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On the automorphism group of a G-structure

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§ 0. Introduction.

A linear Lie group is called *elliptic* if its Lie algebra contains no matrix of rank one. A G-structure is called *elliptic* if G is *elliptic*. (N. B. G is a linear subgroup of $GL(n, \boldsymbol{R})$.) The purpose of this paper is to prove that the globally defined infinitesimal automorphisms of a G-structure (called G-vector field) are given by a system of linear elliptic differential equations if and only if this G-structure is elliptic. (See Lemma for a precise statement.) It follows easily

THEOREM A. *The group of diffeomorphisms of M which leave a given elliptic G-structure invariant is a finite dimensional Lie group, provided M is compact.*

Theorem A is a generalization of the results of Boothby-Kobayashi-Wang [1] and Ruh [8]. (In fact, Ruh's sufficient condition clearly implies that the G-structure in question is elliptic.) Both Lemma and Theorem A are contained implicitly in Guillemin-Sternberg [3]. Still we feel their explicit statements with proofs would be worth publishing because of their importance. Also we shall provide two examples to show that Theorem A is best possible in a sense, following suggestions of Professor S. Kobayashi and Professor S. Sternberg. Also the author wishes to express his thanks to Professor T. Nagano and Professor M. Kuranishi.

§ 1. Let $P(M, \pi, G)$ be any G-structure on M, and $\mathfrak{g}$ be the Lie algebra of G. That is, P is a subbundle with structure group G of the frame bundle of M. A (local) diffeomorphism of M is a *(local) G-automorphism* if and only if it leaves the G-structure $P(M, \pi, G)$ invariant.

Let $\{x^1, \cdots, x^n\}$ be a local coordinate system around $z \in M$, defined on an open neighbourhood U of M. Furthermore, we assume that the neighbourhood U is so small that it admits a local cross-section ϕ from U into P. Let V be an open set of U. A local diffeomorphism f from V into U is a local G-automorphism if and only if there exists a mapping g from V into G such that

$$(1) \qquad (df)(\phi(x)) = \phi(f(x)) \cdot g(x)$$

where (df) means the lift of f to the frame bundle of M. The local cross-section ϕ is expressed by $\phi(x)=\left(x, \sum_i \phi_1^i(x)\left(\frac{\partial}{\partial x^i}\right)_x, \cdots, \sum_i \phi_j^i(x)\left(\frac{\partial}{\partial x^i}\right)_x, \cdots, \sum_i \phi_n^i(x)\left(\frac{\partial}{\partial x^i}\right)_x\right)$, where $\phi_j^i(x)$ $(1\leqq i, j\leqq n)$ are differentiable functions on U. By the definition, we have

$$(df)\phi(x^1, \cdots, x^n)=\left(f(x), \cdots, \sum_{i,k} \phi_j^i(x)\left(\frac{\partial f^k}{\partial x^i}\right)_x\left(\frac{\partial}{\partial x^k}\right)_{f(x)}, \cdots\right)$$

where $f=(f^1, \cdots, f^n)$. Let $g_j^i(x)$ $(1\leqq i, j\leqq n)$ be the (i, j)-entries of the matrix $g(x)$. By (1), we have

$$\sum_{i,k} \phi_j^i(x)\left(\frac{\partial f^k}{\partial x^i}\right)_x\left(\frac{\partial}{\partial x^k}\right)_{f(x)}=\sum_{i,l} \phi_l^i(f(x))\left(\frac{\partial}{\partial x^i}\right)_{f(x)} g_j^l(x).$$

Hence we have;

$$(1)' \qquad \sum_i \phi_j^i(x)\left(\frac{\partial f^k}{\partial x^i}\right)_x=\sum \phi_l^k(f(x))g_j^l(x).$$

Since the matrix $(\phi_j^i(x))_{1\leqq i,j\leqq n}$ is nonsingular, we denote by $(\theta_j^i(x))_{1\leqq i,j\leqq n}$ the inverse matrix of $(\phi_j^i(x))_{1\leqq i,j\leqq n}$. Multiplying (1)$'$ by $\theta_k^h(f(x))$ and summing it up we get $\sum_{i,k} \theta_k^h(f(x))\phi_j^i(x)\left(\frac{\partial f^k}{\partial x^i}\right)_x=\sum_l \delta_l^h g_j^l(x)=g_j^h(x)$. Since the matrix $(g_j^h(x))$ belongs to G, we may write the above equation

$$(2) \qquad \left(\sum_{i,k} \theta_k^h(f(x))\phi_j^i(x)\left(\frac{\partial f^k}{\partial x^i}\right)_x\right)_{1\leqq h,j\leqq n}\in G.$$

A vector field on M is *a G-vector field* of P by definition if and only if it generates local G-automorphisms. Let $\sum_i X^i\frac{\partial}{\partial x^i}$ be the local expression on U of an arbitrary vector field $\mathfrak{X}$ and $\psi_t(|t|<\varepsilon)$ be the local one-parameter group around z which $\mathfrak{X}$ generates. If we take a sufficiently small neighbourhood $V\ni z$, we may assume that $\psi_t(|t|<\varepsilon)$ maps V into U. $\mathfrak{X}$ is a G-vector field if and only if ψ_t satisfies the equation (2) for each $t(|t|<\varepsilon)$. Hence we get

$$(2)' \qquad \left(\sum_{i,k} \theta_k^h(\psi_t(x))\phi_j^i(x)\left(\frac{\partial\psi_t^k}{\partial x^i}\right)_x\right)\in G \text{ for any small } t.$$

The matrix in (2)$'$ is the neutral element of G when t equals 0. Therefore, differentiating (2)$'$ at $t=0$ with respect to the variable t, we get the element of $\mathfrak{g}$. I. e.,

$$\left(\frac{\partial}{\partial t}\left\{\sum_{i,k} \theta_k^h(\psi_t(x))\phi_j^i(x)\left(\frac{\partial\psi_t^k}{\partial x^i}\right)_x\right\}\Big|_{t=0}\right)\in\mathfrak{g}.$$

Therefore,

$$\Big(\sum_{i,k,m}\Big(\frac{\partial\theta_k^h}{\partial x^i}\Big)_x X^m\phi_j^i(x)\delta_i^k+\sum_{i,k}\theta_k^h(x)\phi_j^i(x)\frac{\partial X^k}{\partial x^i}\Big)\in\mathfrak{g}\,.$$

Hence we get

$$(2)'' \qquad \Big(\sum_{i,k}\Big(\theta_k^h(x)\phi_j^i(x)\Big(\frac{\partial X^k}{\partial x^i}\Big)_x+\phi_j^i(x)\Big(\frac{\partial\theta_i^h}{\partial x^k}\Big)_x X^k\Big)\Big)\in\mathfrak{g}\,.$$

Let us choose a set of constants ${}_1C_j^i, \cdots, {}_rC_j^i$, $i, j=1, \cdots, n$ (where r is the codimension of $\mathfrak{g}$ in $\mathfrak{gl}(n, \boldsymbol{R})$) such that

$$(a_j^i)\in\mathfrak{g} \text{ if and only if } \sum_{i,j}{}_\alpha C_j^i a_i^j=0, \quad \alpha=1, \cdots, r\,.$$

Therefore a G-vector field $\mathfrak{X}$ (locally expressed by $\sum X^i\frac{\partial}{\partial x^i}$) satisfies the linear differential equation with unknown functions $X^1, \cdots, X^n$;

$$(3) \qquad \sum_{\substack{i,k\\ j,h}}{}_\alpha C_h^j\Big(\theta_k^h(x)\phi_j^i(x)\Big(\frac{\partial X^k}{\partial x^i}\Big)_x+\phi_j^i(x)\frac{\partial\theta_i^h(x)}{\partial x^k}X^k\Big)=0\,,$$
$$\alpha=1, \cdots, r\,.$$

Let D be the linear differential operator which corresponds to (3). For any n-tuple $\xi=(\xi_1, \cdots, \xi_n)\neq 0$, we denote by $S(x, \xi)_k^\alpha$ $(\alpha=1, \cdots, r\,;\ k=1, \cdots, n)$

$$S(x, \xi)_k^\alpha=\sum_{h,i,j}{}_\alpha C_h^j\theta_k^h(x)\phi_j^i(x)\xi_i\,.$$

The matrix $(S(x, \xi)_k^\alpha)$ is the symbol of D with respect to ξ at $x\in V$. Let D^* be the adjoint operator of D with respect to the usual inner product $\langle(x^i), (y^i)\rangle=\sum_i x^i y^i$. It is well known that the symbol $(\hat{S}(x, \xi)_q^p)$ $(n\times n$ matrix$)$ of the 2nd order linear differential operator D^*D with respect to ξ at x is given by ${}^t(S(x, \xi)_k^\alpha)(S(x, \xi)_k^\alpha)$.

Now we shall prove,

LEMMA. *The 2nd order linear differential operator D^*D is elliptic if and only if $P(M, \pi, G)$ is elliptic i.e. $\mathfrak{g}$ contains no element of rank one.*

PROOF. Only-if part; Suppose the equation $(\hat{S}(x, \xi)_q^p)\mathfrak{a}=0$ holds for some x, ξ and $\mathfrak{a}=(a^i)_{1\leq i\leq n}$. Therefore $\langle(\hat{S}(x, \xi)_q^p)\mathfrak{a}, \mathfrak{a}\rangle=0$. By the definition we get, $\langle(\hat{S}(x, \xi)_q^p)\mathfrak{a}, \mathfrak{a}\rangle=\langle{}^t(S(x, \xi)_k^\alpha)(S(x, \xi)_k^\alpha)\mathfrak{a}, \mathfrak{a}\rangle=\langle(S(x, \xi)_k^\alpha)\mathfrak{a}, (S(x, \xi)_k^\alpha)\mathfrak{a}\rangle$. Hence $(S(x, \xi)_k^\alpha)\mathfrak{a}=0$, i.e. $\sum_{h,i,j,p}{}_\alpha C_h^j\theta_p^h(x)\phi_j^i(x)\xi_i a^p=0$, $\alpha=1, 2, \cdots, r$. Defining $\bar{\xi}_j$ (resp. $\bar{a}^h$), $1\leq j,\ h\leq n$ by $\bar{\xi}_j=\sum_i\phi_j^i(x)\xi_i$, (resp. $\bar{a}^h=\sum_p\theta_p^h(x)a^p$), we get $\sum_{h,j}{}_\alpha C_h^j\bar{\xi}_j\bar{a}^h=0$. By the definition of the constants ${}_\alpha C_k^i$, the matrix $(\bar{\xi}_j\bar{a}^h)_{1\leq j,h\leq n}$ lies in $\mathfrak{g}$. Now a matrix $(\neq 0)$ is of rank one if and only if it can be written as $(\eta_j b^i)$. Since the matrix $(\phi_j^i(x))$ is non-singular, $(\bar{\xi}_1, \cdots, \bar{\xi}_n)$ is not zero. Therefore $(\bar{a}^1, \cdots, \bar{a}^n)$ must be zero if $\mathfrak{g}$ contains no matrix of rank one. Hence $\mathfrak{a}=(a^1, \cdots, a^n)$ is zero, proving that D^*D is elliptic. Conversely suppose D^*D is elliptic. Suppose $\mathfrak{g}$ contains a matrix $(\xi'_i a'^j)$ of rank one, then $\mathfrak{a}'=(a'^j)_{1\leq j\leq n}$ is a solution of $(S(x,\xi')^\alpha_k)\mathfrak{a}'=0$ for any $x\in V$ (here $\xi'=(\xi'_1,\cdots,\xi'_n)\neq 0$). Therefore the symbol $(\hat{S}(x,\xi')^p_q)$ is singular for $\xi'\neq 0$ and for any $x\in V$. This is a contradiction. Q. E. D.

Using the well known fact about elliptic differential operators, we get;

COROLLARY. *If M is compact and if $P(M,\pi,G)$ is elliptic then the vector space of globally defined G-vector field is finite dimensional.*

By Theorem of R. S. Palais [2], [6], and by Corollary above, we have proved Theorem A.

§2. In this section we shall give two examples to show that Theorem A is best possible in a sense.

This example is due to Guillemin-Sternberg [3].

EXAMPLE 1. If G is not elliptic, then the automorphism group of any totally flat G-structure P over n-dim enclidian space $M(n>0)$, is not a Lie transformation group in the sense of Gleason-Palais [7]. Here a G-structure $P(M,\pi,G)$ is called flat, as usual, if M has an atlas whose charts give rise to local sections of P. That is, the local section $(x^i)\rightarrow\left(\frac{\partial}{\partial x^i}\right)$ of the frame bundle defined by each chart is that of P also. Such a chart will be called *admissible*. $P(M,\pi,G)$ is called totally flat if we can take a global admissible chart.

Now we give a proof of the assertion above. A vector field $\mathfrak{X}$ on M is a G-vector field of P if and only if the matrix $(\partial X^i/\partial x^j)$ in terms of any global admissible chart is contained in the Lie algebra $\mathfrak{g}$ of G at each point. Since G is not elliptic, we may assume $\mathfrak{g}$ contains either the matrix

$$\left(\begin{array}{c|c} 1 & 0\cdots 0 \\ \hline 0 & \\ \vdots & 0 \\ 0 & \end{array}\right) \quad \text{or the matrix} \quad \left(\begin{array}{c|c} 0 & 0\cdots 0 \\ \hline 1 & \\ 0 & \\ \vdots & 0 \\ 0 & \end{array}\right).$$

In the first case, for any smooth function $f(x^1)$, a vector field $f(x^1)\frac{\partial}{\partial x^1}$ is a G-vector field by the above remark.

And in the second case, for any smooth function $f(x^1)$, a vector field $f(x^1)\frac{\partial}{\partial x^2}$ is a G-vector field. Therefore the vector space of the G-vector field of P is of infinite dimension. Hence our assertion has been proved.

EXAMPLE 2. Now we shall give the famous non flat example. Let M^{2n+1} be an (orientable) $(2n+1)$-dimensional manifold, on which a 1-form ω with $d\omega$ of maximal rank is given (so-called contact structure). Then the linear differential system $\omega=0$ naturally gives a G-structure. Since $\omega\wedge d\omega$ is not zero, $\omega=0$ is not integrable. Thus that G-structure is non flat. It is easy to see that any G-vector field $\mathfrak{X}$ is the infinitesimal automorphism of the contact structure, i. e.

$$\theta(\mathfrak{X})\omega=f\omega \qquad f\text{: smooth function,}$$

(here $\theta(\mathfrak{X})$ means the Lie derivative with respect to $\mathfrak{X}$) and *vice versa*. It is well-known that the vector space of the infinitesimal automorphism of a contact structure is isomorphic to the vector space (of infinite dimension) of the smooth functions on M^{2n+1} [4]. Therefore the automorphism group of this G-structure is not a Lie transformation group.

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