J. Math. Soc. Japan
Vol. 18, No. 2, 1966

On product spaces and product mappings

By Tadashi ISHII

(Received Nov. 22, 1965)

Recently E. Michael [7] has proved that the topological product of a normal space with a metric space is not normal in general. As for the problem to find a necessary and sufficient condition for a topological space X to possess the property that the product space $X\times Y$ is normal for any metric space Y, K. Morita [13] gave a complete solution by introducing the notion of P-spaces. Furthermore, in his paper [14] K. Morita has defined the notion of basic coverings in the product space $X\times Y$ of a normal space X with a metric space Y, and established a necessary and sufficient condition for $X\times Y$ to be countably paracompact and normal. His theorem reads as follows.

THEOREM A. *The product space $X\times Y$ is countably paracompact and normal if and only if X is countably paracompact and any basic covering of $X\times Y$ has a special refinement.*

On the other hand, H. Tamano [17] has given a result that the product space $X\times Y$ of a paracompact Hausdorff space X with a metric space Y is paracompact if and only if $X\times Y$ is countably paracompact. In an unpublished paper [15] K. Morita has pointed out that this result is true although the proof in [17] is incomplete.

In § 1 of this paper we shall prove mainly the following two results which are related to Theorem A.

(1) In the 'if' part of Theorem A, we can exclude the assumption that X is countably paracompact (Theorem 1.3).

(2) In case X is countably paracompact, if any basic covering of special type in $X\times Y$ has a special refinement, then $X\times Y$ is countably paracompact and normal (Theorem 1.4).

The basic covering of special type in (2) is one obtained from a countable open covering of $X\times Y$ which contains an open dense subset of $X\times Y$ as its element.

In § 2 we investigate the product mappings of two closed mappings. Concerning the matter, it seems that only few results are known until now. The following has been obtained by Z. Frolík [2] and K. Morita [10].

(a) The cartesian product of perfect mappings is also perfect.

(b) The product of two closed mappings is not closed in general.

The main result in § 2 reads as follows: Let R and S be perfectly normal T_1-spaces, and let X and Y be topological spaces each of which is not discrete. If $f: R \rightarrow X$ and $g: S \rightarrow Y$ are closed, continuous and onto mappings, and if the product mapping $f \times g: R \times S \rightarrow X \times Y$ is also closed, then $f^{-1}(x)$ and $g^{-1}(y)$ are both countably compact for every $x \in X$ and $y \in Y$.

In § 3 we shall be concerned with the product space of paracompact Hausdorff spaces X and Y. As is well known, the product space of paracompact Hausdorff spaces X and Y is not normal in general. Recently K. Morita [**12**] has proved the following theorems.

THEOREM B. *Let X be a paracompact normal space which is a countable union of locally compact closed subsets, and let Y be a paracompact normal space. Then the product space $X \times Y$ is paracompact and normal.*

THEOREM C. *Let X be a metric space or more generally an M-space. Then in order that the product space $X \times Y$ be normal for any paracompact normal space Y it is necessary and sufficient that X be a paracompact normal space which is a countable union of locally compact closed subsets.*

He raised an open problem whether Theorem C is true without any restriction on X. For this problem we shall give a negative answer by showing the following results.

(1) Let X be the image under a closed continuous mapping of a locally compact and paracompact Hausdorff space, and let Y be a paracompact Hausdorff space. Then the product space $X \times Y$ is a paracompact Hausdorff space (Theorem 3.2).

(2) Let X be the image under a closed continuous mapping of a locally compact and paracompact Hausdorff space. Then X is a paracompact Hausdorff space, but is not represented as a countable union of locally compact closed subspaces in general (cf. Example in § 3).

The author expresses his hearty thanks to Prof. K. Morita who has been kind enough to give him various suggestions and advices.

§ 1. On the product space of a normal space with a metric space.

Throughout this section we assume that X is a normal space and Y is a metric space. Let $\mathfrak{V} = \{\mathfrak{V}_i\}$ be an open basis of Y such that (i) $\mathfrak{V}_i = \{V_{i\alpha} | \alpha \in \Omega_i\}$ is a locally finite open covering of Y for $i = 1, 2, \cdots$, and (ii) $\{St(y, \mathfrak{V}_i) | i = 1, 2, \cdots\}$ is a basis for neighborhoods at each point y of Y. Let us put

$$W(\alpha_1, \cdots, \alpha_i) = \bigcap_{\nu=1}^{i} V_{\nu\alpha_\nu} \qquad \text{for} \quad \alpha_1 \in \Omega_1, \cdots, \alpha_i \in \Omega_i .$$

According to Morita [**14**], a covering $\mathfrak{G}$ of the product space $X \times Y$ is said to be a basic covering if $\mathfrak{G}$ has the form

$$\mathfrak{G}=\{G(\alpha_1, \cdots, \alpha_i)\times W(\alpha_1, \cdots, \alpha_i) \mid \alpha_\nu \in \Omega_\nu, \nu=1, \cdots, i ; i=1, 2, \cdots\}$$

and if $G(\alpha_1, \cdots, \alpha_i)$ are open subsets of X such that

$$G(\alpha_1, \cdots, \alpha_i)\subset G(\alpha_1, \cdots \alpha_i, \alpha_{i+1}) \qquad \text{for} \quad \alpha_1 \in \Omega_1, \cdots, \alpha_{i+1}\in \Omega_{i+1}.$$

If, for a basic covering $\mathfrak{G}$ of $X\times Y$, there exists a family $\{F(\alpha_1, \cdots, \alpha_i) \mid \alpha_\nu \in \Omega_\nu, \nu=1, \cdots, i ; i=1, 2, \cdots\}$ of closed subsets of X such that

$$F(\alpha_1, \cdots, \alpha_i)\subset G(\alpha_1, \cdots, \alpha_i),$$

$$\cup\{F(\alpha_1, \cdots, \alpha_i)\times W(\alpha_1, \cdots, \alpha_i) \mid \alpha_\nu \in \Omega_\nu, \nu=1, \cdots, i ; i=1, 2, \cdots\}=X\times Y,$$

$\mathfrak{G}$ is said to have a special refinement.

The following lemma is due to Morita [**14**, Theorem 1.2].

LEMMA 1.1. *A basic covering of $X\times Y$ has a special refinement if and only if it is a normal covering.*

Let $\mathfrak{M}=\{M_\lambda \mid \lambda \in \Lambda\}$ be an open covering of $X\times Y$. If we put

$$M(\alpha_1, \cdots, \alpha_i)=\cup\{M(\alpha_1, \cdots, \alpha_i ; \lambda) \mid \lambda\in\Lambda\}$$

where $M(\alpha_1, \cdots, \alpha_i ; \lambda)$ is the union of all the open subsets P of X such that $P\times W(\alpha_1, \cdots, \alpha_i)\subset M_\lambda$, then

$$\{M(\alpha_1, \cdots, \alpha_i)\times W(\alpha_1, \cdots, \alpha_i) \mid \alpha_\nu\in\Omega_\nu ; \nu=1, \cdots, i ; i=1, 2, \cdots\}$$

is a basic covering of $X\times Y$. We shall denote this basic covering by $\mathfrak{B}(\mathfrak{M})$. $\mathfrak{B}(\mathfrak{M})$ has the following property:

$$(\sharp) \qquad \bigcup_{i=1}^{\infty} M(\alpha_1, \cdots, \alpha_i)=X \qquad \text{if} \quad \bigcap_{i=1}^{\infty} W(\alpha_1, \cdots, \alpha_i)\neq\phi.$$

It should be noted that every basic covering which we consider below satisfies the condition ($\sharp$).

The following lemma is essentially due to Morita [**14**, Theorem 2.2].

LEMMA 1.2. *Let $\mathfrak{M}$ be any finite open covering of $X\times Y$. If $\mathfrak{B}(\mathfrak{M})$ has a special refinement, then $\mathfrak{M}$ is a normal covering of $X\times Y$.*

For any basic covering $\mathfrak{G}=\{G(\alpha_1, \cdots, \alpha_i)\times W(\alpha_1, \cdots, \alpha_i) \mid \alpha_\nu\in\Omega_\nu, \nu=1, \cdots, i ; i=1, 2, \cdots\}$, let us put

$$H_i(\mathfrak{G})=\cup\{G(\alpha_1, \cdots, \alpha_i)\times W(\alpha_1, \cdots, \alpha_i) \mid i: \text{fixed}\}.$$

Since $W(\alpha_1, \cdots, \alpha_i)=\cup\{W(\alpha_1, \cdots, \alpha_i, \alpha_{i+1}) \mid \alpha_{i+1}\in\Omega_{i+1}\}$ and $G(\alpha_1, \cdots, \alpha_i)\subset G(\alpha_1, \cdots, \alpha_i, \alpha_{i+1})$, we obtain $H_i(\mathfrak{G})\subset H_{i+1}(\mathfrak{G})$ for each $i=1, 2, \cdots$.

Now we shall prove the following

THEOREM 1.3. *For the product space $X\times Y$ of a normal space X with a* *metric space Y, the following properties are equivalent.*

(1) *Any basic covering $\mathfrak{G}$ of $X\times Y$ has a special refinement.*

(2) *For any basic covering $\mathfrak{G}$ of $X\times Y$, there exists a countable family of open subsets $\{O_i\}$ of $X\times Y$ such that $\bar{O}_i\subset H_i(\mathfrak{G})$, $i=1, 2, \cdots$, and $\bigcup_{i=1}^{\infty} O_i=X\times Y$.*

(3) *$X\times Y$ is countably paracompact.*

(4) *$X\times Y$ is countably paracompact and normal.*

In his unpublished paper [**15**], K. Morita has proved that (3) implies (4). We shall restate his proof here for completeness.

PROOF. (1)→(2). Let $\mathfrak{G}$ be any basic covering of $X\times Y$.

Since $\mathfrak{G}$ is normal by Lemma 1.1, there exists a locally finite open covering $\mathfrak{U}=\{U_\lambda \mid \lambda\in\Lambda\}$ such that $\{\bar{U}_\lambda \mid \lambda\in\Lambda\}$ is a refinement of $\mathfrak{G}$. Let O_i be the union of all the elements of $\mathfrak{U}$ whose closure in $X\times Y$ are contained in some $G(\alpha_1, \cdots, \alpha_j)\times W(\alpha_1, \cdots, \alpha_j)\in\mathfrak{G}$, where $j\leqq i$. Then we have $\bar{O}_i\subset H_i(\mathfrak{G})$ and $\bigcup_{i=1}^{\infty} O_i=X\times Y$.

(2)→(3). Let $\mathfrak{U}=\{U_i\}$ be a countable open covering of $X\times Y$ such that $U_i\subset U_{i+1}$ $(i=1, 2, \cdots)$. Let us put

$$G(\alpha_1, \cdots, \alpha_i)=\bigcup\{P \mid P\times W(\alpha_1, \cdots, \alpha_i)\subset U_i,\ P\colon \text{open}\}\,.$$

Then we can prove that $\mathfrak{B}=\{G(\alpha, \cdots, \alpha_i)\times W(\alpha_1, \cdots, \alpha_i) \mid \alpha_1\in\Omega_1, \cdots, \alpha_i\in\Omega_i\,;\ i=1, 2, \cdots\}$ is a basic covering of $X\times Y$. In fact, it is obvious that $G(\alpha_1, \cdots, \alpha_i)\subset G(\alpha_1, \cdots, \alpha_i, \alpha_{i+1})$ for $\alpha_1\in\Omega_1, \cdots, \alpha_{i+1}\in\Omega_{i+1}$. Hence we shall prove that $\mathfrak{B}$ is a covering of $X\times Y$. Let (x, y) be an arbitrary point of $X\times Y$. Then (x, y) is contained in some U_i. Hence we can find an open set $P\times W(\alpha_1, \cdots, \alpha_j)$ of $X\times Y$ such that

$$(x, y)\in P\times W(\alpha_1, \cdots, \alpha_j)\subset U_i\,.$$

In case $i\geqq j$, we have

$$(x, y)\in G(\alpha_1, \cdots, \alpha_j, \alpha_{j+1}, \cdots, \alpha_i)\times W(\alpha_1, \cdots, \alpha_j, \alpha_{j+1}, \cdots, \alpha_i)$$

for some $\alpha_{j+1}\in\Omega_{j+1}, \cdots, \alpha_i\in\Omega_i$. In case $i<j$, we have

$$(x, y)\in G(\alpha_1, \cdots, \alpha_j)\times W(\alpha_1, \cdots, \alpha_j)\,,$$

because $(x, y)\in U_i\subset U_j$. Hence $\mathfrak{B}$ is a basic covering of $X\times Y$. Moreover from the definition of $\mathfrak{B}$ it follows that $H_i(\mathfrak{B})\subset U_i$. Therefore by (2) there exists a countable family of open subsets $\{O_i\}$ of $X\times Y$ such that $O_i\subset\bar{O}_i$ $H_i(\mathfrak{B})\subset U_i$ and $\bigcup_{i=1}^{\infty} O_i=X\times Y$. Hence $X\times Y$ is countably paracompact by a theorem of Ishikawa [**4**].

(3)→(4). We shall first consider the case when the covering dimension of Y is zero, i.e., $\dim Y=0$. In this case we can assume that for each i $\mathfrak{B}_i$ consists of mutually disjoint open subsets of Y. Let $\mathfrak{G}=\{G(\alpha_1, \cdots, \alpha_i)$

$\times W(\alpha_1, \cdots, \alpha_i) \mid \alpha_\nu \in \Omega_\nu, \nu=1, \cdots, i; i=1, 2, \cdots\}$ be any basic covering of $X \times Y$. Then we have $H_i(\mathfrak{G}) \subset H_{i+1}(\mathfrak{G})$ and $\bigcup_{i=1}^{\infty} H_i(\mathfrak{G}) = X \times Y$. Hence by assumption that $X \times Y$ is countably paracompact, there exists a locally finite open covering $\{U_i\}$ of $X \times Y$ such that $\bar{U}_i \subset H_i(\mathfrak{G})$, $i=1, 2, \cdots$ by Ishikawa [4]. Since $\{W(\alpha_1, \cdots, \alpha_i) \mid \alpha_\nu \in \Omega_\nu, \nu=1, \cdots, i\}$ consists of mutually disjoint open subsets of Y, we see that $\bar{U}_i \cap [G(\alpha_1, \cdots, \alpha_i) \times W(\alpha_1, \cdots, \alpha_i)]$ is a closed subset of $\bar{U}_i$ (and hence a closed subset of $X \times Y$). Hence, if we put

$$L(\alpha_1, \cdots, \alpha_i) = U_i \cap [G(\alpha_1, \cdots, \alpha_i) \times W(\alpha_1, \cdots, \alpha_i)],$$

then $\{L(\alpha_1, \cdots, \alpha_i) \mid \alpha_\nu \in \Omega_\nu, \nu=1, \cdots, i; i=1, 2, \cdots\}$ is a locally finite oenp covering of $X \times Y$ such that

$$\overline{L(\alpha_1, \cdots, \alpha_i)} \subset G(\alpha_1, \cdots, \alpha_i) \times W(\alpha_1, \cdots, \alpha_i).$$

Hence by the proof of [**14**, Theorem 1.2] we see that $\mathfrak{G}$ has a special refinement. Therefore by Lemma 1.2 $X \times Y$ is normal.

Now let Y be any (not necessarily zero-dimensional) metric space. Then, by [**13**, Theorem 2.1], there exists a metric space Y_0 of covering dimension $\leqq 0$ and a closed continuous mapping g from Y_0 onto Y such that $g^{-1}(y)$ is compact for any point y of Y. Let us put $f(x, y_0) = (x, g(y_0))$ for $x \in X$, $y_0 \in Y_0$. Then f is a closed continuous mapping from $X \times Y_0$ onto $X \times Y$ such that $f^{-1}(x, y)$ is compact for any point $(x, y) \in X \times Y$. By assumption $X \times Y$ is countably paracompact, and hence $X \times Y_0$ is countably paracompact. Since $X \times Y_0$ is normal as shown above, $X \times Y$ is also normal.

(4)→(1). This is due to [**14**, Theorem 2.1].

Thus we have completed the proof of Theorem 1.3.

In the case when X is countably paracompact and normal, we can prove the following

Theorem 1.4. *For the product space $X \times Y$ of a countably paracompact normal space X with a metric space Y, the following properties are equivalent.*

(1) *$X \times Y$ is countably paracompact and normal.*

(2) *For any countable open covering $\mathfrak{U}$ of $X \times Y$ which contains a dense open subset of $X \times Y$ as its element, the basic covering $\mathfrak{B}(\mathfrak{U})$ has a special refinement.*

(3) *For any nowhere-dense closed subset Q and any countable collection $\{U_i\}$ of open subsets of $X \times Y$ such that $Q \subset \bigcup_{i=1}^{\infty} U_i$, there exists a countable collection $\{H_i\}$ of open subsets of $X \times Y$ such that $H_i \subset U_i$, $Q \subset \bigcup_{i=1}^{\infty} H_i$ and each H_i is expressed as $\{(x, y) \mid f_i(x, y) > 0\}$ by a continuous function $f_i : X \times Y \to I$ $(=[0, 1])$.*

PROOF. (1)→(2). This is obvious by [14, Theorem 2.1].

(2)→(3). Let Q be any nowhere-dense closed subset of $X\times Y$, and let $\{U_i\}$ be a countable collection of open subsets of $X\times Y$ such that $Q\subset\bigcup_{i=1}^{\infty}U_i$. If we put $\mathfrak{U}=\{X\times Y-Q, U_i\mid i=1,2,\cdots\}$, then by assumption $\mathfrak{B}(\mathfrak{U})$ has a special refinement. Hence by the same argument as in the proof of [14, Theorem 2.2] $\mathfrak{U}$ is a normal covering of $X\times Y$. Therefore by [11, Theorem 1.2] $\mathfrak{U}$ admits as its refinement a locally finite open covering $\{P_\lambda\mid\lambda\in\Lambda\}$ each set of which is expressed as $\{(x,y)\mid\varphi_\lambda(x,y)>0\}$ by a continuous function $\varphi_\lambda: X\times Y\to I$. Let $H_i=\bigcup\{P_\lambda\mid P_\lambda\subset U_i\}$. Then it is obvious that $\{H_i\}$ satisfies the required properties.

(3)→(1). Let $\mathfrak{B}=\{\mathfrak{B}_i\}$ be an open basis of Y such that $\mathfrak{B}_i=\{V_{i\alpha}\mid\alpha\in\Omega_i\}$ is a locally finite open covering of Y with $\delta(V_{i\alpha})$ (=diameter of $V_{i\alpha}$)$<1/2^i$ for $i=1,2,\cdots$; then $\{St(y,\mathfrak{B}_i)\mid i=1,2,\cdots\}$ is a basis of neighborhoods at each point y of Y. Let us pick up one point $y_{i\alpha}$ from each $V_{i\alpha}$, and let us put $Y_i=\{y_{i\alpha}\mid\alpha\in\Omega_i\}$. Then for each i Y_i is a closed discrete subspace of Y, and $\bigcup_{i=1}^{\infty}Y_i$ is dense in Y. Now let $\mathfrak{M}=\{M_j\mid j=1,2,\cdots\}$ be a countable open covering of $X\times Y$. Let $G(\alpha_1,\cdots,\alpha_i;j)$ be the union of all the open subsets P of X such that $P\times W(\alpha_1,\cdots,\alpha_i)\subset M_j$, where $W(\alpha_1,\cdots,\alpha_i)=\bigcap_{\nu=1}^{i}V_{\nu\alpha_\nu}$, and let us put

$$G(\alpha_1,\cdots,\alpha_i)=\bigcup_{j=1}^{\infty}G(\alpha_1,\cdots,\alpha_i;j).$$

Since X is countably paracompact and normal, the product space $X\times Y_k$ is obviously countably paracompact and normal. Hence a basic covering $\{[G(\alpha_1,\cdots,\alpha_i)\times W(\alpha_1,\cdots,\alpha_i)]\cap[X\times Y_k]\mid\alpha_1\in\Omega_1,\cdots,\alpha_i\in\Omega_i;i=1,2,\cdots\}$ of $X\times Y_k$ has a special refinement. Therefore we can find a family $\{H_k(\alpha_1,\cdots,\alpha_i)\mid\alpha_\nu\in\Omega_\nu,\nu=1,\cdots,i;i=1,2,\cdots\}$ of open F_σ-subsets of X such that $H_k(\alpha_1,\cdots,\alpha_i)\subset G(\alpha_1,\cdots,\alpha_i)$ and $\bigcup\{H_k(\alpha_1,\cdots,\alpha_i)\times W(\alpha_1,\cdots,\alpha_i)\mid\alpha_\nu\in\Omega_\nu, \nu=1,\cdots,i;i=1,2,\cdots\}\supset X\times Y_k$. Let us put

$$H(\alpha_1,\cdots,\alpha_i)=\bigcup_{k=1}^{\infty}H_k(\alpha_1,\cdots,\alpha_i).$$

Then $H(\alpha_1,\cdots,\alpha_i)$ are open F_σ-subsets of X, and we see that $H(\alpha_1,\cdots,\alpha_i)\subset G(\alpha_1,\cdots,\alpha_i)$ and $\bigcup\{H(\alpha_1,\cdots,\alpha_i)\times W(\alpha_1,\cdots,\alpha_i)\mid\alpha_\nu\in\Omega_\nu,\nu=1,\cdots,i; i=1,2,\cdots\}\supset X\times\bigcup_{k=1}^{\infty}Y_k$. Since $H(\alpha_1,\cdots,\alpha_i)$ are open F_σ-subspaces of X, they are countably paracompact and normal. Hence for each $H(\alpha_1,\cdots,\alpha_i)$ there exists a countable family $\{H(\alpha_1,\cdots,\alpha_i;j)\mid j=1,2,\cdots\}$ of open F_σ-subsets of X such that

$$H(\alpha_1,\cdots,\alpha_i)=\bigcup\{H(\alpha_1,\cdots,\alpha_i;j)\mid j=1,2,\cdots\},$$

$$H(\alpha_1, \cdots, \alpha_i ; j) \subset G(\alpha_1, \cdots, \alpha_i ; j).$$

If we put

$$L_j = \bigcup \{H(\alpha_1, \cdots, \alpha_i ; j) \times W(\alpha_1, \cdots, \alpha_i) \mid \alpha_\nu \in \Omega_\nu, \nu = 1, \cdots, i ; i = 1, 2, \cdots\},$$

$$Q = X \times Y - \bigcup_{j=1}^{\infty} L_j,$$

then we see that each L_j is contained in M_j, and that L_j is expressed as $\{(x, y) \mid f_j(x, y) > 0\}$ with a continuous function $f_j ; X \times Y \to I$, and moreover Q is a nowhere-dense subset of $X \times Y$. Hence by assumption there exists a countable collection $\{H_j\}$ of open subsets of $X \times Y$ such that $H_j \subset M_j$, $\bigcup_{j=1}^{\infty} H_j \supset Q$ and each H_j is expressed as $\{(x, y) \mid g_j(x, y) > 0\}$ by a continuous function $g_j : X \times Y \to I$. Then $\{L_j \cup H_j \mid j = 1, 2, \cdots\}$ is an open refinement of $\mathfrak{M}$ each set of which is expressed as $\{(x, y) \mid \varphi(x, y) > 0\}$ by a continuous function $\varphi : X \times Y \to I$. Hence $\mathfrak{M}$ is a normal covering of $X \times Y$ (cf. [**13**, Theorem 1.2]). Therefore $X \times Y$ is countably paracompact and normal. This completes the proof of Theorem 1.4.

Now we shall introduce a notion of countable paracompactness in the weak sense. A topological space X is said to be countably paracompact in the weak sense if, for every countable open covering $\mathfrak{U} = \{U_i \mid i = 1, 2, \cdots\}$ of X, there exists a countable family $\{H_i \mid i = 1, 2, \cdots\}$ of open subsets of X such that (i) each H_i is contained in some $U_j \in \mathfrak{U}$, (ii) $H = \bigcup_{i=1}^{\infty} H_i$ is dense in X, and (iii) $\{H_i \mid i = 1, 2, \cdots\}$ is locally finite in H.

Concerning the product space of a countably paracompact normal space with a metric space, we can prove the following theorem by the same argument as in the proof of Theorem 1.4.

THEOREM 1.5. *The product space $X \times Y$ of a countably paracompact normal space X with a metric space Y is countably paracompact in the weak sense.*

We don't know whether a normal space is countably paracompact in the weak sense or not. But, as an example of C. H. Dowker [**1**, p. 221] shows, a normal space which is countably paracompact in the weak sense is not countably paracompact in general.

§2. Closed mappings and the product spaces.

Let R, S, X and Y be topological spaces, and let $f : R \to X$ and $g : S \to Y$ be closed, continuous and onto mappings. If we put

$$h(r, s) = (f(r), g(s)) \qquad \text{for} \qquad (r, s) \in R \times S,$$

then the product mapping $h : R \times S \to X \times Y$ is not necessarily closed, as Morita [**10**] shows. Hence it is desirable to find a necessary and sufficient condition for the product mapping h to be closed.

THEOREM 2.1. *The product mapping h of closed, continuous and onto mappings* $f: R\to X$ *and* $g: S\to Y$ *is closed if and only if, for any point* (x, y) *of* $X\times Y$ *and any open subset* U *of* $R\times S$ *such that* $f^{-1}(x)\times g^{-1}(y)\subset U$, *there exists open subsets* $G\subset R$ *and* $H\subset S$ *such that*

$$f^{-1}(x)\times g^{-1}(y)\subset G\times H\subset U.$$

To prove Theorem 2.1, we shall use the following lemma (cf. [5, Chap. 3, Theorem 12]).

LEMMA 2.2. *A continuous and onto mapping* $f: X\to Y$ *is closed if and only if, for any point* y *of* Y *and any open subset* U *of* X *such that* $f^{-1}(y)\subset U$, *there exists an open neighborhood* $V(y)$ *of* y *such that* $f^{-1}(V(y))\subset U$.

PROOF OF THEOREM 2.1. Since the 'only-if' part is an immediate consequence of lemma 2.2, we shall prove only the 'if' part. For this purpose, it is sufficient to prove that $h(F)$ is closed in $X\times Y$ for any closed subset F of $R\times S$. Let $(x_0, y_0)\in\overline{h(F)}$. Then we can show that

$$[f^{-1}(x_0)\times g^{-1}(y_0)]\cap F\neq\phi,$$

which implies that $(x_0, y_0)\in h(F)$. If otherwise, there exist open subsets $G\subset R$ and $H\subset S$ such that

$$f^{-1}(x_0)\times g^{-1}(y_0)\subset G\times H\subset R\times S-F$$

by assumption. Hence we have

$$[\{X-f(R-G)\}\times\{Y-g(S-H)\}]\cap h(F)=\phi.$$

Since f and g are closed mappings, $X-f(R-G)$ and $Y-g(S-H)$ are open subsets of R and S respectively. From $x_0\in X-f(R-G)$ and $y_0\in Y-g(S-H)$, it follows that $(x_0, y_0)\notin\overline{h(F)}$, which is a contradiction. This completes the proof of Theorem 2.1.

Now we shall define a regular point of $X\times Y$ with respect to a product mapping $h=f\times g: R\times S\to X\times Y$ of closed, continuous and onto mappings $f: R\to X$ and $g: S\to Y$. If a point (x, y) of $X\times Y$ satisfies the following condition (*), we call (x, y) a regular point of $X\times Y$ with respect to a product mapping $h=f\times g$:

(*) For any open subset U of $R\times S$ such that $f^{-1}(x)\times g^{-1}(y)\subset U$, there exist open subsets $G\subset R$ and $H\subset S$ such that

$$f^{-1}(x)\times g^{-1}(y)\subset G\times H\subset U.$$

If (x, y) is not a regular point of $X\times Y$ with respect to a product mapping $f\times g$, we call (x, y) an irregular point of $X\times Y$ with respect to $f\times g$.

By our terminology, we can restate Theorem 2.1 as follows: The product mapping $h=f\times g$ of closed, continuous and onto mappings f and g is closed if and only if every point (x, y) of $X\times Y$ is regular with respect to $f\times g$.

A closed, continuous and onto mapping $f: R\to X$ is said to be perfect if $f^{-1}(x)$ is compact for every $x\in X$. As is well known, the product mapping $f\times g$ of perfect mappings $f: R\to X$ and $g: S\to Y$ is closed (cf. Morita [**10**]), and moreover the cartesian product of perfect mappings is also closed (cf. Frolik [**2**]). The former follows immediately from Theorem 2.1.

For two closed, continuous and onto mappings $f: R\to X$ and $g: S\to Y$, we shall investigate a problem whether, if the product mapping $f\times g$ is closed, f and g are both perfect or not. Of course, if X and Y are discrete spaces, then the answer to this problem is obviously negative. Hence we assume that X and Y are not discrete spaces. In this case we can show in the following example that the answer is also negative.

EXAMPLE. Let R be the space of real numbers with the usual topology, X the quotient space obtained from R by contracting the set I_0 of integers to one point y_0, S the space $\{\alpha\mid\alpha\leqq\Omega\}$ of ordinals in which the open sets are the null set ϕ, the whole space S, $\{\alpha\}$ for all $\alpha(<\Omega)$, and $\{\alpha\mid\gamma<\alpha\leqq\Omega\}$ for all $\gamma(<\Omega)$, where Ω is the first uncountable ordinal.

The space S defined above is a paracompact Hausdorff space, because S is regular and satisfies the Lindelöf property. Let f be the natural mapping of R onto X, and i the identity mapping of S onto itself. Then we can prove that the product mapping $h=f\times i$ is closed, while f is closed and continuous, but not perfect. In fact, let (x, α) be any point of $X\times S$. In case $(x, \alpha)\neq(y_0, \Omega)$, (x, α) is obviously regular with respect to $f\times i$. In case $(x, \alpha)=(y_0, \Omega)$, we see that $h^{-1}(x, \alpha)=f^{-1}(x)\times i^{-1}(\alpha)=I_0\times\{\Omega\}$. Let U be any open subset of $R\times S$ such that $I_0\times\{\Omega\}\subset U$. Then for each integer n we can find an ordinal $\gamma(n)$ such that $\gamma(n)<\Omega$ and $V_n\times\{\alpha\mid\gamma(n)<\alpha\}\subset U$, where V_n is a neighborhood of n in R. If we put $\gamma_0=\sup_n\gamma(n)$, then we have $\gamma_0<\Omega$ and $(\bigcup_n V_n)\times\{\alpha\mid\gamma_0<\alpha\}\subset U$. This shows that (y_0, Ω) is regular with respect to $h=f\times i$.

In case R and S are perfectly normal spaces, we have the following

THEOREM 2.3. *Let R and S be perfectly normal T_1-spaces, and let X and Y be topological spaces each of which is not discrete. If $f: R\to X$ and $g: S\to Y$ are closed, continuous and onto mappings, and if the product mapping $f\times g$ is closed, then $f^{-1}(x)$ and $g^{-1}(y)$ are both countably compact for every $x\in X$ and $y\in Y$.*

PROOF. By assumption X and Y are necessarily T_1-spaces. Now suppose that $f^{-1}(x_0)$ is not countably compact for some point $x_0\in X$. Then there exists an infinite sequence $\{r_i\}$ of points of $f^{-1}(x_0)$ which is discrete and closed in R. For each i, we can find an open neighborhcod $N(r_i)$ such that

$N(r_i) \cap N(r_j) = \phi$ for $i \neq j$. Let y_0 be a point of Y which is not isolated. Then it is obvious that $g^{-1}(y_0)$ is closed but not open in Y. Since Y is perfectly normal, $g^{-1}(y_0)$ is a G_δ-set of Y, and hence we can put $g^{-1}(y_0) = \bigcap_{n=1}^{\infty} G_n$, where G_n are open in Y.

Now let us put

$$W = [R \times S - \bigcup_{i=1}^{\infty} (r_i \times S)] \cup [\bigcup_{i=1}^{\infty} (N(r_i) \times G_i)].$$

Then we have $f^{-1}(x_0) \times g^{-1}(y_0) \subset W$. But we can not find any open subset $P \times Q$ of $R \times S$ such that

$$f^{-1}(x_0) \times g^{-1}(y_0) \subset P \times Q \subset W$$

In fact, if otherwise, then for each i we have

$$r_i \times g^{-1}(y_0) \subset r_i \times Q \subset \bigcup_{j=1}^{\infty} (N(r_j) \times G_j).$$

From $r_i \notin N(r_j)$ for $j \neq i$, it follows that $Q \subset G_i$. Hence we have $g^{-1}(y_0) \subset Q \subset \bigcap_{i=1}^{\infty} G_i$, which shows that $Q = g^{-1}(y_0)$. This contradicts the fact that $g^{-1}(y_0)$ is not open in Y. Hence (x_0, y_0) is an irregular point of $R \times S$ with respect to $f \times g$. This is contradictory to the fact that $f \times g$ is closed. Therefore $f^{-1}(x)$ is countably compact for every $x \in X$. Similarly we can prove that $g^{-1}(y)$ is also countably compact for every $y \in Y$. This completes the proof of Theorem 2.3.

COROLLARY 2.4. *Let R and S be metric spaces, and let X and Y be topological spaces each of which is not discrete. In case $f: R \to X$ and $g: S \to Y$ are closed, continuous and onto mappings, the product mapping $f \times g$ is closed if and only if f and g are both perfect.*

If Y is a single point set, while X contains at least two points, then we can prove the following

THEOREM 2.5. *Let R and X be topological spaces, S a regular T_1-space, Y a single point set, f a closed continuous mapping of R onto X, g a mapping of S to Y. If there exists a point x_0 of X such that $f^{-1}(x_0)$ is a G_δ-set but not open in R, and if the product mapping $f \times g$ is closed, then S is countably compact.*

Since this can be proved by the same argument as in the proof of Theorem 2.4, we omit the proof.

In Theorem 2.5, if $X = R$ and f is an identity mapping, then the product mapping $f \times g$ is identical with the projection of $R \times S$ onto R. But in this case, Hanai [3] has proved the following result: Let X be a topological space such that there exists an F_σ-set which is not closed, and let Y be a topological space. If the projection of the product space $X \times Y$ onto X is closed, then Y is countably compact.

§ 3. On the product of paracompact spaces.

According to Morita [11], for any infinite cardinal number $\mathfrak{m}$, a topological space X is said to be $\mathfrak{m}$-paracompact if any open covering of X with power $\leqq \mathfrak{m}$ (i.e. consisting of at most $\mathfrak{m}$ sets) admits a locally finite open covering as its refinement.

Now we shall prove the following

THEOREM 3.1. *Let $f: R \rightarrow X$ and $g: S \rightarrow Y$ be closed continuous mappings of paracompact Hausdorff spaces R and S onto topological spaces X and Y respectively, and let K be the set of irregular points of $X \times Y$ with respect to the product mapping $h = f \times g$. If the projection $X' = \{x_\lambda \mid \lambda \in \Lambda\}$ of K to X is closed and discrete in X, then the following are valid.*

(a) *If $R \times S$ is paracompact, so is also $X \times Y$.*

(b) *If $R \times S$ is normal, so is also $X \times Y$.*

(c) *If $R \times S$ is $\mathfrak{m}$-paracompact and normal so is also $X \times Y$.*

PROOF. (a) Since X and Y are the images under closed continuous mappings of paracompact Hausdorff spaces R and S respectively, they are also paracompact Hausdorff spaces. Hence for a closed discrete subset X' of X, there is a locally finite collection $\{N_\lambda \mid \lambda \in \Lambda\}$ of open subsets of X such that $x_\lambda \in N_\lambda$ and $\bar{N}_\lambda \cap \bar{N}_\mu = \phi (\lambda \neq \mu)$ (cf. [8, Theorem 1.3 and Lemma in § 3]). Let $\mathfrak{M} = \{M_\alpha \mid \alpha \in \Omega\}$ be any open covering of $X \times Y$. For each point (x_λ, y) there exists an open neighborhood $U_y(x_\lambda) \times V(y)$ of (x_λ, y) such that $U_y(x_\lambda) \times V(y)$ is contained in some M_α of $\mathfrak{M}$ and $U_y(x_\lambda) \subset N_\lambda$. Since Y is a paracompact Hausdorff space, an open covering $\{V(y) \mid y \in Y\}$ of Y has a locally finite partition of unity subordinated to it, i.e., a family $\{g_\sigma^\lambda \mid \sigma \in \Gamma\}$ of real valued continuous functions on Y such that $0 \leqq g_\sigma^\lambda(y) \leqq 1$, $\sum_\sigma g_\sigma^\lambda(y) = 1$, $G_\sigma^\lambda = \{y \mid g_\sigma^\lambda(y) > 0\} \subset$ some $V(y)$, and $\{G_\sigma^\lambda\}$ are locally finite in Y. Hence for a fixed λ we can find a family $\{U_\sigma(x_\lambda) \times G_\sigma^\lambda \mid \sigma \in \Gamma\}$ of open subsets of $X \times Y$ such that $U_\sigma(x_\lambda) \times G_\sigma^\lambda \subset$ some $U_y(x_\lambda) \times V(y)$ and $U_\sigma(x_\lambda) = \{x \mid f_\sigma^\lambda(x) > 0\}$, where $f_\sigma^\lambda: X \rightarrow I$ are continuous functions such that $f_\sigma^\lambda(x_\lambda) = 1$.

Now let us put

$$F_\lambda(x, y) = \sum_\sigma f_\sigma^\lambda(x) g_\sigma^\lambda(y),$$

$$F(x, y) = \sum_\lambda F_\lambda(x, y).$$

Then it is easily verified that $F_\lambda: X \times Y \rightarrow I$ and $F: X \times Y \rightarrow I$ are continuous functions, and moreover

$$\bigcup_\sigma [U_\sigma(x_\lambda) \times G_\sigma^\lambda] = \{(x, y) \mid F_\lambda(x, y) > 0\}.$$

$$F_\lambda(x_\lambda, y)=1 \qquad \text{for every} \quad y\in Y.$$

If we put

$$P_\lambda=\{(x, y)\mid F_\lambda(x, y)>0\},$$

$$Q_\lambda^{(1)}=\{(x, y)\mid F_\lambda(x, y)\geqq 1/2\},$$

$$Q_\lambda^{(2)}=\{(x, y)\mid F_\lambda(x, y)\geqq 1/3\},$$

$$A=X\times Y-\bigcup_\lambda Q_\lambda^{(1)}.$$

then we have

$$Q_\lambda^{(1)}\subset Q_\lambda^{(2)}\subset P_\lambda=\bigcup_\sigma[U_\sigma(x_\lambda)\times G_\sigma^\lambda],$$

$$\bigcup_\lambda Q_\lambda^{(1)}=\{(x, y)\mid F(x, y)\geqq 1/2\},$$

$$\bigcup_\lambda Q_\lambda^{(2)}=\{(x, y)\mid F(x, y)\geqq 1/3\},$$

$$A=\{(x, y)\mid F(x, y)<1/2\}.$$

From $h^{-1}(A)=\{(r, s)\mid F(h(r, s))<1/2\}$, it follows that $h^{-1}(A)$ is an open F_σ-subset of $R\times S$. Hence $h^{-1}(A)$ is paracompact as a subspace of $R\times S$. Furthermore we can prove that the partial mapping $h^*=h\mid h^{-1}(A)$ is closed. In fact, let F be a closed subset of $R\times S$. Then it is easily verified that $\overline{h(F)}-h(F)$ contains only irregular points of $X\times Y$ with respect to h, that is, $\overline{h(F)}-h(F)\subset K$. Since $K\frown A=\phi$, we have

$$h(h^{-1}(A)\frown F)=A\frown h(F)=A\frown\overline{h(F)},$$

which shows that $A\frown h(F)$ is closed in A. Therefore A is paracompact as a subspace of $X\times Y$, because it is the image of a paracompact space $h^{-1}(A)$ under a closed continuous mapping $h^*: h^{-1}(A)\to A$. Hence an open covering $\mathfrak{U}=\{A\frown M_\alpha\mid\alpha\in\Omega\}$ of A admits a locally finite open refinement $\{G_\alpha\mid\alpha\in\Omega\}$ in A. Let us put $L_\alpha=G_\alpha-\bigcup_\lambda Q_\lambda^{(2)}$ for every $\alpha\in\Omega$. Then $\{L_\alpha\mid\alpha\in\Omega\}$ is a family of open subsets of $X\times Y$ which is locally finite in $X\times Y$. If we put

$$\mathfrak{N}=\{L_\alpha, U_\sigma(x_\lambda)\times G_\sigma^\lambda\mid\alpha\in\Omega,\ \sigma\in\Gamma,\ \lambda\in\Lambda\},$$

then $\mathfrak{N}$ is obviously a locally finite open refinement of $\mathfrak{M}$. Hence $X\times Y$ is paracompact.

(b) Proceeding by the same procedure for a finite open covering $\mathfrak{M}=\{M_j\mid j=1, 2, \cdots, k\}$, we obtain an open F_σ-subset $h^{-1}(A)$ of $R\times S$. Since $R\times S$ is normal, $h^{-1}(A)$ is also normal as a subspace of $R\times S$. Hence A is a normal subspace of $X\times Y$, because it is the image under a closed continuous mapping h^* $(=h\mid h^{-1}(A))$ of $h^{-1}(A)$. Therefore an open covering $\mathfrak{U}=\{A\frown M_j\mid j=1, 2, \cdots, k\}$ of A admits a locally finite open refinement $\{G_\alpha\mid\alpha\in\Omega\}$ in A such that each G_α is expressed as $\{(x, y)\mid\varphi_\alpha(x, y)>0\}$ with a continuous function $\varphi_\alpha: A \to I$. Let us put $L_\alpha = G_\alpha - \bigcup_\lambda Q_\lambda^{(2)}$. Then $\{L_\alpha \mid \alpha \in \Omega\}$ is a locally finite collection of open subsets of $X \times Y$ each of which is expressed as $\{(x, y) \mid \psi_\alpha(x, y) > 0\}$ with a continuous function $\psi_\alpha: X \times Y \to I$. If we put

$$\mathfrak{N} = \{L_\alpha,\ U_\sigma(x_\lambda) \times G_\sigma^\lambda \mid \alpha \in \Omega,\ \sigma \in \Gamma,\ \lambda \in \Lambda\}\ ,$$

then $\mathfrak{N}$ is a normal open covering which refines $\mathfrak{M}$ by [**13**, Theorem 1.2]. Hence $X \times Y$ is normal.

(c) This can be proved by the same argument as in the proof of (b) by using the following facts.

(i) An open F_σ-subset of an $\mathfrak{m}$-paracompact normal space is $\mathfrak{m}$-paracompact and normal as a subspace (cf. [**11**, Theorem 1.5]).

(ii) The image under a closed continuous mapping of an $\mathfrak{m}$-paracompact normal space is also $\mathfrak{m}$-paracompact and normal (cf. [**11**, Theorem 1.4]).

Thus we have completed the proof of Theorem 3.1.

The problem whether Theorem 3.1 is true without any restriction on irregular points remains open.

As an application of Theorem 3.1, we can prove the following

THEOREM 3.2. *Let X be the image under a closed continuous mapping f of a locally compact and paracompact Hausdorff space R and let Y be a paracompact Hausdorff space. Then the product space $X \times Y$ is paracompact.*

Since the product space $R \times Y$ of a locally compact and paracompact Hausdorff space R with a paracompact Hausdorff space Y is paracompact (cf. Morita [**12**]), Theorem 3.2 is a direct consequence of Theorem 3.1 and the following lemma which is due to Morita [**9**].

LEMMA 3.3. *Let f be a closed continuous mapping of a locally compact and paracompact Hausdorff space R onto another topological space X. If we denote by X' the set of all points x of X such that $f^{-1}(x)$ is not compact, then X' is a closed discrete subset of X.*

NOTE. In Theorem 3.2, if we denote by i the identity mapping of Y onto itself, then the product mapping $f \times i$ is not closed in general. This is shown by an example of Morita [**10**].

In case X is the image under a closed continuous mapping f of a paracompact Hausdorff space R which is the countable union of locally compact closed subsets, the problem whether Theorem 3.2 is true remains open. Quite recently M. Tsuda [**18**] has proved the following result: Let X be the image under a closed continuous mapping of a paracompact and perfectly normal Hausdorff space which is the countable union of locally compact closed subspaces. Then the product space $X \times Y$ is paracompact and normal for any paracompact and normal space Y.

Now we are in a position to show by an example that the image under a closed continuous mapping of a locally compact and paracompact Hausdorff space is not represented as the countable union of locally compact closed subspaces in general, while it is a paracompact Hausdorff space.

EXAMPLE. Let Ω be an index set such that $|\Omega|>\aleph_0$, where $|\Omega|$ is the cardinal number of Ω. For each index $\alpha\in\Omega$, let A_α be a locally compact Hausdorff space which does not satisfy the Lindelöf property (for example, let A_α be a discrete space whose elements are uncountable), and let C_α be the one-point compactification of A_α such that $C_\alpha=A_\alpha\cup p_\alpha$. Now let X be the disjoint union of C_α's, i.e., $X=\cup\{C_\alpha\mid\alpha\in\Omega\}$, and let Y be the quotient space obtained from X by contracting the set $P=\{p_\alpha\mid\alpha\in\Omega\}$ to a single point y_0. If we define $f:X\to Y$ as the identification mapping, then it is easily verified that f is a closed, continuous and onto mapping. Since X is a paracompact space, Y is also a paracompact Hausdorff space. But we can prove that Y is not represented as the countable union of locally compact closed subspaces, while X is locally compact.

In fact, assume that Y is the countable union $\bigcup_{i=1}^{\infty}K_i$ of locally compact closed subspaces, and let $\{K_{i(k)}\mid k=1,2,\cdots\}$ be the subfamily of $\{K_i\}$ such that $y_0\in K_{i(k)}$ for each k. Let us put $L_k=f^{-1}(K_{i(k)})$ and $g_k=f\mid L_k$ for every k. Then $g_k:L_k\to K_{i(k)}$ is closed and continuous. Hence $\mathfrak{B}g_k^{-1}(y)$ (=the boundary of the set $g_k^{-1}(y)$) is compact for every $y\in K_{i(k)}$ by K. Morita [**9**, Theorem 2], because L_k is locally compact and paracompact, and $K_{i(k)}$ is locally compact. Therefore $\mathfrak{B}g_k^{-1}(y_0)$ is compact in L_k. On the other hand we have

$$\begin{aligned}\mathfrak{B}g_k^{-1}(y_0)&=P\cap\overline{(L_k-P)}\\&=\cup\{\overline{(L_k-P)\cap C_\alpha}\cap P\mid\alpha\in\Omega\}\\&=\cup\{\overline{(L_k-P)\cap C_\alpha}\cap p_\alpha\mid\alpha\in\Omega\}.\end{aligned}$$

Hence, if we put

$$\Gamma_k=\{\alpha\in\Omega\mid p_\alpha\in\overline{(L_k-P)\cap C_\alpha}\},$$

Γ_k is a finite set, and so we have $|\bigcup_{k=1}^{\infty}\Gamma_k|\leqq\aleph_0$. Thus from assumption that $|\Omega|>\aleph_0$, it follows that $\Omega-\bigcup_{k=1}^{\infty}\Gamma_k$ is not empty. For any $\alpha\in\Omega-\bigcup_{k=1}^{\infty}\Gamma_k$ we have

$$p_\alpha\notin\overline{(L_k-P)\cap C_\alpha}\quad\text{for every}\quad k=1,2,\cdots.$$

This implies that, if $\alpha\notin\Omega-\bigcup_{k=1}^{\infty}\Gamma_k$, then

$$p_\alpha\notin\overline{(f^{-1}(K_i)-P)\cap C_\alpha}\quad\text{for every}\quad i=1,2,\cdots.$$

Since $X=\bigcup_{i=1}^{\infty}f^{-1}(K_i)$ and $(f^{-1}(K_i)-P)\cap C_\alpha=f^{-1}(K_i)\cap C_\alpha-p_\alpha$, we see that

$$A_\alpha = C_\alpha - p_\alpha = \bigcup_{i=1}^{\infty} \{\overline{f^{-1}(K_i) \cap C_\alpha - p_\alpha}\} \quad \text{for} \quad \alpha \notin \Omega - \bigcup_{k=1}^{\infty} \Gamma_k .$$

Hence for each $\alpha \notin \Omega - \bigcup_{k=1}^{\infty} \Gamma_k$ A_α is an F_σ-set of a compact space C_α, and so satisfies the Lindelöf property. This is a contradiction. Thus we have completed the proof.

By Theorem 3.2 and the example cited above, we see that the following is not valid: Let X be a paracompact Hausdorff space. If the product space $X \times Y$ is paracompact for any paracompact Hausdorff space Y, then X is the countable union of locally compact closed subsets.

This gives a negative answer for an open problem raised by K. Morita [12].

Finally we shall state a slight generalization of Theorem 3.1 which can be proved by a similar way as in the proof of Theorem 3.1.

Theorem 3.4. *Let $f: R \to X$ and $g: S \to Y$ be closed continuous mappings of paracompact Hausdorff spaces R and S onto topological spaces X and Y respectively, and let K be the set of irregular points of $X \times Y$ with respect to $f \times g$. If $K \subset (\bigcup_\alpha a_\alpha \times Y) \cup (X \times \bigcup_\gamma b_\gamma)$, where $\{a_\alpha \mid \alpha \in \Omega\} \subset X$ and $\{b_\gamma \mid \gamma \in \Gamma\} \subset Y$ are closed and discrete, then the following are valid.*

(a) *If $R \times S$ is paracompact, so is also $X \times Y$.*

(b) *If $R \times S$ is normal, so is also $X \times Y$.*

(c) *If $R \times S$ is $\mathfrak{m}$-paracompact and normal, so is also $X \times Y$.*

As an application of Theorem 3.4, we have the following

Corollary 3.5. *Let R and S be paracompact Hausdorff spaces, $\{A_\alpha \mid \alpha \in \Omega\}$ and $\{B_\gamma \mid \gamma \in \Gamma\}$ discrete collections of closed subsets of R and S respectively, X the quotient space obtained from R by contracting each set A_α to a single point x_α, and Y the quotient space obtained from S by contracting each set B_γ to a single point y_γ. Then the following are valid.*

(a) *If $R \times S$ is paracompact, so is also $X \times Y$.*

(b) *If $R \times S$ is normal, so is also $X \times Y$.*

(c) *If $R \times S$ is $\mathfrak{m}$-paracompact and normal, so is also $X \times Y$.*

Proof. Let $f: R \to X$ and $g: S \to Y$ be identification mappings. Then it is easily verified that f and g are closed, continuous and onto mappings. Hence this corollary is an immediate consequence of Theorem 3.4.

Utsunomiya University

References

[1] C. H. Dowker, On countably paracompact spaces, Canad. J. Math., 1 (1951), 219-224.

[2] Z. Frolik, On the topological product of paracompact spaces, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys., **8** (1960), 747–750.
[3] S. Hanai, Inverse images of closed mappings I, Proc. Japan Acad., **37** (1961), 208–301.
[4] F. Ishikawa, On countably paracompact spaces, Proc. Japan Acad., **31** (1955), 686–689.
[5] J. L. Kelley, General topology, Van Nostrand, New York, 1955.
[6] E. Michael, A note on paracompact spaces, Proc. Amer. Math. Soc., **4** (1953), 831–838.
[7] E. Michael, The product of a normal space and a metric space need not be normal, Bull. Amer. Math. Soc., **69** (1963), 375–376.
[8] K. Morita, On the dimension of normal spaces II, J. Math. Soc. Japan, **2** (1950), 16–33.
[9] K. Morita, On closed mappings, Proc. Japan Acad., **32** (1956), 539–543.
[10] K. Morita, Note on paracompactness, Proc. Japan Acad., **37** (1961), 1–3.
[11] K. Morita, Paracompactness and product spaces, Fund. Math., **50** (1962), 223–236.
[12] K. Morita, On the product of paracompact spaces, Proc. Japan Acad., **39** (1963), 559–563.
[13] K. Morita, Products of normal spaces with metric spaces, Math. Ann., **154** (1964), 365–382.
[14] K. Morita, Products of normal spaces with metric spaces II, Sci. Rep. Tokyo Kyoiku Daigaku, Sect. A, **8** (1964), 87–92.
[15] K. Morita, Note on the product of a normal space with a metric space, unpublished.
[16] H. Tamano, On compactifications, J. Math. Kyoto Univ., **1** (1962), 162–193.
[17] H. Tamano, Note on paracompactness, J. Math. Kyoto Univ., **3** (1963), 137–143.
[18] M. Tsuda, On the normality of certain product spaces, Proc. Japan Acad., **40** (1964), 465–467.