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On semisimple extensions and separable extensions over non commutative rings

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Introduction

In this paper we intend to generalize the semisimplicity and separability for algebras to the case where the basic ring is non commutative. Separable algebras in the sense of Auslander-Goldman [2] and semisimple algebras in the sense of Hattori [5] are separable extensions and semisimple extensions respectively in our sense. For the definitions of these the relative homological algebra introduced by Hochshild [8] is useful.

In § 1 we shall define the *semisimple extension* over non commutative rings and study about some properties of them. In § 2 we study about *separable extensions*. Separable extensions are semisimple extensions (Proposition 2.6). If Λ is an algebra over a commutative ring R and Γ is a subalgebra, then Λ is a separable extension of Γ if, and only if, $\Lambda\otimes_R\Delta$ is a semisimple extension of $\Gamma\otimes_R\Delta$ for every R-algebra Δ (Corollary 2.16). In § 3 some examples are given. The Galois extension of a non commutative ring, in the sense of T. Kanzaki, is a separable extension (Proposition 3.3).

Throughout this paper we assume that all rings have the identity, all subrings contain this element and all modules are unitary.

§ 1. Semisimple extension

The basic notions of relative homological algebra are introduced by G. Hochschild [8]. We recall these briefly.

Let Λ be a ring, Γ a subring of Λ. A sequence of (left) Λ-modules is (Λ, Γ)-exact if it is exact and splits as a sequence of Γ-modules. Consider a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\ & & & & & & \uparrow & & \\ & & & & & & P & & \end{array}$$

where the row is (Λ, Γ)-exact. P is (Λ, Γ)-projective if there exists a Λ-homomorphism of P into M such that the diagram is commutative. For any

Γ-module M', $\Lambda\otimes_\Gamma M'$ is (Λ, Γ)-projective, and a Λ-module P is (Λ, Γ)-projective if, and only if, P is a Λ-direct summand of $\Lambda\otimes_\Gamma P$. Dually (Λ, Γ)-injective modules are defined. For any Γ-module M', $\mathrm{Hom}_\Gamma(\Lambda, M')$ is (Λ, Γ)-injective, and a Λ-module Q is (Λ, Γ)-injective if, and only if, Q is a Λ-direct summand of $\mathrm{Hom}_\Gamma(\Lambda, Q)$.

THEOREM 1.1. *Let Λ be a ring, Γ a subring of Λ. Then the following conditions are equivalent:*

(*a*) *All left Λ-modules are (Λ, Γ)-projective.*

(*b*) *All (Λ, Γ)-exact sequences of left Λ-modules split as Λ-modules.*

(*c*) *Every submodule of a left Λ-module which is a Γ-direct summand is a Λ-direct summand.*

(*d*) *All left Λ-modules are (Λ, Γ)-injective.*

Theorem 1.1 is easily proved from the definition, and we shall omit its proof.

DEFINITION 1. Let Λ be a ring, Γ a subring of Λ. We shall say that Λ is a *left semisimple extension* of Γ if they satisfy the equivalent conditions in Theorem 1.1. If every finitely generated left Λ-module is (Λ, Γ)-projective, then we shall say that Λ is a *weakly left semisimple extension* of Γ. Similarly we can define the (*weakly*) *right semisimple extension* of a ring.

REMARK. Let Λ be an algebra over a commutative ring Γ. If Λ is a weakly left semisimple extension of Γ it is left semisimple in the sense of A. Hattori [**5**].

PROPOSITION 1.2. *Let f be a ring epimorphism of a ring Λ onto a ring Λ_1. If Λ is a* (*weakly*) *left semisimple extension of Γ then Λ_1 is a* (*weakly*) *left semisimple extension of $f(\Gamma)$.*

PROOF. Let M be a left Λ_1-module. Regarding M as a Λ-module, we have a commutative diagram

$$\begin{array}{ccc} \Lambda\otimes_\Gamma M & \xrightarrow{\varphi} & M \\ \big\downarrow f\otimes 1_M & \nearrow \varphi_1 & \\ \Lambda_1\otimes_{f(\Gamma)} M & & \end{array}$$

where φ and φ_1 are naturally defined homomorphisms. Since M is (Λ, Γ)-projective, there exists a Λ-homomorphism $\psi: M\to\Lambda\otimes_\Gamma M$ such that $\varphi\circ\psi=1_M$. Then $\psi_1=(f\otimes 1_M)\circ\psi$ is a Λ_1-homomorphism of M into $\Lambda_1\otimes_{f(\Gamma)} M$ and $\varphi_1\circ\psi_1=1_M$. Therefore M is $(\Lambda_1, f(\Gamma))$-projective and Λ_1 is a left semisimple extension of $f(\Gamma)$. If M is finitely generated over Λ_1, it is also finitely generated over Λ. Therefore Λ_1 is a weakly left semisimple extension of $f(\Gamma)$ if Λ is a weakly left semisimple extension of Γ.

PROPOSITION 1.3. *Let Λ be a ring, Ω and Γ subrings of Λ such that $\Omega\supseteq\Gamma$.* (1) *If Λ is a left semisimple extension of Γ then Λ is a left semisimple ex-*

tension of Ω. (2) If Λ is a left semisimple extension of Ω and Ω is a left semisimple extension of Γ, then Λ is a left semisimple extension of Γ.

PROOF. (1) Let M be a left Λ-module. Then we have a commutative diagram

$$\begin{array}{ccc} \Lambda\otimes_\Gamma M & \xrightarrow{\varphi} & M \\ \eta\downarrow & \nearrow \varphi' & \\ \Lambda\otimes_\Omega M & & \end{array}$$

where φ, φ' and η are naturally defined Λ-homomorphisms respectively. Since M is (Λ, Γ)-projective there exists a Λ-homomorphism $\psi: M\to\Lambda\otimes_\Gamma M$ such that $\varphi\circ\psi=1_M$. Now $\eta\circ\psi$ is a Λ-homomorphism of M into $\Lambda\otimes_\Omega M$ such that $\varphi'\circ(\eta\circ\psi)=\varphi\circ\psi=1_M$.

(2) If Ω is a left semisimple extension of Γ and M is a Λ-module, then the sequence

$$\Omega\otimes_\Gamma M\longrightarrow M\longrightarrow 0$$

Ω-splits. Tensoring Λ over Ω, the sequence

$$\Lambda\otimes_\Gamma M\longrightarrow\Lambda\otimes_\Omega M\longrightarrow 0$$

Λ-splits. Since Λ is a left semisimple extension of Ω, M is a Λ-direct summand of $\Lambda\otimes_\Omega M$. Therefore M is a Λ-direct summand of $\Lambda\otimes_\Gamma M$ and M is (Λ, Γ)-projective.

REMARK. Proposition 1.3 (1) holds for weakly left semisimple extensions. If Λ is Ω-finitely generated, then (2) also holds.

THEOREM 1.4. (1) *If Λ is a weakly left semisimple extension of Γ, and if I is a left ideal of Λ such that ${}_\Gamma I<\oplus_\Gamma\Lambda$ then ${}_\Lambda I<\oplus_\Lambda\Lambda$.* (2) *If Λ is a weakly semisimple extension of Γ, and if A is a two-sided ideal of Λ such that ${}_\Gamma A_\Gamma<\oplus_\Gamma\Lambda_\Gamma$ then there exists a two sided ideal B of Λ such that $\Lambda=A\oplus B$ as rings.*

PROOF. (1) Since the sequence

$$0\longrightarrow I\longrightarrow\Lambda\longrightarrow\Lambda/I\longrightarrow 0$$

Γ-splits and Λ/I is Λ-finitely generated, it splits as Λ-modules.

(2) By (1) there exists a left ideal B such that $\Lambda=A\oplus B$ as left ideals. Let $1=e_1+e_2$, $A=\Lambda e_1$ and $B=\Lambda e_2$. Then $A=e_1A\oplus e_2A<\oplus\Lambda$ as a right Γ-module. Since Λ is right semisimple, too, $e_2A<\oplus\Lambda$ as a right Λ-module. Hence there exists an idempotent e' such that $e_2A=e'\Lambda$. Since $e'=e'^2\in(e_2A)^2\subset e_2ABA=0$, $e_2A=e'\Lambda=0$. Therefore $B\Lambda\subset BA+B=\Lambda e_2A+B=B$. Thus B is a two-sided ideal of Λ. This completes the proof.

COROLLARY 1.5. *Let Λ be weakly semisimple over Γ. Then Λ is indecomposable as rings, if and only if there exists no non trivial idempotent e such that $(1-e)\Lambda e=0$ and $e\Gamma(1-e)=0$.*

PROOF. If there exists e as above, then $\Lambda = \Lambda e \oplus \Lambda(1-e)$ and ${}_{\Gamma}\Lambda(1-e)_{\Gamma} < \oplus_{\Gamma}\Lambda_{\Gamma}$. Hence Λ is decomposable as a ring by Theorem 1.5. The converse is obvious.

PROPOSITION 1.6. *Let Λ be a left semisimple extension of Γ. Then if a left Λ-module M is Γ-projective it is Λ-projective. Dually if a left Λ-module M is Γ-injective it is Λ-injective.*

PROOF. Since M is (Λ, Γ)-projective it is a Λ-direct summand of $\Lambda \otimes_{\Gamma} M$ which is a Λ-projective module. Therefore M is Λ-projective. It is similar for injective modules.

REMARK. If Λ is a weakly left semisimple extension of Γ. Then every finitly generated Λ-module which is Γ-projective is Λ-projective.

COROLLARY 1.7. *If Λ is a left semisimple extension of a semisimple subring Γ, then Λ is a semisimple ring. Consequently Λ is also a right semisimple extension of Γ.*

PROPOSITION 1.8. *Let Λ be a left semisimple extension of Γ, and let M be a left Λ-module. If Λ is right Γ-flat then*

$$\text{l. dim}_{\Lambda} M \leqq \text{l. dim}_{\Gamma} M$$

$$\text{l. inj. dim}_{\Lambda} M \geqq \text{l. inj. dim}_{\Gamma} M .$$

If Λ is left Γ-projective then

$$\text{l. dim}_{\Lambda} M \geqq \text{l. dim}_{\Gamma} M$$

$$\text{l. inj. dim}_{\Lambda} M \leqq \text{l. inj. dim}_{\Gamma} M .$$

In either case l. gl. dim $\Lambda \leqq$ l. gl. dim Γ.

PROOF. If $\text{l. dim}_{\Gamma} M \leqq n$ there exists a Γ-projective resolution of M

$$0 \longrightarrow X_n \longrightarrow \cdots \longrightarrow X_0 \longrightarrow M \longrightarrow 0 .$$

Since Λ is right Γ-flat

$$0 \longrightarrow \Lambda \otimes_{\Gamma} X_n \longrightarrow \cdots \longrightarrow \Lambda \otimes_{\Gamma} X_0 \longrightarrow \Lambda \otimes_{\Gamma} M \longrightarrow 0$$

is a Λ-projective resolution of $\Lambda \otimes_{\Gamma} M$. As M is a Λ-direct summand of $\Lambda \otimes_{\Gamma} M$,

$$\text{l. dim}_{\Lambda} M \leqq \text{l. dim}_{\Lambda} \Lambda \otimes_{\Gamma} M \leqq \text{l. dim}_{\Gamma} M .$$

If Λ is left Γ-projective every Λ-projective resolution of M is a Γ-projective resolution of M. Therefore

$$\text{l. dim}_{\Lambda} M \geqq \text{l. dim}_{\Gamma} M .$$

Similarly we can prove the inequalities of injective dimensions.

COROLLARY 1.9. *Let Λ be a left semisimple extension of a left hereditary ring Γ. If Λ is left Γ-projective or right Γ-flat then Λ is a left hereditary ring.*

PROPOSITION 1.10. *If Γ is left and right Noetherian and Λ is left and right Γ-finitely generated, then Λ is a weakly left semisimple extension of Γ if and only if Λ is a weakly right semisimple extension of Γ.*

The proof is an easy consequence of Prop. 1.3 in [5].

From now on we assume that Λ is an algebra over a commutative ring R and Γ a subalgebra of Λ.

LEMMA 1.11. *Let Λ be an algebra over a commutative ring R and Γ a subalgebra of Λ. If a Λ-module P is (Λ, Γ)-projective then $P\otimes_R\Delta$ is $(\Lambda\otimes_R\Delta, \Gamma\otimes_R\Delta)$-projective for any R-algebra Δ.*

The proof is clear and we shall omit it.

As an easy consequence of the above lemma we have

PROPOSITION 1.12. *Let Λ be an algebra over a commutative ring R and Γ a subalgebra of Λ. If Λ is a (weakly) left semisimple extension of Γ then $\Lambda_S=\Lambda\otimes_R R_S$ is a (weakly) left semisimple extension of $\Gamma_S=\Gamma\otimes_R R_S$, for every multiplicative subset S of R.*

PROOF. Let M' be a Λ_S-module. Regarding M' as a Λ-module M' is (Λ, Γ)-projective. So $M'_S=M'$ is (Λ_S, Γ_S)-projective by the above lemma. For a finitely generated Λ_S-module M' there exists a finitely generated Λ-submodule M of M' such that $M_S=M'$. Since M is (Λ, Γ)-projective M' is (Λ_S, Γ_S)-projective.

PROPOSITION 1.13. *Let R, Λ and Γ be as above. If Λ is left Noetherian and finitely generated as a left Γ-module, then* $\dim_{\Lambda,\Gamma} M=\sup_{\mathfrak{m}}\dim_{\Lambda_\mathfrak{m},\Gamma_\mathfrak{m}} M_\mathfrak{m}$, *where $\mathfrak{m}$ runs over all maximal ideals of R, for every finitely generated Λ-module M.*

PROOF. As Λ is left Noetherian and Γ-finitely generated, we have a (Λ, Γ)-projective resolution P of M such that every P_i is finitely generated. By Lemma 2.4 of [1], we have $R_\mathfrak{m}\otimes\mathrm{Hom}_\Lambda(P_i, B)=\mathrm{Hom}_{\Lambda\otimes R_\mathfrak{m}}(P_i\otimes R_\mathfrak{m}, B\otimes R_\mathfrak{m})$, for any Λ-module B and every maximal ideal $\mathfrak{m}$ of R. Since $P\otimes R_\mathfrak{m}$ is a $(\Lambda_\mathfrak{m}, \Gamma_\mathfrak{m})$-projective resolution of $M\otimes R_\mathfrak{m}$, by Lemma 1.11, passing to homology, we have

$$H(R_\mathfrak{m}\otimes\mathrm{Hom}_\Lambda(P, B))=H(\mathrm{Hom}_{\Lambda_\mathfrak{m}}(P_\mathfrak{m}, B_\mathfrak{m}))$$
$$R_\mathfrak{m}\otimes\mathrm{Ext}^n_{\Lambda,\Gamma}(M, B)=\mathrm{Ext}^n_{\Lambda_\mathfrak{m},\Gamma_\mathfrak{m}}(M_\mathfrak{m}, B_\mathfrak{m}).$$

Since every $\Lambda_\mathfrak{m}$-module B' has a Λ-submodule B such that $B'=B\otimes R_\mathfrak{m}$ if $\dim_{\Lambda,\Gamma} M\leqq n$, $\mathrm{Ext}^{n+1}_{\Lambda_\mathfrak{m},\Gamma_\mathfrak{m}}(M_\mathfrak{m}, B')=0$ for every $\Lambda_\mathfrak{m}$-module B'. Thus we have $\dim_{\Lambda_\mathfrak{m},\Gamma_\mathfrak{m}} M_\mathfrak{m}\leqq n$. Therefore $\dim_{\Lambda,\Gamma} M\geqq\sup_{\mathfrak{m}}\dim_{\Lambda_\mathfrak{m},\Gamma_\mathfrak{m}} M_\mathfrak{m}$. Conversely if $\dim_{\Lambda_\mathfrak{m},\Gamma_\mathfrak{m}} M_\mathfrak{m}\leqq n$ for every $\mathfrak{m}$, we have $R_\mathfrak{m}\otimes\mathrm{Ext}^{n+1}_{\Lambda,\Gamma}(M, B)=0$ for every $\mathfrak{m}$. Then $\mathrm{Ext}^{n+1}_{\Lambda,\Gamma}(M, B)=0$, for any Λ-module B. Hence $\dim_{\Lambda,\Gamma} M\leqq n$. Therefore $\dim_{\Lambda,\Gamma} M=\sup_{\mathfrak{m}}\dim_{\Lambda_\mathfrak{m},\Gamma_\mathfrak{m}} M_\mathfrak{m}$.

COROLLARY 1.14. *Let R, Λ and Γ be as in Proposition 1.13. Then Λ is* *weakly left semisimple over Γ if and only if $\Lambda_{\mathfrak{m}}$ is weakly left semisimple over $\Lambda_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ of R.*

The proof is easy so we omit it.

In connection with Theorem 1.1, we have

THEOREM 1.15. *Let Λ be a ring Γ a subring of Λ. Then the following conditions are equivalent:*

(*a*) *All left (Λ, Γ)-projective modules are Λ-projective.*

(*b*) *All exact sequencences of left Λ-modules are (Λ, Γ)-exact.*

(*c*) *Every Λ-submodule of a Λ-module is a Γ-direct summand.*

(*d*) *Every left ideal of Λ is a Γ-direct summand.*

(*e*) *All left (Λ, Γ)-injective modules are Λ-injective.*

PROOF. $(a) \Rightarrow (b)$. It is sufficient to prove that any short exact sequence of Λ-modules Γ-splits. Consider a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & M & \overset{\alpha}{\longrightarrow} & N & \longrightarrow & 0 \\ & & & & & & {\scriptstyle f}\uparrow & & \\ & & & & & & \Lambda \otimes_{\Gamma} N & & \end{array}$$

where the row is exact and f is the natural map. The map $g: N \to \Lambda \otimes_{\Gamma} N$ defined by $g(n) = 1 \otimes n$ is a Γ-homomorphism and $f \circ g = 1_N$. Since $\Lambda \otimes_{\Gamma} N$ is (Λ, Γ)-projective, it is Λ-projective. Therefore there exists a Λ-homomorphism $h: \Lambda \otimes_{\Gamma} N \to M$ such that $\alpha \circ h = f$. Then $h \circ g$ is a Γ-homomorphism of N into M such that $\alpha \circ (h \circ g) = f \circ g = 1_N$.

$(b) \Rightarrow (a)$. Let M be a (Λ, Γ)-projective module and consider an exact sequence

$$0 \longrightarrow K \longrightarrow P \overset{\alpha}{\longrightarrow} M \longrightarrow 0$$

with P Λ-projective. By the assumption the sequence is (Λ, Γ)-exact. Since M is (Λ, Γ)-projective there exists a Λ-homomorphism $\beta: M \to P$ such that $\alpha \circ \beta = 1_M$. Therefore M is Λ-projective.

$(b) \Rightarrow (c)$. Let N be a submodule of a Λ-module M and consider the exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0 .$$

Since the sequence Γ-splits, N is a Γ-direct summand of M.

$(c) \Rightarrow (d)$. Trivial.

$(d) \Rightarrow (e)$. Let M be a (Λ, Γ)-injective module and I a left ideal of Λ. Since I is a Γ-direct summand of Λ, the sequence

$$0 \longrightarrow I \longrightarrow \Lambda \longrightarrow \Lambda/I \longrightarrow 0$$

is a (Λ, Γ)-exact sequence. Therefore, since M is (Λ, Γ)-injective, every Λ-homomorphism of I into M is extended to a Λ-homomorphism of Λ into M.

and M is Λ-injective.

$(e)\Rightarrow(b)$. The proof is similar to $(a)\Rightarrow(b)$.

§ 2. Separable extension

Let R be a commutative ring. Consider an algebra Λ over R and its subalgebra Γ. Let ι be the natural map of $\Gamma\otimes_R\Lambda^0$ into $\Lambda\otimes_R\Lambda^0$, where Λ^0 is an anti-isomorphic copy of Λ. Then Λ is considered as a left $\Lambda\otimes\Lambda^0$-module. Λ is $(\Lambda\otimes\Lambda^0, \iota(\Gamma\otimes\Lambda^0))$-projective if and only if the sequence

$$(\Lambda\otimes\Lambda^0)\otimes_{\iota(\Gamma\otimes\Lambda^0)}\Lambda\xrightarrow{\varphi}\Lambda\longrightarrow 0$$

where φ sends $(a\otimes b^0)\otimes x$ into axb, $\Lambda\otimes\Lambda^0$-splits. Therefore if Λ is $(\Lambda\otimes\Lambda^0, \iota(\Gamma\otimes\Lambda^0))$-projective, there exists a $\Lambda\otimes\Lambda^0$-homomorphism $\psi:\Lambda\to(\Lambda\otimes\Lambda^0)\otimes_{\iota(\Gamma\otimes\Lambda^0)}\Lambda$ such that $\varphi\circ\psi=1_\Lambda$. Since $(\Lambda\otimes\Lambda^0)\otimes_{\iota(\Gamma\otimes\Lambda^0)}\Lambda$ is isomorphic to $\Lambda\otimes_\Gamma\Lambda$, if we put $\psi(1)=\sum a_i\otimes b_i$ then $\sum a_i\otimes b_i$ satisfies the conditions

(1) $\sum a_ib_i=1$.

(2) $\sum xa_i\otimes b_i=\sum a_i\otimes b_ix$ for all $x\in\Lambda$.

These conditions lead us that, for the definition of the separable extension, Λ need not necessarily be an algebra.

DEFINITION 2. Let Λ be a ring Γ a subring of Λ. We shall say that Λ is a *separable extension* of Γ if there exists an element $\sum a_i\otimes b_i$ in $\Lambda\otimes_\Gamma\Lambda$ satisfying (1) and (2) above.

REMARK 1. If Γ is in the center of Λ and Λ is a separable extension of Γ then Λ is a separable algebra over Γ in the sense of Auslander-Goldman [**2**].

REMARK 2. If Λ is an algebra over a commutative ring R and if Γ is a subalgebra of Λ, then another definition for a separable extension is possible. It is easily proved that Λ is $(\Lambda\otimes_R\Lambda^0, \iota(\Gamma\otimes_R\Lambda^0))$-projective if and only if Λ is $(\Lambda\otimes_R\Lambda^0, \iota(\Gamma\otimes_R\Gamma^0))$-projective. The proof is given in Proposition 2.14.

Let Λ be a ring, Γ a subring of Λ. For a two-sided Λ-module M, we set $M^\Lambda=\{m\in M\mid xm=mx, \text{ for all } x\in\Lambda\}$. M^Λ is isomorphic to $\mathrm{Hom}_{(\Lambda,\Lambda)}(\Lambda, M)$, two-sided Λ-homomorphisms of Λ into M.

Throughout this section φ means the two-sided Λ-homomorphism of $\Lambda\otimes_\Gamma\Lambda$ into Λ defined by $\varphi(x\otimes y)=xy$. We put $A=(\Lambda\otimes_\Gamma\Lambda)^\Lambda$, then $\varphi(A)$ is in the center of Λ. Let C be the center of Λ, then $\varphi(A)=C$ if and only if A contains an element $\sum a_i\otimes b_i$ such that $\sum a_ib_i=1$. So we have

PROPOSITION 2.1. *Λ is a separable extension of Γ if and only if* $\varphi(A)=C$.

Let Λ be a separable extension of Γ. Then the sequence

$$\Lambda\otimes_\Gamma\Lambda\xrightarrow{\varphi}\Lambda\longrightarrow 0$$

splits as a two-sided Λ-module. Therefore $\mathrm{Hom}_{(\Lambda,\Lambda)}(\Lambda, M)\cong M^\Lambda$ is a direct summand of $\mathrm{Hom}_{(\Lambda,\Lambda)}(\Lambda\otimes_{\Gamma}\Lambda, M)$. $\mathrm{Hom}_{(\Lambda,\Lambda)}(\Lambda\otimes_{\Gamma}\Lambda, M)$ is isomorphic to $M^{\Gamma}=\{m\in M\mid \gamma m=m\gamma$, for all $\gamma\in\Gamma\}$ by the map $f\rightarrow f(1\otimes 1)\in M$, for $f\in \mathrm{Hom}_{(\Lambda,\Lambda)}(\Lambda\otimes_{\Gamma}\Lambda, M)$. M^{Λ} is a C-submodule and M^{Γ} is a two-sided Γ'-module respectively, where $\Gamma'=\Lambda^{\Gamma}$, the commutator of Γ in Λ. Thus we have

LEMMA 2.2. *If Λ is a separable extension of Γ then M^{Λ} is a C-direct summand of M^{Γ} for a two-sided Λ-module M.*

If we put $M=\Lambda$, we have

COROLLARY 2.3. *If Λ is a separable extension of Γ then the center C of Λ is a C-direct summand of Γ', and so if $\mathfrak{a}$ is an ideal of C then $\mathfrak{a}\Gamma'\cap C=\mathfrak{a}$.*

PROPOSITION 2.4. *Let f be a ring epimorphism of a ring Λ onto a ring Λ_1. If Λ is a separable extension of Γ then Λ_1 is so over $f(\Gamma)$.*

PROOF. If $\sum a_i\otimes b_i$ satisfies the conditions of separability for Λ and Γ, then $\sum f(a_i)\otimes f(b_i)$ does so for Λ_1 and $f(\Gamma)$.

PROPOSITION 2.5. *Let Λ be a ring, Ω and Γ are subrings of Λ such that $\Omega\supseteqq\Gamma$. (1) If Λ is a separable extension of Γ then Λ is a separable extension of Ω. (2) If Λ over Ω and Ω over Γ are separable extensions respectively then Λ is a separable extension of Γ.*

PROOF. (1). If Λ is a separable extension of Γ then there exists $\sum a_i\otimes b_i\in\Lambda\otimes_{\Gamma}\Lambda$ satisfying the separable conditions. The image of $\sum a_i\otimes b_i$ in $\Lambda\otimes_{\Omega}\Lambda$ satisfies the conditions for Λ and Ω. (2). Let $\sum a_i\otimes b_i\in\Lambda\otimes_{\Omega}\Lambda$ and $\sum\alpha_j\otimes\beta_j\in\Omega\otimes_{\Gamma}\Omega$ satisfy the separable conditions respectively. Then the map $x\otimes y\rightarrow\sum x\alpha_j\otimes\beta_j y$ of $\Lambda\otimes_{\Omega}\Lambda\rightarrow\Lambda\otimes_{\Gamma}\Lambda$ is a well defined map and the image of $\sum a_i\otimes b_i$ in $\Lambda\otimes_{\Gamma}\Lambda$ by this map, $\sum a_i\alpha_j\otimes\beta_j b_i$, satisfies the separable conditions.

PROPOSITION 2.6. *If Λ is a separable extension of Γ then Λ is a left (resp. right) semisimple extension of Γ.*

PROOF. Let M be a left Λ-module, and $\sum a_i\otimes b_i\in\Lambda\otimes_{\Gamma}\Lambda$ satisfy the separable conditions. Consider the sequence

$$\Lambda\otimes_{\Gamma}M\xrightarrow{f}M\longrightarrow 0$$

where f is the natural epimorphism. The map $g: M\rightarrow\Lambda\otimes_{\Gamma}M$ defined by $g(m)=\sum a_i\otimes b_i m$ is a left Λ-homomorphism such that $f\circ g=1_M$. Therefore Λ is a left semisimple extension of Γ. Similary Λ is a right semisimple extension of Γ.

From now on we consider the case of algebras.

PROPOSITION 2.7. *Let R be a commutative ring and let Λ_1 and Λ_2 be R-algebras, Γ_1 and Γ_2 be R-subalgebras of Λ_1 and Λ_2 respectively. If Λ_i is a separable extension of Γ_i for $i=1, 2$ then $\Lambda_1\otimes_R\Lambda_2$ is also a separable extension of $\iota(\Gamma_1\otimes_R\Gamma_2)$ where $\iota(\Gamma_1\otimes_R\Gamma_2)$ is the natural image of $\Gamma_1\otimes_R\Gamma_2$ in $\Lambda_1\otimes_R\Lambda_2$.*

PROOF. Let a_i and $b_i\in\Lambda_1$, α_j and $\beta_j\in\Lambda_2$ satisfy the conditions of separability (1) and (2). Then for $\sum_{ij}(a_i\otimes\alpha_j)\otimes(b_i\otimes\beta_j)\in(\Lambda_1\otimes\Lambda_2)\otimes_{\iota(\Gamma_1\otimes\Gamma_2)}(\Lambda_1\otimes\Lambda_2)$ we have

$$\sum(a_i\otimes\alpha_j)(b_i\otimes\beta_j)=\sum a_ib_i\otimes\sum\alpha_j\beta_j=1\otimes 1\,.$$

Now since $(\Lambda_1\otimes\Lambda_2)\otimes_{\iota(\Gamma_1\otimes\Gamma_2)}(\Lambda_1\otimes\Lambda_2)=(\Lambda_1\otimes\Lambda_2)\otimes_{\Gamma_1\otimes\Gamma_2}(\Lambda_1\otimes\Lambda_2)\cong(\Lambda_1\otimes_{\Gamma_1}\Lambda_1)\otimes_R(\Lambda_2\otimes_{\Gamma_2}\Lambda_2)$, we have for $x\otimes y\in\Lambda_1\otimes\Lambda_2$

$$\begin{aligned}\sum(x\otimes y)(a_i\otimes\alpha_j)\otimes(b_i\otimes\beta_j)&=\sum(xa_i\otimes y\alpha_j)\otimes(b_i\otimes\beta_j)\leftrightarrow\\&\sum(xa_i\otimes b_i)\otimes(y\alpha_j\otimes\beta_j)\\&=\sum(a_i\otimes b_ix)\otimes(\alpha_j\otimes\beta_jy)\leftrightarrow\\&\sum(a_i\otimes\alpha_j)\otimes(b_ix\otimes\beta_jy)\\&=\sum(a_i\otimes\alpha_j)\otimes(b_i\otimes\beta_j)(x\otimes y)\,.\end{aligned}$$

This completes the proof.

If we set $\Lambda_2=\Gamma_2=\Delta$ in the above proposition we have

COROLLARY 2.8. *Let Λ be an algebra over a commutative ring R and Γ a subalgebra of Λ. If Λ is a separable extension of Γ then $\Lambda\otimes_R\Delta$ is a separable extension of $\iota(\Gamma\otimes_R\Delta)$ for any R-algebra Δ.*

In particular if Λ and Δ are subrings of a ring and elementwise commutative and Γ is a subring of $\Lambda\cap\Delta$ (necessarily commutative), and if Λ is a separable extension of Γ, then $\Lambda\Delta$ is a separable extension of Δ.

COROLLARY 2.9. *Let $\Lambda\supseteqq\Gamma$ and Δ be algebras over a commutative ring R. If Δ is separable over R and if Λ is a left semisimple extension of Γ then $\Lambda\otimes\Delta$ is a left semisimple extension of $\iota(\Gamma\otimes R)$.*

PROOF. By Proposition 2.6 and Corollary 2.8 $\Lambda\otimes\Delta$ is a left semisimple extension of $\iota(\Lambda\otimes R)$ which is a left semisimple extension of $\iota(\Gamma\otimes R)$. Therefore $\Lambda\otimes\Delta$ is a left semisimple extension of $\iota(\Gamma\otimes R)$ by Proposition 1.3.

PROPOSITION 2.10. *Let Λ be an algebra over a commutative ring R and Γ a subalgebra of Λ and let Δ be an R-algebra. Assume that Δ has an R-direct summand isomorphic to R. Then if $\Lambda\otimes_R\Delta$ is a separable extension of $\iota(\Gamma\otimes_R\Delta)$, Λ is a separable extension of Γ.*

Firstly we prove the next lemma.

LEMMA 2.11. *Let A be an algebra over a commutative ring R, B a subalgebra of A and C an R-algebra. If M is a left $A\otimes_RC$-module and $(A\otimes_RC,\iota(B\otimes_RC))$-projective, where $\iota(B\otimes_RC)$ is the natural image of $B\otimes_RC$ in $A\otimes C$, then M, as an A-module, is (A, B)-projective.*

PROOF. M is an $(A\otimes C)$-direct summand of $(A\otimes C)\otimes_{B\otimes C}M$. Since $(A\otimes C)\otimes_{B\otimes C}M\cong A\otimes_BM$, M is an A-direct summand of $A\otimes_BM$.

PROOF OF PROPOSITION 2.10. Since $\Lambda\otimes\Delta$ is a separable extension of $\iota(\Gamma\otimes\Delta)$, $\Lambda\otimes\Delta$ is $(\Lambda^e\otimes\Delta^e,(\Gamma\otimes\Lambda^0)\otimes\Delta^e)$-projective. So by Lemma 2.11 $\Lambda\otimes\Delta$ is $(\Lambda^e,\Gamma\otimes\Lambda^0)$-projective. As Λ is a left Λ^e-direct summand of $\Lambda\otimes\Delta$, Λ is

$(\Lambda^e, \Gamma\otimes\Lambda^0)$-projective. Thus Λ is a separable extension of Γ.

If we put $\Gamma=R\cdot 1$, in Proposition 2.10 we have

COROLLARY 2.12. *Let Λ and Δ be R-algebras. If Δ has an R-direct summand isomorphic to R and if $\Lambda\otimes\Delta$ is a separable extension of $\iota(R\otimes\Delta)$ then Λ is a separable algebra over R.*

This corollary is a generalization of Prop. 1.7 in [2], since for any subalgebra Ω of a separable algebra Λ, Λ is a separable extension of Ω.

PROPOSITION 2.13. *Let R be a commutative Noetherian ring and Λ be an R-algebra which is finitely generated as an R-module. Then, for a subalgebra Γ of Λ, Λ is a separable extension of Γ if and only if $\Lambda_{\mathfrak{m}}$ is a separable extension of $\Gamma_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ of R.*

The proof is easy by Proposition 1.13.

PROPOSITION 2.14. *Let Λ be an algebra over a commutative ring R and Γ a subalgebra of Λ. Then the following conditions are equivalent:*

(*a*) *Λ is a separable extension of Γ.*

(*b*) *$\Lambda^e=\Lambda\otimes_R\Lambda^0$ is a separable extension of $\iota(\Gamma^e)$.*

(*c*) *Λ^e is a left (resp. right) semisimple extension of $\iota(\Gamma^e)$.*

(*d*) *Λ^e is a weakly left (resp. right) semisimple extension of $\iota(\Gamma^e)$.*

(*e*) *Λ^e is a left (resp. right) semisimple extension of $\iota(\Gamma\otimes\Lambda^0)$.*

(*f*) *Λ^e is a weakly left (resp. right) semisimple extension of $\iota(\Gamma\otimes\Lambda^0)$.*

First of all we prove the next lemma.

LEMMA 2.15. *Let A be a ring and let B and C be subrings of A such that $B\supseteqq C$.* (1) *If an A-module M is (A, B)-projective as well as (B, C)-projective then M is (A, C)-projective.* (2) *If M is (A, C)-projective then M is (A, B)-projective.*

PROOF. (1) As M is a B-direct summand of $B\otimes_C M$, $A\otimes_B M$ is an A-direct summand of $A\otimes_C M$. On the other hand M is an A-direct summand of $A\otimes_B M$. So M is an A-direct summand of $A\otimes_C M$. (2) is clear.

PROOF OF PROPOSITION 2.14. $(a)\Rightarrow(b)$. Let $\sum a_i\otimes b_i\in\Lambda\otimes_\Gamma\Lambda$ satisfy the separable conditions. Consider the sequence

$$(\Gamma\otimes\Lambda^0)\otimes_{\Gamma\otimes\Gamma^0}\Lambda\xrightarrow{\varphi'}\Lambda$$

where φ' sends $(x\otimes y^0)\otimes z$ to xzy. Define $\psi':\Lambda\to(\Gamma\otimes\Lambda^0)\otimes_{\Gamma\otimes\Gamma^0}\Lambda$, by $\psi'(z)=\sum(1\otimes b_i^0)\otimes za_i=\sum(1\otimes(b_iz)^0)\otimes a_i$. Then ψ' is a $\Gamma\otimes\Lambda^0$-homomorphism such that $\varphi'\circ\psi'=1_\Lambda$. Therefore Λ is $(\Gamma\otimes\Lambda^0, \Gamma\otimes\Gamma^0)$-projective. Since Λ is $(\Lambda\otimes\Lambda^0, \Gamma\otimes\Lambda^0)$-projective, by the above lemma, Λ is $(\Lambda\otimes\Lambda^0, \Gamma\otimes\Gamma^0)$-projective.

$(b)\Rightarrow(c)$. By Proposition 2.6 it is clear.

$(c)\Rightarrow(d)$. Trivial.

$(d)\Rightarrow(f)$. This follows from Proposition 1.3.

$(f)\Rightarrow(a)$. Since Λ is finitely generated $\Lambda\otimes\Lambda^0$-module, Λ is $(\Lambda\otimes\Lambda_0, \Gamma\otimes\Lambda^0)$-projective.

Similarly $(c)\Rightarrow(e)\Rightarrow(a)$ are obvious.

COROLLARY 2.16. *Let Λ be an algebra over a commutative ring R and Γ a subalgebra of Λ. Then Λ is a separable extension of Γ if and only if $\Lambda\otimes_R\Delta$ is a weakly semisimple extension of $\iota(\Gamma\otimes_R\Delta)$ for any R-algebra Δ.*

PROOF. 'Only if' part is given by Cor. 2.8 and Prop. 2.6. For 'if' part, put $\Delta=\Lambda^0$.

For the case of $\Gamma=R\cdot 1$ we have

COROLLARY 2.17. *Let Λ be an algebra over a commutative ring R. Then Λ is separable over R if and only if Λ^e is left (or right) semisimple over R* (cf. Theorem 2.5, [**5**]).

Next proposition is due to Onodera in the case of algebras. Let Λ be a Frobenius extension of Γ, that is,

(i) Λ is a finitely generated projective right Γ-module.

(ii) $\mathrm{Hom}(\Lambda_\Gamma, \Gamma_\Gamma)\cong\Lambda$ as left Γ and right Λ-module, where $\mathrm{Hom}(\Lambda_\Gamma, \Gamma_\Gamma)$ is the set of all right Γ-homomorphisms of Λ into Γ.

Then we have

$$\Lambda\otimes_\Gamma\Lambda\cong\mathrm{Hom}({}_\Gamma\mathrm{Hom}(\Lambda_\Gamma, \Gamma_\Gamma), {}_\Gamma\Lambda)\cong\mathrm{Hom}({}_\Gamma\Lambda, {}_\Gamma\Lambda).$$

The first map is given by $x\otimes y\leftrightarrow(f\to f(x)y)$ for $f\in\mathrm{Hom}(\Lambda_\Gamma, \Gamma_\Gamma)$, and is an isomorphism by virtue of (i). The second map is the one induced by (ii). Furthermore these are two-sided Λ-isomorphisms. If we write the element in $\Lambda\otimes_\Gamma\Lambda$, corresponding to the identity map in $\mathrm{Hom}({}_\Gamma\Lambda, {}_\Gamma\Lambda)$, by $\sum r_i\otimes l_i$ then $\{r_i\}$ and $\{l_i\}$ form a pair of dual base (cf. Onodera [**5**]). It is easily proved that $A=(\Lambda\otimes_\Gamma\Lambda)^\Lambda$ corresponds to $\mathrm{Hom}({}_\Gamma\Lambda_\Lambda, {}_\Gamma\Lambda_\Lambda)=\Gamma'_l$, the left multiplications of the elements of the commutator of Γ. More precisely $\sum r_i\otimes\gamma' l_i$ corresponds to $\gamma'_l\in\Gamma'_l$. Thus we have proved that $\varphi(A)=\sum r_i\Gamma' l_i$. By Proposition 2.1 we have

PROPOSITION 2.18. *Let Λ be a Frobenius extension of Γ and let $\{r_i\}$ and $\{l_i\}$ be a pair of dual base. Then Λ is a separable extension of Γ if and only if $\sum r_i\Gamma' l_i=C$.*

§ 3. Examples

PROPOSITION 3.1. *Let G be a group and H a subgroup of G with finite index, say n. If R is a commutative ring such that $nR=R$, then the group ring RG of G over R is a separable extension of RH.*

PROOF. Let $G=\sum_{i=1}^n g_iH=\sum Hg_i^{-1}$ be a left and a right coset decomposition of G by H respectively, and let $1=nr$, $r\in R$. Then in $RG\otimes_{RH}RG$, $r\sum g_i\otimes g_i^{-1}$

satisfies the separable conditions (1) and (2)

Let $\Lambda=(\Gamma)_n$ be the ring of square matrices of order n over a ring Γ. Let e_{ij} be the matrix with 1 at the intersection of the i-th row and j-th column, and with zero everywhere else. Then Γ is identified with diagonal matrices of the form $\sum \gamma e_{ii}$, $\gamma\in\Gamma$. It is easily proved that the element $\sum e_{i1}\otimes e_{1i}$ $\in \Lambda\otimes_\Gamma \Lambda$ satisfies the separable conditions. We have

PROPOSITION 3.2. *If Λ is the ring of square matrices of order n over a ring Γ then Λ is a separable extension of Γ.*

Lastly we consider the Galois extension. For the definition and some basic properties see for example [7] § 1 or [16]. Let Λ be a Galois extension of Γ with Galois group G. Then there exist x_i and y_i in Λ such that, for $\sigma\in G$,

$$(*)\qquad \sum_i x_i\sigma(y_i)=\begin{cases} 1 & \text{if } \sigma=1 \\ 0 & \text{if } \sigma\neq 1. \end{cases}$$

Multiplying by $\sigma(z)$ on the right and adding on $\sigma\in G$, we have

$$z=\sum_i x_i \operatorname{Tr}(y_i z)$$

for all $z\in\Lambda$, where $\operatorname{Tr}(x)=\sum_{\sigma\in G}\sigma(x)$. Similarly we have

$$z=\sum_i \operatorname{Tr}(zx_i)y_i$$

for all $z\in\Lambda$. In particular we have

$$zx_i=\sum_j x_j \operatorname{Tr}(y_j z x_i)$$

and

$$y_j z=\sum_i \operatorname{Tr}(y_j z x_i)\, y_i\,.$$

Therefore in $\Lambda\otimes_\Gamma\Lambda$

$$\sum_i zx_i\otimes y_i=\sum_{ij} x_j \operatorname{Tr}(y_j z x_i)\otimes y_i=\sum_{ij} x_j\otimes \operatorname{Tr}(y_j z x_i) y_i=\sum_j x_j\otimes y_j z\,.$$

Since $\sum x_i y_i=1$, the element $\sum x_i\otimes y_i$ satisfies the separable conditions. We have

PROPOSITION 3.3. *If Λ is a Galois extension of Γ then Λ is a separable extension of Γ.*

In general $\operatorname{Tr}(\Lambda)=\{\sum_{\sigma\in G}\sigma(x)\mid x\in\Lambda\}$ is in Γ. We assume that $\operatorname{Tr}(\Lambda)=\Gamma$. Let H be a subgroup of G and set $\Omega=\mathfrak{F}(H)$, the H-fixed subring of Λ. Let b be an element in Λ such that $\operatorname{Tr}_H(b)=\sum_{\tau\in H}\tau(b)=1$.

The existence of such an element is assured by [7] Prop. 4. From the relation (*) we have

$$\sum_i x_i \operatorname{Tr}_H(y_i)=1\,.$$

Then

$$\sum_i bx_i \operatorname{Tr}_H(y_i) = b$$

and

(*1) $$\sum_i \operatorname{Tr}_H(bx_i)\operatorname{Tr}_H(y_i) = 1.$$

Next, if $\rho, \tau \in H$ and $\sigma \notin H$, then $\rho^{-1}\sigma\tau \neq 1$, and so

$$\sum_i x_i\rho^{-1}\sigma\tau(y_i) = 0.$$

Adding on τ and multiplying by b we have

$$\sum_i bx_i\rho^{-1}\sigma(\operatorname{Tr}_H(y_i)) = 0$$

and

$$\sum_i \rho(bx_i)\sigma(\operatorname{Tr}_H(y_i)) = 0.$$

Adding on ρ we have

(*2) $$\sum_i \operatorname{Tr}_H(bx_i)\sigma(\operatorname{Tr}_H(y_i)) = 0 \quad \text{if} \quad \sigma \notin H.$$

From (*2) we have also

(*3) $$\sum_i \sigma(\operatorname{Tr}_H(bx_i))\operatorname{Tr}_H(y_i) = 0 \quad \text{if} \quad \sigma \notin H.$$

Let ω be an element of Ω. Then from (*2) we have

(*4) $$\sum_i \operatorname{Tr}_H(bx_i)\sigma(\operatorname{Tr}_H(y_i\omega)) = 0.$$

In (*1) and (*4), if we sum up only those σ_k, where $G = \sum \sigma_k H$, we have

$$\sum \operatorname{Tr}_H(bx_i)\operatorname{Tr}_G(y_i\omega) = \omega.$$

Similarly from (*1) and (*3) we have

$$\sum_i \operatorname{Tr}_G(\omega bx_i)\operatorname{Tr}_H(y_i) = \omega$$

for all $\omega \in \Omega$. Therefore in particular

$$\omega \operatorname{Tr}_H(bx_j) = \operatorname{Tr}_H(\omega bx_j) = \sum_i \operatorname{Tr}_H(bx_i)\operatorname{Tr}_G(y_i \operatorname{Tr}_H(\omega bx_j))$$

and

$$\operatorname{Tr}_H(y_i)\omega = \operatorname{Tr}_H(y_i\omega) = \sum_j \operatorname{Tr}_G(\operatorname{Tr}_H(y_i\omega)bx_j)\operatorname{Tr}_H(y_j).$$

It is easily proved that

$$\operatorname{Tr}_G(y_i \operatorname{Tr}_H(\omega bx_j)) = \sum_{\sigma_k} \sigma_k(\operatorname{Tr}_H(y_i\omega)\operatorname{Tr}_H(bx_j)) = \operatorname{Tr}_G(\operatorname{Tr}_H(y_i\omega)bx_j).$$

Therefore in $\Omega \otimes_\Gamma \Omega$

$$\sum_j \omega \operatorname{Tr}_H(bx_j) \otimes \operatorname{Tr}_H(y_j) = \sum_i \operatorname{Tr}_H(bx_i) \otimes \operatorname{Tr}_H(y_i)\omega.$$

From this and (*1) we have proved that the element $\sum \operatorname{Tr}_H(bx_i) \otimes \operatorname{Tr}_H(y_i)$ satisfies the separability conditions. Thus we have

PROPOSITION 3.4. *Let Λ be a Galois extension of Γ with Galois group G and let H be a subgroup of G. If* $\mathrm{Tr}_G(\Lambda)=\Gamma$ *then the H-fixed subring of Λ is a separable extension of Γ.*

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