2^r \times r!}$$

$\omega =$



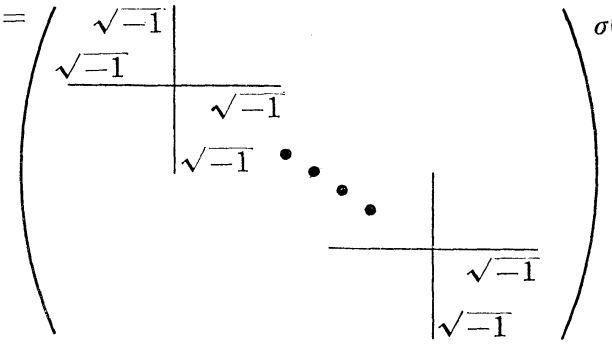
$\sigma(\omega)\omega = 1$

$$T_w^+ \ni a\,, \qquad a^2 = 1 \Rightarrow a = \mathrm{Diag}\,[\underbrace{\pm(1,1), \pm(1,1), \cdots, \pm(1,1)}_{2r}, 1, \cdots, 1]\,.$$

All these elements are congruent with each other mod $(T_w^+ \cap T_w^-)$. The total $= \dfrac{n(n-1)(n-2)\cdots(n-2r+1)}{2^r \times r!}$.

Case $w = (12)(34)\cdots(2\nu-1, 2\nu)$, $n = 2\nu$

$\omega =$



$\sigma(\omega) \cdot \omega = 1$

$$\hat{T}_w^+ = \{T_w^+ \ni a \ \ a^2 = 1\} = \{a = \mathrm{Diag}\,[\pm(1,1), \pm(1,1), \cdots, \pm(1,1)]\}$$

$$\mathfrak{h}_u^- = \{\mathrm{Diag}\,[\sqrt{-1}t_1\ \sqrt{-1}t_1\ \sqrt{-1}t_3\ \sqrt{-1}t_3 \cdots \sqrt{-1}t_{2\nu-1}\ \sqrt{-1}t_{2\nu-1}]$$
$$\text{where } \sum_{j=1}^{\nu} t_{2j-1} = 0\}$$
$$\mathfrak{h}_u^- \curvearrowright T_w^- = \{\mathrm{Diag}\,[\lambda_1\ \lambda_1\ \lambda_3\ \lambda_3 \cdots \lambda_{2\nu-1}\ \lambda_{2\nu-1}]\ \ \lambda_1\lambda_3 \cdots \lambda_{2\nu-1} = 1\}\,.$$

Therefore any element of $\hat{T}_w^+$ is congruent modulo T_w^- to either $\mathrm{Diag}\,[1\ 1 \cdots 1] = 1$ or $\mathrm{Diag}\,[-1\ -1, 1 \cdots 1]$ which is not congruent to each other. The total $= 2\times\dfrac{n(n-1)(n-2)\cdots(n-2\nu+1)}{2^\nu\times\nu!}$. Thus, the number of $[\Psi_0]$ is equal to

$$\delta = 1+\binom{n}{2}+\frac{1}{2!}\frac{n(n-1)(n-2)(n-3)}{2^2}+\cdots+\frac{1}{r!}\frac{n(n-1)\cdots(n-2r+1)}{2^r}$$
$$+\cdots+\begin{cases}\dfrac{n(n-1)\cdots(n-2\nu+1)}{\nu!\times 2^\nu} & n=2\nu+1\\[2ex] \dfrac{2n(n-1)\cdots(n-2\nu+1)}{\nu!\times 2^\nu} & n=2\nu\,.\end{cases}$$

For example $n=2$ $\delta=3$; $n=3$ $\delta=4$; $n=4$ $\delta=13$. It is when and only when $w[\sqrt{-1}t_1\ \sqrt{-1}t_2 \cdots \sqrt{-1}t_n] = w[\sqrt{-1}t_n \cdots \sqrt{-1}t_1]$ that $\mathfrak{s}$ defined by (40) has the smallest dimension, and $\mathfrak{s}$ has the smallest dimension in only one case if $n=2\nu+1$ and in two cases if $n=2\nu$. The G-orbits in $B\backslash G^c$ corresponding to these $\mathfrak{s}$ are open in $B\backslash G^c$ and isomorphic to homogeneous spaces $S\backslash G$, where S is a fundamental Cartan subgroup of G (see (47)).

These manifolds are very important in the theory of unitary representations as Gelfand and Graev have shown in [**11**]. They have shown, when $n=4$, that these manifolds have explicit parametric representations. When $n=2\nu$, it is easily generalized as follows: Let $\boldsymbol{P}^{n-1}$ be the projective space of $(n-1)$ complex dimension, $z^{(i)} = (z_1^{(i)} : z_2^{(i)} : \cdots : z_n^{(i)})$ be ν homogeneous coordinates $1 \leqq i \leqq \nu$, and $z = (z^{(1)}\ z^{(2)} \cdots z^{(\nu)}) \in \underbrace{\boldsymbol{P}^{n-1}\times\boldsymbol{P}^{n-1}\times\cdots\times\boldsymbol{P}^{n-1}}_{\nu} = \boldsymbol{Q}$,

$$\Delta(z) = \text{the determinant}\begin{pmatrix} z_1^{(1)} & z_2^{(1)} & \cdots & z_n^{(1)}\\ z_1^{(2)} & z_2^{(2)} & \cdots & z_n^{(2)}\\ \vdots & \vdots & & \\ z_1^{(\nu)} & z_2^{(\nu)} & \cdots & z_n^{(\nu)}\\ \bar{z}_1^{(1)} & \bar{z}_2^{(1)} & \cdots & \bar{z}_n^{(1)}\\ \vdots & \vdots & & \\ \bar{z}_1^{(\nu)} & \bar{z}_2^{(\nu)} & \cdots & \bar{z}_n^{(\nu)}\end{pmatrix}.$$

We define $\boldsymbol{Q}^+$, $\boldsymbol{Q}^-$

$$\boldsymbol{Q}^+ = \{z\in\boldsymbol{Q};\ \Delta(z) > 0\}$$
$$\boldsymbol{Q}^- = \{z\in\boldsymbol{Q};\ \Delta(z) < 0\}\,.$$

Then $G = SL(n, \boldsymbol{R})$ operates transitively on $\boldsymbol{Q}^+$ or $\boldsymbol{Q}^-$ by defining

$$z \to (z \cdot g)_i = \sum_{j=1}^{n} z_j g_{ji} \quad \text{for} \quad g \in G \qquad g = (g_{ij}).$$

These two manifolds $\boldsymbol{Q}^+, \boldsymbol{Q}^-$ are isomorphic to $S\backslash G$ where S is a fundamental Cartan subgroup. When $n = 2\nu+1$, $B\backslash G^c$ has the only one open dense G-orbit.

Finally we give a table showing the dimension of G-orbits in $B\backslash G^c$ when $n=4$. dim $B\backslash G^c = 12$

dimension	number of G-orbits
12	2
11	1
10	2
9	2
8	2
7	3
6	1

2. $G^c = SL(n, \boldsymbol{C})$, $G_u = SU(n)$, $n = 2\nu$

$$\tilde{\sigma}: \mathfrak{g}^c \ni \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \to \begin{pmatrix} \bar{X}_{22} & -\bar{X}_{21} \\ -\bar{X}_{12} & \bar{X}_{11} \end{pmatrix} \qquad X_{ij},\ \nu \times \nu \text{ degree}$$

$$\mathfrak{h} = \{\mathrm{Diag}\,[t_1 \cdots t_\nu\ \bar{t}_1 \cdots \bar{t}_\nu],\ \sum_{j=1}^{\nu} \mathrm{Re}\, t_j = 0\}$$

is a fundamental Cartan subalgebra of $\mathfrak{g}$

$$\mathfrak{h}_u = \{\mathrm{Diag}\,[\sqrt{-1}t_1 \cdots \sqrt{-1}t_\nu\ \sqrt{-1}t_{\nu+1} \cdots \sqrt{-1}t_n]\ \sum_{j=1}^{\nu} t_j = 0,\ t_j \text{ real}\}$$

$$T_u^+ \curvearrowright \mathfrak{h}_u \cap \mathfrak{k} = \mathfrak{h}_t, \qquad T_u^- \curvearrowright \mathfrak{h}_u \cap \sqrt{-1}\mathfrak{p} = \sqrt{-1}\mathfrak{h}_\mathfrak{p}$$

$$\mathfrak{h}_t = \{\mathrm{Diag}\,[\sqrt{-1}t_1 \cdots \sqrt{-1}t\ -\sqrt{-1}t_1 \cdots -\sqrt{-1}t_\nu]\ t_i \text{ real}\}$$

$$T_u^+ \ni a, \qquad a^2 = 1 \Rightarrow a = \mathrm{Diag}\,[\pm 1 \cdots \pm 1\ \pm 1 \cdots \pm 1]$$

the i-th and $(i+\nu)$-th elements have the same sign.

These elements do not always belong to Θ_0. For example, we take $a = \mathrm{Diag}\,[\underbrace{-1, 1 \cdots 1}_{\nu}, -1, \cdots, 1]$. The elements in $\sqrt{-1}\mathfrak{p}$ which commute with a have a reduced form:

$$\begin{pmatrix} \sqrt{-1}\tau & 0 & & \\ & X'_{11} & & X'_{12} \\ 0 & & \sqrt{-1}\tau & \\ & X'_{21} & & X'_{22} \end{pmatrix}$$

where $-\bar{X}'_{11}=X'_{22}$, $X'_{21}=\bar{X}'_{12}=$skew symmetric of degree $\nu-1$. The set of elements of $\mathfrak{g}_u$ which commute with a is a subalgebra $\mathfrak{g}'_u$ of $\mathfrak{g}_u$. $\mathfrak{g}'$ is σ-invariant.

$$\mathfrak{g}'=\mathfrak{k}'+\sqrt{-1}\mathfrak{p}' \qquad \sqrt{-1}\mathfrak{p}'=\mathfrak{g}'\cap\sqrt{-1}\mathfrak{p}\,.$$

Let $\sqrt{-1}\mathfrak{h}'_{\mathfrak{p}}$ be any maximal abelian subalgebra contained in $\sqrt{-1}\mathfrak{p}'$ and let $\sqrt{-1}\mathfrak{h}'_{\mathfrak{p}}\frown T_u^{-\prime}$. If $a\in\Theta_0$, then $a\in T_u^{-\prime}$, because a lies in the centre of G' which is an analytic group corresponding to $\mathfrak{g}'$ and because $\sqrt{-1}\mathfrak{h}_{\mathfrak{p}}$ is also maximal abelian in $\sqrt{-1}\mathfrak{p}$.

Any element of such $\sqrt{-1}\mathfrak{h}'_{\mathfrak{p}}$, for example, has a form

$$H=\mathrm{Diag}\,[\sqrt{-1}\tau,\ \sqrt{-1}t_2\cdots\sqrt{-1}t_\nu,\ \sqrt{-1}\tau,\ \sqrt{-1}t_2\cdots\sqrt{-1}t_\nu]$$

when $\left(\sum_{j=2}^{\nu}t_j\right)+\tau=0$.

But a can not be written in the form $\exp H$, and so a does not lie in Θ_0. We have found that $\Theta\neq\Theta_0$ and so $\Phi\neq\Phi_0$. We must make in experiment whether $\omega a\in\Psi$ is contained in Ψ_0 or not.

$$H\in\mathfrak{h}_u\,,\qquad H=\mathrm{Diag}\,[\sqrt{-1}t_1\ \sqrt{-1}t_2\cdots\sqrt{-1}t_n]$$

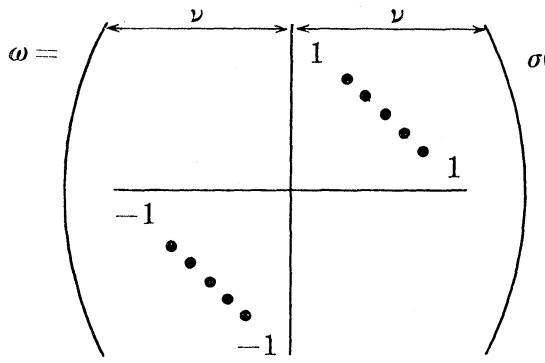
$$\sigma(H)=\mathrm{Diag}\,[-\sqrt{-1}t_{\nu+1}\cdots-\sqrt{-1}t_n\ -\sqrt{-1}t_1\cdots-\sqrt{-1}t_\nu]$$

$$w(H)=\mathrm{Diag}\,[\sqrt{-1}t_{i_1}\ \sqrt{-1}t_{i_2}\cdots\sqrt{-1}t_{i_n}]$$

$$\sigma\cdot w(H)=\mathrm{Diag}\,[-\sqrt{-1}t_{i_{\nu+1}}\cdots-\sqrt{-1}t_{i_n}\ -\sqrt{-1}t_{i_1}\cdots-\sqrt{-1}t_{i_\nu}]\,.$$

$-\sigma\cdot w\in W$ and the conjugacy of $\sigma\cdot w$ is equivalent to that of $-\sigma\cdot w$ under W.

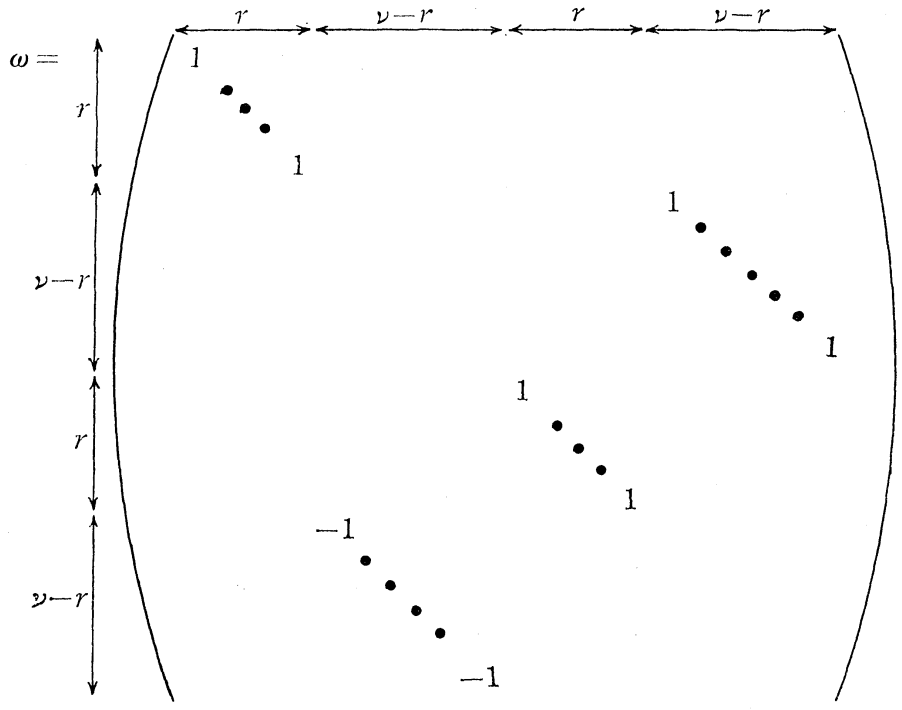
Case 1) $-\sigma\cdot w=1$, $w=\begin{pmatrix}1 & 2\cdots\nu & \nu+1\cdots 2\nu\\ \nu+1 & \nu+2\cdots 2\nu & 1\cdots\nu\end{pmatrix}$



$\mathfrak{h}_u^+ = \{0\}$, $\mathfrak{h}_u^- = \{0\}$, $T_w^+ = \{1\}$. Therefore $\hat{T}_w^+ = \phi$.

Case 2) $-\sigma \cdot w = (1, \nu+1) \cdots (r, \nu+r)$, $r < \nu$.

$$w = \begin{pmatrix} 1\ 2 \cdots r & r+1 \cdots \nu & \nu+1 \cdots \nu+r & \nu+r+1 \cdots 2\nu \\ 1\ 2 \cdots r & \nu+r+1 \cdots 2\nu & \nu+1 \cdots \nu+r & r+1 \cdots \nu \end{pmatrix}$$



$$\sigma(\omega) \cdot \omega = \mathrm{Diag}\,[\underbrace{1 \cdots 1}_{r}\ \underbrace{-1 \cdots -1}_{\nu-r}\ \underbrace{1 \cdots 1}_{r}\ \underbrace{-1 \cdots -1}_{\nu-r}]$$

$$\mathfrak{h}_u^+ = \{\mathrm{Diag}\,[\sqrt{-1}t_1 \cdots \sqrt{-1}t_r\ 0 \cdots 0\ -\sqrt{-1}t_1 \cdots -\sqrt{-1}t_r\ 0 \cdots 0]\}\ .$$

But $T_w^+ \ni (\sigma(\omega) \cdot \omega)^{-1}$, therefore $\hat{T}_w^+ = \phi$.

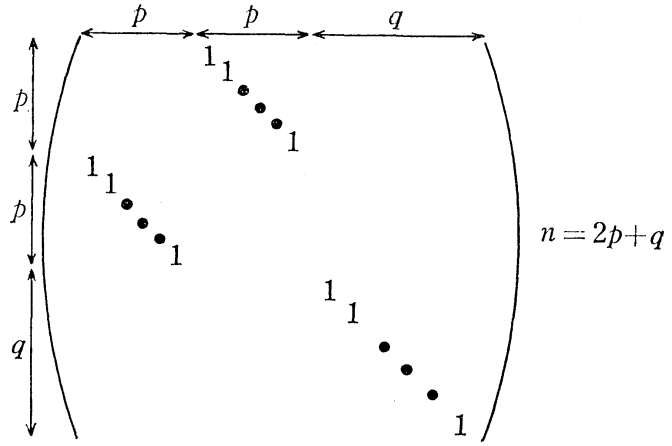
Case 3) $-\sigma \cdot w = (1\ \nu+1)(2\ \nu+2) \cdots (\nu\ 2\nu)$, $w = 1$, $\sigma(\omega) \cdot \omega = 1$

$$\mathfrak{h}_u^+ = \{\mathrm{Diag}\,[\sqrt{-1}t_1 \cdots \sqrt{-1}t_\nu\ -\sqrt{-1}t_1 \cdots -\sqrt{-1}t_\nu]\ \ t_i \text{ real}\}$$

$$\mathfrak{h}_u^- = \{\mathrm{Diag}\,[\sqrt{-1}t_1 \cdots \sqrt{-1}t_\nu\ \sqrt{-1}t_1 \cdots \sqrt{-1}t_\nu]\ \sum_{i=1}^{\nu} t_i = 0,\ t_i \text{ real}\}\,.$$

$\hat{T}_w^+ = \mathrm{Diag}\,[\pm 1 \cdots \pm 1\ \pm 1 \cdots \pm 1]$ i-th and $i+\nu$-th are of the same sign. In this case $\sqrt{-1}\mathfrak{h}_\mathfrak{p}$ is also maximal abelian in $\sqrt{-1}\mathfrak{p}$, therefore by Proposition 7, $\hat{T}_w^+ \cap \Theta_0 = \hat{T}_w^+ \cap T_u^- = \hat{T}_w^+ \cap T_w^-$. Any element in $\hat{T}_w^+ \cap \Theta_0$ is congruent to each other modulo T_w^-. The number of $[\mathcal{W}_0]$ is equal to $\dfrac{(2\nu)!}{\nu!\,2^\nu}$. An open orbit in $B\backslash G^C$ is precisely the one defined by $w=1$, and so there is only one.

3. $G^C = SL(n, \boldsymbol{C})$, $G_u = SU(n)$, $\tilde{\sigma}(X) = -J_{pq}\,{}^t\bar{X} J_{pq}^{-1}$, $X \in SL(n, \boldsymbol{C})$ (or $\mathfrak{sl}(n, \boldsymbol{C})$), where

$J_{pq} =$ 

$$X = \begin{pmatrix} \overset{\longleftrightarrow}{X_{11}}^{p} & \overset{\longleftrightarrow}{X_{12}}^{p} & \overset{\longleftrightarrow}{X_{13}}^{q} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{pmatrix} \in \mathfrak{g}^C, \qquad \tilde{\sigma}(X) = -\begin{pmatrix} {}^t\bar{X}_{22} & {}^t\bar{X}_{12} & {}^t\bar{X}_{32} \\ {}^t\bar{X}_{21} & {}^t\bar{X}_{11} & {}^t\bar{X}_{31} \\ {}^t\bar{X}_{23} & {}^t\bar{X}_{13} & {}^t\bar{X}_{33} \end{pmatrix}$$

$$\mathfrak{g} = \{X = \begin{pmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & -{}^t\bar{X}_{11} & X_{23} \\ -{}^t\bar{X}_{23} & -{}^t\bar{X}_{13} & X_{33} \end{pmatrix};\ X_{12}, X_{21} \in \mathfrak{u}(p),\ X_{33} \in \mathfrak{u}(q)\}$$

$$\mathfrak{k} = \{X = \begin{pmatrix} X_{11} & X_{12} & X_{13} \\ X_{12} & X_{11} & X_{13} \\ -{}^t\bar{X}_{13} & -{}^t\bar{X}_{13} & X_{33} \end{pmatrix};\ X_{11}, X_{12} \in \mathfrak{u}(p),\ X_{13} \in \mathfrak{M}(p, q),\ X_{33} \in \mathfrak{u}(q)\}$$

$$\mathfrak{p} = \{X = \begin{pmatrix} X_{11} & X_{12} & X_{13} \\ -X_{12} & -X_{11} & -X_{13} \\ {}^t\bar{X}_{13} & -{}^t\bar{X}_{13} & 0 \end{pmatrix};\ X_{11} \in \mathrm{He}(p),\ X_{12} \in \mathfrak{u}(p),\ X_{13} \in \mathfrak{M}(p, q)\}$$

$$\mathfrak{h}=\{H=\mathrm{Diag}\,[t_1+\sqrt{-1}u_1,\cdots,t_p+\sqrt{-1}u_p,-t_1+\sqrt{-1}u_1,\cdots,-t_p+\sqrt{-1}u_p,$$
$$\sqrt{-1}u_{p+1},\cdots,\sqrt{-1}u_{p+q}],\ 2\sum_{j=1}^{p}u_j+\sum_{j=1}^{q}u_{p+j}=0,\ t_j,u_j\ \text{real}\}$$

$$\mathfrak{h}_u=\{H=\mathrm{Diag}\,[\sqrt{-1}(t_1+u_1),\cdots,\sqrt{-1}(t_p+u_p),$$
$$\sqrt{-1}(-t_1+u_1),\cdots,\sqrt{-1}(-t_p+u_p),\sqrt{-1}u_{p+1},\cdots,\sqrt{-1}u_{p+q}]$$
$$\text{where } t_j,u_j=\text{real},\ 2\sum_{j=1}^{p}u_j+\sum_{j=1}^{q}u_{p+j}=0\}$$

$$Z\cap\Theta=\{1\},\ n=\text{odd},\ =\{\pm1\},\ n=\text{even},\ Z\cap\Theta_0=T_u^-\cap\Theta_0=\{1\}$$
$$\text{if } q>0,\ =\{\pm1\}\ \text{if}\ q=0.$$

We put $t_1+u_1=\tau_1,\cdots,t_p+u_p=\tau_p,-t_1+u_1=\tau_{p+1},\cdots,-t_p+u_p=\tau_{2p},u_{p+1}=\tau_{2p+1},\cdots,u_{p+q}=\tau_n$. Then $\sigma\,\mathrm{Diag}\,[\sqrt{-1}\tau_1,\cdots,\sqrt{-1}\tau_n]=\mathrm{Diag}\,[\underbrace{\sqrt{-1}\tau_{p+1},\cdots,\sqrt{-1}}_{p}\tau_{2p},\underbrace{\sqrt{-1}\tau_1,\cdots,\sqrt{-1}}_{p}\tau_p,\sqrt{-1}\tau_{2p+1},\cdots,\sqrt{-1}\tau_n]$,

$$\sigma=\begin{pmatrix}1 & 2\cdots p & p+1\cdots 2p & 2p+1\cdots n\\ p+1 & p+2\cdots 2p & 1\cdots p & 2p+1\cdots n\end{pmatrix}\in W$$

for any $w\in W$, $w=\begin{pmatrix}1&2&\cdots&n\\ i_1&i_2&\cdots&i_n\end{pmatrix}$, $\sigma\cdot w\,\mathrm{Diag}\,[\sqrt{-1}\tau_1,\cdots,\sqrt{-1}\tau_n]=\mathrm{Diag}\,[\tau_{i_{p+1}},\cdots,\tau_{i_{2p}},\tau_{i_1},\cdots,\tau_{i_p},\tau_{i_{2p+1}},\cdots,\tau_{i_n}]$ representative in conjugate class of order 2:

$$\sigma\cdot w=id,\ (1,p+1),(1,p+1)(2,p+2),\cdots,(1,p+1)(2,p+2)$$
$$\cdots(p,2p),(1,p+1)\cdots(p,2p)(2p+1,2p+2),\cdots,(1,p+1)$$
$$\cdots(p,2p)(2p+1,2p+2)\cdots(2p+2\nu-1,2p+2\nu),\ \text{where}\ \nu=\left[\frac{q}{2}\right].$$

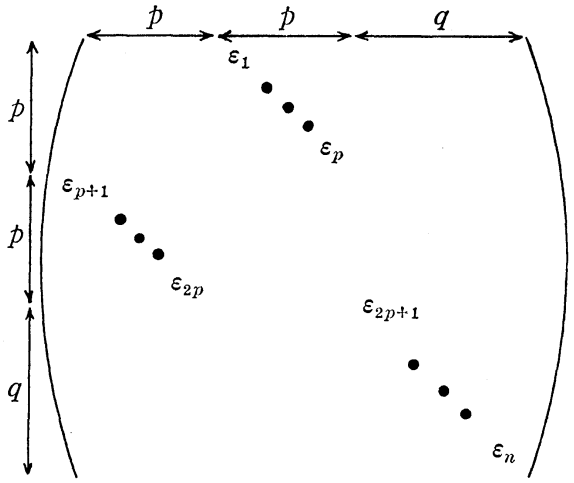
Case (1) $\sigma\cdot w=id$, $W=\begin{pmatrix}1 & 2\cdots p & p+1\cdots 2p & 2p+1\cdots n\\ p+1 & p+2\cdots 2p & 1\cdots p & 2p+1\cdots n\end{pmatrix}$

$$\omega=\begin{pmatrix}\overbrace{1&&&}^{p}&&&\\ &\ddots&&&\overbrace{\quad}^{p}&&\overbrace{\quad}^{q}\\ &&1&&&&\\ &-1&&\ddots&&&\\ &&&&1&&\\ &&-1&&&\ddots&\\ &&&&&&1\end{pmatrix},\quad \sigma(\omega)\omega=1$$

$$\mathfrak{g}_u\ni X=\begin{pmatrix}X_{11}&X_{12}&X_{13}\\ X_{21}&X_{22}&X_{23}\\ X_{31}&X_{32}&X_{33}\end{pmatrix}\to\sigma(X)=\begin{pmatrix}X_{22}&X_{21}&X_{23}\\ X_{12}&X_{11}&X_{13}\\ X_{32}&X_{31}&X_{33}\end{pmatrix}$$

$\mathfrak{h}_u^+ = \mathfrak{h}_u$, $\mathfrak{h}_u^- = \{0\}$, $T_w^+ = T$, $\hat{T}_w^+ = \{\mathrm{Diag}\,[\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n]$ whose determinant $= 1$, $\varepsilon_i = \pm 1\}$

$\Psi \ni \omega a =$



$J_{pq} \cdot \omega a = \mathrm{Diag}\,[\varepsilon_1, \cdots, \varepsilon_p, \varepsilon_{p+1}, \cdots, \varepsilon_{2p}, \varepsilon_{2p+1}, \cdots, \varepsilon_n]$

where $\varepsilon_i = \pm 1$. If $\omega a \in \Psi_0$, $\sigma \cdot \mathrm{Ad}\,\omega a$ is conjugate to σ under $\mathrm{Ad}\,G_u$. Therefore the real semi-simple algebra which $\sigma \cdot \mathrm{Ad}\,\omega a$ defines is isomorphic to $\mathfrak{g}$. Let $\mathfrak{g}_u = \mathfrak{g}_u^+ + \mathfrak{g}_u^-$ where $\mathfrak{g}_u = \{X \in \mathfrak{g}_u, \sigma \cdot \mathrm{Ad}\,\omega a\ X = X\}$, $\mathfrak{g}_u^- = \{X \in \mathfrak{g}_u, \sigma \cdot \mathrm{Ad}\,\omega a\ X = -X\}$. Suppose the number of ε_i which is equal to -1 is m, $\dim \mathfrak{g}_u^+ = m^2 + (n-m)^2 - 1$. On the other, $\dim \mathfrak{k} = p + (n-p)^2 - 1$.

If $\dim \mathfrak{g}_u^+ = \dim \mathfrak{k}$, $m = p$ or $n-p$. As the determinant of ωa is equal to 1.

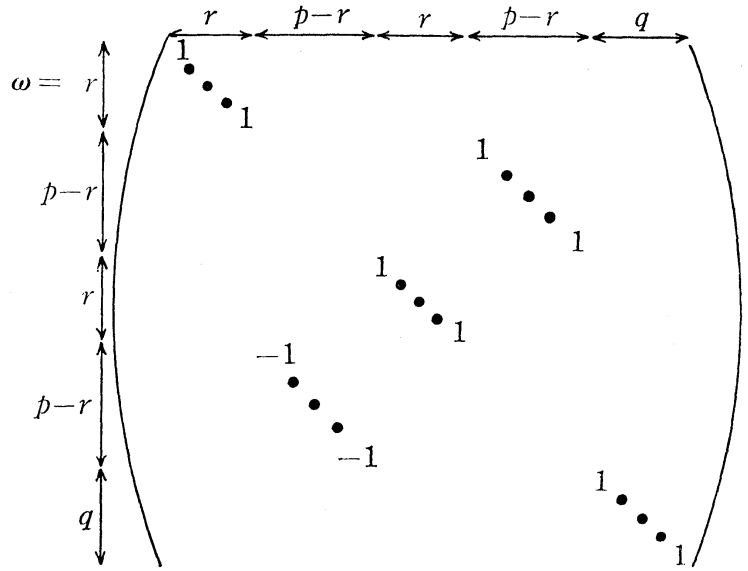
$$(-1)^p \varepsilon_1 \varepsilon_2 \cdots \varepsilon_n = 1, \qquad (-1)^{p+m} = 1, \qquad p+m \equiv 0 \bmod (2\boldsymbol{Z})$$

$$n = \mathrm{odd} \Rightarrow m = p, \qquad n = \mathrm{even} \Rightarrow m = p \ \text{ or } \ m = n-p.$$

$J_{pq} \cdot \omega a$ is conjugate to J_{pq} under G_u when and only when $\omega a \in \Theta_0$. This is equivalent to $m = p$. Therefore $\omega a \in \Psi_0 \Leftrightarrow m = p$. The total number $= \binom{n}{p}$.

Case (ii) $\sigma \cdot w = (1, p+1) \cdots (r, p+r)$ $1 \leqq r \leqq p$

$$w = \begin{pmatrix} 1\ 2 \cdots r & r+1 \cdots p & p+1 \cdots p+r & p+r+1 \cdots 2p & 2p+1 \cdots n \\ 1\ 2 \cdots r & p+r+1 \cdots 2p & p+1 \cdots p+r & r+1 \cdots p & 2p+1 \cdots n \end{pmatrix}$$

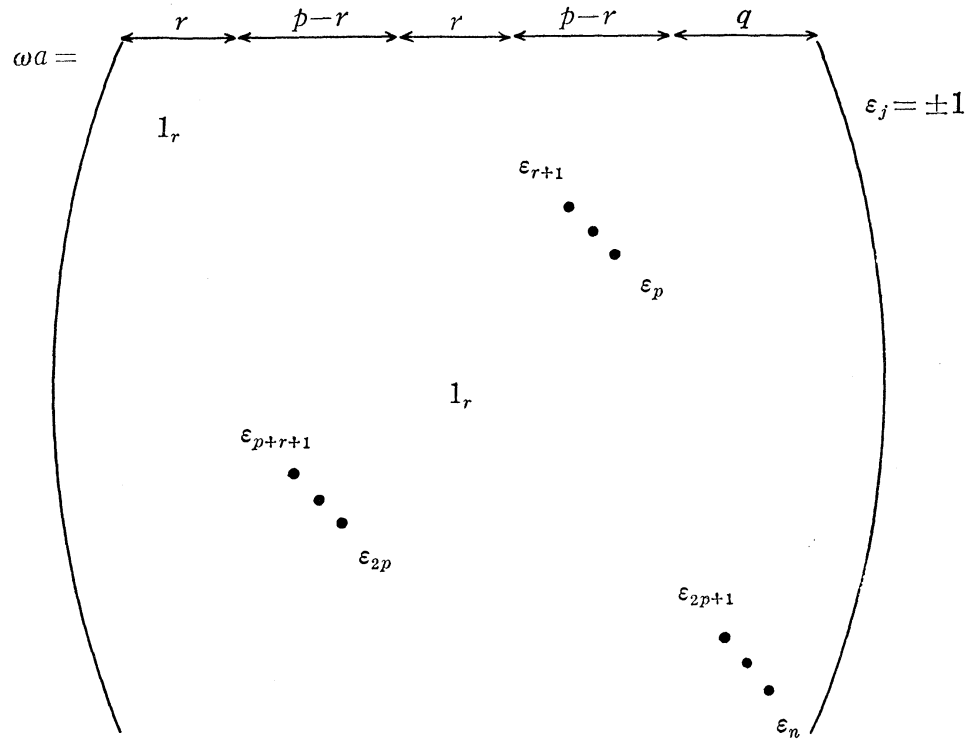


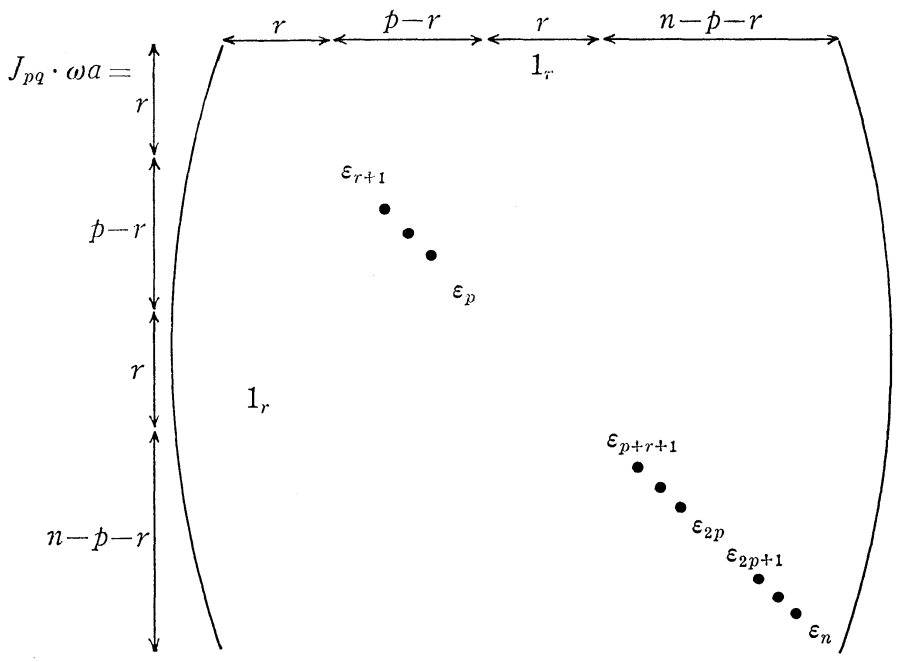
$$\mathfrak{h}_u^+ = [\sqrt{-1}t_1 \cdots \sqrt{-1}t_r\ \sqrt{-1}t_{r+1} \cdots \sqrt{-1}t_p\ \sqrt{-1}t_1 \cdots \sqrt{-1}t_r$$
$$\sqrt{-1}t_{p+r+1} \cdots \sqrt{-1}t_{2p}\ \sqrt{-1}t_{2p+1} \cdots \sqrt{-1}t_n]$$
$$\mathfrak{h}_u^- = [\sqrt{-1}t_1 \cdots \sqrt{-1}t_r\ 0 \cdots 0\ -\sqrt{-1}t_1 \cdots -\sqrt{-1}t_r\ 0 \cdots 0\ 0 \cdots 0].$$

Any element a' of $\hat{T}_w^+$ is congruent to

(*) $$a = \mathrm{Diag}[\underbrace{1\ 1 \cdots 1}_{r}\ \varepsilon_{r+1} \cdots \varepsilon_p\ \underbrace{1 \cdots 1}_{r}\ \varepsilon_{p+r+1} \cdots \varepsilon_{2p}\ \varepsilon_{2p+1} \cdots \varepsilon_n]$$

mod T_w^- $a^2 = 1$. None of these two elements are congruent mod T_w^-.





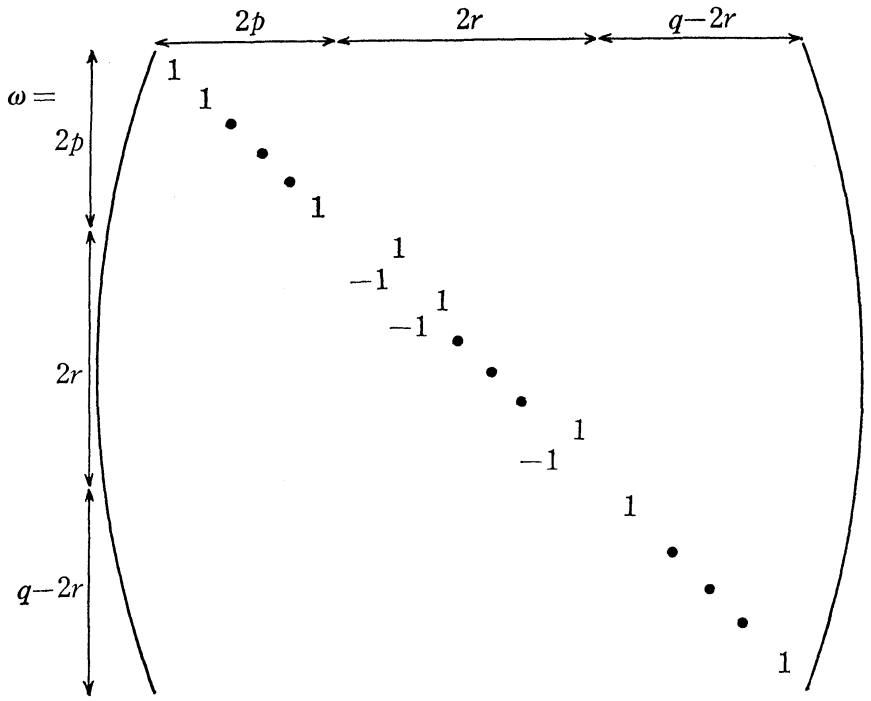
$$(-1)^{p-r}\varepsilon_{r+1}\cdots\varepsilon_p\,\varepsilon_{p+r+1}\cdots\varepsilon_{2p}\,\varepsilon_{2p+1}\cdots\varepsilon_n=1\,.$$

Let $m=$the number of ε_i which is equal to -1. Then, $\dim \mathfrak{g}_u^+=(r+m)^2+(n-r-m)^2-1$, $(-1)^{n+p-r}=1$, $m+p-r\equiv 0 \bmod 2\boldsymbol{Z}$ where $\mathfrak{g}_u^++\mathfrak{g}_u^-=\mathfrak{g}_u$ is a Cartan decomposition with respect to $J_{pq}\cdot \mathrm{Ad}\,\omega a$. If $\dim \mathfrak{g}_u^+=\dim \mathfrak{k}$, $r+m=p$ or $n-p$, that is, $m=p-r$ or $m=n-p-r$. The signature of the symmetric matrix $J_{pq}\cdot\omega a$ must be equal to that of J_{pq}, which occurs only when $r+m=p$. Thus $\omega a\in \mathit{\Psi}_0$ of the form $(*)\Leftrightarrow r+m=p$ $(m=p-r)$. The number of such elements $=\binom{n-2r}{p-r}$.

Case (iii)

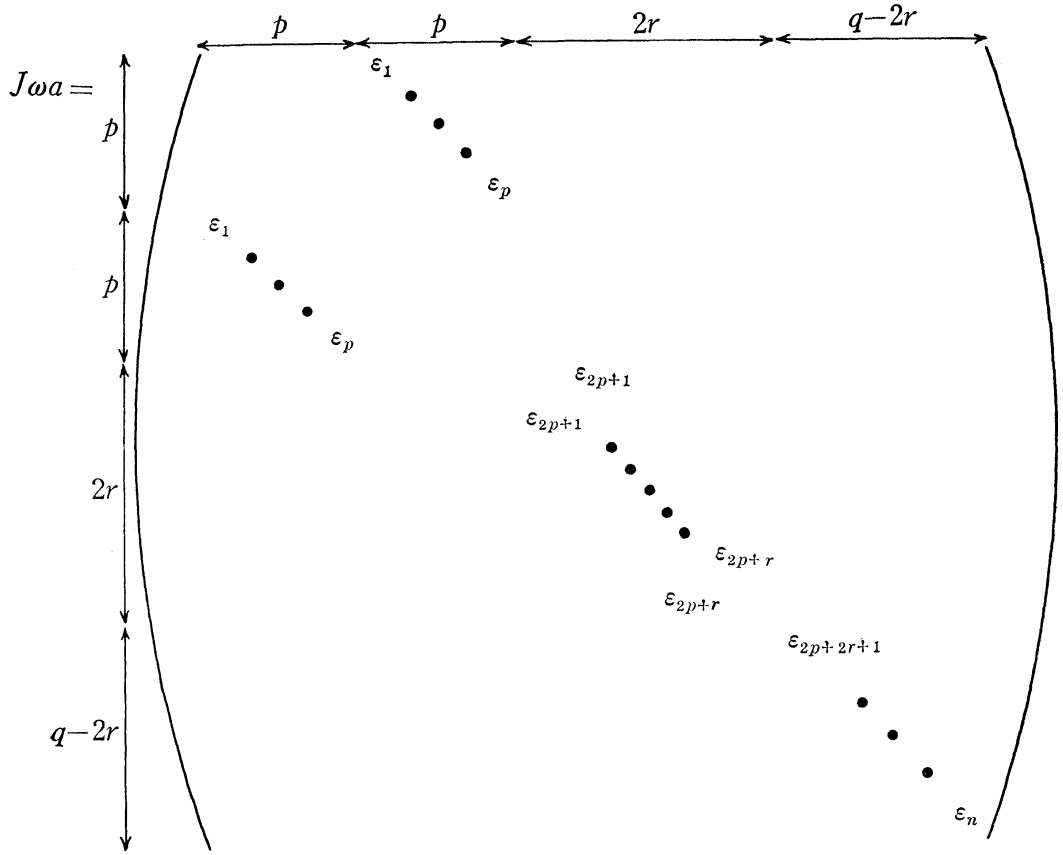
$$\sigma\cdot w=(1,\,p+1)\cdots(p,\,2p)(2p-1,\,2p+2)\cdots(2p+2r-1,\,2p+2r)\qquad r\geqq 1$$

$$w=\begin{pmatrix}1&2&\cdots&2p&2p+1&2p+2&\cdots&2p+2r+1&2p+2r\\1&2&\cdots&2p&2p+2&2p+1&\cdots&2p+2r&2p+2r+1\end{pmatrix}$$



$$\mathfrak{h}_u^+ = \{\mathrm{Diag}\,[\sqrt{-1}t_1 \cdots \sqrt{-1}t_p\ \sqrt{-1}t_1 \cdots \sqrt{-1}t_p\ \sqrt{-1}t_{2p+1}\ \sqrt{-1}t_{2p+1}$$
$$\cdots\ \sqrt{-1}t_{2p+2r-1}\ \sqrt{-1}t_{2p+2r-1}\ \sqrt{-1}t_{2p+2r+1} \cdots \sqrt{-1}t_n]\}\,.$$

For any $\omega a \in \Psi$,

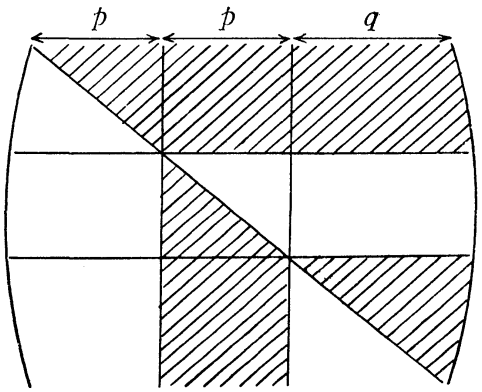


The signature of $J\cdot\omega a$ is not equal to that of J. Therefore $\omega a\notin\mathfrak{W}_0$. The total number of $[\mathfrak{W}_0]$ is equal to

$$\binom{n}{p}+\binom{n}{2}\binom{n-2}{p-1}+\cdots+\frac{1}{r!}\binom{n}{2}\binom{n-2}{2}\cdots\binom{n-2r+2}{2}\binom{n-2r}{p-r}$$

$$+\cdots+\frac{1}{p!}\binom{n}{2}\binom{n-2}{2}\cdots\binom{n-2p+2}{2}=\sum_{r=0}^{p}\frac{n}{2^{p-r}r!(n-2p+r)!}\,.$$

We take, as a Borel subgroup B, the subgroup of elements of the form:



The part of oblique lines is not zero, the remaining part is zero.

$$\mathfrak{b}=\mathfrak{h}^C+\sum_{\alpha\in\boldsymbol{P}}\boldsymbol{C}E_\alpha\,,\mathfrak{h}_R=\{\mathrm{Diag}\,[t_1+u_1,\cdots,t_p+u_p,-t_1+u_1,\cdots,-t_p+u_p,u_{2p+1},\cdots,u_n]$$

$$2\sum_{j=1}^{p}u_j+\sum_{j=1}^{q}u_{2p+1}=0,\ t_j,u_j\ \text{real}\}\,,$$

$\boldsymbol{P}=\{(t_i+u_i)-(t_j+u_j)\ 1\leqq i<j\leqq p,\ (+t_i+u_i)-(-t_j+u_j)\ 1\leqq i\leqq p,\ 1\leqq j\leqq p,\ (-t_i+u_i)-(-t_j+u_j)\ 1\leqq j<i\leqq p,\ t_i+u_i-u_{2p+j}\ 1\leqq i\leqq p,\ 1\leqq j\leqq q,\ -(-t_i+u_i)+u_{2p+j}\ 1\leqq i\leqq p,\ 1\leqq j\leqq q,\ u_{2p+1}-u_{2p+j}\ 1\leqq i<j\leqq q\}$,

$$\tilde{\sigma}\,\mathrm{Diag}\,[t_1+u_1,\cdots,t_p+u_p,-t_1+u_1,\cdots,-t_p+u_p,u_{2p+1},\cdots,u_n]$$

$$=\mathrm{Diag}\,[t_1-u_1,\cdots,t_p-u_p,-t_1-u_1,\cdots,-t_p-u_p,u_{2p+1},\cdots,u_n]$$

$\mathfrak{h}^C\curvearrowright A^C$, $\mathfrak{n}^C=N^C$, $\tilde{\mathfrak{n}}^C=\tilde{N}^C$, $B=A^C\cdot N^C$, $\tilde{\mathfrak{n}}^C=\sum_{\alpha\in\boldsymbol{P}}\boldsymbol{C}E_{-\alpha}$. If in (44), $\mathfrak{n}_1=0$, the $w\cdot\tilde{\sigma}$ transforms $\boldsymbol{P}$ into $-\boldsymbol{P}$. Let $w\cdot\tilde{\sigma}=\tau_0$ on $\mathfrak{h}_R=\sqrt{-1}\mathfrak{h}_u$, $\tau_0(\boldsymbol{P})=-\boldsymbol{P}$, Then $w\cdot\sigma=-\tau_0$ on $\mathfrak{h}_u$, where σ is the restriction of $\tilde{\sigma}$ to $\mathfrak{g}_u$. Because $w\cdot\sigma$ is contained in W, $-\tau_0\in W$. On the other $-\tau_0(\boldsymbol{P})=\boldsymbol{P}$. Therefore $-\tau_0=1$ and $w\cdot\tilde{\sigma}=1$, $w=\sigma$ on $\mathfrak{h}_u$, $\sigma\cdot w=1$. The elements of $[\mathfrak{W}_0]$ corresponding to this w, have been calculated in Case (i), and the number is equal to $\binom{n}{p}$. Therefore the number of open orbits is equal to $\binom{n}{p}$.

4. $G^c=S_p(\nu, \boldsymbol{C})$, $G=S_p(\nu, \boldsymbol{R})$, $G_u=SU_p(\nu)$, $\mathfrak{g}^c=\left\{\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & -{}^tX_{11}\end{pmatrix};\ X_{11}\in\mathfrak{gl}(\nu, \boldsymbol{C});\ X_{12},\ X_{21}\in S_y(\nu, \boldsymbol{C})\right\}$, $\tilde{\sigma}$: $X\in\mathfrak{g}^c\to\overline{X}$. $\mathfrak{g}=\left\{\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & -{}^tX_{11}\end{pmatrix};\ X_{11}\in\mathfrak{gl}(\nu, \boldsymbol{R});\ X_{12},\ X_{21}\in S_y(\nu, \boldsymbol{R})\right\}$, $\mathfrak{k}=\left\{\begin{pmatrix} X_{11} & X_{12} \\ -X_{12} & X_{11}\end{pmatrix};\ X_{11}\in\theta(\nu),\ X_{12}\in S_y(\nu, \boldsymbol{R})\right\}$, $\mathfrak{p}=\left\{\begin{pmatrix} X_{11} & X_{12} \\ X_{12} & -X_{11}\end{pmatrix};\ X_{11},\ X_{12}\in S_y(\nu, \boldsymbol{R})\right\}$, $\mathfrak{h}=\mathfrak{h}_\mathfrak{p}=\{\mathrm{Diag}\,[t_1, \cdots, t_\nu, -t_1, \cdots, -t_\nu],\ t_j\in\boldsymbol{R}\}$, $\mathfrak{h}_u=\sqrt{-1}\,\mathfrak{h}$.

W (Weylgroup) $=\mathfrak{S}_\nu\cdot(\boldsymbol{Z}_2)^\nu$ (semi-direct, while $\mathfrak{S}_\nu$ is the symmetric group of order ν, $(\boldsymbol{Z}_2)^\nu$ is the direct sum of ν cyclic groups of order 2). $P\cdot A\in W$ operates on $\mathfrak{h}$ as follows:

$$P\cdot A(t_1, t_2, \cdots, t_\nu)=(\varepsilon_{i_1}t_{i_1}, \varepsilon_{i_2}t_{i_2}, \cdots, \varepsilon_{i_\nu}t_{i_\nu})$$

where $P\cdot(t_1, t_2, \cdots, t_\nu)=(t_{i_1}, t_{i_2}, \cdots, t_{i_\nu})$, $A\cdot(t_1, t_2, \cdots, t_\nu)=(\varepsilon_1t_1, \varepsilon_2t_2, \cdots, \varepsilon_\nu t_\nu)$ $\varepsilon_j=\pm1$. $P^{-1}AP(t_1, t_2, \cdots, t_\nu)=(\varepsilon_{i_1'}t_1, \varepsilon_{i_2'}t_2, \cdots, \varepsilon_{i_\nu'}t_\nu)$, putting

$$P^{-1}=\begin{pmatrix}1, 2, \cdots, \nu\\ i_1, i_2, \cdots, i_\nu\end{pmatrix}^{-1}=\begin{pmatrix}1, 2, \cdots, \nu\\ i_1', i_2', \cdots, i_\nu'\end{pmatrix},$$

$$\mathfrak{h}_u\ni H=\mathrm{Diag}\,[\sqrt{-1}t_1, \cdots, \sqrt{-1}t_\nu, -\sqrt{-1}t_1, \cdots, \sqrt{-1}t_\nu]$$
$$\leftrightarrow(t_1, t_2, \cdots, t_\nu)\qquad\text{(coordinate of } H).$$

Suppose $(P\cdot A)^2=1$, then $PAPA=P^2P^{-1}APA=1$, therefore $P^2=1$, $P^{-1}APA=1$. P is written as the sum of transpositions, for example,

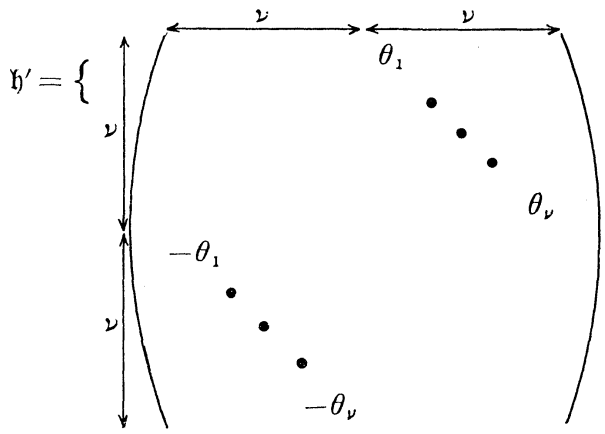
$$P=(j_1, j_2)(j_3, j_4), \cdots, (j_{2r-1}, j_{2r}).$$

$$(P^{-1}AP\cdot A)(t_1, t_2, \cdots, t_\nu)=(\varepsilon_{i_1'}\varepsilon_1t_1, \varepsilon_{i_2'}\varepsilon_2t_2, \cdots, \varepsilon_{i_\nu'}\varepsilon_\nu t_\nu)=(t_1, t_2, \cdots, t_\nu);$$

Consequently, $\varepsilon_{i_1'}\varepsilon_i=1$, $\varepsilon_{i_2'}\varepsilon_2=1, \cdots, \varepsilon_{i_\nu'}\varepsilon_\nu=1$. Since $P^2=1$, $i_1=i_1'$, $i_2=i_2'$, $\cdots, i_\nu=i_\nu'$, and so $\varepsilon_{i_s}\varepsilon_s=1$ $1\leqq s\leqq\nu$. Thus $\varepsilon_{j_1}=\varepsilon_{j_2}$, $\varepsilon_{j_3}=\varepsilon_{j_4}, \cdots, \varepsilon_{j_{2r-1}}=\varepsilon_{j_{2r}}$ and for other indices j, ε_j-arbitrary. The characterization of conjugate classes of order 2 in W is the following (see [18]): Let the compliment of the set $\{j_1, j_2, j_3, j_4, \cdots, j_{2r-1}, j_{2r}\}$ in $\{1, 2, \cdots, \nu\}$ be $\{k_1, k_2, \cdots, k_{\nu-2r}\}$, and the number of ε_{k_s} $1\leqq s\leqq\nu-2r$ which are equal to 1 be ρ. If PA and $P'A'$ in W are conjugate to each other, then (r, ρ) coincides with (r', ρ'). Conversely if $(r, \rho)=(r', \rho')$, then PA and $P'A'$ are conjugate in W. We choose representative in each conjugate class, $P=(12)(34)\cdots(2r-1\ 2r)$

$$\nu-2r+1\begin{cases} A=(\underbrace{1\cdots1}_{2r}\quad 1\cdots1)\\ A=(\underbrace{1\cdots1}_{2r}\quad 1\cdots-1)\\ A=(\underbrace{1\cdots1}_{2r}\quad -1\cdots-1)\qquad 0\leqq2r\leqq\nu.\end{cases}$$

Now we will ask if Θ_0 coincides with Θ.



Cartan subalgebra of $\mathfrak{g}$ and lies in $\mathfrak{k}$. The analytic subgroup T_u corresponding to this $\mathfrak{h}'$, is:

$$\left\{ \begin{matrix} \cos\theta_1 & & & & \sin\theta_1 & & & \\ & \cos\theta_2 & & & & \sin\theta_2 & & \\ & & \ddots & & & & \ddots & \\ & & & \cos\theta_\nu & & & & \sin\theta_\nu \\ -\sin\theta_1 & & & & \cos\theta_1 & & & \\ & -\sin\theta_2 & & & & \cos\theta_2 & & \\ & & \ddots & & & & \ddots & \\ & & & -\sin\theta_\nu & & & & \cos\theta_\nu \end{matrix} \right\} = \xi \Big\} = T_u \subset K \subset G_u .$$

Since Z (centre of G_u) is contained in T_u, $T_u Z \cap \Theta = T_u \cap \Theta$. Let $\xi \in T_u \cap \Theta$, then $\xi^2 = 1$, has the following form: $\xi = \mathrm{Diag}[\pm 1, \pm 1, \cdots, \pm 1, \pm 1, \cdots, \pm 1]$ where i-th and $(i+\nu)$-th elements are equal. On the other hand,

$$X=\begin{pmatrix} \sqrt{-1}\theta_1 & & & & \\ & \sqrt{-1}\theta_2 & & & \\ & & \ddots & & \\ & & & & \sqrt{-1}\theta_\nu \\ \sqrt{-1}\theta_1 & & & & \\ & \sqrt{-1}\theta_2 & & & \\ & & \ddots & & \\ & & & & \sqrt{-1}\theta_\nu \end{pmatrix}$$

is an element of $\sqrt{-1}\mathfrak{p}$.

Thus $\exp X=$

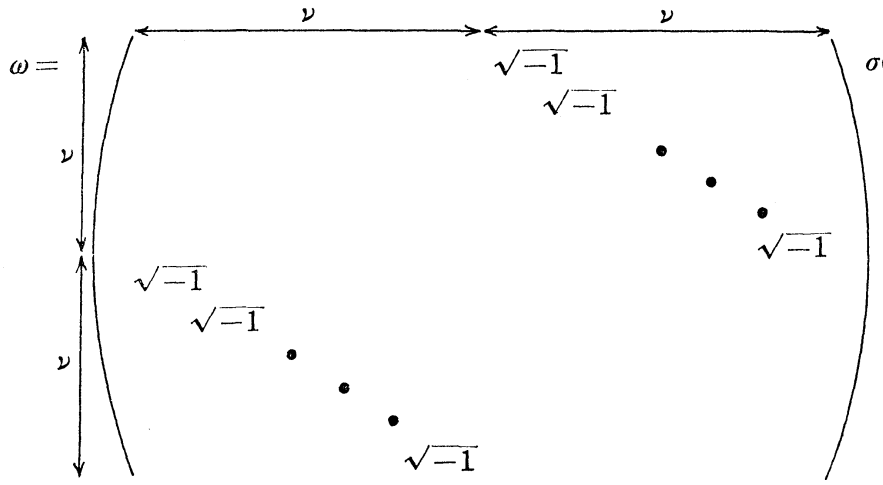
$$\begin{pmatrix} \cos\theta_1 & & & & \sqrt{-1}\sin\theta_1 & & & \\ & \cos\theta_2 & & & & \sqrt{-1}\sin\theta_2 & & \\ & & \ddots & & & & \ddots & \\ & & & \cos\theta_\nu & & & & \sqrt{-1}\sin\theta_\nu \\ \sqrt{-1}\sin\theta_1 & & & & \cos\theta_1 & & & \\ & \sqrt{-1}\sin\theta_2 & & & & \cos\theta_2 & & \\ & & \ddots & & & & \ddots & \\ & & & \sqrt{-1}\sin\theta_\nu & & & & \cos\theta_\nu \end{pmatrix} \in \Theta_0 .$$

We can write ξ in the form $\exp X$, by taking $\theta_i=\pi$ or 0. Therefore by Proposition 6, $\Theta=\Theta_0$. Now, to return to the point, we take $\mathfrak{h}_u=\sqrt{-1}\mathfrak{h}$.

$$\sigma(H)=-H \quad \text{for any} \quad H\in\mathfrak{h}_u .$$

For $w\in W$, $(\sigma\cdot w)^2=1$ is equivalent to $w^2=1$. $\sigma\cdot w$ is conjugate to $\sigma\cdot w_1$ if and only if w is conjugate to w_1.

In case $\sigma\cdot w(t_1,\cdots,t_\nu)=(t_1,t_2,\cdots,t_\nu)$, $w(t_1,\cdots,t_\nu)=(-t_1,-t_2,\cdots,-t_\nu)$,



$\mathfrak{h}_u^+ = \mathfrak{h}_u$, $\mathfrak{h}_u^- = \{0\}$, $T_w^+ = T$, $T_w^- = \{1\}$, $\hat{T}_u^+ = \{\mathrm{Diag}\,[\underbrace{\pm 1\ \pm 1 \cdots \pm 1}_{\nu}\ \ \underbrace{\pm 1\ \pm 1 \cdots \pm 1}_{\nu}]$ i-th and $i+\nu$ the elements are equal$\}$. None of elements of $\hat{T}_w^+$ is not congruent with each other.

Therefore $[\Psi_0] = [\Psi] \supset \hat{T}_w^+$. The total $= 2^\nu$. This is just equal to the number of open orbits in $B\backslash G^c$.

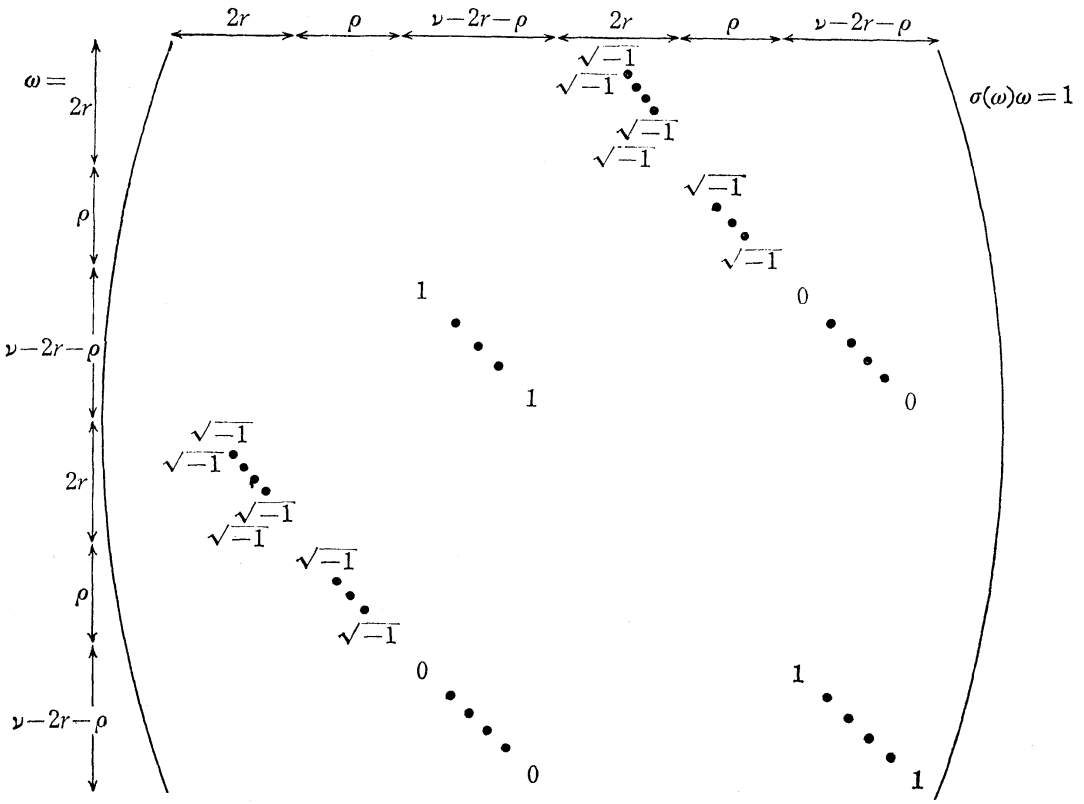
In most general case: $2r+\rho \leqq \nu$

$$\sigma \cdot w(H) = (\underbrace{t_2, t_1, t_4, t_3, \cdots, t_{2r}, t_{2r-1}}_{2r}, \underbrace{t_{2r+1}, \cdots, t_{2r+\rho}}_{\rho}, -t_{2r+\rho+1}, \cdots, -t_\nu),$$

$$w(H) = (-t_2, -t_1, -t_4, -t_3, \cdots, -t_{2r}, -t_{2r-1}, -t_{2r+1}, \cdots, -t_{2r+\rho}, t_{2r+\rho+1}, \cdots t_\nu),$$

$$\mathfrak{h}_u^+ = \{(\underbrace{t_1, t_1, t_3, t_3, \cdots, t_{2r-1}, t_{2r-1}}_{2r}, \underbrace{t_{2r+1}, \cdots, t_{2r+\rho}}_{\rho}, \underbrace{0 \cdots 0}_{\nu-2r-\rho})\},$$

$$\mathfrak{h}_u^- = \{(\underbrace{t_1, -t_1, t_3, -t_3, \cdots, t_{2r-1}, -t_{2r-1}}_{2r}, \underbrace{0 \cdots 0}_{\rho}, \underbrace{t_{2r+\rho+1}, \cdots, t_\nu}_{\nu-2r-\rho})\},$$



$$\hat{T}_w^+ = \{\mathrm{Diag}\,[\overbrace{\varepsilon_1, \varepsilon_1, \varepsilon_2, \varepsilon_2, \cdots, \varepsilon_r, \varepsilon_r}^{2r}, \overbrace{\varepsilon_{2r+1}, \cdots, \varepsilon_{2r+\rho}}^{\rho}, \overbrace{1 \cdots 1}^{\nu-2r-\rho},$$

$$\varepsilon_1, \varepsilon_1, \varepsilon_2, \varepsilon_2, \cdots, \varepsilon_r, \varepsilon_r, \varepsilon_{2r+1}, \cdots, \varepsilon_{2r+\rho}, 1 \cdots 1]\} \qquad \varepsilon_j = \pm 1 .$$

Any element of $\hat{T}_w^+$ is congruent to one of those elements

$$\mathrm{Diag}\,[\underbrace{1, 1 \cdots 1}_{2r}, \varepsilon_{2r+1}, \cdots, \varepsilon_{2r+\rho}, \underbrace{1 \cdots 1}_{\nu-2r-\rho}, \underbrace{1 \cdots 1}_{2r}, \varepsilon_{2r+1}, \cdots, \varepsilon_{2r+\rho}, 1 \cdots 1],$$

which are not congruent among themselves. Therefore the number of $\hat{T}_w^+ \cap [\Psi_0]$ is equal to 2^ρ. The total

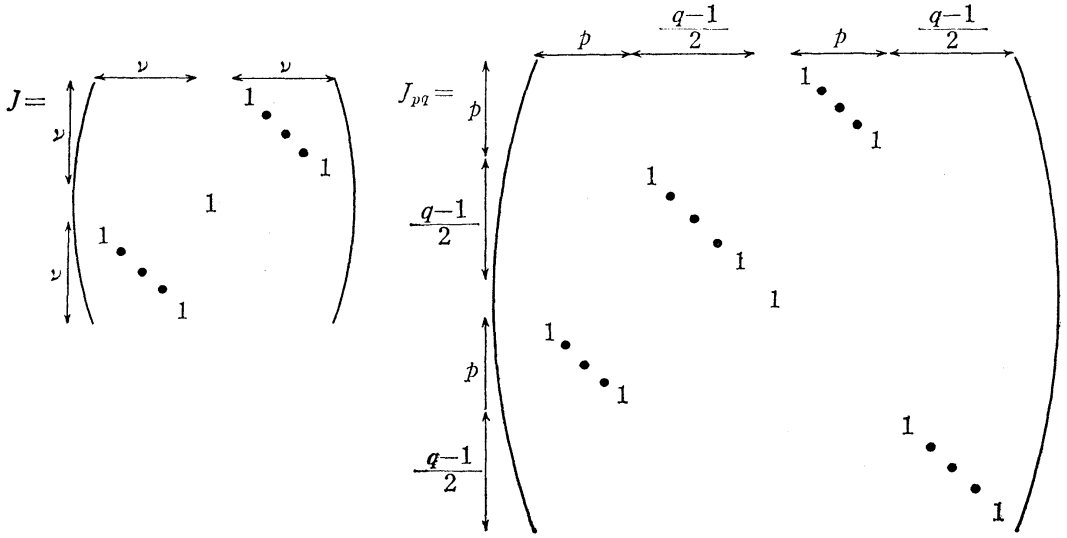
$$= \frac{1}{r!}\binom{\nu}{2}\binom{\nu-2}{2} \cdots \binom{\nu-2r+2}{2} \times \binom{\nu-2r}{\rho} \times 2^\rho = \frac{\nu!\, 2^\rho}{r!\, 2^r \rho!\, (\nu-2r-\rho)!} .$$

Finally, we have obtained our conclusion that the number of $[\Psi_0]$ is equal to

$$\sum_{r=0}^{[\frac{\nu}{2}]} \sum_{\rho=0}^{\nu-2r} \frac{\nu!\, 2^\rho}{r!\, 2^r \rho!\, (\nu-2r-\rho)!} = \sum_{r=0}^{[\frac{\nu}{2}]} \frac{\nu!\, 3^{\nu-2r}}{r!\, 2^r (\nu-2r)!} = (\sqrt{-2})^{-\nu} H_\nu\Big(\frac{\sqrt{-3}}{\sqrt{-2}}\Big)$$

where $H_\nu(z)$ is the Hermite polynomial of degree ν.

5. $G^c = \{g \in SL(n, \boldsymbol{C})\,;\, {}^t g J g = J\}$ $n = 2\nu+1 = 2p+q$, $G_u = G^c \cap SU(n)$, $G = \{g \in G^c\,;\, {}^t\bar{g} J_{pq} g = J_{pq}\}$, where



$\mathfrak{g}^C=\{X\in\mathfrak{gl}(n,\boldsymbol{C}),\ {}^tXJ+JX=0\}$, σ (involutive automorphism of $\mathfrak{g}_u$):

$$X\to J_{pq}^{-1}XJ_{pq}\quad(X\in\mathfrak{g}_u),$$

$$\mathfrak{g}_u=\mathfrak{g}^C\cap\mathfrak{gl}(n)=\left\{\begin{pmatrix} X_{11} & \mathfrak{x}_1 & X_{12} \\ -{}^t\bar{\mathfrak{x}}_1 & 0 & -{}^t\mathfrak{x}_1 \\ X_{12} & \bar{\mathfrak{x}}_1 & -{}^tX_{11}\end{pmatrix};\quad X_{11}\in\mathfrak{u}(\nu),\ X_{12}\in\mathcal{O}(\nu,\boldsymbol{C}),\ \mathfrak{x}_1\in\boldsymbol{C}^\nu\right\}.$$

$$\mathfrak{g}_u=\mathfrak{k}+\sqrt{-1}\mathfrak{p},\qquad \mathfrak{g}=\mathfrak{k}+\mathfrak{p},$$

$$\mathfrak{k}=\left\{X=\begin{pmatrix} Y_{11} & Y_{12} & \mathfrak{y}_1 & Y_{13} & \bar{Y}_{12} \\ -{}^t\bar{Y}_{12} & Y_{22} & \mathfrak{y}_2 & -{}^t\bar{Y}_{12} & Y_{24} \\ -{}^t\mathfrak{y}_1 & -{}^t\mathfrak{y}_2 & 0 & -{}^t\mathfrak{y}_1 & -{}^t\mathfrak{y}_2 \\ Y_{13} & Y_{12} & \mathfrak{y}_1 & -{}^tY_{11} & \bar{Y}_{12} \\ -{}^tY_{12} & \bar{Y}_{24} & \mathfrak{y}_2 & -{}^tY_{12} & -{}^tY_{22}\end{pmatrix};\ \begin{array}{l} Y_{11},\ Y_{13}\in\mathcal{O}(p),\\ Y_{12}\in\mathfrak{M}(p,\nu-p,\boldsymbol{C}),\\ \mathfrak{y}_1\in\boldsymbol{R}^p,\ \mathfrak{y}_2\in\boldsymbol{C}^{\nu-p},\\ Y_{22}\in\mathfrak{u}(\nu-p),\\ Y_{24}\in\mathcal{O}(\nu-p,\boldsymbol{C})\end{array}\right\}.$$

$$\sqrt{-1}\mathfrak{p}=\left\{X=\begin{pmatrix} \sqrt{-1}Y_{11} & Y_{12} & \sqrt{-1}\mathfrak{y}_1 & \sqrt{-1}Y_{13} & -\bar{Y}_{12} \\ -{}^t\bar{Y}_{12} & 0 & 0 & {}^t\bar{Y}_{12} & 0 \\ \sqrt{-1}{}^t\mathfrak{y}_1 & 0 & 0 & -\sqrt{-1}{}^t\mathfrak{y}_1 & 0 \\ -\sqrt{-1}Y_{13} & -Y_{12} & -\sqrt{-1}\mathfrak{y}_1 & -\sqrt{-1}{}^tY_{11} & \bar{Y}_{12} \\ {}^tY_{12} & 0 & 0 & -{}^tY_{12} & 0\end{pmatrix};\right.$$

$$\left.Y_{11}\in Sy(p,\boldsymbol{R}),\quad Y_{12}\in\mathfrak{M}(p,\nu-p,\boldsymbol{C}),\quad Y_{13}\in\mathcal{O}(p),\quad \mathfrak{y}_1\in\boldsymbol{R}^p\right\}.$$

$\sqrt{-1}\mathfrak{h}_p$ (=maximal abelien subalgebra of $\sqrt{-1}\mathfrak{p}$)

$$= \{\mathrm{Diag}\,[\underbrace{\sqrt{-1}t_1, \cdots, \sqrt{-1}t_p}_{p}, \underbrace{0 \cdots 0}_{\frac{q-1}{2}}, 0, \underbrace{-\sqrt{-1}t_1, \cdots, -\sqrt{-1}t_p}_{p}, \underbrace{0 \cdots 0}_{\frac{q-1}{2}}]\}\,.$$

$$\mathfrak{h}_u = \mathfrak{h}_t + \sqrt{-1}\mathfrak{h}_\flat = \{\mathrm{Diag}\,[\sqrt{-1}t_1, \cdots, \sqrt{-1}t_\nu, 0, -\sqrt{-1}t_1, \cdots, -\sqrt{-1}t_\nu] \quad t_j \in \boldsymbol{R}\}\,.$$

For $H = \mathrm{Diag}\,[\sqrt{-1}t_1, \cdots, \sqrt{-1}t_\nu, 0, -\sqrt{-1}t_1, \cdots, -\sqrt{-1}t_\nu] \in \mathfrak{h}_u$ whose coordinate is $(t_1, t_2, \cdots, t_\nu) \in \boldsymbol{R}^\nu$, $\sigma(H) = (-t_1, \cdots, -t_p, t_{p+1}, \cdots, t_\nu)$. The Weyl group is the same as that of C_ν type. For any $w \in W$, $\sigma \cdot w \in W$, and so $\sigma \cdot w$ and w can be written in the following way: $\sigma \cdot w = P' \cdot A'$, $w = P \cdot A$, $P, P' \in \mathfrak{S}_\nu$ (= symmetric group of degree ν); $A, A' \in \boldsymbol{Z}_2^\nu$ (ν products of cyclic groups of order 2).

The conjugate classes of $\sigma \cdot w$ of order 2 are:

$$\sigma \cdot w = P' \cdot A',\ P' = (12)(34) \cdots (2r-1, 2r) \in \mathfrak{S}_\nu\,,$$
$$A' = (\underbrace{1 \cdots 1}_{2r}, \underbrace{-1 \cdots -1}_{s}, \underbrace{1 \cdots 1}_{\nu-2r-s}) \in \boldsymbol{Z}_2^\nu\,,$$

that is,

$$\sigma w:\ (t_1, t_2, \cdots, t_{2r-1}, t_{2r}, t_{2r+1}, \cdots, t_{2r+s}, \cdots, t_\nu)$$
$$\to (\underbrace{t_2, t_1}, \underbrace{t_4, t_3}, \cdots, \underbrace{t_{2r}, t_{2r-1}}, -t_{2r+1}, \cdots, -t_{2r+s}, t_{2r+s+1}, \cdots, t_\nu)\,,$$
$$T_w^+ = (\underbrace{\lambda_1, \lambda_1}, \cdots, \underbrace{\lambda_r, \lambda_r}, \underbrace{1 \cdots 1}_{s}, \underbrace{\lambda_{2r+s+1}, \cdots, \lambda_\nu}_{\nu-2r-s})\,,$$
$$T_w^- = (\lambda_1, \lambda_1^{-1}, \cdots, \lambda_r, \lambda_r^{-1}, \underbrace{\lambda_{2r+1}, \cdots, \lambda_{2r+s}}_{s}, \underbrace{1 \cdots 1}_{\nu-2r-s})\,.$$

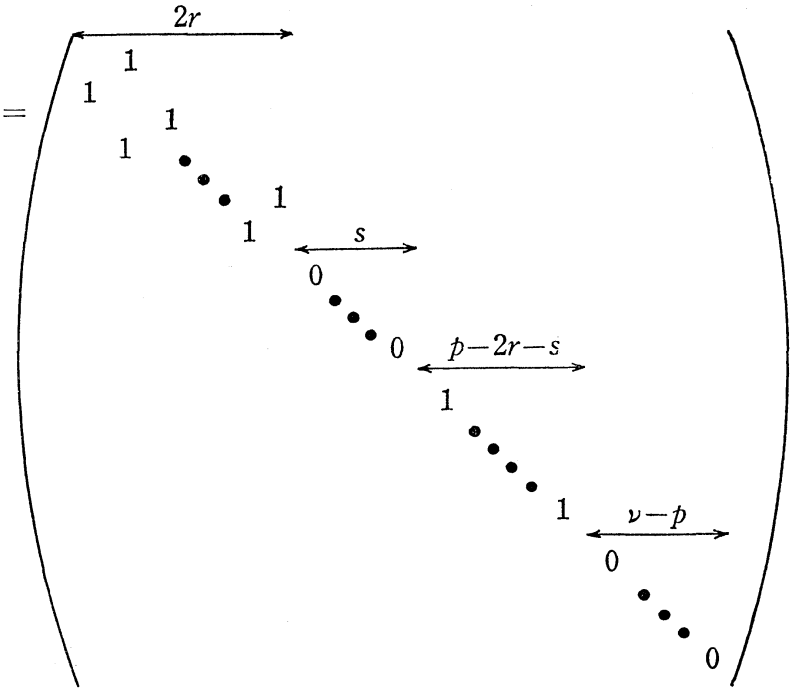
Each element of $\hat{T}_w^+$ is congruent mod T_w^- to one and only one of the elements:

(**) $$t = (\underbrace{1 \cdots 1}_{2r}, \underbrace{1 \cdots 1}_{s}, \underbrace{\lambda_{2r+s+1}, \cdots, \lambda_\nu}_{\nu-2r-s}) \qquad \lambda_j^2 = 1,\ 2r+s \leqq j \leqq \nu\,.$$

We consider three cases, separately, case a) $p \geqq 2r+s$, b) $2r+s > p \geqq 2r$, c) $2r > p$.

Case a) $\omega = \begin{pmatrix} X_{11} & 0 & X_{12} \\ 0 & (-1)^{p-s} & 0 \\ X_{12} & 0 & X_{11} \end{pmatrix}$, where $X_{11} = \mathrm{Diag}\,[\underbrace{0 \cdots 0}_{2r}, \underbrace{1 \cdots 1}_{s}, \underbrace{0 \cdots 0}_{p-2r-s}, \underbrace{1 \cdots 1}_{\nu-p}]$,

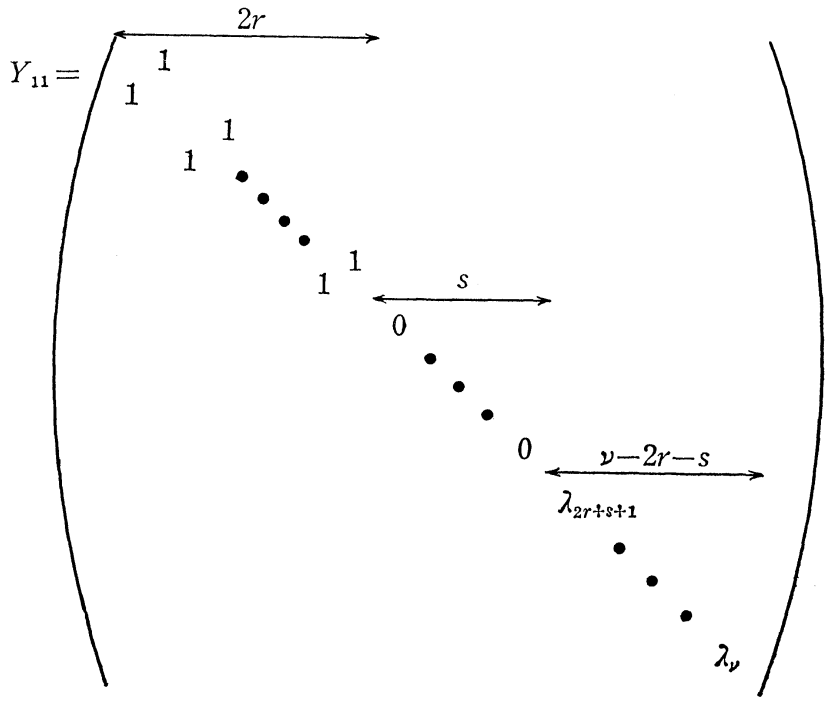
$X_{12} =$



$t_0^{-1} = \sigma(\omega) \cdot \omega = 1$, $[\Psi] \ni \omega t = \begin{pmatrix} X_{11} & 0 & X_{12} \\ 0 & (-1)^{p-s} & 0 \\ X_{21} & 0 & X_{22} \end{pmatrix} \cdot t$, t has the form (**).

Then

$$\omega t \cdot J_{pq} = \begin{pmatrix} Y_{11} & 0 & Y_{12} \\ 0 & (-1)^{p-s} & 0 \\ Y_{21} & 0 & Y_{22} \end{pmatrix} \text{ where}$$

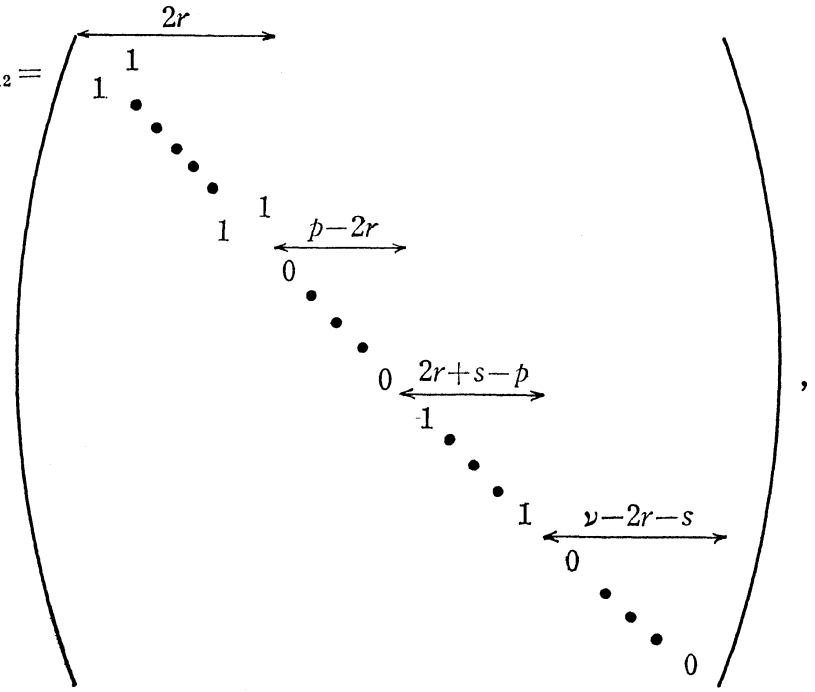


$$Y_{12}=\mathrm{Diag}\,[\underbrace{0\cdots 0}_{2r},\ \underbrace{1\cdots 1}_{s},\ \underbrace{0\cdots 0}_{\nu-2r-s}].$$

Let ι be the number of λ_j $2r+s+1\leqq j\leqq \nu$ which are equal to -1. The $\omega t\cdot J_{pq}$ and J_{pq} are conjugate with respect to G_u if and only if $p=2r+s+2\iota$ ($p-s=$ even) or $p=2r+s+2\iota+1$ ($p-s=$ odd). This can be proved by an elementary method or by the Remark in §2. Therefore the number of such elements is equal to $\begin{pmatrix}\nu-2r-s\\ \left[\frac{p-2r-s}{2}\right]\end{pmatrix}$.

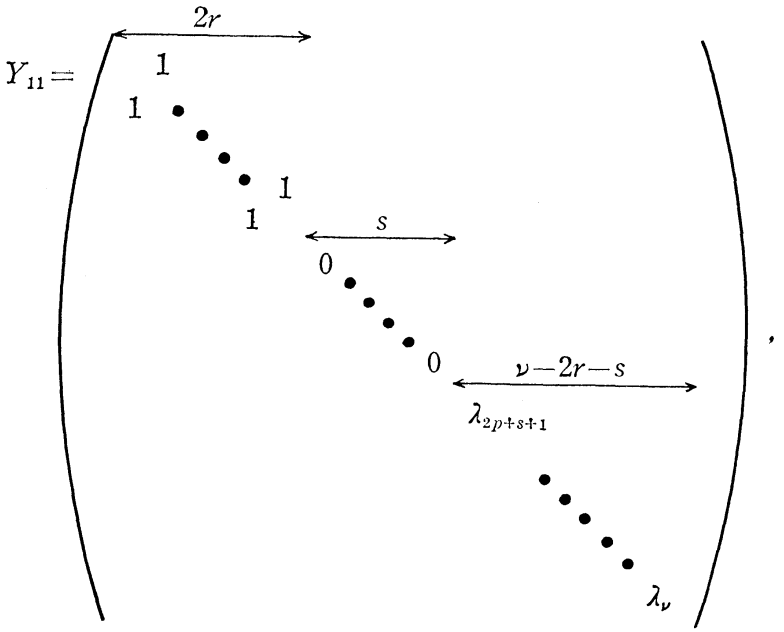
Case b) $\omega=\begin{pmatrix} X_{11} & 0 & X_{12}\\ 0 & (-1)^{s-p} & 0\\ X_{12} & 0 & X_{11}\end{pmatrix}$, where $X_{11}=\mathrm{Diag}\,[\underbrace{0\cdots 0}_{2r},\ \underbrace{1\cdots 1}_{p-2r},$

$\underbrace{0\cdots 0}_{2r+s-p},\ \underbrace{1\cdots 1}_{\nu-2r-s}]$,

$X_{12} =$ 

,

$t_0^{-1} = \sigma(\omega)\omega = 1,\ \omega t \cdot J_{pq} = \begin{pmatrix} Y_{11} & 0 & Y_{12} \\ 0 & (-1)^{s-p} & 0 \\ Y_{12} & 0 & Y_{11} \end{pmatrix}$, where

$Y_{11} =$ 

,

$$Y_{12}=\mathrm{Diag}\,[\underbrace{0\cdots 0}_{2r},\ \underbrace{1\cdots 1}_{s},\ \underbrace{0\cdots 0}_{\nu-2r-s}].$$

In this case, any $\iota\geqq 0$ can not satisfy $2r+s+2\iota=p$ ($s-p=$even) or $2r+s+2\iota+1=p$ ($s-p=$odd). Therefore, no such ωt lies in Ψ_0.

Case c) For $\omega t\in[\Psi]$,

$$\omega t\cdot J_{pq}=\begin{pmatrix} Y_{11} & 0 & Y_{12} \\ 0 & \varepsilon & 0 \\ Y_{12} & 0 & Y_{22} \end{pmatrix},\qquad \varepsilon=1 \text{ or } -1$$

according as $p=$even or odd respectively.

$$Y_{11}=\begin{matrix} \overset{2r}{\longleftrightarrow} & & & & & & & & \\ 1 & & & & & & & & \\ & 1 & & & & & & & \\ & & \ddots & & & & & & \\ & & & 1 & & & & & \\ & & & & 1 & & & & \\ & & & & & \overset{s}{\longleftrightarrow} & & & \\ & & & & & 0 & & & \\ & & & & & & \ddots & & \\ & & & & & & & \overset{\nu-2r-s}{\longleftrightarrow} & \\ & & & & & & & \lambda_{2r+s+1} & \\ & & & & & & & & \ddots \\ & & & & & & & & \lambda_\nu \end{matrix},$$

$$Y_{12}=\mathrm{Diag}\,[\underbrace{0\cdots 0}_{2r},\ \underbrace{1\cdots 1}_{s},\ \underbrace{0\cdots 0}_{\nu-2r-s}].$$

In this case, also, $\omega t\cdot J_{pq}$ is not conjugate to J_{pq}. Therefore, $\omega t\cdot J_{pq}\notin\Psi_0$. Thus, the number of $[\Psi_0]$ is equal to:

$$\sum_{\substack{p\geqq 2r+s\\ r,s\geqq 0}}\binom{\nu-2r-s}{\left[\frac{p-2r-s}{2}\right]}\times\frac{\nu!}{r!\,2^r(\nu-2r)!}\times\binom{\nu-2r}{s}.$$

If $\sigma\cdot w=1$ $r=s=0$, and therefore the number of open orbits is equal to $\binom{\nu}{\left[\frac{p}{2}\right]}$.

G is not connected. Let G_0 be its connected component of the identity. Then $[G: G_0]=2$. $G=G\cup\gamma G_0$ for a suitable $\gamma\in G$. But a fundamental Cartan subgroup S of G_0 is compact and is also a maximal torus of G_u. Open orbits are all isomorphic to $S\backslash G=S\backslash G_0\cup\gamma^{-1}S\gamma\backslash G_0$ (disjoint sum), so that the number of open orbits of G_0 in $B\backslash G_C$ is $\binom{\nu}{\left[\frac{p}{2}\right]}\times 2$.

In particular, if $p=1$, there is only one G-open orbit which is separated into two G_0-orbits. This manifold seems to be closely related to the discrete series of infinitesimal irreducible unitary representations of G_0 constructed by T. Hirai [**20**].

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