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Regular points and Green functions in Markov processes

By Mamoru KANDA

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§ 0. Introduction.

Our aim of this paper is to investigate the regular points of multi-dimensional standard processes having an adequate Green function $G(x, y)$ with the condition (S);

(S). *There exists* $\alpha \in (0, d)$ $(d \geqq 3)$ *such that for any compact set* K *given, there exist* $\delta > 0$ *and* $C_1 C_2 \in (0, \infty)$ *such that*

$$C_1 |x-y|^{-\alpha} \geqq G(x, y) \geqq C_2 |x-y|^{-\alpha}$$

for $|x-y| < \delta$ *and* $x, y \in K$.

In case $d=2$, *we include the following case:*

$$C_1 \log \frac{1}{|x-y|} \geqq G(x, y) \geqq C_2 \log \frac{1}{|x-y|}.$$

In § 1, for an adequate Green function with the condition (S), we shall construct a standard process in Dynkin's sense with

$$E_x\Big(\int_0^\zeta f(x_t)dt\Big) = Gf(x)$$

by modifying Ray's theory. [Th. 1.1.]

In § 2 and § 3, we shall apply the result of § 1 to the uniformly elliptic operators of the forms

i)
$$D^s u = \sum_{i \cdot j=1}^{d} \frac{\partial}{\partial x_i}\Big(a_{ij} \frac{\partial u}{\partial x_j}\Big),$$

where $\{a_{ij}\}$ are bounded, measurable and symmetric,

ii).
$$D^* u = \sum_{i \cdot j=1} \frac{\partial^2}{\partial x_i \partial x_j}(a_{ij} \cdot u) - \sum_{i=1}^{d} \frac{\partial}{\partial x_i}(a_i \cdot u),$$

where $\{a_{ij}\}$, $\{a_i\}$ are bounded Hölder continuous, and in addition W. Littman's condition (L) is assumed:

(L)
$$-\int_\Omega Dv(x)dx \geqq 0$$

for every non-negative C^2-function v with compact support in a ball Ω, where D is a formal adjoint operator of D^*. The continuity of the paths of the process connected with D^s will be proved in § 2.

In § 4 if the standard process having a Green function with the property (S) satisfies an additional condition (R):

$$(R) \qquad P_x(\sigma_A < \zeta) = \int_{\bar{A}} G(x, y)\mu_A(dy),$$

we shall prove the Wiener test (Th. 4.1) by the same idea as in Ito-Mckean [7] and in S. Watanabe [18] and using this we can see that given two processes with the Green functions satisfying the condition (S) with the same index α, a point is regular for one if and only if it is so for the other. [Th. 4.2.]

In § 5, by verifying the condition (R), we shall show that a point is regular for the canonical diffusion processes connected with D, its dual processes connected with D^* and minimal diffusion processes connected with D^s, if and only if it is regular for the Brownian motion. This result corresponds to that of R. M. Hervé [4] in the case of the differential operator D and of W. Littman, J. Stampacchia and F. Weinberger [11] in the case of the differential operator D^s.

When the coefficients of D are assumed to be only continuous, the above result does not always hold, as is shown by an example in § 7. In addition we shall show that no such example exists for the 3-dimensional rotation-invariant process connected with D with continuous coefficients.

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§ 1. Construction of a multi-dimensional standard process from a Green function.

Let us first introduce some preliminary notions and notations.

Let Ω denote a domain in the d-dimensional space $R^d (d \geqq 2)$. We shall consider the following space of functions defined on Ω.

C_k is the space of continuous functions with compact support in Ω.

C_0 is the space of continuous functions vanishing at infinity (with respect to the one-point compactification of Ω).

DEFINITION 1.1. A function $G(x, y) : \Omega \times \Omega \to (0, \infty]$ is called a Green function if it satisfies the following four conditions.

(G. 1). *$G(x, x) = \infty$ and $G(x, y)$ is continuous in (x, y) as far as $x \neq y$.*

(G. 2). *$f \in C_k$ implies $Gf(x) \equiv \int G(x, y) f(y) dy \in C_0$.*

(*G.* 3). *$Gf(x), f \in C_k$ separate any two points on Ω.*

(*G.* 4). (*the weak principle of the positive maximum*). *If $m \equiv \sup_{x \in \Omega} Gf(x)$ is strictly positive, m equals $\sup_{x \in S} Gf(x)$, where $S = \overline{\{x ; f(x) > 0\}}$.*

We shall often impose the condition (S) on the singularity of $G(x, y)$ on $x = y$.

THEOREM 1.1. *Given a Green function $G(x, y)$ satisfying the condition (S), we can construct a unique standard Markov process (in Dynkin's sense) $X = (x_t, \zeta, M_t, P_x)$ with*

$$E_x\Big(\int_0^\zeta f(x_t)dt\Big) = Gf(x). \tag{1.1}$$

PROOF. Using a standard method (see D. B. Ray [13], G. Lion [10]) we can construct a family of linear operators $\{G^\lambda\}_{\lambda>0}$ satisfying the following conditions

(1.1. *A*) *G^λ maps C_0 into C_0,*

(1.1. *B*) $\| \lambda G^\lambda \| \leqq 1$,*

(1.1. *C*) $\lambda ; \mu > 0$, $(\mu - \lambda) G^\lambda G^\mu = G^\lambda - G^\mu$, (*resolvent equation*),

(1.1. *D*) $Gf = G^\lambda(\lambda Gf + f) = G^\lambda f + \lambda G G^\lambda f$, $f \in C_K$;

G^λ, $\lambda > 0$ are called resolvent operators. Using the separation assumption (*G.* 3), we can see that for any $f \in C_0$ there exists a bounded measurable function $\hat{f}$ such that

$$\lim_{k \uparrow \infty} kG^k f = \hat{f}. \tag{1.2}$$

Furthermore, in case f belongs to $\overline{G(C_K)} = \overline{\{Gf, f \in C_K\}}$ we have

$$\hat{f} = f. \tag{1.3}$$

Therefore by applying Ray's theory [13] (cf. also H. Kunita-H. Nomoto [8]), we can construct a Markov process which may have branching points. Note that there exist positive measures of total mass $\leqq 1$, $\{\mu(x, dy), x \in \Omega\}$ such that

$$\hat{f} = \lim_{k \uparrow \infty} kG^k f(x) = \int_\Omega f(y) \mu(x, dy), \qquad f \in C_0. \tag{1.4}$$

$\mu(x, E)$ is called the branching measure at x and x is called a branching point if $\mu(x, \{x\}^c) > 0$.

We shall later use the following property of the branching measure.

If A is the set of all branching points,

$$\mu(x, A) = 0 \qquad \text{for every } x. \tag{1.5}$$

Furthermore, if $x \in \Omega - A$, we have $\hat{f}(x) = f(x)$ as was proved by G. Lion [10].

To see that there is no branching point we shall prove,

* $\| \cdot \|$ is the norm of $C_0 : \| f \| = \sup_x | f(x) |$.

PROPOSITION 1.

$$\lim_{k\to\infty} kG^k f(x) = f(x), \qquad x\in\Omega, \qquad f\in C_0. \tag{1.6}$$

PROOF. Now assume that there exists a point $x_0 \in \Omega$ belonging to A. Let $U(x_0)$ be a neighborhood $\{x; |x-x_0|<r\}$ of x_0 and $g(x)$ be a continuous function such that

$$\begin{aligned} g(x)&=1, & x&\in Q,\\ 0\leqq g(x)&\leqq 1, & x&\in Q'-Q,\\ g(x)&=0, & x&\in \Omega-Q', \end{aligned} \tag{1.7}$$

where $Q=\{x;\ |x-x_0|<r'\}$ and $Q'=\{x;\ |x-x_0|<2r'\}(2r'<r)$. Then we can select a sufficiently large compact set K such that

$$\int_{\Omega-K}\int_{\Omega} G(x, y)g(y)dy\mu(x_0, dy) < \frac{1}{3}\int G(x_0, y)g(y)dy, \tag{1.8}$$

for sufficiently small any r'. Indeed, by the condition (S) we have

$$\int_{\Omega} G(x_0, y)g(y)dy \geqq \text{const}\int_{\Omega}|x_0-y|^{-\alpha}g(y)dy \geqq \text{const}\int_{Q}|x_0-y|^{-\alpha}dy$$

and

$$\sup_{x\in Q'}\int G(x, y)g(y)dy \leqq \sup_{x\in Q'}\text{const}\int_{Q'}|x-y|^{-\alpha}dy = \text{const}\int_{Q'}|x_0-y|^{-\alpha}dy.$$

Hence, if we choose a large compact set K such that $\mu(x_0, \Omega-K)$ is sufficienly small, noting that there exists an absolute constant M such that

$$1\leqq \frac{\int_{Q'}|x_0-y|^{-\alpha}dy}{\int_{Q}|x_0-y|^{-\alpha}dy} < M,$$

we have by the weak principle of the positive maximum the left-hand side of

$$(1.8) \quad \leqq \sup_{x\in\Omega-K}\int_{\Omega} G(x, y)g(y)dy\mu(x_0, \Omega-K) \leqq \sup_{x\in Q'}\int_{\Omega} G(x, y)g(y)dy\mu(x_0, \Omega-K)$$

$<$ the right-hand side of (1.8).

Using $(G.\ 1)$ and (S) we can obtain constants $C_1, C_2>0$ depending only on K such that

$$C_1\cdot|x_1-x_2|^{-\alpha} \geqq G(x_1, x_2) > C_2\cdot|x_1-x_2|^{-\alpha},\ x_1, x_2\in K,$$

by change C_1 and C_2 in the condition (S). Furthermore, by (1.5), it holds $\mu(x_0, \{x_0\})=0$, so we can select $U(x_0)$ sufficiently small such that

$$\mu(x_0, U(x_0)) < \frac{C_2}{C_1}\frac{1}{3M}, \tag{1.9}$$

where M is an absolute constant which depends only on the dimension d and will be determined later. Hereafter we shall fix K and $U(x_0)$. By choosing r' sufficiently small, we have

$$\sup_{K-U(x_0)\ni x}\int_\Omega G(x, y)g(y)dy < \frac{1}{3}\int_\Omega G(x_0, y)g(y)dy. \tag{1.10}$$

Indeed it holds

$$\sup_{x\in K-U(x_0)}\int_\Omega G(x, y)g(y)dy < C_1(r-2r')^{-\alpha}|Q'|,$$

$$\int_\Omega G(x_0, y)g(y)dy > C_2(2r')^{-\alpha}|Q|.$$

As r' is sufficiently small, we have $\frac{1}{3}C_2(2r')^{-\alpha}|Q| > C_1(r-2r')^{-\alpha}|Q'|$. So we obtain (1.10). In the following, we shall show that there exists a constant M depending only on the dimension d such that

$$\frac{\sup_{x\in U(x_0)}\int_\Omega G(x, y)g(y)dy}{\int_\Omega G(x_0, y)g(y)dy} > \frac{C_1}{C_2}M. \tag{1.11}$$

Indeed we have

$$\text{the left-hand side of (1.11)} < \frac{C_1 \sup_{x\in U(x_0)}\int_\Omega |x-y|^{-\alpha}g(y)dy}{C_2\int_\Omega |x_0-y|^{-\alpha}g(y)dy}$$

$$< \frac{C_1 \sup_{x\in U(x_0)}\int_{Q'} |x-y|^{-\alpha}dy}{C_2\int_Q |x_0-y|^{-\alpha}dy} = \frac{C_1\int_{Q'}|x_0-y|^{-\alpha}dy}{C_2\int_Q |x_0-y|^{-\alpha}dy} \leqq \frac{C_1}{C_2}M.$$

From (1.9), (1.11), (1.10) and (1.8), we obtain

$$\int_\Omega\int_\Omega G(x, y)g(y)dy\mu(x_0, dx) = \int_{U(x_0)}\int_\Omega G(x, y)g(y)dy\mu(x_0, dx)$$

$$+\int_{K-U(x_0)}\int_\Omega G(x, y)g(y)dy\mu(x_0, dx) + \int_{\Omega-K}\int_\Omega G(x, y)g(y)dy\mu(x_0, dx)$$

$$< \frac{C_1}{C_2}M\int_\Omega G(x_0, y)g(y)dy \cdot \frac{C_2}{C_1}\frac{1}{3M} + \frac{1}{3}\int_\Omega G(x_0, y)g(y)dy$$

$$+\frac{1}{3}\int_\Omega G(x_0, y)g(y)dy = \int_\Omega G(x_0, y)g(y)dy$$

in contradiction with (1.3). Hence we have $A=\phi$.

To see that the process obtained above is a standard process we need only prove

PROPOSITION 2. *If we set*

$$G(C_0)=\{G^{\lambda}f\,;\,f\in C_0,\ \lambda>0\}\,,$$

$G(C_0)$ is dense in C_0 with respect to the uniform norm.

PROOF. From the results of D. B. Ray [**13**] (cf. G. Lion [**10**]), when f belongs to the following function class;

$$E_{\lambda}=\{f\in C_0,\ \text{non-negative},\ {}^{\forall}k\geqq 0,\ kG^{k+\lambda}f\leqq f\}\,,$$

kG^kf increases to f monotonically as $k\uparrow\infty$. By (1.1. A) $kG^kf\in C_0$ and by Proposition 1 $\hat{f}=f\in C_0$, and so by the Dini's theorem, we have $\lim_{k\uparrow\infty}kG^kf(x)=f(x)$, uniformly in x. Therefore, for any $f\in\tilde{E}=\{f\in C_0, f=f_1-f_2, f_i\in\bigcup_{\lambda>0}E\}$, we see that the convergence is uniform. To complete the proof of our proposition, we have only to note that $\tilde{E}$ is dense in C_0, which is shown in [**10**].

By the above results we can apply the Hille-Yosida theorem to construct a semi-group $\{T_t\}_{t\geqq 0}$ which is strong continuous and sub-Markov on C_0 such that

$$G^{\lambda}f=\int_0^{\infty}e^{-\lambda t}T_tf\,dt\,.$$

The transition probability $P(t, x, \Gamma)$ corresponding to this semi-group is continuous in the sense that

$$\lim_{t\downarrow 0}P(t, x, U)=1,\ U \text{ open set}\ni x$$

by Dynkin [**2**], lemma 2.10. Following Dynkin [**2**], Th. 3.7, we can construct a bounded Markov process whose almost all paths are right continuous and have left limits. Furthermore, by Dynkin [**2**], Th. 3.10, it is strong Markov, so that by Dynkin [**2**] Th. 3.13, we find that it has quasi-left-continuity. Thus the process obtained above is a standard process.

REMARK. Under the condition (S), (G. 3) is satisfied necessarily. For any two points x_0, y_0 such that $|x_0-y_0|=r$, let Q, Q' be sufficiently small balls

$$Q=\{y\,;\,x_0-y|<r'\}\quad\text{and}\quad Q'=\{y\,;\,|x_0-y|<2r'\}(2r'<r)\,.$$

Then we can construct a potential $Gg(x)=\int_{\Omega}G(x, y)g(y)dy$ which separates x_0 and y_0 by choosing an adequate function $g(x)$ having the form (1.7).

§ 2. A diffusion process connected with the self-adjoint elliptic operator of second order.

In this section we shall consider the following differential operator in the d-dimensional space $R^d(d\geqq 3)$

$$D^s u = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right),$$

where $\{a_{ij}\}$ are symmetric with respect to $i \cdot j$, bounded and measurable, D^s is assumed to be uniformly elliptic. For this operator on a ball Ω, W. Littman, G. Stampacchia and F. Weinberger [12] have shown that there exists a Green function $G(x, y)$ having the condition (S) with $\alpha = d-2$, which is a weak solution of $-D^s G = \delta_y$ in the sense of [12]. (G. 1)~(G. 3) are proved in [12], P. 64~P. 67. (G. 4) is proved as follows.

For any $f \in C_K(\Omega)$, $Gf(x)$ is a solution of $-D^s Gf(x) = f(x)$, so we have by the definition

$$\sum \int_{\Omega} a_{ij} \frac{\partial}{\partial x_i} Gf(x) \cdot \phi_{x_j} dx = \int_{\Omega} f \cdot \phi dx,$$

where $\phi \in H_0^{1,2}(\Omega)$. Let S be $S = \overline{\{x; f(x) > 0\}}$ and ϕ be a non-negative function with compact support in $\Omega - S$ belonging to $C^\infty(\Omega)$. Then we have

$$\int_{\Omega} \sum_{i,j=1}^{d} a_{ij} \frac{\partial}{\partial x_i} Gf(x) \cdot \phi_{x_j} dx = \int_{\Omega} f(x) \cdot \phi(x) dx = \int_{\Omega - S} f(x) \cdot \phi(x) dx \leqq 0.$$

Hence Gf is a D^s-subsolution in $\Omega - S$, and so we can apply G. Stampacchia's maximum theorem [16] to Gf-m where $m = \sup_{x \in S} Gf(x)$, which is clearly non positive on $\partial(\Omega - S)$. Then $Gf \leqq m$ on $\Omega - S$, and so

$$m \geqq Gf(x), \qquad {}^\forall x \in \Omega.$$

From this Green function we can construct a standard process by Theorem 1.3. We call this process the minimal process associated with D^s and is denoted by X^s.

We are going to prove the continuity of the sample paths of this process. Let us first observe the following fact for a standard process in general.

LEMMA 2.1. *If for an arbitrary ball $Q \subset \Omega$ and a point $x_0 \in \Omega - \bar{Q}$, there exist functions f_1, f_2 with compact supports in $\Omega - \bar{Q}$, measurable, such that Gf_1, Gf_2 are bounded measurable and*

i) $Gf_1(x) \geqq Gf_2(x)$ *for* $x \in \Omega$

ii) $Gf_1(x) = Gf_2(x)$ *for* $x \in Q$

iii) $Gf_1(x) > Gf_2(x)$ *for some neighborhood* $U(x_0)$ *of* x_0,

then the harmonic measure concentrates on the boundary of Q, that is,

$$P_x(x_{\tau_Q} \in \Omega - \bar{Q}) = 0,$$

where $\tau_Q = \inf(t \geqq 0,\ x_t \notin Q)$.

PROOF. By Dynkin's formula we have

$$E_x Gf_i(x_{\tau_Q}) = Gf_i(x) \quad \text{for } x \in Q,\ i = 1, 2. \tag{2.1}$$

Now we suppose that the harmonic measure $P_x(x_{\tau_Q} \in dy)$ has strictly positive mass on a neighborhood $U(x_0)$ of x_0. Then we have

$$E_x Gf_1(x_{\tau_Q}) > E_x Gf_2(x_{\tau_Q}).$$

This contradicts (2.1).

THEOREM 2.1. *There exists a continuous standard process $X=(x_t, \zeta, M_t, P_x)$ on Ω whose generator is D^s.*

PROOF. We have only to show the continuity of the sample paths. For any ball $Q \Subset \Omega$ and any point $x_0 \in \Omega - \bar{Q}$, let us consider the following function $g_a(x)$ (a; positive constant) which is used in [12] for other purpose,

$$g_a(x) = \begin{cases} \dfrac{1}{2a} \sum_{i,j=1}^{d} a_{ij} \dfrac{\partial}{\partial x_i} G_{x_0}(x) \cdot \dfrac{\partial}{\partial x_j} G_{x_0}(x), & a \leqq G_{x_0}(x) \leqq 3a, \\ 0, & \text{otherwise,} \end{cases}$$

where $G_{x_0}(x) = G(x, x_0)$. Then we have

$$\int_\Omega G(x, y) g_a(y) dy = \begin{cases} G_{x_0}(x), & G_{x_0}(x) \leqq a, \\ G_{x_0}(x) - \dfrac{1}{4a}(G_{x_0}(x) - a)^2, & a \leqq G_{x_0}(x) \leqq 3a, \\ 2a, & G_{x_0}(x) \geqq 3a. \end{cases}$$

If we fix a sufficiently large compact subset K of Ω in the condition (S), there exists a constant $C > 0$ such that

$$Cr^{2-d} > G(x, x_0) \qquad \text{for any } x \in \bar{Q},$$

where r denotes the distance between x_0 and Q. Hence if we select a constant a such that $a \geqq Cr^{2-d}$, we have $g_a(x) = 0$ in Q. Let us set $f_1(x) = g_{3a}(x)$ and $f_2(x) = g_a(x)$, then f_1 and f_2 satisfies the conditions i), ii), iii) in lemma 2.1, so we have

$$P_x(x_{\tau_Q} \in \Omega - \bar{Q}) = 0$$

for each ball Q. This means that almost all sample paths are continuous from Courrege and Priouret [1] and R. Kondo [unpublished].

§ 3. The dual process of the canonical diffusion process.

Let us consider the following differential operator D^* in R^d

$$D^* u = \sum_{i,j}^{d} \frac{\partial}{\partial x_i \partial x_j}(a_{ij} \cdot u) - \sum_{i=1}^{d} \frac{\partial}{\partial x_i}(a_i \cdot u)$$

where $\{a_{ij}\}$, $\{a_j\}$ are Hölder continuous and bounded and $\{a_{ij}\}$ is strictly positive definite. D^* is the formal adjoint operator of the strictly elliptic operator D

$$Du = \sum_{i,j}^{d} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} u + \sum_{i=1}^{d} a_i \frac{\partial u}{\partial x_i} .$$

The Markov process whose generator is D is called a (minimal) *canonical diffusion process* X [2]. Hereafter we shall assume W. Littman's condition

$$(L) \qquad -\int_{\Omega} Dv(x)dx \geqq 0$$

for every non-negative C^2-function v with compact support in Ω.

Let us set

$$G^*(x, y) = G^{\Omega}(y, x) ,$$

where $G^{\Omega}(x, y)$ is the Green function in a ball Ω of D.

THEOREM 3.1. *There exists a standard Markov process X^* in a ball in the sense of theorem* 1.1 *with respect to* $G^*(x, y)$.

PROOF. It is sufficient to prove that $G^*(x, y)$ is the Green function with the condition (S). The property $(G. 1)$ is obvious, because it is true for $G(y, x)$. The property $(G. 4)$ is proved by using the following W. Littman's theorem [11], theorem B. p. 210. Let Q be a smooth domain in Ω.

W. Littman's theorem: if under the condition (L)

$$(3.1) \qquad \int_{\Omega} u(x)Dv(x)dx \geqq 0$$

holds for all non-negative v in $C^2(Q)$ with compact support in Q where u is locally integrable in Q, and if, in addition, for some compact subdomain $Q' \subset Q$ we have

$$0 \leqq M \equiv \operatorname*{ess\,sup}_{x \in Q} u(x) = \operatorname*{ess\,sup}_{x \in Q'} u(x) ,$$

then $u = M$ almost everywhere in Q.

Indeed, let $u(x)$ be $\int G^*(x, y)f(y)dy$, where $f \in C_K$, and let S be $\overline{\{x; f(x) > 0\}}$. Then for any smooth domain $Q \subset \Omega - S$ and any non-negative C^2-function v with compact support in Q we have

$$\int_Q \int_{\Omega} G^*(x, y)f(y)dyDv(x)dx = \int_{\Omega} f(y)dy \int_{\Omega} G^{\Omega}(y, x)Dv(x)dx$$

$$= -\int_{\Omega} f(y)v(y)dy = -\int_Q f(y)v(y)dy \geqq 0 .$$

Hence, if we set $m = \sup_{x \in S} u(x)$, then $u(x) - m \vee o$ satisfies (3.1) by the condition (L). Suppose $A = \sup_{x \in \Omega} u(x) > m \vee o$, then this supremum is attained at some point z in $\Omega - S$ because $u \in C_0(\Omega)$. ($u \in C_0(\Omega)$ follows from $(G. 2)$ which is proved later without using $(G. 4)$.) Let z be such a point. Then for any smooth domain Q inside $\Omega - S$ and containing z the hypothesis of Littman's theorem is satisfied, and hence $u=A$ inside Q, that is, $u=A$ on the part of ∂S. But this contradicts $u\in C_0(\Omega)$.

To verify the condition (S), it suffices to prove it for the Green function $G(x, y)$ in R^d of the operator D. We can show this by using a theorem in D. Gilbarg and J. Serrin [3], which is an extension of the so-called maximum principle, but here we shall prove it, using the estimate of the fundamental solution $p(t, x, y)$ in R^d of $Dp=\frac{\partial p}{\partial t}$ with $\lim_{|x|\to\infty} p(t, x, y)=0$:

$$G(x, y)=\int_0^\infty p(t, x, y)dt,\ x, y\in R^d,$$

$$p(t, x, y)\leqq Mt^{-d/2}e^{-\frac{\alpha(y-x)^2}{t}},$$

$$p(t, x, y)\geqq M_1t^{-d/2}e^{-\frac{\alpha_1|y-x|^2}{t}}-M_2t^{-\frac{d}{2}+\lambda}e^{-\frac{\alpha_2|y-x|^2}{t}},$$

where $M, \alpha, M_1, M_2, \alpha_1, \alpha_2, \lambda$ are positive constants [6]. The proof is as follows. We define $p_1(t, x, y)$, $p_2(t, x, y)$ by $p_1(t, x, y)=M_1t^{-d/2}e^{-\alpha_1|y-x|^2/t}$, $p_2(t, x, y)=M_2t^{(-d/2)+\lambda}e^{-\alpha_2|y-x|^2/t}$, and choose constants δ, r_1, C_2' such that

$$\delta=\left(\frac{1}{4}\frac{\alpha_2}{\alpha_1}\frac{M_1}{M_2}\right)^{1/\lambda},$$

(3.2)

$$r_1=\left(\frac{\delta^{(d/2)-1}}{2\left(\frac{2}{d-2}\right)\alpha/\Gamma\left(\frac{d}{2}\right)}\right)^{1/d-2}$$

$$C_2'=\frac{1}{4}\frac{M_1}{\alpha_1}\Gamma(d/2).$$

Then, from the following estimate,

$$\int_0^\infty p_1(t, x, y)dt=M_1\frac{\Gamma(d/2)}{\alpha_1}\frac{1}{|x-y|^{d-2}}$$

$$\int_\delta^\infty p_1(t, x, y)dt<M_1\int_\delta^\infty t^{-d/2}dt=M_1\left(\frac{2}{d-2}\right)\delta^{-d/2+1}$$

we have

$$\int_0^\delta p_1(t, x, y)dt>M_1\frac{\Gamma(d/2)}{\alpha}\frac{1}{|x-y|^{d-2}}-M_1\left(\frac{2}{d-2}\right)\delta^{-\frac{d}{2}+1}$$

and from (3.2) we have

$$\int_0^\delta p_1(t, x, y)dt\geqq\frac{1}{2}M_1\frac{\Gamma(d/2)}{\alpha_1}\frac{1}{|x-y|^{d-2}},\qquad \text{for } |x-y|<r_1.$$

Hence, noting $p(t, x, y)\geqq p_1(t, x, y)-p_2(t, x, y)$, we have for $|x-y|<r_1$

$$\int_0^\infty p(t, x, y)dt \geqq \int_0^\delta p(t, x, y)dt$$

$$\geqq \frac{1}{2} M_1 \frac{\Gamma(d/2)}{\alpha_1} \frac{1}{|x-y|^{d-2}} - \delta^\lambda \int_0^\infty t^{-d/2} e^{-\frac{\alpha_2|y-x|^2}{t}} dt \cdot M_2$$

$$= \frac{1}{2} M_1 \frac{\Gamma(d/2)}{\alpha_1} \frac{1}{|x-y|^{d-2}} - \delta^\lambda M_2 \frac{\Gamma(d/2)}{\alpha_2} \frac{1}{|x-y|^{d-2}}$$

$$= \frac{1}{4} M_1 \frac{\Gamma(d/2)}{\alpha} \frac{1}{|x-y|^{d-2}} = C_2' \frac{1}{|x-y|^{d-2}} .$$

It is obvious that $C_1 \frac{1}{|x-y|^{d-2}} \geqq G(x, y)$, $C_1 > 0$. Therefore the condition (S) is satisfied for $\alpha = d-2$ ($d \geqq 3$).

To prove the property (G. 2) we have only to show

$$\lim_{x \to a} G^*(x, y) = \lim_{x \to a} G(y, x) = 0, \qquad a \in \partial\Omega .$$

In the following we shall use the notion "(super) harmonic (X) in G" for brevity, which means "(super) harmonic in an open set G with respect to a Markov process X" according to Dynkin's book [2]. Noting that $G^\Omega(x, y) = G(x, y) - E_x G(x_{\tau_\Omega}, y)$, we have only to show for $x \in \Omega$

$$\text{(3.3)} \qquad \lim_{y_m \to y} E_x G(x_{\tau_\Omega}, y_m) = G(x, y),\ y_m \in \Omega,\ y \in \partial\Omega .$$

First, by Fatou's lemma we have

$$\text{(3.4)} \qquad \varliminf_{y_m \to y} E_x G(x_{\tau_\Omega}, y_m) \geqq E_x \varliminf_{y_m \to y} G(x_{\tau_\Omega}, y_m)$$

$$= E_x G(x_{\tau_\Omega}, y),\ y_m \in \Omega,\ y \in \partial\Omega,\ x \in \Omega .$$

On the other hand, as $G(x, y)$ is superharmonic (X) in x, we have

$$\text{(3.5)} \qquad \varlimsup_{y_m \to y} E_x G(x_{\tau_\Omega}, y_m) \leqq \varlimsup_{y_m \to y} G(x, y_m) = G(x, y),\ x \in \Omega .$$

Therefore, if we can prove $E_x G(x_{\tau_\Omega}, y) = G(x, y)$, we obtain (3.3) from (3.4) and (3.5). Let $y \in \partial\Omega$ and let 0 be a center of Ω. If we choose a sequence $\{y_n\}$ on the half line $\overrightarrow{0Y} \cap \bar{\Omega}^c$ which converges to y as m tends to infinity, we have

$$G(u, y_n) \leqq \frac{C_1}{|u-y_n|^{d-2}} \leqq \frac{C_1}{|u-y|^{d-2}} \leqq \frac{C_1}{C_2} G(u, y)$$

for all $u \in \partial\Omega$ by the property (S) of $G(x, y)$, and $G(u, y_n)$ converges $G(u, y)$ as n tends to infinity. Hence, noting $E_x(G(x_{\tau_\Omega}, y) < \infty$ by (3.4) and (3.5), we have by Lebesgue's convergence theorem

$$\text{(3.6)} \qquad \lim_{n \to \infty} E_x G(x_{\tau_\Omega}, y_n) = E_x G(x_{\tau_\Omega}, y) .$$

Noting that $E_x G(x_{\tau_\Omega}, y_n) = G(x, y_n)$, $x \in \Omega$, because $G(x, y)$ is harmonic (X) in

$R^d-\{y\}$, we get $E_xG(x_{\tau_\Omega}, y)=G(x, y)$. Thus (3.3) was proved. The property (*G.* 3) follows from the remark of § 1.

REMARK. Let A^* be the strong infinitesimal operator of X^*. Then a function $u(x)\in C_0(\Omega)$ such that

$$A^*u(x)=-f(x) \text{ in } \Omega \text{ for } f\in C_0(\Omega)$$

is a weak solution of $D^*u(x)=f(x)$ in W. Littman's sense, that is: $u(x)$ is locally integrable in Ω and it satisfies

$$\int_\Omega u(x)Dv(x)dx=-\int_\Omega f(x)v(x)dx$$

for all v in $C^2(\Omega)$ with compact support in Ω.

§ 4. Wiener test and regular points.

Throughout this section, we shall assume that we are given a Green function $G(x, y)$ which satisfies the condition (S) and the standard process $X=(x_t, \zeta, M_t, P_x)$ corresponding to G by Theorem 1.1. In addition we shall assume the following condition (R).

(R). *If A is an analytic set with compact closure, there exists a finite measure μ_A concentrating on $\bar{A}$ such that*

$$P_x(\sigma_A<\zeta)=\int_{\bar{A}} G(x, y)\mu_A(dy),$$

where $\sigma_A=\inf(t>0,\ x_t\in A)=\zeta$ *if* $x_t\notin A$ *for every* $t>0$.
The condition (R) corresponds to the so-called Riesz's representation theorem. We shall discuss the validity of (R) in § 5. A point x is said to be a regular point of an analytic set B for the process X, if it holds

$$P_x(\sigma_B=0)=1$$

for the probability law P_x of the path of the process X starting at x.

Our aim of this section is to prove the following results:

THEOREM 4.1. *The Wiener test which determines whether a point is regular or not holds for the above standard process $X=(x_t, \zeta, M_t, P_x)$, that is: let B be an analytic set and let x be its boundary point and set*

$$B_k=\left\{y;\ \frac{1}{2^k}<|y-x|\leqq\frac{1}{2^{k-1}}\right\}\cap B.$$

Then, x is a regular point of B for the process X, if and only if

$$\sum_{k=1}^{\infty} 2^{k\alpha}C(B_k)=\infty, \tag{4.1}$$

where $C(B_k)=\mu_{B_k}(\bar{B})$ *(capacity of B_k).*

THEOREM 4.2. *Let X_1^α and X_2^α be two standard processes corresponding to the Green functions G_1 and G_2 which satisfy the condition (S) for the same α and assume the condition (R). Then a point $x \in \Omega$ is a regular point of an analytic set $B \subset \Omega$ for the process X_1^α, if and only if it is a regular point of B for the process X_2^α.*

To prove Theorem 4.1 we shall first prepare several lemmas.

LEMMA 4.1. *Let 0_n be a sequence of balls with the common center z such that $0_n \downarrow z$ as $n \uparrow \infty$. Then*

$$\lim_{n\to\infty} \sup_{x\in\Omega-0_1} P_x(\sigma_{0_n} < \zeta) = 0 . \tag{4.2}$$

PROOF. We fix a compact set $K \subset \Omega$ which contains every 0_n. Then by the conditions (S) and (R), we have

$$C_2 r_n^{-\alpha} \mu_{0n}(\bar{0}_n) \leqq \int_{\bar{0}_n} G(z, y) \mu_{0n}(dy) = P_z(\sigma_{0n} < \zeta) \leqq 1 ,$$

where r_n is the radius of $\bar{0}_n$. Hence we have $\mu_{0n}(0_n) \downarrow 0$ as $n \uparrow \infty$. On the otherhand, it holds

$$\sup_{x\in\Omega-0_1} P_x(\sigma_{0n} < \zeta) \leqq C_1 |r_1 - r_n|^{-\alpha} \mu_{0n}(\bar{0}_n) + a\mu_{0n}(\bar{0}_n) ,$$

where a is a constant such that $\sup_{\substack{x\in\Omega-K \\ y\in\bar{0}_1}} G(x, y) = a$. Hence we have

$$\sup_{x\in\Omega-0_1} P_x(\sigma_{0n} < \zeta) < \frac{C_1 |r_1 - r_n|^{-\alpha}}{C_2 r_n^{-\alpha}} + a\mu_{0n}(\bar{0}_n) \to 0 \text{ as } n \to \infty .$$

REMARK 1. We see easily that a point x is not a regular point of $\{x\}$ for X.

LEMMA 4.2. *A point x is a regular point of B for X, if and only if*

$$P_x(\overline{\lim_{k\uparrow\infty}} B_k^*) > 0 , \tag{4.3}$$

where $B_k^* = \{\sigma_{B_k} < \zeta\}$.

PROOF. i) Suppose that x is not a regular point of B for X and that (4.3) holds. Noting that

$$P_x\Big\{ \bigcup_{n=1}^{\infty} (0 < {}^\forall t < \sigma_{0_n^c}, \, x_t \notin B) \Big\} = P_x(\sigma_B > 0)^{*)} ,$$

where $0_n = \Big\{ y : |y-x| < \frac{1}{2^n} \Big\} \cap \Omega$, we see that for any given $\varepsilon > 0$ there exists a number n_0 such that

$$P_x(0 < {}^\forall t < \sigma_{0_{n_0}^c}, \, x_t \notin B) \geqq 1 - \varepsilon^{**)} . \tag{4.4}$$

*) Notice that $P_x(\sigma_{0_n^c} \downarrow 0) = 1$ and see remark 1.

**) $P_x(\sigma_B > 0) = 1$ if x is not a regular point of B by Blumenthal's $0-1$ law.

On the other hand, it holds that

$$P_x(\sigma_{G_n}<\zeta)=P_x(0<{}^\forall t<\sigma_{0^c_{n0}},\ x_t\notin B,\ \sigma_{G_n}<\zeta)+P_x(0<{}^\exists t<\sigma_{0^c_{n0}},\ x_t\in B,\ \sigma_G<\zeta)$$

$$=E_x(P_{x_{\sigma_{0^c_n}}}(\sigma_{G_n}<\zeta),\ 0<{}^\forall t<\sigma_{0^c_{n0}},\ x_t\notin B)+P_x(0<{}^\exists t<\sigma_{0^c_{n0}},\ x_t\in B,\ \sigma_{G_n}<\zeta)$$

for each $n>n_0$ where $G_n=\left\{y;\ |y-x|\leqq\frac{1}{2^n}\right\}\cap B$. Hence, by (4.4) and lemma 4.1, we have

$$(4.5)\qquad P_x(\sigma_{G_n}<\zeta)=\sup_{z\in\Omega-0_{n0}}P_z(\sigma_{G_n}<\zeta)+\varepsilon\leqq 2\varepsilon$$

for sufficiently large n. As ε is arbitrary, (4.5) contradicts (4.3). Hence x is a regular point of B for X, if (4.3) holds.

ii) Suppose that

$$P_x(\overline{\lim_{n\to\infty}}B_n^*)=0\,.$$

Then we can easily show that x is not a regular point of B for X.

The following Lamperti's lemma [**9**] is used to prove the next lemma.

Let the sequence of events $\{E_k,\ k=1,2,\cdots\}$ *satisfy the following conditions,*

$$\text{i)}\qquad \sum_{k=1}^{\infty}P_x(E_k)=\infty\,,$$

ii) *there exist positive constants* N *and* C *such that* $P_x(E_n\cap E_m)\leqq CP_x(E_n)P_x(E_m)$ *for all* $n>m>N$. *Then* $P_x(\overline{\lim_{k\to\infty}}E_k)>0$.

LEMMA 4.3. *A point x is a regular point of B for X, if and only if*

$$(4.7)\qquad \sum_{k=1}^{\infty}P_x(B_k^*)=\infty\,.$$

PROOF. We first notice that $\phi(r)=r^{-\alpha}$ posseses the following properties;

α) *$\phi(r)$ is continuous except at* $r=0$.

β) $\phi(r)\uparrow\infty$ *as* $r\downarrow 0$.

γ) *There exists a positive constant M independent of r such that*

$$(4.8)\qquad \frac{\phi\left(\frac{r}{2}\right)}{\phi(r)}\leqq M\,.$$

i) If $\sum_{k=1}^{\infty}P_x(B_k^*)<\infty$, we have

$$P_x(\overline{\lim_{k\uparrow\infty}}B_k^*)=0$$

by Borel-Cantelli lemma. Hence x is not a regular point of B.

ii) If $\sum_{k=1}^{\infty}P_x(B_k^*)=\infty$, either $\sum_{k=1}^{\infty}P_x(B_{2k}^*)$ or $\sum_{k=1}^{\infty}P_x(B_{2k+1}^*)$ diverges. We suppose that the former diverges. When $k>j$, we have

$$P_x(B_{2k}^*\cap B_{2j}^*)=P_x(\sigma_{B_{2k}}<\zeta,\ \sigma_{B_{2j}}<\zeta)$$
$$=P_x(\sigma_{B_{2k}}<\sigma_{B_{2j}}<\zeta)+P_x(\sigma_{B_{2j}}<\sigma_{B_{2k}}<\zeta)$$
$$=E_x(P_{x_{\sigma_{B_{2k}}}}(\sigma_{B_{2j}}<\zeta),\ \sigma_{B_{2k}}<\sigma_{B_{2j}},\ \sigma_{B_{2k}}<\zeta)$$
$$+E_x(P_{x_{\sigma_{B_{2j}}}}(\sigma_{B_{2k}}<\zeta),\ \sigma_{B_{2j}}<\sigma_{B_{2k}},\ \sigma_{B_{2j}}<\zeta)$$
$$\leqq E_x(P_{x_{\sigma_{B_{2k}}}}(\sigma_{B_{2j}}<\zeta),\ \sigma_{B_{2k}}<\zeta)+E_x(P_{x_{\sigma_{B_{2j}}}}(\sigma_{B_{2k}}<\zeta),\ \sigma_{B_{2j}}<\zeta)\,.$$

Noting that the distance between B_{2j} and B_{2k} exceeds $\frac{1}{2^{2j+1}}$, $\frac{1}{2^{2k-1}}$, we get by the condition (R)

$$P_y(B_{2k}^*)=\int_{\bar{B}_{2k}}G(y,z)\mu_{B_{2k}}(dz) \tag{4.9}$$
$$\leqq C_1\int_{\bar{B}_{2k}}\phi(|y-z|)\mu_{B_{2k}}(dz)\leqq C_1\phi\Big(\frac{1}{2^{2j+1}}\Big)C(B_{2k})$$

for each $y\in\bar{B}_{2j}$. Similarly we have

$$P_y(B_{2j}^*)\leqq C_1\phi\Big(\frac{1}{2^{2j+1}}\Big)C(B_{2j}) \tag{4.10}$$

for each $y\in\bar{B}_{2k}$. On the other hand it holds

$$P_x(B_{2k}^*)>C_2\phi\Big(\frac{1}{2^{2k-1}}\Big)C(B_{2k})\geqq C_2\phi\Big(\frac{1}{2^{2j-1}}\Big)C(B_{2k}) \tag{4.11}$$

and similarly

$$P_x(B_{2j}^*)>C_2\phi\Big(\frac{1}{2^{2j-1}}\Big)C(B_{2j})\,. \tag{4.12}$$

Therefore, we have by (4.9) and (4.11)

$$P_y(B_{2k}^*)<\frac{C_1}{C_2}\frac{\phi\Big(\frac{1}{2^{2j-1}}\Big)}{\phi\Big(\frac{1}{2^{2j-1}}\Big)}P_x(B_{2k}^*) \tag{4.13}$$

for each $y\in\bar{B}_{2j}$. Similarly we have by (4.10) and (4.12)

$$P_y(B_{2j}^*)<\frac{C_1}{C_2}\frac{\phi\Big(\frac{1}{2^{2j+1}}\Big)}{\phi\Big(\frac{1}{2^{2j-1}}\Big)}P_x(B_{2j}^*),\ \text{for each } y\in\bar{B}_{2k}\,.$$

Hence by (4.9) and (4.1) we obtain

$$P_x(B_{2j}^*\cap B_{2k}^*)\leqq 2\frac{C_1}{C_2}M^2P_x(B_{2j}^*)P_x(B_{2k}^*)\,.$$

$\{B_{2k}^*\}$ satisfies ii) of Lamperti's lemma and so

$$P_x(\overline{\lim_{k\uparrow\infty}} B_k^*) > 0.$$

By lemma 4.2, we see that x is a regular point of B for X.

PROOF OF THEOREM 4.1. By using the computation in Lemma 2.3, we get

$$\frac{C_2}{M}\phi\left(\frac{1}{2^k}\right)C(B_k) \leqq P_x(B_k^*) \leqq C_1\phi\left(\frac{1}{2^k}\right)C(B_k).$$

Hence (4.1) is equivalent to (4.7). This proves the Theorem.

In the sequel we shall prove Theorem 4.2. Let C_i $(i=1, 2)$ be the capacity of X_i^α $(i=1, 2)$.

LEMMA 4.4. *Let A be an open set with compact closure, and K be a compact set containing $\bar{A}$ in the condition* (S). *Then there exist positive constants depending only on K such that*

$$k_2C_1(S) < C_2(A) < k_1C_1(G)$$

for an open set $G \supset \bar{A}$ and a compact set $S \subset A$.

PROOF. If we set

$$L^* = \{\text{measure } \mu, \int G_1(x, y)\mu(dy) \leqq 1$$

$$\text{on } K, \text{ support of } \mu \subset \bar{A}\}, \text{ then we have}$$

$$\mu(\bar{A}) = \int_{\bar{A}} P_x^1(\sigma_G < \zeta)\mu(dx)$$

for every $\mu \in L^*$ and an open set G such that $K \supset G \supset \bar{A}$. Noting the conditions (R) and (S), we obtain

$$(4.14) \qquad \mu(\bar{A}) \leqq \int_{\bar{A}}\int_{\bar{G}} G_1(x, y)\mu_G^1(dy)\mu(dx)$$

$$\leqq \frac{C_1}{C_2}\int_{\bar{G}}\int_{\bar{A}} G_1(y, x)\mu(dx)\mu_G^1(dy)$$

$$\leqq \int_{\bar{G}} \frac{C_1}{C_2}\mu_G^1(dy) \leqq \frac{C_1}{C_2}C_1(G).$$

On the other hand, we can show that there exist constants $k_1', k_2' > 0$ such that

$$(4.15) \qquad 1/k_1'G_1(x, y) < G_2(x, y) < 1/k_2''G_1(x, y), \; x, y \in K$$

from the condition (S). Therefore, $1/k_1'\mu_A^2(dy)$ belongs to L^* and so it holds $C_2(A) < k_1'\frac{C_1}{C_2}C_1(G)$ by (4.14). Hence we have

$$(4.16) \qquad C_2(A) \leqq k_1C_1(G)$$

for any open set $G \supset \bar{A}$. If we set

$$L^{**}=\left\{\text{measure } \mu;\ \int G_1(x,y)\mu(dy)\geqq 1 \text{ inside } A, \text{ support of } \mu\subseteqq \bar{A}\right\},$$

then we have for every $\mu\in L^{**}$ and a compact set $S\subset A$

$$\mu(\bar{A})\geqq\int_{\bar{A}}P_x^1(\sigma_s<\zeta)\mu(dx),$$

and by the same reason as in (4.14), we get

$$\mu(\bar{A})\geqq\int_{\bar{A}}\int_S G_1(x,y)\mu_S^1(dy)\mu(dx)$$

$$\geqq\frac{C_2}{C_1}\int_S\mu_S^1(dy)>\frac{C_2}{C_1}C_1(S).$$

As $1/k_2'\mu_A^2(dy)$ belongs to L^{**} by (4.15), we see

$$C_2(A)\geqq k_2C_1(S) \tag{4.17}$$

where $k_2=\frac{C_2}{C_1}k_2'$. The conclusion follows from (4.16) and (4.17).

PROOF OF THEOREM 4.2. As we may assume that X_i^α $(i=1,2)$ is a standard process in a bounded domain Ω, $E_x(\zeta)=\int_\Omega G(x,y)dy$ is finite, and so we have $P_x(\zeta<\infty)=1$, (remark $\zeta\leqq\tau_\Omega$). Hence we can take open sets G_k, $\hat{G}_k$, $\hat{\hat{G}}_k$ for each B_k such that $\bar{B}_k\subset G_k$, $\bar{G}_k\subset\hat{G}_k$, $\bar{\hat{G}}_k\subset\hat{\hat{G}}_k$

$$P_x^1(\sigma_{B_k}<\zeta)+\frac{1}{2^k}>P_x^1(\sigma_{\hat{G}_k}<\zeta)>P_x^1(\sigma_{B_k}<\zeta), \tag{4.19}$$

$$P_x^2(\sigma_{B_k}<\zeta)+\frac{1}{2^k}>P_x^2(\sigma_{\hat{G}_k}<\zeta)>P_x^2(\sigma_{B_k}<\zeta),$$

and, if we denote the distance between Q and R by $|Q,R|$,

$$\phi(|x,\hat{\hat{G}}_k|)<2\phi\left(\frac{1}{2^k}\right),$$

$$\phi(\sup_{y\in\hat{\hat{G}}_k}|x-y|)>\frac{1}{2}\phi\left(\frac{1}{2^{k-1}}\right). \tag{4.20}$$

For each G_k which satisfies (4.19), $\sum_{k=1}^\infty P_x^i(B_k^*)$ diverges, if and only if $\sum_{k=1}^\infty P_x^i(\sigma_{G_k}<\zeta)$ $(i=1,2)$ diverges. Furthermore, from (4.20) we can see that $\sum_k^\infty\phi\left(\frac{1}{2^k}\right)C_i(G_k)$ diverges, if and only $\sum_k^\infty P_x^i(\sigma_{G_k}<\zeta)$ diverges. As it follows from lemma 4.4 that

$$\sum_k^\infty\phi(1/2^k)C_1(G_k)=\infty\Rightarrow\sum_k^\infty\phi(1/2^k)C_2(\hat{G}_k)=\infty\Rightarrow\sum_k^\infty\phi(1/2^k)C_1(\hat{\hat{G}}_k)=\infty,$$

we obtain

$$(4.21)\qquad \sum_k^{\infty} P_x^1(B_k^*)=\infty \Rightarrow \sum_k^{\infty} P_x^2(\sigma_{\hat{G}_k}<\zeta)=\infty \Rightarrow \sum_k^{\infty} P_x^1(\sigma_{\hat{G}_k}<\zeta)=\infty\,.$$

Hence by (4.19) and (4.21) we have

$$\sum_k^{\infty} P_x^1(B_k^*)=\infty \Leftrightarrow \sum_k^{\infty} P_x^2(B_k^*)=\infty\,.$$

This means that x is a regular point of B for X_1^{α}, if and only if x is a regular point of B for X_2^{α}.

§ 5. Regular points for the multi-dimensional standard processes connected with the differential operator of second order.

In this section we are concerned with the canonical diffusion process X connected with D, its dual process constructed by theorem 3.1 and the minimal diffusion process X^s connected with the self-adjoint operator D^s in § 2.

Our aim is to prove the following theorem.

THEOREM 5.1. *Let B be an analytic set with compact closure. Then a point is a regular point of B for X, X^* or X^s, if and only if x is a regular point of B for the Brownian motion,*

PROOF. To prove this by theorem 4.2, we have only to show that the condition (R) is satisfied. For X^* and X^s, the condition (R) is easily verified by using Hunt's theory because of their dual property (under the condition (L), the dual process of X^* is X and the dual process of X^s is X^s itself). But without the condition (L), it is not obvious in the case of a canonical diffusion process. Hence we need to prove the following lemma.

LEMMA 5.1. *Let X be a canonical diffusion process in R^d ($d \geqq 3$). Then the condition (R) is satisfied for X, that is:*

$$P_x(\sigma_A<\infty)=\int_{\bar{A}} G(x, y)\mu_A(dy),\ \textit{for every}$$

analytic set A with compact closure, where μ_A is a uniquely determined measure concentrating on $\bar{A}$.

PROOF. Remark that $P_x(\sigma_A<\infty)$ is X-excessive (see. Dynkin [**2**]) and harmonic (X) in $R^d-\bar{A}$. First we have for every open set Q with compact closure

$$(5.1)\qquad P_x(\sigma_A<\infty)=g(x)+\int_Q G(x, y)\mu(dy),\ \ x\in Q\,,$$

where $g(x)$ is harmonic (X) in R^d and μ is a measure on Q.

Indeed, the proof of (5.1) follows the same lines as that of Schur [**14**], if we prove the following proposition.

PROPOSITION 5.1. *Choose an open ball Q containing a fixed point x and let $\{T_t^Q\}$ be a semi-group of a stopped canonical diffusion process X^Q on Q (see, Dynkin [2]). Then*

$$h_s(x, y) \equiv T_s^Q G_y(x)$$

is continuous in y.

PROOF. Since $h_s(x, y) = T_s^Q[G_y(x) - E_x G_y(x_{\tau_Q})] + T_s^Q[E_x G_y(x_{\tau_Q})]$, we shall show that the rigth-hand side is continous.

i) We first prove that $T_s^Q[E_x G_y(x_{\tau_Q})]$ is continuous in y. If we fix a point $x \in Q$, we have

$$\begin{aligned} T_s^Q[E_x G_y(x_{\tau_Q})] &= E_x^Q[E_{x_s} G_y(x_{\tau_Q}), s < \tau_Q] \\ &+ E_x^Q[G(x_{\tau_Q}, y), s \geqq \tau_Q] = E_x[E_{x_s} G(x_{\tau_Q}, y), s < \tau_Q] \\ &+ E_x[G(x_{\tau_Q}, y), s \geqq \tau_Q] = E_x[G(x_{\tau_Q}, y)]. \end{aligned}$$

Hence it suffices to show that $E_x G(x_{\tau_Q}, y)$ is continuous. If we fix a point $y \in R^d - \bar{Q}$, then $G(\cdot, y)$ is harmonic (X) in $R^d - y$, and so we have $G(x, y) = E_x G(x_{\tau_Q}, y)$, where $x \in Q$ and $y \in R^d - \bar{Q}$. Therefore $E_x G(x_{\tau_Q}, y)$ is continuous in $R^d - \bar{Q}$. From (3.6), we have

$$\lim_{y_m \to y} E_x G(x_{\tau_Q}, y_m) = E_x G(x_{\tau_Q}, y) = G(x, y),$$

where $y_m \in R^d - \bar{Q}, y \in \partial Q$. Thus $E_x G(x_{\tau_Q}, y)$ is continuous in $R^d - Q$. If y_1, y_2 belong to Q, we have

$$|E_x\{G(x_{\tau_Q}, y_1) - G(x_{\tau_Q}, y_2)\}| \leqq \sup_{z \in \partial Q} |G(z, y_1) - G(z, y_2)| \to 0 \text{ as } y_1 \to y_2$$

as and from (3.2) we obtain

$$\lim_{y_m \to y} E_x G(x_{\tau_Q}, y_m) = G(x, y), y_m \in Q, y \in \partial Q.$$

Therefore $E_x G(x_{\tau_Q}, y)$ is continuous in R^d.

ii) For each $x, y \in Q$, we have

$$\begin{aligned} T_s^Q[G_y(x) - E_x G_y(x_{\tau_Q}) &= T_s^Q[G_y^Q(x)] = \int G^Q(z, y) P^Q(s, x, z) dz \\ &= \int_0^\infty P^Q(t+s, x, y) dt = \int_s^\infty P^Q(t, x, y) dt. \end{aligned}$$

When $y \in Q^c$, we see that $G_y(x) = E_x G_y(x_{\tau_Q})$ and for an arbitrary sequence $\{y_m\}$ in Q such that $y_m \to y \in \partial Q$, we have $\lim_{m \to \infty} \{G(x, y_m) - E_x G_{y_m}(x_{\tau_Q})\} = 0$. Thus $T_s^Q[G_y(x) - E_x G_y(x_{\tau_Q})]$ is continuous in R^d. We have proved the lemma.

Hence Schur's argument [15] carries over to the present case of the canonical diffusion process, if only we prove the following proposition.

PROPOSITION 5.2. *Let Q be a bounded domain with sufficiently smooth* *boundary and μ_i, $i=1, 2$ be finite measures with the same compact support in Q. Then if*

$$\int_Q G^Q(x, y)\mu_1(dy)=\int_Q G^Q(x, y)\mu_2(dy)$$

holds for all $x\in Q$, we have

$$\mu_1=\mu_2 .$$

PROOF. It suffices to show that for any open set $\omega\subset Q$, we have

$$\int_\omega G^Q(x, y)\mu_1(dy)=\int_\omega G^Q(x, y)\mu_2(dy) .$$

Let $h(x)=\int_Q G^Q(x, y)\mu_1(dy)=\int_Q G^Q(x, y)\mu_2(dy)$ and h_ω be defined as $\inf_{f\in G} f(x)$ where $H=\{f$; positive superharmonic (X_Q) in $Q, f-h$; superharmonic (X_Q) in $\omega\}$. If we set $I^i_\omega(x)=\int_\omega G^Q(x, y)\mu_i(dy)$ $(i=1, 2)$, we have

$$I^i_\omega=h_\omega \quad (i=1, 2)$$

following Hervé [4] Prop. |7.|. As the proof is short, we repeat it here:

$h-I^i_\omega$ is harmonic (X_Q) in ω, so I_ω belongs to H. Hence we have $I^i_\omega\geqq h_\omega$. Let K be a compact set included in ω. Then for any $h'\in G$, $h'-I_K$ is superharmonic (X_Q) in $Q-K$. As $h'-h$ is superharmonic (X_Q) in ω, $h'-I_K=h'-h+I_{D-K}$ is superharmonic (X_Q) in ω. Hence $h'-I_K$ is superharmonic (X_Q) in Q. Noting $\liminf_{x\to a} h'-I_K\geqq 0$, $x\in Q$, $a\in\partial Q$, we have $h'\geqq I_K$.

Thus, we have proved the theorem.

REMARK. When $d=2$, theorem 5.1 is hold by taking $G^Q(x, y)$ (Q ;sufficiently smooth bounded domain) instead of $G(x, y)$.

§ 6. Regular points for some isotropic diffusions.

In this section we shall treat a uniformly elliptic differential operator on a closed ball $\bar{\Omega}$ with radius h such that

$$Du(x)=\sum_{i,j=1}^d a_{ij}(x)\frac{\partial^2}{\partial x_i\partial x_j}u(x) , \tag{6.1}$$

where $x=(x_1, x_2, \cdots, x_d)\in\bar{\Omega}$ and the coefficients a_{ij} are bounded continuous and symmetric. H. Tanaka [16] has shown that there exists a continuous standard process $X=(x_t, \zeta, M_t, P_x)$ with semigroup $\{T_t\}$ such that

$$\lim_{t\to 0} t^{-1}\|T_t f(x)-f(x)\|=\sum_{i,j=1}^d a_{ij}(x)\frac{\partial^2 f(x)}{\partial x_i\partial x_j} , \text{ for each } f\in C^2$$

with compact support in Ω.

In what follows we shall treat this process. We shall assume $d=2$ or 3 for simplicity.

By isotropy it is meant that transition probabilities are invariant under all orthogonal transformations $\{g\}$ that leave the origin fixed; that is

$$P(t, x, E) = P(t, gx, gE).$$

The following lemma was proved in a little different form by Wentzell in the case of the differential operator such that

$$D = \sum_{i,j=1}^{d} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i \frac{\partial}{\partial x_i},$$

where $x = (x_1, \cdots, x_d)$. In the case of (6.1) we get more detailed results.

LEMMA 6.1. *Assume that the process X defined above is isotropic. In case* $d = 3$,

$$f(r, \theta, \varphi) = a(r)\frac{\partial^2 f}{\partial r^2} + \frac{2b(r)}{r}\frac{\partial f}{\partial r} + \frac{b(r)}{r^2}\Big(\frac{\partial^2 f}{\partial \theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2 f}{\partial \varphi^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial f}{\partial \theta}\Big), \tag{6.2}$$

where $x = (x_1, x_2, x_3) = (r, \theta, \varphi)^{*)} \neq (0, \theta, \varphi)$

$$a_{ij}(x) = \delta_{ij} b(r) + \{a(r) - b(r)\} \frac{x_i x_j}{r^2}.$$

In case $d = 2$, under the assumptions of isotropy and reflection invariance, we have

$$f(r, \theta) = a(r)\frac{\partial^2 f}{\partial r^2} + \frac{b(r)}{r}\frac{\partial f}{\partial r} + \frac{b(r)}{r^2}\frac{\partial^2 f}{\partial \theta^2}, \tag{6.3}$$

where $x = (x_1, x_2) = (r, \theta) \neq (0, \theta)$

$$a_{ij}(x) = \delta_{ij} b(r) + \{a(r) - b(r)\} \frac{x_i x_j}{r^2}.$$

Moreover, by the continuity and boundedness of the cofficients a_{ij} and uniform ellipticity, we can show that $a(r)$ and $b(r)$ are positive bounded continuous function of r on $[0, h)$, and

$$\lim_{r \to 0} \{a(r) - b(r)\} = 0.$$

When the operator D is expressed by polar coordinates, the form of infinitesimal operator is given by (6.2) and (6.3) except at the origin. Hence, in order to see the behaviour of the process X at the origin, it is necessary to investigate the boundary conditions of the radial process X_r on $[0, h)$, which is defined by $X_r(t) = |x_t|$. It is known that the infinitesimal operator A_r of X_r

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*) (r) is a point on the radial coordinate space $(0, h)$.
(θ, φ) is a point on the spherical coordinate space S^{n-1}.

has a form

$$A_r f(r) = a(r)\frac{\partial^2 f}{\partial r^2} + \frac{(d-1)}{r} b(r)\frac{\partial f}{\partial r}, \tag{6.4}$$

for $f \in c^2(0, h)$.

THEOREM 6.1. *Consider the radial process X_r defined above on $[0, h)$. Then the boundary 0 can be neither "natural" nor "exit" in Feller's sense.*

PROOF. If we assume that it is natural or exit, we find that the point 0 is a trap with respect to the original process X, as 0 is a reflecting barrier. Hence we have $Af(0)=0$ for every function $f \in D(A)$*). On the other hand, a function $f(x)=x_1^2+x_2^2+x_3^2$ where $x=(x_1, x_2, x_3)$ belongs to $D(A)$, obviously and $Df(0)=2(a_{11}(0)+a_{22}(0)+a_{33}(0))>0$ by uniform ellipticity of D. This yields a contradiction.

When $d=2$, we shall show by an example that there exists a process X whose radial process X_r has a regular boundary 0. Consider the operator D on the disk Ω with radius e^{-3} such that

$$D = \sum_{i,j=1}^{d} a_{ij}\frac{\partial^2}{\partial x_i \partial x_j}, \qquad x=(x_1, x_2), \tag{6.5}$$

$$a_{ij}(x) = \delta_{ij} + \left(\frac{\log r}{2+\log r} - 1\right)\frac{x_i x_j}{r^2}\ ^{**)}.$$

Then the generator of X_r has the form

$$D_r = \frac{\log r}{2+\log r}\,\frac{1}{r(\log r)^2}\,\frac{\partial}{\partial r}\left\{r(\log r)^2\frac{\partial}{\partial r}\right\}.$$

Hence it holds

$$\sigma = \int_0^{e^{-3}}\int_y^{e^{-3}} (2+\log x)\frac{x(\log x)^2}{\log x}dx\frac{dy}{y(\log y)^2} < \infty,$$

$$\mu = \int_0^{e^{-3}}\int_y^{e^{-3}} \frac{dx}{x(\log x)^2}\,\frac{y(\log y)^2}{\log y}(2+\log y)dy < \infty,$$

which shows that the boundary 0 is regular in Feller's sense. This example illustrates the following important remark; the point 0 is a regular point of the set $\{0\}$ for the original process X which corresponds to (6.5). In the case of Brownian motion, this never occurs. Hence we see that the Hölder continuity of a_{ij} plays an essential role in the proof of Theorem 5.1. However, in case $d=3$, we cannot construct such type of counter examples, as is shown by the following.

THEOREM 6.2. *In case $d=3$, the boundary 0 is always entrance.*

PROOF. Keeping (6.4) in mind, we see that the boundery 0 is entrance, if

*) $D(A)$ denotes the domain of definitions of A.

**) Remark that D is uniformly elliptic in Ω.

and only if $\sigma=\infty$ and $\mu<\infty$ where

$$\sigma=\iint_{0<y<x<c} dm(x)ds(y),$$

$$\mu=\iint_{0<y<x<c} ds(x)dm(y),$$

$$s(x)=\int_c^x e^{-B(y)}dy,$$

$$m(x)=\int_c^x \frac{1}{a(y)} e^{B(y)}dy,$$

$$B(x)=\int_c^x \frac{2b(y)}{ya(y)} dy$$

c : some fixed constant in $(0, h)$.

By Theorem 6.1, 0 cannot be natural nor exit. Hence it suffices to show $6=\infty$. Without loss of generality we may assume

$$\frac{1}{2}<\frac{b(r)}{a(r)}<\frac{3}{2},$$

for any $r\in(0, c]$, because c can be chosen sufficiently small. (It is here that we use the properties of $a(r)$ and $b(r)$ mentioned in Lemma 6.1.) Hence, noting $x<c$, we have

$$3\log x-3\log c\leqq B(x)\leqq 2\frac{1}{2}\int_c^x \frac{1}{y}dy.$$

Therefore, it holds that

$$\sigma\geqq\iint_{0<y<x<c} dm(x)e^{-\log y+\log c}dy$$

$$\geqq e^{\log c}\int_0^c\int_y^c \frac{1}{M}e^{3\log x-3\log c}dx\, e^{-\log y}dy$$

$$=\frac{e^{-2\log c}}{M}\int_0^c \frac{1}{4}(c^4-y^4)\frac{1}{y}dy=\infty,$$

where M is an upper bound of $a(r)$. This completes the proof of Theorem 6.2.

Nagoya University

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