J. Math. Soc. Japan
Vol. 18, No. 4, 1966

Complex of differential forms

By Krishna TEWARI

(Received Jan. 24, 1966)
(Revised July 13, 1966)

1. Introduction

We know that for the finitely generated extension fields of the ground field, the complex of differential forms is isomorphic to a universal complex. Therefore, it seems of interest to investigate the universality of the complex of differential forms of a unitary commutative R-algebra A, R being a commutative ring with unit. This paper is an attempt to find conditions under which the two objects—complex of differential forms of A and a univeral complex over A are same. The main theorems of the paper are

(1) If (U, d) is a universal complex over A such that U_1 is a finitely generated projective A-module then (U, d) and $(A(D), \delta)$ are isomorphic, $(A(D), \delta)$ being the complex of differential forms of A.

(2) If A is a finitely generated algebra over a noetherian commutative ring R such that A is a hereditary ring and if U_1 is reflexive then the complex of differential forms of A is universal.

Since for certain algebras, the two complexes—the complex of differential forms and the universal complexes—are isomorphic, it is interesting to see that they differ quite widely in other cases. The algebra considered here is the algebra $K\{x\}$ of formal power series in one indeterminate x over a field K. We have proved that if (V, ∂) is a universal complex over $K\{x\}$ then V_1 cannot be finitely generated free $K\{x\}$-module whereas the $K\{x\}$-module $D_K(K\{x\})$ of K-derivations of $K\{x\}$ is a free module with basis consisting of one element; thus V_1 cannot be isomorphic to the $K\{x\}$-dual $D_K^*(K\{x\})$.

Throughout this paper R will be a commutative ring with unit.

2. Basic definitions

A *complex* over A is a pair (X, d) where X is an anticommutative regularly graded A-algebra [**1**] and $d: X \to X$ is an R-linear mapping such that (i) $dX_n \subseteqq X_{n+1} \forall n \geqq 0$: (ii) $d(xx') = dx \cdot x' + (-1)^n x \cdot dx'$ for all $x \in X_n$ and $x' \in X$ $(n \geqq 0)$: and (iii) $dd = 0$. If (X, d) and (Y, δ) are two complexes over A then a *complex homomorphism* $f: (X, d) \to (Y, \delta)$ is an A-algebra homomorphism from X into

Y such that (i) $f(X_n) \subseteqq Y_n \forall n \geqq 0$; and (ii) $f_0 d = \delta \circ f$. A complex (U, d) over A is called *universal* if given any complex (V, δ) over A there exists a unique complex homomorphism $f: (U, d) \to (V, \delta)$.

An R-linear mapping $\zeta: A \to A$ is called an *R-derivation* of A if and only if $d(ab) = da \cdot b + a \cdot db$ for all $a, b \in A$. It is well known that the set D of all R-derivations of A is an A-module.

The *alternating differential forms of degree n of A* are (i) $a \in A$ for $n = 0$; (ii) the alternating multilinear forms [1] of degree n of the A-module D for $n \geqq 1$. We denote the set of alternating differential forms of degree n of A by $A_n(D)$ and put $A(D) = \sum_{n \geqq 0} A_n(D)$ (dir). Then $A(D)$ is an anticommutative regularly graded A-algebra [1]. The multiplication in $A(D)$ being given by,

For $\varphi \in A_n(D), \psi \in A_m(D), (\varphi \wedge \psi)(\zeta_1, \zeta_2, \cdots, \zeta_{m+n}) = \sum_\sigma \eta(\sigma, \sigma^*) \varphi(t(\sigma)) \psi(t(\sigma^*))$

where $\zeta_1, \zeta_2, \cdots, \zeta_{m+n}$ are in D; $\sigma = (i_1, i_2, \cdots, i_n)$ with $i_1 < i_2 < \cdots < i_n$; σ^* is the complementary sequence $(j_1, j_2, \cdots, j_m)$ with $j_1 < j_2 < \cdots < j_m$; $t(\sigma) = (\zeta_{i_1}, \zeta_{i_2}, \cdots, \zeta_{i_n})$; $t(\sigma^*) = (\zeta_{j_1}, \zeta_{j_2}, \cdots, \zeta_{j_m})$ and $\eta(\sigma, \sigma^*) = (-1)^{N(\sigma, \sigma^*)}$ where $N(\sigma, \sigma^*) =$ number of pairs (i, j) with $i \in \sigma, j \in \sigma^*, i > j$.

Let $\delta: A(D) \to A(D)$ be given by

$$(\delta\varphi)(\zeta_1, \zeta_2, \cdots, \zeta_{n+1}) = \sum_{i=1}^{n+1} (-1)^{i+1} \zeta_i(\varphi(\zeta_1, \zeta_2, \cdots, \hat{\zeta}_i, \cdots, \zeta_{n+1}))$$
$$- \sum_{r<s} (-1)^{r+s\pm 1} \varphi([\zeta_r, \zeta_s], \zeta_1, \zeta_2, \cdots, \hat{\zeta}_r, \cdots, \hat{\zeta}_s, \cdots, \zeta_{n+1})$$

where $\varphi \in A_n(D)$ arbitrary; $\zeta_1, \zeta_2, \cdots, \zeta_{n+1}$ are any elements of D and $[\zeta_r, \zeta_s]$ is the derivation $\zeta_r \zeta_s - \zeta_s \zeta_r$. It is known [1] that δ is an R-linear mapping such that (i) $\delta(A_n(D)) \subseteqq A_{n+1}(D)$ for all $n \geqq 0$ (ii) $\delta(\varphi \wedge \psi) = (\delta\varphi) \wedge \psi + (-1)^n \varphi \wedge (\delta\psi)$ for all $\varphi \in A_n(D), \psi \in A(D), n \geqq 0$. ('$\wedge$' being the grassmann product in $A(D)$); and (iii) $\delta\delta = 0$. Therefore $(A(D), \delta)$ is a complex over A, called the complex of differential forms of A.

3. Universality of the complex of differential forms

First we shall prove that if (U, d) is a universal complex over A such that the module U_1 of homogeneous elements of degree 1 of U is finitely generated and projective then $(A(D), \delta)$ is a universal complex over A. For this we need the following machinary.

LEMMA 3.1. *Let M be an A-module and let L be a direct summand of M. If the natural homomorphism λ_M of M into its bidual M^{**} is an isomorphism then so is $\lambda_L: L \to L^{**}$ where L^{**} is the bidual of L.*

PROOF. Proof follows immediately from the fact that the association of the bidual M^{**} with an A-module M and of $f^{**}: M^{**} \to N^{**}$ with any A-

module homomorphism $f: M \to N$ is a covariant functor from the category of all A-modules into itself.

Next we make the following observations:

1. Recall [1] that with every A-module M we can associate its exterior algebra $E(M)$; and with every A-module homomorphism $f: M \to N$, N being an A-module, we can associate the A-algebra homomorphism $\bar{f}: E(M) \to E(N)$ such that $\bar{f}(E_n(M)) \subseteqq E_n(N)(n \geqq 0)$ and such that $\bar{f}$ extends f. If $\mathfrak{M}$ denotes the category of all A-modules and their homomorphisms: and if $\mathcal{G}$ denotes the category of all graded A-algebras and their homomorphisms then the function $T: \mathfrak{M} \to \mathcal{G}$ given by $T(M) = E(M)$ and $T(f) = \bar{f}$ for all M and f in $\mathfrak{M}$ is a covariant functor.

2. The association of the A-module M^* with an A-module M and of the A-module homomorphism $f^*: N^* \to M^*$ with any A-module homomorphism $f: M \to N$, (N being an A-module) is a contravariant functor.

3. For any A-module homomorphism f from an A-module M into an A-module N, let $f^n: M^n \to N^n$ denote the mapping $(x_1, x_2, \cdots, x_n) \to (f(x_1), f(x_2), \cdots, f(x_n))$. Then for any φ in $A_n(N)(A_n(N)$ being the A-module of the alternating multilinear forms of degree n of N), $\varphi \circ f^n$ belongs to $A_n(M)$. Let $f'': A(N) \to A(M)$ be given by $f''(\varphi) = \varphi \circ f^n$ for each $\varphi \in A_n(N)$, $n \geqq 0$. Then f'' is an A-algebra homomorphism such that $f''(A_n(N)) \subseteqq A_n(M)$. Moreover, let $f': E(N^*) \to E(M^*)$ be induced by $f^*: N^* \to M^*$. Then the diagram

$$\begin{array}{ccc} E(N^*) & \xrightarrow{f'} & E(M^*) \\ \tau_N \downarrow & & \downarrow \tau_M \\ A(N) & \xrightarrow{f''} & A(M) \end{array}$$

commutes, where for any A-module M, $\tau_M: E(M^*) \to A(M)$ is given by $\tau_M(\varphi_1, \varphi_2, \cdots, \varphi_n) = \varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n$, φ_i in M^* $(1 \leqq i \leqq n)$ and '$\wedge$' denotes the grassmann product [1] in $A(M)$.

Observations 1, 2 and 3 immediately lead to the following Lemma.

LEMMA 3.2. *Let M be any A-module and L be a direct summand in M. Then if τ_M is an isomorphism, τ_L is also an isomorphism.*

To prove the following lemma let us recall [4] that a complex over (U, d) is universal if and only if (U_1, d_0) is a universal derivation module of A and U is the exterior algebra of A. By a *derivation module* of A we mean a pair (M, δ) where M is an A-module and $\delta: A \to M$ is an R-linear mapping such that $\delta(ab) = \delta ab + a\delta b$, for all a, b in A. A *derivation module* (M, δ) of A is called *universal* if and only if given any other derivation module (N, ∂) of A there exists a unique A-homomorphism $f: M \to N$ such that $f \circ \delta = \partial$.

LEMMA 3.3. *Let (U, d) be a universal complex over A. Then U_1^* is iso-*

morphic to D.

PROOF. Let $f \in U_1^*$ be arbitrary. Then $f = U_1 \to A$ is an A-module homomorphism. It can be easily seen that $f \circ d_0$ is an R-derivation and hence $f \circ d_0 \in D$. Now we consider the mapping $\varphi: U_1^* \to D$ given by $\varphi(f) = f \circ d_0$ for all f in U_1^*. Clearly, φ is an A-module homomorphism. Also, $f \circ d_0 = 0$ implies $f(d_0 A) = 0$ and so $f(U_1) = 0$ since U_1 is generated by dA as an A-module. Therefore, $f \circ d_0 = 0$ implies $f = 0$ which shows that φ is one-one. It remains to show that φ is onto. For this we note that for any $\delta \in D$, (A, δ) is a derivation module of A. By universality of (U_1, d_0) there exists a unique A-module homomorphism $f_\delta: U_1 \to A$ such that $f_\delta \circ d_0 = \delta$. Since $f_\delta \in U_1^*$, we have that $\varphi(f_\delta) = f_\delta \circ d_0 = \delta$ and this proves the ontoness of φ. Hence the lemma is proved.

THEOREM 3.1. *Let (U, d) be a universal complex over A. If U_1 is a finitely generated projective A-module then $(A(D), \delta)$ is isomorphic to (U, d).*

PROOF. Since $(A(D), \delta)$ is a complex over A, in view of the universality of (U, d) there exists a unique complex homomorphism $f: (U, d) \to (A(D), \delta)$. Let $g: U_1 \to A_1(D)$ be the restriction of f to U_1. Then $g \circ d_0 = \delta_0$ on A. We claim that g is an isomorphism. Recall that U_1 being a finitely generated projective A-module, is a direct summand of a free A-module F with finite basis. For F the natural homomorphism $\lambda_F: F \to F^{**}$ is an isomorphism; and so, by lemma 4.1 $\lambda_{U_1}: U_1 \to U_1^{**}$ is also an isomorphism. In view of lemma 4.3, U_1^* is isomorphic to D and the isomorphism $\varphi: U_1^* \to D$ is given by $\varphi(f) = f \circ d_0$. Let $\varphi^{-1}: D \to U_1^*$ be the inverse isomorphism. Then $\varphi^{-1}(\partial) = f_\partial$ for each ∂ in D, where $f_\partial \in U_1^*$, is such that $f_\partial \circ d_0 = \partial$. Let $(\varphi^{-1})^*: U_1^{**} \to D^* = A(D)$ be the mapping induced by φ^{-1}. Then $(\varphi^{-1})^*$ is given by $(\varphi^{-1})^*(f) = f \circ \varphi^{-1}$ for each f in U_1^{**}, and $(\varphi^{-1})^*$ is an isomorphism. Thus $(\varphi^{-1})_\delta^* \lambda_{U_1}$ is an isomorphism of the A-module U_1 with the A-module $D^* = A_1(D)$. Moreover, for each a in A, $(\varphi^{-1})^* \circ \lambda_{U_1}(d_0 a) = \lambda_{U_1}(d_0 a) \circ \varphi^{-1}$. Therefore, for an arbitrary ∂ in D, $(\lambda_{U_1}(d_0 a) \circ \varphi^{-1})\partial = \lambda_{U_1}(d_0 a)\ (\varphi^{-1}(\partial)) = \varphi^{-1}(\partial)\ (d_0 a) = \partial a$, that is $(\varphi^{-1})^* \lambda_{U_1}(d_0 a)$ is a mapping of D in A given by $\partial \to \partial a$ for all ∂ in D. But by the defininition of $\delta: A(D) \to A(D)$, $(\delta a)\partial = \partial a$ for each $a \in A$, $\partial \in D$. Therefore $(\varphi^{-1})^* \circ \lambda_{U_1}(d_0 a) = \delta a$, for all $a \in A$; that is $(\varphi^{-1})^* \circ \lambda_{U_1} \circ d_0 = \delta$ on A. Hence $(\varphi^{-1})^* \circ \lambda_{U_1} \circ d_0 = g \circ d_0$ on A. Since U_1 is generated by $d_0 A$ as an A-module, we have that $(\varphi^{-1})^* \circ \lambda_{U_1} = g$ on U_1 and hence $g: U_1 \to D^* = A_1(D)$ is an isomorphism. Now, we recall that g extends to a unique A-algebra isomorphism $\bar{g}: E(U_1) \to E(D)^*$. Since $E(U_1) = U$ we get that $g: U \to E(D^*)$ is an isomorphism. Now recall that U_1 finitely generated and projective A-module implies U_1^* is a finitely generated projective A-module. Therefore D is a finitely generated projective A-module. Thus D is a direct summand of a finite free A-module, say P. We know [1] that for P, $E(P^*)$ is isomorphic to $E^*(P) = A(P)$ i.e. $\tau_p\ E(P^*) \to A(P)$ is an isomorphism. Therefore, by lemma 4.2, $\tau_D: E(D^*)\to A(D)$ is also an isomorphism. Therefore, $\bar{g}$ induces an A-algebra isomorphism $h: U\to A(D)$. Now $h|U_1=\bar{g}|U_1=g=f|U_1$ and since U_1 generates the complex (U, d) we have that $h=f$ on U. Hence it follows that $f:(U, d)\to(A(D), \delta)$ is an isomorphism.

DEFINITION 3.1. A ring A is called *hereditary* if every submodule of a projective A-module is again projective.

PROPOSITION 3.1. *Let R be a commutative noetherian ring with unit and let A be a finitely generated R-algebra. Suppose A is a hereditary ring. If (U, d) is a universal complex over A such that U_1 is reflexive (i.e. the natural homomorphism $\lambda_M: M\to M^{**}$ is an isomorphism) then (U, d) is isomorphic to $(A(D), \delta)$.*

PROOF. If A is generated by $a_1, a_2, \cdots, a_n$ then the mapping $\partial\to(\partial a_1, \partial a_2, \cdots, \partial a_n)$ gives an A-monomorphism $D\to A^n$. Since A is noetherian and hereditary D is finitely generated projective. Hence, in view of lemma 3.3 U_1^* is finitely generated projective A-module. Therefore, the dual U_1^{**} of U_1^* is also finitely generated projective A-module. Since U_1 is reflexive, U_1 is finitely generated projective A-module. Hence the result follows from theorem 3.1.

REMARK. If U_1 is finitely presented and flat then U_1 is finitely generated projective and so theorem 3.1 gives the isomorphism of two complexes in this case.

Now we shall show that if $K\{x\}$ is the K-algebra of formal power series in one indeterminate x over a field K, then the complex of dfferential forms of $K\{x\}$ is not universal. To prove this it is enough to show that if (V, ∂) is a universal complex over $K\{x\}$ then V_1 is not isomorphic to the dual of the $K\{x\}$-module of all K-derivations on $K\{x\}$. Since the $K\{x\}$-module of all K-derivations of $K\{x\}$ is a free module with basis consisting of one element, its dual is also a free module with basis consisting of one element. But, as we shall see in the following, V_1 is an infinitely generated free $K\{x\}$-moduule.

Let $S\subsetneqq A$ be a multiplicatively closed subset of A and let A_S denote the generalized algebra of quotients of A with respect to S. Now, if X is an anticommutative regularly graded A-algebra, then $X_s=A_{SA}\otimes X$ is an anticommutative regularly graded A_S-algebra. Moreover, if (X, d) is a complex over A, then there exists a unique derivation $d_S: X_S\to X_S$ such that (X_S, d_S) is a complex over A_S. Actually $d_S: X_S\to X_S$ is given by $d_S\left(\frac{x}{s}\right)=\frac{sdx+(-1)^n xds}{s^2}$ for each homogeneneous $\frac{x}{s}$ of degree n in X_S.

LEMMA 3.4. *If (U, d) is a universal complex over A then (U_S, d_S) is a universal complex over A_S.*

PROOF. Let (V, Δ) be any complex over A_S. We wish to show that there exists a unique complex homomorphism from (U_S, d_S) into (V, Δ). For this recall that for each $n \geqq 1$, V_n can be made into an A-module by way of natural homomorphism $\mu: A \rightarrow A_S$. Let ${}_{(\mu)}V_n$ be the A-module thus obtained $(n \geqq 1)$. Then (W, Δ') with $W_0 = A$, $W_n = {}_{(\mu)}V_n (n \geqq 1)$ and $\Delta'_0 = \Delta_0 \circ \mu$, $\Delta'_n = \Delta_n$ $(n \geqq 1)$ is a complex over A. By universality of (U, d) there exists a unique complex homomorphism $f: (U, d) \rightarrow (W, \Delta')$. Consider $f_S: U_S \rightarrow W_S$ given by $f_S\left(\frac{\mu}{s}\right) = \frac{f(\mu)}{s}$ for each $\mu \in U$, $s \in S$. Then it can be easily checked that f_S is a complex homomorphism from (U_S, d_S) to (W_S, Δ'_S). Moreover, f_S is unique. Since $W_S = A_S \bigotimes_A W = A_S + \sum_{n \geqq 1} A_S \bigotimes_A {}_{(\mu)}V_n = A_S + \sum_{n \geqq 1} V_n$ and since Δ'_S is the same as Δ we have that f_S is a unique homomorphism from (U_S, d_S) to (V, Δ) and this proves the lemma.

Now let (V, ∂) be a universal complex over $K\{x\}$ and let S be the set of all non-zero elements of $K\{x\}$. Then (V_S, ∂_S) is a universal complex over $K((x))$ which is the field of quotients of the integral domain $K\{x\}$. Since the degree of transcendence of $K\{x\}$ over K is infinite, the dimension of $(V_S)_1$ over $K((x))$ is infinite. Since $V_S = K((x)) \otimes V$, we have that V_1 is infinitely generated over $K\{x\}$. Or, in other words V_1 cannot be a finite free $K\{x\}$ module. Hence V_1 is not isomorphic to the dual of the $K\{x\}$-module of K-derivations on $K\{x\}$; and therefore, the complex of differential forms of $K\{x\}$ is not universal.

S. U. N. Y. Buffalo, New York, U. S. A.
and
I. I. T. Kanpur (U. P.), India

References

[1] G. Chevalley, Fundamental Concepts of Algebra, Academic Press, 1956.
[2] E. Kähler, Algebra und Differential-rechnung, Bericht über die Mathematiker-Togung in Berlin., Vol. 14, Bis. 18, 1953.
[3] K. Tewari, Universal Complexes over an Algebra, J. Indian Math. Soc. (to appear).
[4] K. Tewari, Derivation Modules and Complexes over an Algebra, J. Indian Math. Soc. (to appear).