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On the large perturbation by a class of non-selfadjoint operators

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Introduction

We shall be concerned with the non-selfadjoint operator in the Hilbert space $\mathfrak{H}$ of abstract functions $f(x)$

$$L_1 f \equiv L_0 f + Vf = xf(x) + \int_a^b v(x, y) f(y) dy, \tag{0.1}$$

where $[a, b]$ may be an infinite interval. More precisely, let Ω be a fixed Hilbert space, where the inner product and the norm are denoted by $f \cdot \bar{g}$ and $|f| = (f \cdot \bar{f})^{1/2}$, respectively. $\mathfrak{H}$ is the Hilbert space of all measurable functions $f(x)$ $(x \in [a, b])$ with values in Ω such that $\int_a^b |f(x)|^2 dx < +\infty$. We denote the inner product and the norm in $\mathfrak{H}$ by

$$\langle f(x), g(x) \rangle = \int_a^b f(x) \cdot \overline{g(x)} dx \quad \text{and} \quad \|f\| = \langle f(x), f(x) \rangle^{1/2},$$

respectively. Then, $L_0 f \equiv xf(x)$ is defined in the domain $\mathfrak{D}$ of all functions $f(x) \in \mathfrak{H}$ satisfying $\int_a^b (1+|x|)^2 |f(x)|^2 dx < +\infty$. We regard the operator L_1 as a disturbed operator of L_0 by the perturbation V.

The above operator (0.1) was initially studied by Friedrichs ([3]). He supposed that V is *small* enough to obtain the resolvent of L_1 by the Neumann series in terms of the resolvent of L_0, and proved, among other things, the existence of a bounded operator U establishing the similarity of L_1 to L_0. Namely,

$$L_1 = U L_0 U^{-1}. \tag{0.2}$$

Recently Faddeev ([2]) treated the same problem without assuming the *smallness* of the operator V, and showed that the results of Friedrichs are essentially true even in that case. He assumed however that V is a symmetric operator. Our aim is to extend his results to the case where V is no longer symmetric. As we shall show, his method can be applied in the study of the behavior of the resolvent of L_1 without any significant modification. However, we should mention that in our case some serious difficulties occur when we consider the integral equation introduced by Faddeev, and these difficulties make our results considerably weak ones (see Example in § 1).

Let us explain briefly the outline of our proof.

In § 2, we first prove that there exists a class of subintervals $\varDelta$ of the real segment $[a, b]$, to which corresponds the spectral resolution $E_1(\varDelta)$ of L_1. It should be natural to define $E_1(\varDelta)$ by using the formula

$$E_1(\varDelta)=\lim_{\varepsilon\to+0}\frac{1}{2\pi i}\int_{\varDelta}\{R_1(\lambda+i\varepsilon)-R_1(\lambda-i\varepsilon)\}d\lambda\,, \tag{0.3}$$

where $R_1(z)=(L_1-zI)^{-1}$. However, we define $E_1(\varDelta)$ explicitly by using the wave operators $W^{(\pm)}(\varDelta)$ and $Z^{(\pm)}(\varDelta)$, and show that the right side of (0.3) converges to $E_1(\varDelta)$ in the weak topology of operators. Then we can prove the relations

$$E_1(\varDelta)^2=E_1(\varDelta),\qquad L_1E_1(\varDelta)=E_1(\varDelta)L_1\,. \tag{0.4}$$

Next we show the relation

$$W^{(\pm)}(\varDelta)L_0=L_1W^{(\pm)}(\varDelta)\,. \tag{0.5}$$

The similarity of L_1 to L_0 is then established, by restricting them to the invariant subspaces $E_1(\varDelta)\mathfrak{H}$ and $E_0(\varDelta)\mathfrak{H}$, respectively, where $E_0(\varDelta)$ is the resolution of the identity of L_0. We use a stationary method developed by Friedrichs and Faddeev to construct the wave operators $W^{(\pm)}(\varDelta)$ and $Z^{(\pm)}(\varDelta)$.

The operators $W^{(\pm)}(\varDelta)$ and $Z^{(\pm)}(\varDelta)$ are not in general related to the time dependent scattering theory since we can not say the uniqueness of the solution to the equation

$$i\frac{\partial}{\partial t}u(t)=L_1u(t)\,,\qquad u(0)=u_0(u_0\in\mathfrak{H})\,. \tag{0.6}$$

However, if we assume for example $\mathrm{Im}\,[V]$ is bounded, then $-iL_1$ becomes the infinitesimal generator of the group $\exp\{-iL_1t\}$ $(-\infty<t<+\infty)$. In this case $W^{(\pm)}(\varDelta)$ and $Z^{(\pm)}(\varDelta)$ are used to the study of the asymptotic behaviors for $t\to\pm\infty$ of the solution $u(t)$ to equation (0.6). The scattering theory of L_1 is discussed in § 3.

In § 4, we discuss the completeness of the spectral resolution $E_1(\varDelta)$ of L_1. We consider this problem in the case when V is a product operator such that $V=B^*A$. In this case we can use satisfactorily the Cauchy integral formula. However, we require in our proof that formula (0.3) has a meaning for any subinterval $\varDelta$ of $[a, b]$. Although this is a well-known fact for selfadjoint operators, it seems rather difficult to verify this when a non-selfadjoint operator is concerned with. In this paper, we add this as an assumption.

Finally, in § 5, we remark that some additional properties can be obtained if we restrict V to be dissipative.

We should mention that the works of J. Schwartz ([8]) and Kato ([**6**]) have some contact points with our study. J. Schwartz investigated the Friedrichs model (0.1) without assuming the *smallness* condition on the non-selfadjoint operator V. He proved the existence and the completeness of the spectral resolution $E_1(\Delta)$ of L_1 by reducing these problems to the results of Friedrichs for small perturbation. Our results to the same problems are not much clear in comparison with those of J. Schwartz. However, we proved them by using a direct method. Kato ([**6**]) treated the perturbation theory of general non-selfadjoint operators. Roughly speaking, his assumptions are that L_0 is a closed operator with spectrum on the real line, that V is a *small* perturbation by a product operator, i. e., $V=\kappa B^*A$, where $|\kappa|$ is chosen sufficiently small, that $A(L_0-zI)^{-1}$ and $B(L_0^*-zI)^{-1}$ are uniformly bounded when z moves in a neighborhood of the real line, and that $A(L_0-zI)^{-1}B^*$ is also uniformly bounded. Under these assumptions he proved among other things that there exist the wave operators $W^{(\pm)}$ and $Z^{(\pm)}$ which establish the similarity of L_1 to L_0. We use in § 4 a modification of Kato's method.

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1. Preliminaries

The operator V is defined by

$$[Vf](x)=\int v(x, y)f(y)dy\,, \tag{1.1}$$

where $v(x, y)$ is a completely continuous operator in Ω for any x and y in $[a, b]$ (hereafter we shall write simply $\int$, if the integration is taken over $[a, b]$). We denote by $\bar{v}(x, y)$ and $|v(x, y)|$ the adjoint operator (in Ω) and the norm of $v(x, y)$, respectively. Throughout the present paper the kernel $v(x, y)$ of V is assumed to satisfy the following conditions[1]:

(B) $$|v(x, y)-v(x+\Delta x, y+\Delta y)|$$
$$\leqq \text{const}\,(1+|x|+|y|)^{-\delta_0}(|\Delta x|^{\gamma_0}+|\Delta y|^{\gamma_0}),\ \frac{1}{2}<\gamma_0<1\,;$$

(C) $$v(x, a)=v(x, b)=v(a, y)=v(b, y)=0 \text{ if } a \text{ or } b \text{ is finite.}$$

We first prove the following theorem.

THEOREM 1.1. *Let V be an integral operator satisfying the condition* (A). *Then the operator L_1 defined by* (0.1) *has the following properties:*

(1) *L_1 is a closed operator with the domain $\mathfrak{D}\equiv\mathfrak{D}(L_0)$. Let L_1^* and V^* be the adjoint operators of L_1 and V, respectively. Then $L_1^*=L_0+V^*$, and L_1^* has also $\mathfrak{D}$ as the domain of definition.*

(2) *The essential spectrum of L_1 consists of the real segment $[a, b]$, outside of which L_1 has at most a countable number of discrete eigenvalues to which correspond finite dimensional root subspaces*[2]. *The same properties are true for L_1^*.*

(3) *A complex number $z\in[a, b]$ is a discrete eigenvalue of L_1 if and only if the complex conjugate $\bar{z}$ is a discrete eigenvalue of L_1^*. The root subspaces corresponding to z and $\bar{z}$ are of the same dimension.*

PROOF. By the condition (A), it is not difficult to see that $\mathfrak{D}(V)\supset\mathfrak{D}$. So the integral operator $V(L_0-zI)^{-1}$ with the kernel $v(x, y)(y-z)^{-1}$ is defined everywhere in $\mathfrak{H}$ for any complex number $z\notin[a, b]$. According to the relation

$$L_1-zI=L_0+V-zI=\{I+V(L_0-zI)^{-1}\}(L_0-zI)\,, \tag{1.2}$$

we see that L_1 is a closed operator with the domain $\mathfrak{D}$. Since

$$[V^*f](x)=\int v^*(x, y)f(y)dy \qquad (f\in\mathfrak{D})\,, \tag{1.1*}$$

where $v^*(x, y)=\bar{v}(y, x)$, by the same reasoning, L_0+V^* is also a closed operator with the domain $\mathfrak{D}$. Next, let us prove the relation $L_1^*=L_0+V^*$. It follows easily from (A) that $v(x, y)(y-z)^{-1}$ is a kernel of the Hilbert-Schmidt type and

$$\iint\frac{|v(x, y)|^2}{|y-z|^2}dxdy\leqq\text{const}\iint\frac{(1+|x|+|y|)^{-2\delta_0}}{|y-z|^2}dxdy$$
$$\leqq\text{const}\,|\text{Im}\,z|^{-1}\int(1+|x|)^{-2\delta_0+1}dx\int(1+|y|)^{-1}|y-z|^{-1}dy\,.$$

Hence there exists a complex number z such that

2) A function $f(x)\in\mathfrak{H}$ is called a generalized eigenfunction of L_1 corresponding to an eigenvalue λ, if there exists a natural number n such that $(L_1-\lambda I)^n f=0$. The set of all generalized eigenfunctions of L_1 which correspond to λ is called the root subspace of L_1, corresponding to λ.

$$\| V(L_0-zI)^{-1} \| < 1 , \qquad \| V^*(L_0-\bar{z}I)^{-1} \| < 1 .$$

For arbitrarily fixed such a value z, L_1-zI has a bounded inverse. Moreover we have from (1.2) that

$$(L_1-zI)^{-1*} = \sum_{n=0}^{\infty} [V(L_0-zI)^{-1}]^{*n}(L_0-\bar{z}I)^{-1} .$$

On the other hand, since

(1.2)* $\quad L_0+V^*-\bar{z}I=(L_0-\bar{z}I)\{I+(L_0-\bar{z}I)^{-1}V^*\} \subseteqq (L_0-\bar{z}I)\{I+[V(L_0-zI)^{-1}]^*\}$,

we have similarly that

$$(L_0+V^*-\bar{z}I)^{-1} = \sum_{n=0}^{\infty} [V(L_0-zI)^{-1}]^{*n}(L_0-\bar{z}I)^{-1} .$$

Hence we find tne relation $(L_1-zI)^{-1*}=(L_0+V^*-\bar{z}I)^{-1}$, which implies $L_1^* = L_0+V^*$. The assertion (1) is thus proved. The remainder of the above theorem follows readily from Theorem 5.1 of [5] if we notice the complete continuity of $V(L_0-zI)^{-1}$. The theorem is proved.

Now we introduce, following Faddeev, the operator $V_1(z)$ with the domain $\mathfrak{D}$, satisfying the equation[3)]

(1.3) $$\{I+VR_0(z)\}V_1(z)=V, \quad R_0(z)=(L_0-zI)^{-1} \quad (z \notin [a, b]) .$$

LEMMA 1.1. *The operator $I+VR_0(z)$ is invertible in $\mathfrak{H}$ with the bounded inverse if $z \notin [a, b]$ is not an eigenvalue of L_1.*

PROOF. Let $f+VR_0(z)f=0$, $f \in \mathfrak{H}$. Then, putting $\phi=R_0(z)f \in \mathfrak{D}$, we have $(L_0-zI)\phi+V\phi=0$. Namely $(L_1-zI)\phi=0$. This implies that z is an eigenvalue of L_1. Thus, if we recall that $VR_0(z)$ is completely continuous in $\mathfrak{H}$, this proves our assertion.

Let us define the oprators

(1.4) $$V_1(z)=\{I+VR_0(z)\}^{-1}V ,$$

(1.5) $$R_1(z)=R_0(z)-R_0(z)V_1(z)R_0(z) .$$

Then $R_1(z)$ gives exactly the resolvent of L_1. Let us prove this. We first notice that the range of $R_1(z)$ is contained in $\mathfrak{D}$. So, we can apply to the second side of (1.5) the operator L_1-zI from the left. Then we see easily $(L_1-zI)R_1(z)=I$. Next applying L_1-zI from the right, we get

$$\begin{aligned} R_1(z)(L_1-zI) &= I+R_0(z)\{V-V_1(z)-V_1(z)R_0(z)V\} \\ &= I+R_0(z)\{V-V_1(z)-[I+VR_0(z)]^{-1}VR_0(z)V\} \\ &= I+R_0(z)\{V-V_1(z)-V+V_1(z)\}=I \qquad (\text{in } \mathfrak{D}) . \end{aligned}$$

3) This is a fundamental equation in the stationary approach to scattering theory. Faddeev has developed in [1] a method for the scattering theory of the non-relativistic three body problem starting with this equation. This equation can be applied to the relativistic quantum theory (see [7]).

Conversely, using the resolvent $R_1(z)$ of L_1, $V_1(z)$ is represented as follows:

(1.6) $$V_1(z)=V-VR_1(z)V\,.$$

In fact, there exist always the relations

(1.7) $$R_1(z)=R_0(z)-R_1(z)VR_0(z)\,,$$

(1.8) $$R_1(z)=R_0(z)-R_0(z)VR_1(z)\,,$$

which are called the second resolvent equations. Applying $V_1(z)$ to (1.7) from the right, we get

$$R_0(z)V_1(z)=R_1(z)\{I+VR_0(z)\}V_1(z)=R_1(z)V\,.$$

Namely we have

(1.9) $$R_0(z)V_1(z)=R_1(z)V\,,$$

which yields relation (1.6). Finally, applying $R_0(z)$ to (1.6) from the right, we have

(1.10) $$VR_1(z)=V_1(z)R_0(z)\,.$$

Next, let us define the operators

(1.4)* $$V_1^*(z)=\{I+V^*R_0(z)\}^{-1}V^*\,,$$

(1.5)* $$R_1^*(z)=R_0(z)-R_0(z)V_1^*(z)R_0(z)\,.$$

Then we see similarly that $R_1^*(z)$ is the resolvent of $L_1^*=L_0+V^*$. If $R_1(z)$ exists for some $z\notin[a,b]$, then we see from (3) of Theorem 1.1 that $R_1^*(\bar{z})$ exists and we have

(1.11) $$R_1^*(\bar{z})=R_1(z)^*\,.$$

Since

(1.6)* $$V_1^*(z)=V^*-V^*R_1^*(z)V^*\,,$$

comparing this formula with (1.4)* and taking (1.11) into account, we have

(1.12) $$V_1^*(z)f=V_1(\bar{z})^*f\,,\qquad f\in\mathfrak{D}\,,$$

where $V_1(z)^*$ is the adjoint operator of $V_1(z)$. Finally we get

(1.13) $$[R_1(\bar{z})V]^*=V^*R_1(\bar{z})^*=V_1^*(z)R_0(z)\,.$$

From now on, we shall examine whether the solution $V_1(z)$ of equation (1.3) be an integral operator generated by a smooth kernel similar to $v(x,y)$, even when z approaches the continuous spectrum of L_1. For this aim, we can apply the method developed by Faddeev ([2]) in the case when V is symmetric.

The operator $VR_0(z)$ is an integral operator such that

(1.14) $$[VR_0(z)f](x)=\int\frac{v(x,y)f(y)}{y-z}dy\,,\qquad f\in\mathfrak{H}\,.$$

The right side becomes singular when z approaches to the real segment $[a, b]$. In order to give a meaning to such a type of singular integrals, we require the Hölder conditions on $f(x)$.

Let $\mathfrak{B}_{\delta,\gamma}$ be the space of all continuous functions $f(x)$ ($x\in[a, b]$) with values in Ω, satisfying

(1.15) $$\|f\|_{\delta,\gamma}\equiv\sup_{x,|\Delta x|<1}(1+|x|)^{\delta}\Big\{|f(x)|+\frac{|f(x)-f(x+\Delta x)|}{|\Delta x|^{\gamma}}\Big\}<+\infty\,,$$

where $\delta\geqq 0$ and $0<\gamma<1$. $\mathfrak{B}_{\delta,\gamma}$ is a Banach space. Let Π be a complex plane with a slit along the real segment (a, b), where the points on the upper edge are distinguished from those on the lower edge and these points are denoted by $\lambda+i0$ and $\lambda-i0$ ($\lambda\in(a, b)$), respectively. We denote by $\tilde{\Pi}$ the set composed of Π and those points $\lambda+i0$ and $\lambda-i0$ ($\lambda\in(a, b)$).

Define the operator

(1.16) $$[T(z)f](x)=\int\frac{v(x,y)f(y)}{y-z}dy\qquad(z\in\Pi)$$

acting on the space $\mathfrak{B}_{\delta,\gamma}$. Then we get the following estimates due essentially to Friedrichs ([3])[4].

(1.17)
$$\begin{aligned}&|[T(z)f](x)|\leqq\text{const}\,(1+|x|)^{-\delta_0}(1+|z|)^{-\delta'}\|f\|_{\delta,\gamma}\,,\\&|[T(z)f](x)-[T(z+\Delta z)f](x+\Delta x)|\\&\qquad\leqq\text{const}\,(1+|x|)^{-\delta_0}(1+|z|)^{-\delta'}\|f\|_{\delta,\gamma}(|\Delta x|^{\gamma_0'}+|\Delta z|^{\gamma'})\,,\end{aligned}$$

where $\delta'(<\delta)$ and $\gamma_0'(<\gamma_0)$ can be chosen as close to $\min(\delta, 1)$ and γ_0, respectively, as we wish and $\gamma'=\min(\gamma_0,\gamma)$. Namely $T(z)$ is a bounded mapping from $\mathfrak{B}_{\delta,\gamma}$ into $\mathfrak{B}_{\delta_0,\gamma_0'}$. Now let $\delta<\delta_0$ and $\gamma<\gamma_0$. Then, since we can choose γ_0' such that $\gamma<\gamma_0'<\gamma_0$, we find that $T(z)$ is a completely continuous operator defined in $\mathfrak{B}_{\delta,\gamma}$, following the proof of Faddeev ([2]). It follows immediately from (1.17) that

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4) These estimates are obtained by applying, with the aid of Lemma 2.1 to be given later, the following estimates due to Faddeev (see [**1**]; Lemma 1.3 of page 105): let $\varphi(x)\in\mathfrak{B}_{\delta,\gamma}$ then the function $\Phi(z)=\int\varphi(x)(x-z)^{-1}dx\,(z\in\Pi)$ satisfies

(*)
$$\begin{aligned}&|\Phi(z)|\leqq\text{const}(1+|z|)^{-\delta'}\|\varphi\|_{\delta,\gamma}\,,\\&|\Phi(z)-\Phi(z+\Delta z)|\leqq\text{const}(1+|z|)^{-\delta'}\|\varphi\|_{\delta,\gamma}|\Delta z|^{\gamma}\,.\end{aligned}$$

We remark that the above estimates will play an important role in the following. Sometime we shall use them without explicit mention.

$$\begin{aligned} &\| T(z)\|_{\delta,\gamma} \leqq \text{const}\,(1+|z|)^{-\delta'}\,, \\ &\| T(z)-T(z+\Delta z)\|_{\delta,\gamma} \leqq \text{const}\,(1+|z|)^{-\delta'}|\Delta z|^{\mu}\,, \end{aligned} \tag{1.18}$$

where $0<\mu<\frac{\gamma(\gamma_0-\gamma)}{\gamma_0}$.[5] We can say more. The above estimates show that, by the passage to the limit, $T(\lambda\pm i0)$ exist and they are also completely continuous in $\mathfrak{B}_{\delta,\gamma}$ as the uniform limits of completely continuous operators. They satisfy (1.18) with the same constants.

Now consider the integral equations

$$f+T(z)f=f_0 \qquad (f_0\in\mathfrak{B}_{\delta,\gamma})\,, \tag{1.19}$$

$$\varphi+T(z)\varphi=0\,. \tag{1.20}$$

Since $T(z)$ is completely continuous in $\mathfrak{B}_{\delta,\gamma}$ $(0<\delta<\delta_0,\ 0<\gamma<\gamma_0)$ for any fixed $z\in\tilde{\Pi}$, we can apply the Fredholm alternative. We call a value z for which (1.20) has a non-trivial solution a *singular point* of $T(z)$. We denote by $\sigma(T)$ the set of all the *singular points* of $T(z)$.

As we can see easily, $\sigma(T)$ is independent of the choice of δ and γ. So, as for the investigation of the property of the *singular points*, we can fix δ and γ such that $\frac{1}{2}<\delta<\delta_0$ and $\frac{1}{2}<\gamma<\gamma_0$. In such a case, we see that $f\in\mathfrak{B}_{\delta,\gamma}$ belongs to $\mathfrak{H}$. Consequently, it follows from Lemma 1.1 that, if $z\in\sigma(T)$ is on neither edge of (a,b)[6], then it is a discrete eigenvalue of L_1. The converse is also true.

The following lemma is due to Faddeev ([2]; Lemmas 3.6, 3.9).

LEMMA 1.2. *$\sigma(T)$ is closed and bounded in $\tilde{\Pi}$.*

In the case when V is symmetric, Faddeev has proved moreover ([2]; Lemmas 3.8, 3.11) that $\lambda+i0$ $(\lambda\in(a,b))$ belongs to $\sigma(T)$ if and only if λ is an eigenvalue of L_1, and then $\lambda-i0\in\sigma(T)$. This fact is proved by using the property that the solution $\varphi(x)$ of (1.20) for $z=\lambda+i0$ satisfies $\varphi(\lambda)=0$, and the assumption that $\gamma>\frac{1}{2}$.

In the case where V is no longer symmetric, we cannot deduce the above result, as the following example shows.

EXAMPLE. Let $\lambda\in(a,b)$, and $v(x)$ be a real valued smooth function whose support is compact and contained in (a,b). Suppose furthermore $v(\lambda)^2=\frac{1}{\pi}$ and $v(\lambda-\xi)=v(\lambda+\xi)$ for all ξ. Put $v(x,y)=iv(x)v(y)$ and consider

5) This follows from Lemma 1.2 of page 104 of [1] if we notice that the range of $T(z)$ belongs to $\mathfrak{B}_{\delta_0,\gamma_0'}$.

6) In the case when $z=a$ or b, we see easily that this value is also an eigenvalue of L_1 since $v(a,y)=v(b,y)=0$.

$$L_1 f = xf(x) + iv(x)\int v(y)f(y)dy, \qquad f \in L^2(a, b).$$

In this case equation (1.20) becomes

$$\varphi(x) + iv(x)\int \frac{v(y)\varphi(y)}{y-z}dy = 0.$$

This equation has, for $z=\lambda+i0$, a non-trivial solution $v(x)$. In fact, we have

$$v(x) + iv(x)\left\{P\int \frac{v(y)^2}{y-z}dy + \pi i v(\lambda)^2\right\}^{7)}$$

$$= v(x) + iv(x)\left\{0 + \pi i \cdot \frac{1}{\pi}\right\} = 0$$

taking account of the symmetric property of $v(y)^2$ with respect to $y=\lambda$. On the other hand, the value $z=\lambda$ is not an eigenvalue of L_1. In fact, let $\psi(x) \in \mathfrak{D}$ be a solution of the equation

$$(\lambda - x)\psi(x) = iv(x)\int v(y)\psi(y)dy.$$

Put $\int v(y)\psi(y)dy \equiv C$. Then $(\lambda-x)\psi(x) = iCv(x)$. Since $v(\lambda)\neq 0$, $\psi(x)$ does not belong to $L^2(a, b)$ unless $C=0$. Thus $\psi(x)\equiv 0$.

Incidentally, we can show an example of a selfadjoint operator L_1 such that $\lambda \in [a, b]$ is an eigenvalue of L_1. In fact, let λ be an arbitrary point in $[a, b]$. Then we can find a real valued function $v(x) \in C^1$, whose support is compact and contained in (a, b), satisfying $v(\lambda)=0$ and $\int (\lambda-x)^{-1}v(x)^2 dx = 1$. Then, putting $\psi(x) = (x-\lambda)^{-1}v(x)$ and

$$L_1 f = xf(x) + v(x)\int v(y)f(y)dy,$$

we have

$$(L_1 - \lambda I)\psi = (x-\lambda)\psi(x) + v(x)\int (x-\lambda)^{-1}v(x)^2 dx$$

$$= v(x) - v(x) = 0.$$

Thus $\psi(x)$ is an eigenfunction of L_1 corresponding to the eigenvalue λ.

Let us continue for the moment to investigate the property of the *singular points* of $T(z)$. We introduce another operator

$$\text{(1.16)*} \qquad [T^*(z)f](x) = \int \frac{v^*(x, y)f(y)}{y-z}dy \quad (z \in \Pi), \quad f \in \mathfrak{B}_{\delta,r},$$

where $v^*(x, y) = \bar{v}(y, x)$, and denote the set of the *singular points* of $T^*(z)$ by

7) We denote by P the Cauchy principal value.

$\sigma(T^*)$. Then it follows from (3) of Theorem 1.1 that $z \notin (a, b)$ belongs to $\sigma(T^*)$ if and only if $\bar{z} \in \sigma(T)$ (cf. Foot-note 6).

This property can be extended to z lying both edges of (a, b). Namely we have the

LEMMA 1.3. *The value* $z=\lambda\pm i0$ $(\lambda\in(a, b))$ *belongs to* $\sigma(T^*)$ *if and only if* $\bar{z}=\lambda\mp i0$ *belongs to* $\sigma(T)$.

PROOF. It suffices to show the lemma in the case when $z=\lambda+i0$. Let us denote by $\mathfrak{B}'_{\delta,r}$ the dual of $\mathfrak{B}_{\delta,r}$; that is, the space of all continuous linear functionals defined on $\mathfrak{B}_{\delta,r}$. The dual operator $T(z)'$ of $T(z)$ is defined by

$$[T(z)'F](f)=F(T(z)f), \quad F\in\mathfrak{B}'_{\delta,r}, \quad f\in\mathfrak{B}_{\delta,r}.$$

Then, applying the Fredholm theory, we see that the homogeneous equation

$$\Psi+T(z)'\Psi=0 \tag{1.20$'$}$$

has a non-trivial solution in $\mathfrak{B}'_{\delta,r}$ if and only if $z\in\sigma(T)$.

It follows immediately from conditions (A) and (B) on $v(x, y)$ that, for any $\alpha\in\Omega$, $v(x, y)\alpha$ belongs to $\mathfrak{B}_{\delta,r}$ as a function of x and

$$\|v(\cdot, y)\alpha\|_{\delta,r}\leqq \text{const}\,(1+|y|)^{-\delta_0+\delta}|\alpha|. \tag{1.21}$$

Let $\lambda-i0\in\sigma(T)$ and Ψ be a corresponding solution of (1.20)$'$. Then we can define a continuous linear functional on Ω by $\Psi(v(\cdot, y)\alpha)$ for every fixed y. Namely, there exists a Ω-valued function $\varphi^*(x)$ such that

$$\Psi(v(\cdot, y)\alpha)=\alpha\cdot\overline{\varphi^*(y)}. \tag{1.22}$$

Here we claim that $\varphi^*(y)\not\equiv 0$. In fact, we have for any $f\in\mathfrak{B}_{\delta,r}$

$$\Psi(f)=-\Psi(T(\lambda-i0)f)=-\lim_{\varepsilon\to+0}\int\frac{\Psi(v(\cdot, y)f(y))}{y-(\lambda-i\varepsilon)}dy.$$

Suppose $\varphi^*(y)\equiv 0$, then this implies $\Psi(v(\cdot, y)\alpha)=0$ for any y and $\alpha\in\Omega$. Hence $\Psi(v(\cdot, y)f(y))=0$. This implies $\Psi(f)=0$ for any $f\in\mathfrak{B}_{\delta,r}$, contrary to our assumption. Now since

$$\begin{aligned}\Psi(v(\cdot, y)\alpha)&=-\Psi(T(\lambda-i0)v(\cdot, y)\alpha)\\&=-\lim_{\varepsilon\to+0}\Psi\Big(\int\frac{v(\cdot, u)v(u, y)\alpha}{u-(\lambda-i\varepsilon)}du\Big),\end{aligned}$$

it follows that

$$\alpha\cdot\overline{\varphi^*(y)}=-\lim_{\varepsilon\to+0}\int\frac{\Psi(v(\cdot, u)v(u, y)\alpha)}{u-(\lambda-i\varepsilon)}du.$$

Using again (1.22), we have

$$\alpha\cdot\overline{\varphi^*(y)}=-\lim_{\varepsilon\to+0}\int\frac{v(u, y)\alpha\cdot\overline{\varphi^*(u)}}{u-(\lambda-i\varepsilon)}du.$$

It is easy to see that $\varphi^*(y) \in \mathfrak{B}_{\delta_0-\delta,\gamma}$. Thus we have

$$(1.20)^* \qquad \varphi^*(y) = -\lim_{\varepsilon \to +0} \int \frac{\bar{v}(u,y)\varphi^*(u)}{u-(\lambda+i\varepsilon)} du = -[T^*(\lambda+i0)\varphi^*](y).$$

Namely $\lambda+i0 \in \sigma(T^*)$.

Conversely, using a solution $\varphi^*(x)$ of the last equation, we can define a linear functional acting on $\mathfrak{B}_{\delta,\gamma}$ by

$$\Psi(f) \equiv \int \frac{f(y) \cdot \overline{\varphi^*(y)}}{y-(\lambda-i0)} dy, \qquad f \in \mathfrak{B}_{\delta,\gamma}.$$

Then it is not difficult to see that this satisfies equation (1.20)′ for $z=\lambda-i0$. Namely $\lambda-i0 \in \sigma(T)$.

The proof of the above lemma is complete.

Next, we remark the following:

LEMMA 1.4. *In order that a value $\lambda \in (a, b)$ be an eigenvalue of L_1 or L_1^*, it is necessary and sufficient that both $\lambda+i0$ and $\lambda-i0$ belong to $\sigma(T)$.*

PROOF. Let $\lambda+i0$ and $\lambda-i0$ belong to $\sigma(T)$. Then, by virtue of the above lemma, we see that they also belong to $\sigma(T^*)$. Let $\varphi(x)$ and $\varphi^*(x)$ be solutions of (1.20) and (1.20)*, respectively, corresponding to the *singular point* $z=\lambda+i0$. Then we have

$$\begin{aligned} 0 &= \int \frac{\varphi^*(x) \cdot \overline{\varphi(x)}}{x-(\lambda+i0)} dx + \int \frac{\varphi^*(x)}{x-(\lambda+i0)} \cdot \overline{\left[\int \frac{v(x,y)\varphi(y)}{y-(\lambda+i0)} dy\right]} dx^{8)} \\ &= \int \frac{\varphi^*(x) \cdot \overline{\varphi(x)}}{z-(\lambda+i0)} dx - \int \frac{\varphi^*(y) \cdot \overline{\varphi(y)}}{y-(\lambda-i0)} dy \\ &= 2\pi i \varphi^*(\lambda) \cdot \overline{\varphi(\lambda)}. \end{aligned}$$

This implies that $\varphi(\lambda)=0$ or $\varphi^*(\lambda)=0$. Since $\gamma > \frac{1}{2}$, we see on putting $\psi(x) = (x-\lambda)^{-1}\varphi(x)$ and $\psi^*(x) = (x-\lambda)^{-1}\varphi^*(x)$ that $\psi(x)$ or $\psi^*(x)$ belongs to $\mathfrak{D}$. Suppose $\psi(x) \in \mathfrak{D}$, then we see

$$(L_0-\lambda I)\psi = \varphi(x) = -\int v(x,y)\psi(y)dy = -V\psi.$$

Namely $(L_1-\lambda I)\psi=0$. This shows that λ is an eigenvalue of L_1. Similarly, if we suppose $\psi^*(x) \in \mathfrak{D}$, then λ becomes an eigenvalue of L_1^*.

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8) In fact this is seen from the relation

$$\begin{aligned} 0 &= \langle R_0(\lambda+i\varepsilon)\varphi^*, \varphi+T(\lambda+i\varepsilon)\varphi\rangle \\ &= \int \frac{\varphi^*(x) \cdot \overline{\varphi(x)}}{x-(\lambda+i\varepsilon)} dx + \lim_{\varepsilon' \to +0} \int \frac{\varphi^*(x)}{x-(\lambda+i\varepsilon)} \overline{\left[\int \frac{v(x,y)\varphi(y)}{y-(\lambda+i\varepsilon')} dy\right]} dx, \end{aligned}$$

by letting $\varepsilon \to +0$ and noting by virtue of estimates (1.17) that the orders of the integration and the limit procedure can be arbitrarily changed.

Conversely, let $\lambda \in (a, b)$ be an eigenvalue of L_1, and $\psi(x)$ be a corresponding eigenfunction. Then, since

$$(x-\lambda)\psi(x)+\int v(x, y)\psi(y)dy=0\,,$$

we see on putting $\varphi(x)=(x-\lambda)\psi(x)$ that $\varphi \in \mathfrak{B}_{\delta,\gamma}$ and $\varphi(\lambda)=0$. This implies that $\varphi(x)$ is a solution of (1.20) for $z=\lambda+i0$ as well as $z=\lambda-i0$. The same reasoning can be applied in the case when λ is an eigenvalue of L_1^*.

The proof is thus complete.

We return to the non-homogeneous equation (1.19). If $z \in \tilde{\Pi}$ does not belong to $\sigma(T)$, then we can define the operator $B(z)$ by setting

$$\{I+T(z)\}^{-1}=I+B(z)\,. \tag{1.23}$$

It follows from (1.18) that $B(z)$ is a bounded operator in $\mathfrak{B}_{\delta,\gamma}$, for any δ and γ such that $0<\delta<\delta_0$ and $0<\gamma<\gamma_0$. Moreover

$$\begin{aligned} &\|B(z)\|_{\delta,\gamma} \leqq \text{const}\,(1+|z|)^{-\delta'}\,, \\ &\|B(z)-B(z+\Delta z)\|_{\delta,\gamma} \leqq \text{const}\,(1+|z|)^{-\delta'}|\Delta z|^{\mu}\,, \end{aligned} \tag{1.24}$$

where δ' and μ are the same exponents as given in (1.18). The constants in the right sides are independent of $z \in \tilde{\Pi}$, whenever z is not in a neighbohood of $\sigma(T)$.

Now, let $v_1(\cdot, y\,;\, z)\alpha$ be the solution of (1.19) with f_0 replaced by $v(\cdot, y)\alpha$, $\alpha \in \Omega$. Then this is represented by

$$v_1(x, y\,;\, z)\alpha=[\{I+B(z)\}v(\cdot, y)\alpha](x)\,. \tag{1.25}$$

LEMMA 1.5. *$v_1(x, y\,;\, z)$ is a completely continuous operator in Ω and vanishes when x or y is on the boundary of $[a, b]$. Moreover the following estimates hold.*

$$\begin{aligned} &|v_1(x, y\,;\, z)| \leqq \text{const}\,(1+|x|+|y|)^{-\delta_1}\,, \\ &|v_1(x, y\,;\, z)-v_1(x+\Delta x, y+\Delta y\,;\, z+\Delta z)| \\ &\qquad \leqq \text{const}\,(1+|x|+|y|)^{-\delta_1}(|\Delta x|^{\gamma_1}+|\Delta y|^{\gamma_1}+|\Delta z|^{\mu_1})\,, \end{aligned} \tag{1.26}$$

where $\delta_1(<\delta_0)$, $\gamma_1(<\gamma_0)$ and $\mu_1\left(<\frac{\gamma_0}{4}\right)$ can be chosen as close to δ_0, γ_0 and $\frac{\gamma_0}{4}$, respectively, as we wish. The constants in the right sides are independent of $z \in \tilde{\Pi}$, whenever z is not in a neighborhood of $\sigma(T)$.

PROOF. The first half of the assertion is evident. Applying (1.21) and (1.24) to the identity (1.25), we get

$$\begin{aligned} &|v_1(x, y\,;\, z)| \leqq \text{const}\,(1+|x|)^{-\delta}(1+|y|)^{-\delta_0+\delta}\,, \\ &|v_1(x, y\,;\, z)-v_1(x+\Delta x, y+\Delta y\,;\, z)| \\ &\qquad \leqq \text{const}\,(1+|x|)^{-\delta}(1+|y|)^{-\delta_0+\delta}(|\Delta x|^{\gamma}+|\Delta y|^{\gamma})\,. \end{aligned}$$

Since these inequalities are valid for any δ and γ such that $0<\delta<\delta_0$ and $0<\gamma<\gamma_0$ with the constants which depend only on the choice of δ and γ, we can substitute in the right sides the function $(1+|x|+|y|)^{-\delta}$ for $(1+|x|)^{-\delta}(1+|y|)^{-\delta_0+\delta}$. As for the Hölder continuity concerning the variable z, we have

$$\begin{aligned}|v_1(x, y; z)-v_1(x, y; z+\Delta z)|&\\ \leqq \mathrm{const}\,(1+|x|)^{-\delta}\|B(z)-B(z+\Delta z)\|_{\delta,\gamma}\|v(\cdot, y)\|_{\delta,\gamma}&\\ \leqq \mathrm{const}\,(1+|x|)^{-\delta}(1+|y|)^{-\delta_0+\delta}|\Delta z|^{\mu_1},&\end{aligned}$$

where $\mu_1<\frac{\gamma(\gamma_0-\gamma)}{\gamma_0}$. It is easy to see that $\max\frac{\gamma(\gamma_0-\gamma)}{\gamma_0}=\frac{\gamma_0}{4}$. Hence we have estimates (1.26) and the lemma is proved.

REMARK. As will be proved in the following section, we can say more: $v_1(x, y; z)$ is a Hölder continuous function of z with the exponent μ_1 which can be chosen as close to γ_0 as we wish. Namely, we can replace μ_1 by γ_1 in the second inequality of (1.26).

Since we can choose $\delta_1>\frac{1}{2}$, the integral operator represented by the kernel $v_1(x, y; z)$ has $\mathfrak{D}$ as its domain of definition. It is obvious that this integral operator is a solution of equation (1.3). Hence by Lemma 1.1 we see that this coincides with the operator $V_1(z)$ when z is in the resolvent set of L_1. Moreover the kernel $v_1(x, y; \lambda\pm i0)$ has a meaning and satisfies (1.26) if $\lambda+i0$ or $\lambda-i0$ is not in $\sigma(T)$. We denote by $V_1(\lambda\pm i0)$ the boundary operator generated by $v_1(x, y; \lambda\pm i0)$.

Applying the same reasoning, we obtain the kernel $v_1^*(x, y; z)$ which generates the operator $V_1^*(z)$. This kernel is estimated in the same way as $v_1(x, y; z)$ in (1.26). By virtue of (1.12) and Lemma 1.3, we find finally that they are related to each other through

$$v_1^*(x, y; z)=\bar{v}_1(y, x; \bar{z}),\quad z\in\tilde{\Pi},\quad z\notin\sigma(T^*)\,. \tag{1.27}$$

2. The similarity of L_1 to L_0

First we state a definition.

DEFINITION. We denote by $\mathfrak{M}_{\delta,\gamma}(\delta\geqq 0,\ \gamma>0)$ the set of all the functions $j(x, y)$ with values in the set of completely continuous operators in Ω satisfying that

(A) $$|j(x, y)|\leqq \mathrm{const}\,(1+|x|+|y|)^{-\delta};$$

(B) $$|j(x, y)-j(x+\Delta x, y+\Delta y)| \leqq \mathrm{const}\,(1+|x|+|y|)^{-\delta}(|\Delta x|^{\gamma}+|\Delta y|^{\gamma});$$

(C) $$j(a, y)=j(b, y)=j(x, a)=j(x, b)=0\quad \text{if } a \text{ or } b \text{ is finite.}$$

It is obvious that $\mathfrak{M}_{\delta,\gamma}$ becomes a Banach space with respect to the norm

$$\|j\|_{\mathfrak{M}_{\delta,\gamma}}\equiv \sup_{\substack{x,y\\ |\Delta x|,|\Delta y|\leqq 1}} (1+|x|+|y|)^{\delta}\Big\{|j(x, y)| + \frac{|j(x, y)-j(x, y+\Delta y)|}{|\Delta y|^{\gamma}}+\frac{|j(x, y)-j(x+\Delta x, y)|}{|\Delta x|^{\gamma}}\Big\}. \tag{2.1}$$

Clearly $v(x, y)\in\mathfrak{M}_{\delta_0,\gamma_0}$ and $v_1(x, y; z)\in\mathfrak{M}_{\delta_1,\gamma_1}$ $(0<\delta_1<\delta_0,\ 0<\gamma_1<\gamma_0)$ for any $z\in\tilde{\Pi}$ which is not in $\sigma(T)$.

We need the following lemma.

Lemma 2.1. *Let $j(x, y)\in\mathfrak{M}_{\delta,\gamma}$ and $f(x)\in\mathfrak{B}_{\delta',\gamma'}$ $(0<\delta'\leqq\delta,\ 0<\gamma'\leqq\gamma)$. Then the function $\varphi(x, y)=j(x, y)f(y)$ belongs to $\mathfrak{B}_{\delta',\gamma'}$ for any fixed x and satisfies*

$$\begin{aligned}&\|\varphi(x,\cdot)\|_{\delta',\gamma'}\leqq 2(1+|x|)^{-\delta}\|j\|_{\mathfrak{M}_{\delta,\gamma}}\|f\|_{\delta',\gamma'},\\ &\|\varphi(x,\cdot)-\varphi(x+\Delta x,\cdot)\|_{\delta',(1-\varepsilon)\gamma'}\\ &\qquad\leqq 2(1+|x|)^{-\delta}\|j\|_{\mathfrak{M}_{\delta,\gamma}}\|f\|_{\delta',\gamma'}|\Delta x|^{\varepsilon\gamma},\end{aligned} \tag{2.2}$$

where ε is an arbitrary constant such that $0<\varepsilon<1$.

Proof. By virtue of (1.15) and (2.1), it follows that

$$\begin{aligned}\sup_{y}(1+|y|)^{\delta'}|\varphi(x, y)| &\leqq \sup_{y}|j(x, y|\cdot\|f\|_{\delta',\gamma'}\\ &\leqq(1+|x|)^{-\delta}\|j\|_{\mathfrak{M}_{\delta,\gamma}}\|f\|_{\delta',\gamma'},\end{aligned}$$

$$\begin{aligned}\sup_{y,|\Delta y|\leqq 1}(1+|y|)^{\delta'}\frac{|\varphi(x, y)-\varphi(x, y+\Delta y)|}{|\Delta y|^{\gamma'}}&\leqq\sup_{y,|\Delta y|\leqq 1}(1+|y|)^{\delta'}\\ &\quad\times\Big\{\frac{|j(x, y)-j(x, y+\Delta y)|}{|\Delta y|^{\gamma'}}|f(y)|+\frac{|f(y)-f(y+\Delta y)|}{|\Delta y|^{\gamma'}}|j(x, y)|\Big\}\\ &\leqq\sup_{y,|\Delta y|\leqq 1}\|f\|_{\delta',\gamma'}\Big\{|j(x, y)|+\frac{|j(x, y)-j(x, y+\Delta y)|}{|\Delta y|^{\gamma'}}\Big\}\\ &\leqq(1+|x|)^{-\delta}\|f\|_{\delta',\gamma'}\|j\|_{\mathfrak{M}_{\delta,\gamma}}.\end{aligned}$$

These prove the first inequality of (2.2). The second inequality can be proved similarly if we notice the inequality

$$\begin{aligned}&|\varphi(x, y)-\varphi(x+\Delta x, y)-\varphi(x, y+\Delta y)+\varphi(x+\Delta x, y+\Delta y)|\\ &\quad\leqq(1+|x|)^{-\delta}(1+|y|)^{-\delta'}\|j\|_{\mathfrak{M}_{\delta,\gamma}}\|f\|_{\delta',\gamma'}|\Delta x|^{\varepsilon\gamma}|\Delta y|^{(1-\varepsilon)\gamma'}.\end{aligned}$$

The lemma is proved.

Making use of this lemma, we can prove the

Lemma 2.2. *The following estimate holds for any δ such that $0<\delta<\delta_0$.*

$$\begin{aligned}&|v_1(x, y; z)-v_1(x, y; z+\Delta z)|\\ &\quad\leqq\text{const}\,(1+|x|)^{-\delta}(1+|y|)^{-\delta_0+\delta}(1+|z|)^{-\delta_0'}|\Delta z|^{\gamma_1}.\end{aligned} \tag{2.3}$$

$\delta'(<\delta)$ and $\gamma_1(<\gamma_0)$ can be chosen as close to δ and γ_0 as we wish.

REMARK. By the same reasoning as (1.26) was proved, (2.3) can be re-written with an arbitrary constant δ_1 such that $0<\delta_1<\delta_0$ as follows:

$$|v_1(x, y; z)-v_1(x, y; z+\Delta z)| \leqq \text{const}\,(1+|x|+|y|)^{-\delta_1}|\Delta z|^{\gamma_1}. \tag{2.3}'$$

PROOF. For any $\alpha \in \Omega$, $v_1(x, y; z)\alpha$ satisfies the equation

$$v_1(x, y; z)\alpha = v(x, y)\alpha - [T(z)v_1(\cdot, y; z)\alpha](x).$$

So we have

$$[\{I+T(z+\Delta z)\}\{v_1(\cdot, y; z)-v_1(\cdot, y; z+\Delta z\}\alpha](x)$$
$$=[\{T(z)-T(z+\Delta z)\}v_1(\cdot, y; z)\alpha](x).$$

Namely

$$v_1(x, y; z)\alpha - v_1(x, y; z+\Delta z)\alpha \tag{2.4}$$
$$=[\{I+B(z+\Delta z)\}\{T(z)-T(z+\Delta z)\}v_1(\cdot, y; z)\alpha](x).$$

We put

$$\varphi(x, u) = v(x, u)v_1(u, y; z)\alpha.$$

Then, since $v_1(\cdot, y; z)\alpha \in \mathfrak{B}_{\delta,\gamma}(0<\delta<\delta_0,\ 0<\gamma<\gamma_0)$ and

$$\|v_1(\cdot, y; z)\alpha\|_{\delta,\gamma} \leqq \text{const}\,(1+|y|)^{-\delta_0+\delta}|\alpha|,$$

we find by Lemma 2.1 that

$$\|\varphi(x, \cdot)\|_{\delta,\gamma} \leqq \text{const}\,(1+|x|)^{-\delta_0}(1+|y|)^{-\delta_0+\delta}|\alpha|,$$
$$\|\varphi(x, \cdot)-\varphi(x+\Delta x, \cdot)\|_{\delta,(1-\varepsilon)\gamma}$$
$$\leqq \text{const}\,(1+|x|)^{-\delta_0}(1+|y|)^{-\delta_0+\delta}|\Delta x|^{\varepsilon\gamma_0}|\alpha|.$$

This shows that $\|\varphi(x, \cdot)\|_{\delta,\gamma}$ belongs to $\mathfrak{B}_{\delta_0,\varepsilon\gamma_0}$. Now we put

$$\Phi(x, z) = [T(\cdot)v_1(\cdot, y; z)\alpha](x, z) \equiv \int \frac{\varphi(x, u)}{u-z}\,du.$$

Then we can apply the second estimate of (*) in the foot-note 4) to get

$$|\Phi(x, z)-\Phi(x, z+\Delta z)| \leqq \text{const}\,(1+|z|)^{-\delta'}\|\varphi(x, \cdot)\|_{\delta,\gamma}|\Delta z|^{\gamma}|\alpha|,$$
$$|\Phi(x, z)-\Phi(x+\Delta x, z)-\Phi(x, z+\Delta z)+\Phi(x+\Delta x, z+\Delta z)|$$
$$\leqq \text{const}\,(1+|z|)^{-\delta'}\|\varphi(x, \cdot)-\varphi(x+\Delta x, \cdot)\|_{\delta,(1-\varepsilon)\gamma}|\Delta z|^{(1-\varepsilon)\gamma}|\alpha|.$$

Thus, $\Phi(x, z)-\Phi(x, z+\Delta z)$ belongs to $\mathfrak{B}_{\delta_0,\varepsilon\gamma_0}$ as a function of x and

$$\|\Phi(\cdot, z)-\Phi(\cdot, z+\Delta z)\|_{\delta_0,\varepsilon\gamma_0} \leqq \text{const}\,(1+|z|)^{-\delta'}(1+|y|)^{-\delta_0+\delta}|\Delta z|^{(1-\varepsilon)\gamma}|\alpha|.$$

Hence we find finally from (2.4) that

$$
\begin{aligned}
&|v_1(x, y; z)\alpha - v_1(x, y; z+\Delta z)\alpha| \\
&\qquad \leqq \text{const}\,\{1+\|B(z+\Delta z)\|_{\delta,\varepsilon\gamma}\}\,\|\Phi(\cdot, z)-\Phi(\cdot, z+\Delta z)\|_{\delta,\varepsilon\gamma}(1+|x|)^{-\delta} \\
&\qquad \leqq \text{const}\,(1+|x|)^{-\delta}(1+|y|)^{-\delta_0+\delta}(1+|z|)^{-\delta'}|\Delta z|^{(1-\varepsilon)\gamma}|\alpha|.
\end{aligned}
$$

Since $\varepsilon>0$ can be chosen as small as we wish, this proves (2.3). The proof is thus complete.

Let $\Delta=(\alpha, \beta)$ be a (possibly infinite) subinterval of $[a, b]$ such that its closure is disjoint from a neighborhood of the set $\sigma(T)$ (from Lemma 1.3, we see that it is also disjoint from a neighborhood of $\sigma(T^*)$). Let d be the minimal distance of Δ from $\sigma(T)$, and let $\tau(x)$ be a scalar valued C^∞-function, identically equal to 1 for $x\in\Delta=(\alpha, \beta)$ and identically equal to zero for $x\notin\left(\alpha-\frac{d}{2}, \beta+\frac{d}{2}\right)$ if α or β is finite. Put

$$u(x, y; \varepsilon)=\tau(x)v_1(x, y; x+i\varepsilon), \tag{2.5}$$

$$w(x, y; \varepsilon)=\tau(y)v_1(x, y; y-i\varepsilon). \tag{2.6}$$

Then it is readily seen from estimates (1.26) and (2.3)′ that these kernels belong to the class $\mathfrak{M}_{\delta_1,\gamma_1}$, where we fix δ_1 and γ_1 such that $\frac{1}{2}<\delta_1<\delta_0$ and $\frac{1}{2}<\gamma_1<\gamma_0$. Moreover the norms $\|u(\varepsilon)\|_{\mathfrak{M}_{\delta_1,\gamma_1}}$ and $\|w(\varepsilon)\|_{\mathfrak{M}_{\delta_1,\gamma_1}}$ are uniformly bounded in ε such that $|\varepsilon|<1$ is sufficiently small, and

$$\|u(\varepsilon)-u(\varepsilon')\|_{\mathfrak{M}_{\delta_1,\gamma_1}}\leqq \text{const}\,|\varepsilon-\varepsilon'|^{\nu}, \tag{2.7}$$

$$\|w(\varepsilon)-w(\varepsilon')\|_{\mathfrak{M}_{\delta_1,\gamma_1}}\leqq \text{const}\,|\varepsilon-\varepsilon'|^{\nu}, \tag{2.8}$$

where ν is a constant such that $0<\nu<\frac{\gamma_1(\gamma_0-\gamma_1)}{\gamma_0}$.

Now we define the following operators acting in $\mathfrak{H}$.

$$[\tilde{H}(\Delta, \varepsilon_1, \varepsilon_2)f](x)=\int\frac{u(x, y; \varepsilon_1)f(y)}{y-(x+i\varepsilon_2)}dy, \tag{2.9}$$

$$[\tilde{K}(\Delta, \varepsilon_1, \varepsilon_2)f](x)=\int\frac{w(x, y; \varepsilon_1)f(y)}{x-(y-i\varepsilon_2)}dy, \tag{2.10}$$

where $\varepsilon_k\neq 0$ $(k=1, 2)$ are chosen sufficiently small. We will show that the strong limits of these operators exist and are bounded operators in $\mathfrak{H}$. For this aim we use the following lemma.

LEMMA 2.3. *Let $j(x, y)$ be a kernel belonging to $\mathfrak{M}_{\delta,\gamma}$ for some $\delta>0$ and $\gamma>0$. Then*

$$[J_\varepsilon f](x)=\int\frac{j(x, y)f(y)}{y-x+i\varepsilon}dy \qquad (\varepsilon\neq 0) \tag{2.11}$$

defines a bounded mapping of $\mathfrak{H}$ into itself. Moreover there exists a positive *constant M independent of* ε *such that* $\|J_\varepsilon f\| \leqq M\|f\| \cdot \|j\|_{\mathfrak{M}_{\delta,\gamma}}$.

PROOF. Let $\rho(x)$ be a scalar valued C^∞-function, identically equal to zero for $|x| \geqq 1$ and identically equal to 1 for $|x| \leqq \frac{1}{2}$. Then the integral defining $[J_\varepsilon f](x)$ can be written as

$$\int \frac{j(x, y)f(y)}{y-x+i\varepsilon} dy = j(x, x)\int \frac{\rho(x-y)f(y)}{y-x+i\varepsilon} dy + \int \frac{\{j(x, y)-j(x, x)\rho(x-y)\}f(y)}{y-x+i\varepsilon} dy .$$

Since $|j(x, x)| \leqq \|j\|_{\mathfrak{M}_{\delta,\gamma}}$, the uniform boundedness (with respect to ε) of the mapping defined by the first term follows immediately from the Prancherel theorem[9]. On the other hand, if we note $(1+|x|+|y|)^{-\delta} \leqq (1+|x-y|)^{-\delta}$, it follows from (A) and (B) of Definition that

$$\left| \frac{\{j(x, y)-j(x, x)\rho(x-y)\}f(y)}{y-x+i\varepsilon} \right| \leqq \text{const} \{|\rho(x-y)| \cdot |x-y|^{\gamma-1}+(1+|x-y|)^{-1-\delta}\} |f(y)| \cdot | \|j\|_{\mathfrak{M}_{\delta,\gamma}} ,$$

where the constant does not depend on ε. Since

$$\int \{|\rho(x)| \cdot |x|^{\gamma-1}+(1+|x|)^{-1-\delta}\} dx = K < +\infty ,$$

we have

$$\int dx \left(\int \left| \frac{\{j(x, y)-j(x, x)\rho(x-y)\}f(y)}{y-x+i\varepsilon} \right| dy \right)^2 \leqq \text{const}\, K^2 \|f\|^2 \|j\|^2_{\mathfrak{M}_{\delta,\gamma}} .$$

The lemma is proved.

As we have remarked, $\|u(\varepsilon_1)\|_{\mathfrak{M}_{\delta_1,\gamma_1}}$ and $\|w(\varepsilon_1)\|_{\mathfrak{M}_{\delta_1,\gamma_1}}$ are bounded in ε_1. So, it follows easily from the above lemma that the operator norms $\|\tilde{H}(\Delta, \varepsilon_1, \varepsilon_2)\|$ and $\|\tilde{K}(\Delta, \varepsilon_1, \varepsilon_2)\|$ in $\mathfrak{H}$ are bounded uniformly with respect to ε_1 and ε_2.

Next let us prove that these operators converge as $\varepsilon_1, \varepsilon_2 \to \pm 0$ at each point of a certain dense subset of $\mathfrak{H}$. Suppose $f(x) \in \mathfrak{B}_{\delta,\gamma}$ $(0<\delta \leqq \delta_1,\ 0<\gamma \leqq \gamma_1)$. Then by Lemma 2.1 $u(x, y; \varepsilon_1)f(y) \in \mathfrak{B}_{\delta,\gamma}$ for each fixed x and ε_1, and we have by using estimates (2.2) and (*) in the foot-note 4)

$$|[\tilde{H}(\Delta, \varepsilon_1, \varepsilon_2)f](x)| \leqq \text{const}\,(1+|x+i\varepsilon_2|)^{-\delta'} \|u(x, \cdot\,; \varepsilon_1)f(\cdot)\|_{\delta,\gamma} \leqq \text{const}\,(1+|x|)^{-\delta'-\delta_1} \|u(\varepsilon_1)\|_{\mathfrak{M}_{\delta_1,\gamma_1}} \|f\|_{\delta,\gamma} ,$$

9) In fact, denoting the Fourier transformation by $\mathfrak{F}$, we have

$$\mathfrak{F}\left[\int \frac{\rho(x-y)f(y)}{y-x+i\varepsilon} dy\right] = \frac{-1}{\sqrt{2\pi}} \int \frac{\rho(x)e^{-i\xi x}}{x-i\varepsilon} dx\, \hat{f}(\xi),\ \hat{f} = \mathfrak{F}f .$$

We see easily $\left| \int \frac{\rho(x)e^{-i\xi x}}{x-i\varepsilon} dx \right| \leqq$ const, where the constant is independent of ε and ξ.

$$|[\tilde{H}(\Delta, \varepsilon_1, \varepsilon_2)f](x)-[\tilde{H}(\Delta, \varepsilon_1, \varepsilon_2')f](x)|$$
$$\leqq \text{const}\,(1+|x|)^{-\delta'-\delta_1}\|u(\varepsilon_1)\|_{\mathfrak{M}_{\delta_1,r_1}}\|f\|_{\delta,r}|\varepsilon_2-\varepsilon_2'|^{r},$$

where $\delta'(<\delta)$ can be chosen as close to δ as we wish. The constants in the inequalities are independent of ε_1 and ε_2. Moreover, applying (2.7), we have similarly

$$|[\tilde{H}(\Delta, \varepsilon_1, \varepsilon_2)f](x)-[\tilde{H}(\Delta, \varepsilon_1', \varepsilon_2)f](x)|$$
$$\leqq \text{const}\,(1+|x|)^{-\delta'-\delta_1}\|f\|_{\delta,r}\|u(\varepsilon_1)-u(\varepsilon_1')\|_{\mathfrak{M}_{\delta_1,r_1}}$$
$$\leqq \text{const}\,(1+|x|)^{-\delta'-\delta_1}\|f\|_{\delta,r}|\varepsilon_1-\varepsilon_1'|^{\nu}.$$

These inequalities show that $(1+|x|)^{\delta'+\delta_1}[\tilde{H}(\Delta, \varepsilon_1, \varepsilon_2)f](x)$ converges as ε_1, $\varepsilon_2\to\pm0$ uniformly in x. Since $\delta'+\delta_1>\frac{1}{2}$, we can now say that $[\tilde{H}(\Delta, \varepsilon_1, \varepsilon_2)f](x)$ converges with respect to the norm of $\mathfrak{H}$, whenever $f(x)\in\mathfrak{B}_{\delta,r}$. By the same reasoning, $[\tilde{K}(\Delta, \varepsilon_1, \varepsilon_2)f](x)$ converges strongly in $\mathfrak{H}$. We denote the limit operators from $\mathfrak{B}_{\delta,r}$ into $\mathfrak{H}$ by $\tilde{H}(\Delta, \pm0, \pm0)$ and $\tilde{K}(\Delta, \pm0, \pm0)$, respectively.

Here we choose δ such that $\frac{1}{2}<\delta<\delta_1$. Then, since the set $\mathfrak{B}_{\delta,r}$ is dense in $\mathfrak{H}$, we see taking account of the uniform boundedness of $\tilde{H}(\Delta, \varepsilon_1, \varepsilon_2)$ and $\tilde{K}(\Delta, \varepsilon_1, \varepsilon_2)$ that their limits $\tilde{H}(\Delta, \pm0, \pm0)$ and $\tilde{K}(\Delta, \pm0, \pm0)$ are also bounded operators defined on the whole space $\mathfrak{H}$.

Now we shall consider the operators

$$[H(\Delta, \varepsilon_1, \varepsilon_2)f](x)=\begin{cases}\displaystyle\int\frac{v_1(x, y; x+i\varepsilon_1)f(y)}{y-(x+i\varepsilon_2)}dy, & x\in\Delta\\ 0, & x\notin\Delta,\end{cases}\tag{2.12}$$

$$[K(\Delta, \varepsilon_1, \varepsilon_2)f](x)=\int_\Delta\frac{v_1(x, y; y-i\varepsilon_1)f(y)}{x-(y-i\varepsilon_2)}dy.\tag{2.13}$$

Let $E_0(\Delta)$ be the resolution of the identity of L_0:

$$\langle E_0(\Delta)f, g\rangle=\lim_{\varepsilon\to+0}\frac{1}{2\pi i}\int_\Delta\langle\{R_0(\lambda+i\varepsilon)-R_0(\lambda-i\varepsilon)\}f, g\rangle d\lambda \tag{2.14}$$
$$=\int_\Delta f(\lambda)\cdot\overline{g(\lambda)}d\lambda.$$

Namely

$$[E_0(\Delta)f](\lambda)=\begin{cases}f(\lambda), & \lambda\in\Delta\\ 0, & \lambda\notin\Delta.\end{cases}$$

Then, by the definitions of H and $\tilde{H}$, we get $H(\Delta, \varepsilon_1, \varepsilon_2)=E_0(\Delta)H(\Delta, \varepsilon_1, \varepsilon_2)$. Hence, letting ε_1, $\varepsilon_2\to\pm0$, we have

$$H(\Delta, \varepsilon_1, \varepsilon_2)f\to E_0(\Delta)\tilde{H}(\Delta, \pm0, \pm0)f\equiv H(\Delta, \pm0, \pm0)f \quad (\text{in } \mathfrak{H}).\tag{2.15}$$

Similarly we get $K(\varDelta, \varepsilon_1, \varepsilon_2)=\tilde{K}(\varDelta, \varepsilon_1, \varepsilon_2)E_0(\varDelta)$ and

$$K(\varDelta, \varepsilon_1, \varepsilon_2)f\to\tilde{K}(\varDelta, \pm 0, \pm 0)E_0(\varDelta)f\equiv K(\varDelta, \pm 0, \pm 0)f \qquad (\text{in } \mathfrak{H}). \tag{2.16}$$

We shall next consider the starred operators. Analogously to (2.12) and (2.13) we put

$$[H^*(\varDelta, \varepsilon_1, \varepsilon_2)f](x)=\begin{cases}\displaystyle\int\frac{v_1^*(x, y\,;\, x+i\varepsilon_1)f(y)}{y-(x+i\varepsilon_2)}dy\,, & x\in\varDelta\\ 0\,, & x\notin\varDelta\,,\end{cases} \tag{2.12*}$$

$$[K^*(\varDelta, \varepsilon_1, \varepsilon_2)f](x)=\int_{\varDelta}\frac{v_1^*(x, y\,;\, y-i\varepsilon_1)f(y)}{x-(y-i\varepsilon_2)}dy\,. \tag{2.13*}$$

Then we can follow exactly the same line of argument to obtain the analogous results. Moreover, taking relation (1.27) into account, we find

$$H(\varDelta, \varepsilon_1, \varepsilon_2)^*=K^*(\varDelta, \varepsilon_1, \varepsilon_2),\quad K(\varDelta, \varepsilon_1, \varepsilon_2)^*=H^*(\varDelta, \varepsilon_1, \varepsilon_2)\,. \tag{2.17}$$

In order to express these results in simple terms we set

$$Y^{(\pm)}(\varDelta)=E_0(\varDelta)\tilde{H}(\varDelta, \pm 0, \pm 0),\quad X^{(\pm)}(\varDelta)=\tilde{K}(\varDelta, \pm 0, \pm 0)E_0(\varDelta)\,, \tag{2.18}$$

$$Y^{*(\pm)}(\varDelta)=E_0(\varDelta)\tilde{H}^*(\varDelta, \pm 0, \pm 0),\quad X^{*(\pm)}(\varDelta)=\tilde{K}^*(\varDelta, \pm 0, \pm 0)E_0(\varDelta)\,. \tag{2.18*}$$

Here we take either the upper signs or the lower signs throughout. Then what we have proved is the

LEMMA 2.4. *$Y^{(\pm)}(\varDelta)$, $X^{(\pm)}(\varDelta)$, $Y^{*(\pm)}(\varDelta)$ and $X^{*(\pm)}(\varDelta)$ are all bounded operators acting in $\mathfrak{H}$, and for any $f(x)\in\mathfrak{H}$*

$$H(\varDelta, \varepsilon_1, \varepsilon_2)f\to Y^{(\pm)}(\varDelta)f,\quad K(\varDelta, \varepsilon_1, \varepsilon_2)f\to X^{(\pm)}(\varDelta)f\,, \tag{2.19}$$

$$H^*(\varDelta, \varepsilon_1, \varepsilon_2)f\to Y^{*(\pm)}(\varDelta)f,\quad K^*(\varDelta, \varepsilon_1, \varepsilon_2)f\to X^{*(\pm)}(\varDelta)f \tag{2.19*}$$

strongly in $\mathfrak{H}$ as $\varepsilon_1, \varepsilon_2\to\pm 0$. Moreover we have

$$Y^{(\pm)}(\varDelta)^*=X^{*(\pm)}(\varDelta),\quad X^{(\pm)}(\varDelta)^*=Y^{*(\pm)}(\varDelta)\,, \tag{2.20}$$

$$Y^{(\pm)}(\varDelta)=E_0(\varDelta)Y^{(\pm)}(\varDelta),\quad X^{(\pm)}(\varDelta)=X^{(\pm)}(\varDelta)E_0(\varDelta)\,. \tag{2.21}$$

Now, we define the operators $Z^{(\pm)}(\varDelta)$ and $W^{(\pm)}(\varDelta)$ by setting

$$Z^{(\pm)}(\varDelta)=E_0(\varDelta)-Y^{(\pm)}(\varDelta),\quad W^{(\pm)}(\varDelta)=E_0(\varDelta)-X^{(\pm)}(\varDelta)\,. \tag{2.22}$$

Then in analogy to selfadjoint operators, we can expect that they establish the similarity of L_1 to L_0. The remainder of this section is devoted to the verification of this assertion.

We first prove the following lemma:

LEMMA 2.5. *$Z^{(\pm)}(\varDelta)$ are the left inverses of $W^{(\pm)}(\varDelta)$:*

$$Z^{(\pm)}(\varDelta_1)W^{(\pm)}(\varDelta_2)=E_0(\varDelta_1\cap\varDelta_2)\,, \tag{2.23}$$

where $\varDelta_1$, $\varDelta_2$ are arbitrarily given two subintervals of $[a, b]$ such that they are *both disjoint from a neighborhood of* $\sigma(T)$.

PROOF. It follows from (2.22) that

$$Z^{(\pm)}(\Delta_1)W^{(\pm)}(\Delta_2)-E_0(\Delta_1\cap\Delta_2)$$
$$=Y^{(\pm)}(\Delta_1)X^{(\pm)}(\Delta_2)-Y^{(\pm)}(\Delta_1)E_0(\Delta_2)-E_0(\Delta_1)X^{(\pm)}(\Delta_2).$$

So, we will show that the right side is equal to zero. For the first term, we have

$$\langle Y^{(\pm)}(\Delta_1)X^{(\pm)}(\Delta_2)f, g\rangle=\langle Y^{(\pm)}(\Delta_1)X^{(\pm)}(\Delta_2)E_0(\Delta_2)f,\ E_0(\Delta_1)g\rangle$$
$$=\lim_{\varepsilon,\varepsilon'\to\pm 0}\langle H(\Delta_1,\varepsilon,\varepsilon)K(\Delta_2,\varepsilon',\varepsilon')E_0(\Delta_2)f,\ E_0(\Delta_1)g\rangle.$$

On the other hand, we see easily that

$$V_1(z)R_0(z)R_0(z')V_1(z')=(z'-z)^{-1}V_1(z)+(z-z')^{-1}V_1(z')\text{[10]}.$$

Fixing x and y arbitrarily in Δ_1 and Δ_2, respectively, and putting $z=x+i\varepsilon$ and $z'=y-i\varepsilon'$, we now compare the value at (x, y) of the kernels of both sides of the above relation. Then, since the kernel of $H(\Delta_1,\varepsilon,\varepsilon)K(\Delta_2,\varepsilon',\varepsilon')$ is represented by

$$\int\frac{v_1(x, u\,;\,x+i\varepsilon)}{u-(x+i\varepsilon)}\,\frac{v_1(u, y\,;\,y-i\varepsilon')}{u-(y-i\varepsilon')}\,du,\quad x\in\Delta_1,\ y\in\Delta_2,$$

we have

$$E_0(\Delta_1)H(\Delta_1,\varepsilon,\varepsilon)K(\Delta_2,\varepsilon',\varepsilon')E_0(\Delta_2)$$
$$=E_0(\Delta_1)\{H(\Delta_1,\varepsilon,\varepsilon+\varepsilon')+K(\Delta_2,\varepsilon',\varepsilon+\varepsilon')\}E_0(\Delta_2).$$

Thus

$$\langle Y^{(\pm)}(\Delta_1)X^{(\pm)}(\Delta_2)f, g\rangle$$
$$=\lim_{\varepsilon,\varepsilon'\to\pm 0}\langle H(\Delta_1,\varepsilon,\varepsilon+\varepsilon')E_0(\Delta_2)f,\ E_0(\Delta_1)g\rangle$$
$$+\lim_{\varepsilon,\varepsilon'\to\pm 0}\langle K(\Delta_2,\varepsilon',\varepsilon+\varepsilon')E_0(\Delta_2)f,\ E_0(\Delta_1)g\rangle$$
$$=\langle Y^{(\pm)}(\Delta_1)E_0(\Delta_2)f, g\rangle+\langle X^{(\pm)}(\Delta_2)f,\ E_0(\Delta_1)g\rangle.$$

Namely we have

$$Y^{(\pm)}(\Delta_1)X^{(\pm)}(\Delta_2)=Y^{(\pm)}(\Delta_1)E_0(\Delta_2)+E_0(\Delta_1)X^{(\pm)}(\Delta_2).$$

The lemma is thus proved.

Next, let us define the operator

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10) In fact by (1.6), (1.9) and (1.10)

$$(z-z')V_1(z)R_0(z)R_0(z')V_1(z')=(z-z')VR_1(z)R_1(z')V$$
$$=V\{R_1(z)-R_1(z')\}V=-\{V-VR_1(z)V\}+\{V-VR_1(z')V\}$$
$$=-V_1(z)+V_1(z').$$

(2.24)$^+$ $$E_1(\varDelta)=W^{(+)}(\varDelta)Z^{(+)}(\varDelta).$$

Then we can prove the

LEMMA 2.6. *Let* $f, g\in\mathfrak{B}_{\delta,\gamma}$ $\left(\delta>\frac{1}{2}, \gamma>0\right)$. *Then*

(2.25) $$\langle E_1(\varDelta)f, g\rangle=\lim_{\varepsilon\to+0}\frac{1}{2\pi i}\int_\varDelta\langle\{R_1(\lambda+i\varepsilon)-R_1(\lambda-i\varepsilon)\}f, g\rangle d\lambda.$$

Moreover we have

(2.24)$^-$ $$E_1(\varDelta)=W^{(-)}(\varDelta)Z^{(-)}(\varDelta).$$

PROOF. We wish to show

(2.26) $$\begin{aligned}J&\equiv\lim_{\varepsilon\to+0}\frac{1}{2\pi i}\int_\varDelta\langle\{R_1(\lambda+i\varepsilon)-R_1(\lambda-i\varepsilon)\}f, g\rangle d\lambda\\&=\lim_{\varepsilon\to+0}\int_\varDelta\{f(x)-[H(\varDelta, \pm\varepsilon, \pm\varepsilon)f](x)\}\cdot\overline{\{g(x)-[H^*(\varDelta, \pm\varepsilon, \pm\varepsilon)g](x)\}}dx.\end{aligned}$$

For this aim, we start from the relation

$$R_1(\lambda+i\varepsilon)-R_1(\lambda-i\varepsilon)=2i\varepsilon R_1(\lambda\mp i\varepsilon)R_1(\lambda\pm i\varepsilon).$$

Taking (1.7) and (1.8) into account, we see that the right side is equal to

$$\begin{aligned}&2i\varepsilon\{I-R_1(\lambda\mp i\varepsilon)V\}R_0(\lambda\mp i\varepsilon)R_0(\lambda\pm i\varepsilon)\{I-VR_1(\lambda\pm i\varepsilon)\}\\&=\{I-R_1(\lambda\mp i\varepsilon)V\}\{R_0(\lambda-i\varepsilon)-R_0(\lambda-i\varepsilon)\}\{I-VR_1(\lambda\pm i\varepsilon)\}.\end{aligned}$$

Thus

(2.26a) $$\begin{aligned}J=\lim_{\varepsilon\to+0}\frac{1}{2\pi i}\int_\varDelta\langle&\{R_0(\lambda+i\varepsilon)-R_0(\lambda-i\varepsilon)\}\{I-VR_1(\lambda\pm i\varepsilon)\}f,\\&\{I-V^*R_1(\lambda\mp i\varepsilon)^*\}g\rangle d\lambda.\end{aligned}$$

It follows from (1.10) and (1.13) that

$$[VR_1(\lambda+i\varepsilon)f](x)=\int\frac{v_1(x, y; \lambda+i\varepsilon)f(y)}{y-(\lambda+i\varepsilon)}dy\equiv f_\varepsilon(x, \lambda),$$

$$[V^*R_1(\lambda-i\varepsilon)^*g](x)=\int\frac{v_1^*(x, y; \lambda+i\varepsilon)g(y)}{y-(\lambda+i\varepsilon)}dy\equiv g_\varepsilon(x, \lambda).$$

Without loss of generality we can assume that $\frac{1}{2}<\delta<\delta_0$ and $0<\gamma<\gamma_0$. Then, since $v_1(x, y; \lambda+i\varepsilon)$ and $v_1^*(x, y; \lambda+i\varepsilon)$ belong to the class $\mathfrak{M}_{\delta_1,\gamma_1}$ ($\delta<\delta_1<\delta_0$, $\gamma<\gamma_1<\gamma_0$) for each fixed $\lambda\in\varDelta$ and sufficiently small ε, the same reasoning as we used in §1 when we get (1.18) shows that both $f_\varepsilon(x, \lambda)$ and $g_\varepsilon(x, \lambda)$ belong to $\mathfrak{B}_{\delta_1,\gamma_1'}$ and satisfy

$$\left.\begin{aligned}\|f_\varepsilon(\cdot, \lambda)\|_{\delta,\gamma}&\leqq\text{const}\,(1+|\lambda|)^{-\delta'}\|f\|_{\delta,\gamma}\\\|g_\varepsilon(\cdot, \lambda)\|_{\delta,\gamma}&\leqq\text{const}\,(1+|\lambda|)^{-\delta'}\|g\|_{\delta,\gamma}\end{aligned}\right.\quad\left(\frac{1}{2}<\delta'<\min(\delta, 1)\right),$$

where the constants are independent of ε. For brevity we put

$$\sigma_\varepsilon(\lambda, f, g)=\langle\{R_0(\lambda+i\varepsilon)-R_0(\lambda-i\varepsilon)\}f, g\rangle-2\pi i f(\lambda)\cdot\overline{g(\lambda)},$$

and prove on the basis of the above inequalities and estimate (*) in the footnote 4) that

$$|\sigma_\varepsilon(\lambda, f, g_{\pm\varepsilon})|\leqq \text{const}\,(1+|\lambda|)^{-2\delta'}\varepsilon^{\gamma'},$$

$$|\sigma_\varepsilon(\lambda, f_{\pm\varepsilon}, g)|\leqq \text{const}\,(1+|\lambda|)^{-2\delta'}\varepsilon^{\gamma'}, \qquad (0<\gamma'\leqq\gamma).$$

$$|\sigma_\varepsilon(\lambda, f_{\pm\varepsilon}, g_{\pm\varepsilon})|\leqq \text{const}\,(1+|\lambda|)^{-2\delta'}\varepsilon^{\gamma'}.$$

Let us consider for example $\sigma_\varepsilon(\lambda, f, g_{+\varepsilon})$. By the definition of σ_ε, we get

$$\begin{aligned}\sigma_\varepsilon(\lambda, f, g_{+\varepsilon})&=\langle\{R_0(\lambda+i\varepsilon)-R_0(\lambda-i\varepsilon)\}f(x), g_{+\varepsilon}(x, \lambda)\rangle_x-2\pi i f(\lambda)\cdot\overline{g_{+\varepsilon}(\lambda, \lambda)}\\&=\lim_{\varepsilon'\to+0}\langle\{R_0(\lambda+i\varepsilon)-R_0(\lambda+i\varepsilon')\}f(x), g_{+\varepsilon}(x, \lambda)\rangle_x\\&\quad-\lim_{\varepsilon'\to+0}\langle\{R_0(\lambda-i\varepsilon)-R_0(\lambda-i\varepsilon')\}f(x), g_{+\varepsilon}(x, \lambda)\rangle_x.\end{aligned}$$

Since

$$\|f(\cdot)\overline{g_{+\varepsilon}(\cdot, \lambda)}\|_{2\delta,\gamma}\leqq \text{const}\,(1+|\lambda|)^{-\delta'},$$

clearly we have

$$|\sigma_\varepsilon(\lambda, f, g_{+\varepsilon})|\leqq \text{const}\,(1+|\lambda|)^{-\theta}\|f(\cdot)\overline{g_{+\varepsilon}(\cdot, \lambda)}\|_{2\delta,\gamma}\,\varepsilon^{\gamma},$$

where $0<\theta<\min(1, 2\delta)=1$, and hence θ can be chosen as close to 1 as we wish. Thus we can choose $\theta=\delta'$ and then obtain

$$|\sigma_\varepsilon(\lambda, f, g_{+\varepsilon})|\leqq \text{const}\,(1+|\lambda|)^{-2\delta'}\varepsilon^{\gamma'}. \tag{2.26b}$$

On the other hand, noting that for $\lambda\in\Delta$

$$f_{\pm\varepsilon}(\lambda, \lambda)=[H(\Delta, \pm\varepsilon, \pm\varepsilon)f](\lambda), \quad g_{\pm\varepsilon}(\lambda, \lambda)=[H^*(\Delta, \pm\varepsilon, \pm\varepsilon)g](\lambda),$$

and remembering the relation

$$\lim_{\varepsilon\to+0}\frac{1}{2\pi i}\int_\Delta\langle\{R_0(\lambda+i\varepsilon)-R_0(\lambda-i\varepsilon)\}f, g\rangle d\lambda=\int_\Delta f(\lambda)\cdot\overline{g(\lambda)}d\lambda,$$

we see from (2.26a) that

$$\begin{aligned}J=&\lim_{\varepsilon\to+0}\frac{1}{2\pi i}\int_\Delta\{f(\lambda)-[H(\Delta, \pm\varepsilon, \pm\varepsilon)f](\lambda)\}\cdot\overline{\{g(\lambda)-[H^*(\Delta, \pm\varepsilon, \pm\varepsilon)g](\lambda)\}}d\lambda\\&+\lim_{\varepsilon\to+0}\frac{1}{2\pi i}\int_\Delta\{-\sigma_\varepsilon(\lambda, f, g_{\pm\varepsilon})-\sigma_\varepsilon(\lambda, f_{\pm\varepsilon}, g)+\sigma_\varepsilon(\lambda, f_{\pm\varepsilon}, g_{\pm\varepsilon})\}d\lambda.\end{aligned}$$

By the Lebesgue theorem, however, inequality (2.26b) shows that the second term on the right side is zero. Thus (2.26) is proved.

If we recall finally that, as $\varepsilon\to+0$,

$$\left.\begin{array}{l} H(\Delta, \pm\varepsilon, \pm\varepsilon)f \to Y^{(\pm)}(\Delta)f \\ H^*(\Delta, \pm\varepsilon, \pm\varepsilon)g \to Y^{*(\pm)}(\Delta)g = X^{(\pm)}(\Delta)^*g \end{array}\right. \quad (\text{in } \mathfrak{H}),$$

then we conclude from (2.26) simultaneously relations (2.25) and $(2.24)^-$, and the proof is complete.

REMARK. In the case when V is symmetric, we know from the beginning the existence of the resolution of the identity $E_1(\Delta)$ of L_1 satisfying relation (2.25). So, taking $V^*R_1(\lambda\mp i\varepsilon)^* = VR_1(\lambda\pm i\varepsilon)$ into account, we see that $Z^{(\pm)}(\Delta) = E_0(\Delta) - Y^{(\pm)}(\Delta)$ are the isometric operators from $E_1(\Delta)\mathfrak{H}$ onto $E_0(\Delta)\mathfrak{H}$, and $Z^{(\pm)}(\Delta)^* = W^{(\pm)}(\Delta)$. However, they are not always true in our case, since $E_1(\Delta)$ defined by $(2.24)^+$ does not give the orthogonal projection.

With the aid of the above two lemmas, we can now derive some fundamental properties of $E_1(\Delta)$.

THEOREM 2.1. *The operator $E_1(\Delta)$ defined by $(2.24)^+$ is a projection (not necessarily orthogonal) which is permutable with L_1. Namely*

$$E_1(\Delta)^2 = E_1(\Delta)\,, \tag{2.27}$$

$$E_1(\Delta)L_1 \subseteqq L_1E_1(\Delta)\,. \tag{2.28}$$

Moreover, if Δ_1 and Δ_2 are two subintervals of $[a, b]$ such that they neither intersect a neighborhood of $\sigma(T)$, nor overlap each other, then

$$E_1(\Delta_1)E_1(\Delta_2) = 0\,, \tag{2.29}$$

$$E_1(\Delta_1+\Delta_2) = E_1(\Delta_1)+E_1(\Delta_2)\,. \tag{2.30}$$

PROOF. It follows from (2.23) that

$$E_1(\Delta)^2 = W^{(+)}(\Delta)Z^{(+)}(\Delta)W^{(+)}(\Delta)Z^{(+)}(\Delta) = W^{(+)}(\Delta)E_0(\Delta)Z^{(+)}(\Delta)\,.$$

Since $W^{(+)}(\Delta)E_0(\Delta) = W^{(+)}(\Delta)$, this implies (2.27). In order to verify (2.28) it suffices to show $E_1(\Delta)R_1(z) = R_1(z)E_1(\Delta)$. This follows immediately from (2.25). (2.29) follows from the relation $Z^{(+)}(\Delta_1)W^{(+)}(\Delta_2) = 0$, which is evident by (2.23). Finally (2.30) is obtained by the additivities of $W^{(+)}(\Delta)$ and $Z^{(+)}(\Delta)$ for Δ; that is,

$$\begin{array}{l} W^{(+)}(\Delta_1+\Delta_2) = W^{(+)}(\Delta_1)+W^{(+)}(\Delta_2)\,, \\ Z^{(+)}(\Delta_1+\Delta_2) = Z^{(+)}(\Delta_1)+Z^{(+)}(\Delta_2)\,. \end{array} \tag{2.31}$$

The theorem is proved.

We are now ready to show the similarity of L_1 to L_0. First we prove the following lemma:

LEMMA 2.7. *The following relations hold.*

$$W^{(\pm)}(\Delta)L_0 \subseteqq L_1W^{(\pm)}(\Delta), \quad L_0Z^{(\pm)}(\Delta) \supseteqq Z^{(\pm)}(\Delta)L_1\,. \tag{2.32}$$

PROOF. We have only to show that

(2.33) $$W^{(\pm)}(\Delta)R_0(z)=R_1(z)W^{(\pm)}(\Delta)^{11)}.$$

Since $X^{(\pm)}(\Delta)^*=Y^{*(\pm)}(\Delta)$, we have

$$\begin{aligned}\{R_1(z)W^{(\pm)}(\Delta)\}^* &= \{E_0(\Delta)-Y^{*(\pm)}(\Delta)\}R_1(z)^* \\ &= E_0(\Delta)R_1(z)^*-\operatorname*{s-lim}_{\varepsilon\to\pm 0} H^*(\Delta,\varepsilon,\varepsilon)R_1(z)^*.\end{aligned}$$

By (2.12)* and the relation $R_1(z)^*=R_1^*(\bar z)$ it follows that

$$\begin{aligned}&[H^*(\Delta,\varepsilon,\varepsilon)R_1(z)^*f](x)\\ &\quad=[E_0(\Delta)V^*R_1^*(\lambda+i\varepsilon)R_1^*(\bar z)f](x)|_{\lambda=x}\\ &\quad=[(\lambda+i\varepsilon-\bar z)^{-1}E_0(\Delta)V^*\{R_1^*(\lambda+i\varepsilon)-R_1^*(\bar z)\}f](x)|_{\lambda=x}\\ &\quad=[R_0(z+i\varepsilon)^*H^*(\Delta,\varepsilon,\varepsilon)f](x)-[R_0(z+i\varepsilon)^*E_0(\Delta)V^*R_1(z)^*f](x).\end{aligned}$$

Thus, by passing to the limit as $\varepsilon\to\pm 0$, we have

$$\begin{aligned}&\{R_1(z)W^{(\pm)}(\Delta)\}^*\\ &\quad=E_0(\Delta)R_1(z)^*+R_0(z)^*E_0(\Delta)V^*R_1(z)^*-R_0(z)^*Y^{*(\pm)}(\Delta)\\ &\quad=E_0(\Delta)\{R_1(z)+R_1(z)VR_0(z)\}^*-R_0(z)^*Y^{*(\pm)}(\Delta)\\ &\quad=E_0(\Delta)R_0(z)^*-R_0(z)^*Y^{*(\pm)}(\Delta)\\ &\quad=R_0(z)^*\{E_0(\Delta)-Y^{*(\pm)}(\Delta)\}\\ &\quad=\{W^{(\pm)}(\Delta)R_0(z)\}^*.\end{aligned}$$

This implies (2.33) and the lemma is proved.

Now, (2.28) shows that $E_1(\Delta)\mathfrak{H}$ is an invariant subspace of L_1. So, we can consider L_0 and L_1 as the operators acting on $E_0(\Delta)\mathfrak{H}$ and $E_1(\Delta)\mathfrak{H}$, respectively. Moreover, we see easily from (2.23) and (2.24)$^+$ that $W^{(\pm)}(\Delta)$ are 1−1 bounded mappings of $E_0(\Delta)\mathfrak{H}$ onto $E_1(\Delta)\mathfrak{H}$, and

(2.34) $$Z^{(\pm)}(\Delta)=W^{(\pm)}(\Delta)^{-1}.$$

Hence we have the following theorem.

THEOREM 2.2. *Let Δ be an arbitrary (possibly infinite) subinterval of $[a,b]$, which does not intersect with a neighborhood of $\sigma(T)$. If we restrict L_0 and L_1 to $E_0(\Delta)\mathfrak{H}$ and $E_1(\Delta)\mathfrak{H}$, respectively, then they are similar to each other. The similarity is established by the operators $W^{(\pm)}(\Delta)$; that is,*

(2.35) $$L_1=W^{(\pm)}(\Delta)L_0W^{(\pm)}(\Delta)^{-1}.$$

In concluding this section we remark that L_1 is *partly diagonalizable.* Namely, let $f\in\mathfrak{D}\cap E_1(\Delta)\mathfrak{H}$, then for any $g\in\mathfrak{H}$

11) In order to deduce $R_0(z)Z^{(\pm)}(\Delta)=Z^{(\pm)}(\Delta)R_1(z)$ from (2.33), multiply (2.33) by $Z^{(\pm)}(\Delta)$ from the left and the right and note that $Z^{(\pm)}(\Delta)E_1(\Delta)=Z^{(\pm)}(\Delta)$.

(2.36) $$\langle L_1 f, g\rangle=\int_\Delta \lambda d\langle E_1(\lambda)f, g\rangle, \quad \Delta=(\alpha, \beta),$$

where $E_1(\lambda)=E_1((\alpha, \lambda))$. In fact, by (2.35), we have

$$\langle L_1 f, g\rangle=\langle W^{(\pm)}(\Delta)L_0 Z^{(\pm)}(\Delta)f, g\rangle$$

$$=\int_\Delta \lambda d\langle E_0(\lambda)Z^{(\pm)}(\Delta)f, W^{(\pm)}(\Delta)^* g\rangle, \ E_0(\lambda)=E_0((\alpha, \lambda)).$$

By virtue of (2.31), however, it is easy to see that

$$W^{(\pm)}(\Delta)E_0(\lambda)Z^{(\pm)}(\Delta)=W^{(\pm)}((\alpha, \lambda))Z^{(\pm)}((\alpha, \lambda))=E_1(\lambda).$$

This implies (2.36).

3. Scattering theory

From the point of view of physical application, it is interesting to investigate the asymptotic behavior for $t\to\pm\infty$ of the solution $u(t)$ to the Schrodinger equation

(3.1) $$i\frac{\partial}{\partial t}u(t)=L_1 u(t), \quad u(0)=u_0 \qquad (u_0\in\mathfrak{H}).$$

In this section we wish to develop the time-dependent scattering theory, restricting the initial data u_0 to the space $E_1(\Delta)\mathfrak{H}$ which was studied in the previous section.

In order to discuss the solution $u(t)$ to (3.1), it will be required that there exists always a unique solution for any initial data $u_0\in\mathfrak{D}$. For this purpose only, we assume in this section the following additional condition[12].

CONDITION. *$-iL_1$ is an infinitesimal generator of the group* $\exp\{-iL_1 t\}$ $(-\infty<t<+\infty)$.

We remark that, in the case where $V-V^*$ is a bounded operator in $\mathfrak{H}$, the above condition is satisfied.

Now the solution $u(t)$ is obtained by

(3.2) $$u(t)=\exp\{-iL_1 t\}u_0, \qquad u_0\in\mathfrak{D}.$$

It is easily seen from (2.28) that $E_1(\Delta)\mathfrak{H}\cap\mathfrak{D}=E_1(\Delta)\mathfrak{D}$. Let

(3.3) $$u_0\in E_1(\Delta)\mathfrak{D}.$$

12) If we restrict $u(t)$ to a class of solutions $u(t)$ such that $\|u(t)\|\leqq\exp\{\theta|t|\}$ ($\theta>0$ may depend on u), then the uniqueness of the solution to (3.1) can be verified directly by using the Laplace transforms. Therefore, in this case, it is not necessary to impose the above condition. On the other hand, if we restrict from the beginning the operator L_1 to $E_1(\Delta)\mathfrak{H}$, then the uniqueness of the solution $u(t)\in E_1(\Delta)\mathfrak{H}$ follows easily from (2.36).

Then, as a simple computation shows, we can represent $u(t)$ in the form

$$(3.4)\qquad \begin{aligned} u(t) &= W^{(\pm)}(\Delta)\exp\{-iL_0t\}Z^{(\pm)}(\Delta)u_0 \\ &= \exp\{-iL_0t\}Z^{(\pm)}(\Delta)u_0 - X^{(\pm)}(\Delta)\exp\{-iL_0t\}Z^{(\pm)}(\Delta)u_0 . \end{aligned}$$

We wish to show that, as t tends to $\pm\infty$, the last term tends to 0 *strongly* in $\mathfrak{H}$, where we take either the upper signs or the lower signs throughout.

More precise formulation is given in the following lemma.

LEMMA 3.1. *Let $f(x)$ be any element of $E_0(\Delta)\mathfrak{H}$, then*

$$(3.5)\qquad \lim_{t\to\pm\infty} X^{(\pm)}(\Delta)\exp\{-iL_0t\}f = 0 \qquad (in\ \mathfrak{H}).$$

Before proving this lemma, let us notice the following fact.

LEMMA 3.2. (1) *For any $f\in\mathfrak{H}$,*

$$(3.6)\qquad \lim_{t\to\pm\infty}\int\frac{\exp\{-iyt\}f(y)}{x-(y\mp i0)}dy = 0 \qquad (in\ \mathfrak{H}).$$

(2) *Let $h(y)$ be an $\mathfrak{H}$-valued summable function of y. Then*

$$(3.7)\qquad \lim_{t\to\pm\infty}\int h(y)\exp\{-iyt\}dy = 0 \qquad (in\ \mathfrak{H}).$$

PROOF. The integral in (3.6) is nothing but

$$\Big[\text{v. p.}\frac{1}{x\pm i0}\Big]*\exp\{-ixt\}f(x) .$$

We consider the Fourier transform of this expression. As is well-known

$$\mathfrak{F}\Big[\text{v. p.}\frac{1}{x\pm i0}\Big] = \mp 2\pi i Y(\pm\xi),$$

where $Y(\xi)$ is the Heaviside function, and

$$\mathfrak{F}[\exp\{-ixt\}f(x)] = \hat{f}\Big(\xi+\frac{t}{2\pi}\Big).$$

Since $\lim_{t\to\pm\infty}\Big\|Y(\pm\xi)\hat{f}\Big(\xi+\frac{t}{2\pi}\Big)\Big\| = 0$, (3.6) is proved by applying the Plancherel theorem. (3.7) is a generalization of the Riemann-Lebesgue theorem.

PROOF OF LEMMA 3.1. Since $\|X^{(\pm)}(\Delta)\exp\{-iL_0t\}\| = \|X^{(\pm)}(\Delta)\|$, it suffices to prove the lemma under the assumption that $f(x)$ is continuous and has a compact support contained entirely in Δ. From the argument given in § 2, it is easily seen that

$$[X^{(\pm)}(\Delta)f](x) = \operatorname*{s-lim}_{\varepsilon\to+0}\int\frac{v_1(x, y\,;\, y\mp i0)f(y)}{x-(y\mp i\varepsilon)}dy .$$

Since $\exp\{-iL_0t\}f(x) = \exp\{-ixt\}f(x)$, we take a smooth function $\alpha(x)$ which takes the value 1 on the support of $f(x)$ and whose support is compact and contained entirely in Δ. Then, putting

$$\alpha(y)v_1(x, y; y\mp i0)=\varphi(x, y),$$

we see that

$$[X^{(\pm)}(\Delta)\exp\{-iL_0t\}f](x)=\int\frac{\varphi(x, y)f(y)}{x-(y\mp i0)}\exp\{-iyt\}dy.$$

We remark that the kernel $\varphi(x, y)$ generates an integral operator which belongs to the class $\mathfrak{M}_{\delta_1,\gamma_1}$, where $\delta_1>\frac{1}{2}$, $\gamma_1>\frac{1}{2}$.

Now the above integral can be decomposed as

$$\varphi(x, x)\int\frac{\exp\{-iyt\}f(y)}{x-(y\mp i0)}dy+\int\frac{\{\varphi(x, y)-\varphi(x, x)\}f(y)}{x-y}\exp\{-iyt\}dy.$$

As to the first integral, we apply (1) of Lemma 3.2 with the result

$$\begin{aligned}&\left\|\varphi(x, x)\int\frac{\exp\{-iyt\}f(y)}{x-(y\mp i0)}dy\right\|\\ &\qquad\leqq\sup_x|\varphi(x, x)|\cdot\left\|\int\frac{\exp\{-iyt\}f(y)}{x-(y\mp i0)}dy\right\|\to 0\quad(t\to\pm\infty).\end{aligned}$$

To the second integral we can apply (2) of Lemma 3.2. To see this, put

$$h(x, y)=\frac{\{\varphi(x, y)-\varphi(x, x)\}f(y)}{x-y}.$$

As an $\mathfrak{H}$-valued function of y, $h(\cdot, y)$ is continuous and has a compact support. In fact, let us decompose this into two parts:

$$h(x, y)=\left(\frac{\{\varphi(x, y)-\varphi(x, x)\}f(y)}{x-y}\right)_{|x-y|\leqq\varepsilon}+(\;\;,,\;\;)_{|x-y|\geqq\varepsilon}.$$

The second term is, for fixed ε, an $\mathfrak{H}$-valued continuous function of y. On the other hand, using the inequality

$$|\varphi(x, y)-\varphi(x, x)|\leqq\text{const}\,(1+|y|)^{-\delta_1}|x-y|^{\gamma_1}\qquad(|x-y|\leqq 1),$$

one can obtain the $\mathfrak{H}$-norm of the first term is estimated by

$$\text{const}\left(\int_{|x|\leqq\varepsilon}|x|^{2(\gamma_1-1)}dx\right)^{1/2}(1+|y|)^{-\delta_1}|f(y)|.$$

Since $\gamma_1>\frac{1}{2}$, this can be made arbitrarily small by making ε sufficiently small. The lemma is proved.

We have thus proved the following:

THEOREM 3.1. *For any* $u_0\in E_1(\Delta)\mathfrak{H}$, *put*

$$f^{(\pm)}=Z^{(\pm)}(\Delta)u_0\qquad(\in E_0(\Delta)\mathfrak{H}).\tag{3.8}$$

Then the solution $u(t)$ *to equatian* (3.1) *with the initial value* u_0 *satisfies the* *following asymptotic conditions.*

(3.9) $$\lim_{t\to\pm\infty} \| u(t) - \exp\{-iL_0 t\} f^{(\pm)} \| = 0 .$$

Finally, if we define the operator

(3.10) $$S(\Delta) = Z^{(+)}(\Delta) W^{(-)}(\Delta) ,$$

then we see from (3.8) and (3.9) that $S(\Delta)$ gives the so-called scattering operator. Actually, the relation

(3.11) $$f^{(+)} = S(\Delta) f^{(-)}$$

follows immediately from relation (2.24)$^+$. Moreover we have from (2.32)

(3.12) $$S(\Delta) L_0 \subseteq L_0 S(\Delta) .$$

We can verify that $S(\Delta)$ is a multiplicative operator

(3.13) $$[S(\Delta) f](x) = \{1 - 2\pi i v_1(x, x; x+i0)\} f(x), \qquad x \in \Delta .$$

In order to show this we use the relation

$$\begin{aligned} Z^{(+)}(\Delta) W^{(-)}(\Delta) &= \{E_0(\Delta) - Y^{(+)}(\Delta)\}\{E_0(\Delta) - X^{(-)}(\Delta)\} \\ &= E_0(\Delta) - Y^{(+)}(\Delta) E_0(\Delta) - E_0(\Delta) X^{(-)}(\Delta) + Y^{(+)}(\Delta) X^{(-)}(\Delta) , \end{aligned}$$

where $Y^{(+)}(\Delta) = H(\Delta, +0, +0)$ and $X^{(-)}(\Delta) = K(\Delta, -0, -0)$. Since we have

$$\begin{aligned} Y^{(+)}(\Delta) X^{(-)}(\Delta) &= E_0(\Delta) Y^{(+)}(\Delta) X^{(-)}(\Delta) E_0(\Delta) \\ &= H(\Delta, +0, +0) E_0(\Delta) + E_0(\Delta) K(\Delta, -0, +0) \end{aligned}$$

by using the same method as we proved Lemma 2.5, it follows that

$$S(\Delta) = E_0(\Delta)\{I + K(\Delta, -0, +0) - K(\Delta, -0, -0)\} .$$

Namely, if $f(x)$ is a smooth function with a compact support contained in Δ, we have

$$\begin{aligned} &[S(\Delta) f](x) - [E_0(\Delta) f](x) \\ &= \lim_{\varepsilon\to+\infty} \int \left\{ \frac{v_1(x, y; y+i0)}{x-(y-i\varepsilon)} - \frac{v_1(x, y; y+i0)}{x-(y+i\varepsilon)} \right\} f(y) dy \\ &= -2\pi i v_1(x, x; x+i0) f(x) . \end{aligned}$$

This proves (3.13). Analogously to (3.10) we put

(3.10)* $$S^*(\Delta) = Z^{*(+)}(\Delta) W^{*(-)}(\Delta) .$$

Then we see similarly

(3.13)* $$[S^*(\Delta) f](x) = \{1 - 2\pi i v_1^*(x, x; x+i0)\} f(x). \qquad x \in \Delta .$$

Applying (2.20), we see easily that as an operator defined in $E_0(\Delta)\mathfrak{H}$ $S^*(\Delta)^*$ is the bounded inverse of $S(\Delta)$, namely

$$S(\Delta)S^*(\Delta)^* = S^*(\Delta)^*S(\Delta) = E_0(\Delta)\,. \tag{3.14}$$

4. Perturbation by a product operator

In this section we wish to show the completeness of the spectral resolution $E_1(\Delta)$ obtained in § 2. For this aim we require some additional assumptions on V. We suppose that V is a product operator of the form

$$V = B^*A\,, \tag{4.1}$$

where A and B are both integral operators belonging to the class $\mathfrak{M}_{\delta_2,\gamma_2}$ $\left(\frac{3}{4} < \delta_2 < 1,\, 0 < \gamma_2 < 1\right)$ introduced in Definition of § 2. We denote the kernels of A and B by $a(x, y)$ and $b(x, y)$, respectively.

Let us notice that $V = B^*A$ is a special case of the operator which was investigated in the previous sections. In fact, since the domain $\mathfrak{D}(A)$ of A, for example, includes the set

$$\left\{f \in \mathfrak{H}\,;\ \int (1+|x|)^{-1/4}|f(x)|\,dx < +\infty\right\} \supset \left\{f \in \mathfrak{H}\,;\ \int |f(x)|\,dx < +\infty\right\},$$

it follows from the inequality

$$\int |f(x)|\,dx \leqq \left\{\int (1+|x|)^{-2}dx \int (1+|x|)^2|f(x)|^2 dx\right\}^{1/2}$$

that $\mathfrak{D} \subset \mathfrak{D}(A)$. The image $A\mathfrak{D}$ also belongs to $\mathfrak{D}(A)$. For, if we set $g(x) = \int a(x, y)f(y)dy$. then we have $g \in \mathfrak{H}$ and

$$|g(x)| \leqq \text{const}\,(1+|x|)^{-\delta_2}\int |f(y)|\,dy\,,$$

and therefore $(1+|x|)^{-1/4}g(x)$ is summable. The above argument equally applies to B^*, and hence B^*A becomes an integral operator, defined on $\mathfrak{D}$, generated by the kernel

$$v(x, y) = \int b^*(x, u)a(u, y)du, \quad b^*(x, u) = \bar{b}(u, x)\,.$$

We see easily $V \in \mathfrak{M}_{\delta_3,\gamma_3}$, where $\frac{1}{2} < \delta_3 < 2\delta_2 - 1$ and $0 < \gamma_3 \leqq \gamma_2$.

Following several authors[13] we now consider the operator $Q(z)$ defined by

13) We denote by $[R_0(z)B^*]$ the adjoint operator of $BR_0(z)^*$, which is bounded uniformly for $z \in \tilde{\Pi}$ as we have shown by Lemma 2.3. The operator $Q(z)$ was initially considered in [8] under similar conditions on A and B. Kato [6] developed his interesting method investigating carefully the property of the operator $Q(z)$.

(4.2) $$Q(z)=A[R_0(z)B^*].$$

Once we prove that $Q(z)$, $z\in\Pi$, is everywhere defined bounded operator such that $I+Q(z)$ has the everywhere defined bounded inverse, then it can be easily shown that

(4.3) $$R_1(z)=R_0(z)-[R_0(z)B^*]\{I+Q(z)\}^{-1}AR_0(z),\qquad z\in\Pi.$$

In fact, as we see easily (cf. Lemma 4.1), the invertibility (in $\mathfrak{H}$) of the operators $I+B^*AR_0(z)$ and $I+AR_0(z)B^*$ is equivalent. Since the trivial identity $\{I+B^*AR_0(z)\}B^*=B^*\{I+AR_0(z)B^*\}$ implies $B^*\{I+AR_0(z)B^*\}^{-1}\supseteqq\{I+B^*AR_0(z)\}^{-1}B^*$, we have $V_1(z)=\{I+B^*AR_0(z)\}^{-1}B^*A=B^*\{I+AR_0(z)B^*\}^{-1}A$. Now (4.3) is nothing but (1.5).

We next verify the above mentioned property of $Q(z)$. It is easy to see that $Q(z)$ is an integral operator with the kernel

$$q(x,y;z)=\int\frac{a(x,u)b^*(u,y)}{u-z}du.$$

By the use of inequality (1.17), $q(x,y;z)$ is estimated as follows:

(4.4)
$$|q(x,y;z)|\leqq\text{const}\,(1+|x|)^{-\delta_3}(1+|y|)^{-\delta_3}(1+|z|)^{-\alpha_3},$$
$$|q(x,y;z)-q(x+\Delta x,y+\Delta y;z+\Delta z)|$$
$$\leqq\text{const}\,(1+|x|)^{-\delta_3}(1+|y|)^{-\delta_3}(1+|z|)^{-\alpha_3}(|\Delta x|^{\gamma_3}+|\Delta y|^{\gamma_3}+|\Delta z|^{\gamma_3}),$$

where $0<\alpha_3<2(\delta_2-\delta_3)$. The first inequality shows that

$$\iint|q(x,y;z)|^2dxdy\leqq\text{const}\,(1+|z|)^{-2\alpha_3}.$$

This implies that $Q(z)$ is a completely continuous operator in $\mathfrak{H}$ for any $z\in\Pi$. Moreover it follows from (4.4) that

(4.5)
$$\|Q(z)\|\leqq\text{const}\,(1+|z|)^{-\alpha_3},$$
$$\|Q(z)-Q(z+\Delta z)\|\leqq\text{const}\,(1+|z|)^{-\alpha_3}|\Delta z|^{\gamma_3}.$$

By the second inequality, we see that there exist the uniform limits $Q(\lambda\pm i0)$, $\lambda\in[a,b]$. They are also completely continuous and satisfy the inequalities (4.5) with the same constants.

We call a value $z\in\tilde{\Pi}$, for which the homogeneous equation

(4.6) $$\tilde{\varphi}+Q(z)\tilde{\varphi}=0\qquad(\tilde{\varphi}\in\mathfrak{H})$$

has a non-trivial solution, a *singular point* of $Q(z)$. Then we have the

LEMMA 4.1. *$z\in\tilde{\Pi}$ is a singular point of $Q(z)$ if and only if it is a singular point of $T(z)$, i.e., $z\in\sigma(T)$.*

PROOF. Let $z\in\sigma(T)$, and $\varphi(x)$ be a corresponding non-trivial solution of

(1.20) : $\varphi+T(z)\varphi=\varphi+B^*AR_0(z)\varphi=0$. Applying $AR_0(z)$ from the left and putting $\tilde{\varphi}\equiv AR_0(z)\varphi$, we have formally $\tilde{\varphi}+Q(z)\tilde{\varphi}=0$. Clearly $\tilde{\varphi}\in\mathfrak{H}$ and $\tilde{\varphi}\neq 0$. For $\tilde{\varphi}=0$ yields $\varphi=-B^*AR_0(z)\varphi=-B^*\tilde{\varphi}=0$, which contradicts our assumption. Convesely if $\tilde{\varphi}\in\mathfrak{H}$ satisfies (4.6), then $\varphi\equiv B^*\tilde{\varphi}\in\mathfrak{B}_{\delta,\gamma_2}$ $\left(0<\delta<\delta_2-\frac{1}{2}\right)$ and $\varphi\neq 0$. Moreover, $\varphi=-B^*Q(z)\tilde{\varphi}=-B^*AR_0(z)\varphi=-VR_0(z)\varphi$. This implies that $\varphi+T(z)\varphi=0$. Thus $z\in\sigma(T)$, and this finishes the proof of the lemma.

Thus we see that $I+Q(z)$ has a bounded inverse for any $z\in\tilde{\Pi}$ except the *singular points* of $T(z)$. Moreover, there exists a positive constant M independent of z such that $\|\{I+Q(z)\}^{-1}\|\leqq M$, when z runs over any domain (in $\tilde{\Pi}$) which is disjoint from a neighborhood of $\sigma(T)$. In fact, $I+Q(z)$ is holomorphic in such a domain, and if $|z|$ is large, then $\{I+Q(z)\}^{-1}$ is obtained by the Neumann series in view of (4.5). We then define the operator

$$A_1(z)=\{I+Q(z)\}^{-1}A \qquad (z\in\tilde{\Pi}), \tag{4.7}$$

which has $\mathfrak{D}(A)$ as its domain. Then we see from (4.3) that

$$AR_1(z)=A_1(z)R_0(z). \tag{4.8}$$

$A_1(z)$ is represented by a kernel $a_1(x, y; z)$ which can be estimated in a way similar to $q(x, y; z)$ in (4.4) if we replace the constants δ_3, γ_3 and α_3 by $\delta(<\delta_3)$, $\gamma(<\gamma_3)$ and $\alpha(<\gamma_3)$, respectively, where δ, γ and α can be chosen as close to the respective constants δ_3, γ_3 and γ_3 as we wish.

We can now give a sufficient condition for the completeness of the spectral resolution $E_1(\Delta)$. We denote the eigenvalues of L_1 by $\lambda_\nu(\nu=1, 2, \cdots)$, and denote by $P_\nu(\nu=1, 2, \cdots)$ the projections on the corresponding root subspaces. If λ_ν is isolated, then P_ν is represented as follows (see, e. g., [5]):

$$P_\nu=-\frac{1}{2\pi i}\oint_{\Gamma(\lambda_\nu)}R_1(z)dz, \tag{4.9}$$

where $\Gamma(\lambda_\nu)$ is a small circle with centre λ_ν. Suppose that there exists no *singular point* of $Q(z)$ on either edge of (a, b). Moreover we suppose that a and b are not singular points of $Q(z)$ if a or b is finite. Then $E_1(\Delta)$ is defined for any subinterval Δ of (a, b). We can verify that L_1 has at most a finite number of discrete eigenvalues. In fact, we know from (2) of Theorem 1.1 that the accumulation points of discrete eigenvalues are on the edges of $[a, b]$. It follows from Lemma 4.1 that they are singular points of $Q(z)$. Hence, the above assumption implies the finiteness of the discrete eigenvalues of L_1. By virtue of Lemma 1.4 we see similarly that there exists no eigenvalue of L_1 in $[a, b]$. Let $P_0=\Sigma_0 P_\nu$ be the sum of the projections corresponding to the real eigenvalues lying outside $[a, b]$, and let $P_+=\Sigma_+P_\nu$ and $P_-=\Sigma_-P_\nu$ be the sums of the projections corresponding to the non-real eigenvalues with positive and negative imaginary parts, respectively.

THEOREM 4.1. *Suppose that*

(1) $V=B^*A$, *where* $A, B\in\mathfrak{M}_{\delta_2,\gamma_2}$ $\left(\frac{3}{4}<\delta_2<1,\ 0<\gamma_2<1\right)$;

(2) *there exist no singular points of* $Q(z)$ *on either edge of* (a, b) *and the points* a *and* b *are also non-singular points of* $Q(z)$.

Then the spectral resolution $E_1(\Delta)$ *is complete. Namely,*

$$E_1((a, b))=I-\sum_{\nu=1}^{N} P_\nu\,, \tag{4.10}$$

where P_ν *is the projection given by* (4.9), *and* N *is the number of the discrete eigenvalues of* L_1.

PROOF. For the sake of simplicity we prove (4.10) in the case where a is finite and $b=+\infty$. The same reasoning can be applied to the other cases. We start from the following relation

$$\begin{aligned}&\langle\{R_0(\lambda+i\varepsilon)-R_0(\lambda-i\varepsilon)\}f, g\rangle-\langle\{R_1(\lambda+i\varepsilon)-R_1(\lambda-i\varepsilon)\}f, g\rangle\\&\quad=\langle R_0(\lambda+i\varepsilon)VR_1(\lambda+i\varepsilon)f, g\rangle-\langle R_0(\lambda-i\varepsilon)VR_1(\lambda-i\varepsilon)f, g\rangle\,,\end{aligned} \tag{4.11}$$

which follows from the second resolvent equation. We choose f and g in the class $\mathfrak{B}_{\delta_3,\gamma_3}$. Let $r>|a|$ be a large constant such that all the eigenvalues of L_1 lie in the disk $\{z:|z|<r\}$, and let us take the integral of both sides of (4.11) over $(-r, r)$.

It follows immediately from (2.14) that

$$\lim_{\varepsilon\to+0}\int_{-r}^{r}\langle\{R_0(\lambda+i\varepsilon)-R_0(\lambda-i\varepsilon)\}f, g\rangle d\lambda=2\pi i\langle E_0((a, r))f, g\rangle\,.$$

For dealing with the second term on the left side of (4.11), we suppose that ε is sufficiently small so that the imaginary parts of all non-real eigenvalues are greater than ε. Then by means of the Cauchy integral formula, we have

$$\begin{aligned}&\int_{-r}^{r}\langle\{R_1(\lambda+i\varepsilon)-R_1(\lambda-i\varepsilon)\}f, g\rangle d\lambda=\int_{-r}^{a-s}+\int_{a-s}^{r}\\&\quad=2\pi i\Sigma_0\langle P_\nu f, g\rangle+\int_{a-s}^{r}\langle\{R_1(\lambda+i\varepsilon)-R_1(\lambda-i\varepsilon)\}f, g\rangle d\lambda\\&\qquad+\left(\int_{a-s-i\varepsilon}^{a-s+i\varepsilon}-\int_{-r-i\varepsilon}^{-r+i\varepsilon}\right)\langle R_1(z)f, g\rangle dz\,,\end{aligned}$$

where $s>0$ is a small constant such that there exists no eigenvalue of L_1 in $[a-s, a]$. It is easy to see that the last term converges to zero as $\varepsilon\to+0$. Hence it follows that

$$\begin{aligned}&\lim_{\varepsilon\to+0}\int_{-r}^{r}\langle\{R_1(\lambda+i\varepsilon)-R_1(\lambda-i\varepsilon)\}f, g\rangle d\lambda\\&\quad=2\pi i\Sigma_0\langle P_\nu f, g\rangle+2\pi i\langle E_1((a, r))f, g\rangle\end{aligned}$$

if one takes relation (2.25) into account.

Next let us consider the right-hand side of (4.11). Using again the Cauchy integral formula, we have

$$\int_{-r}^{r}\langle R_0(\lambda+i\varepsilon)VR_1(\lambda+i\varepsilon)f, g\rangle d\lambda$$

$$=\Sigma_+\oint_{\Gamma(\lambda_\nu)}\langle R_0(z)VR_1(z)f, g\rangle dz-\theta_+(r, \varepsilon),$$

$$\int_{-r}^{r}\langle R_0(\lambda-i\varepsilon)VR_1(\lambda-i\varepsilon)f, g\rangle d\lambda$$

$$=-\Sigma_-\oint_{\Gamma(\lambda_\nu)}\langle R_0(z)VR_1(z)f, g\rangle dz+\theta_-(r, \varepsilon),$$

where

$$\theta_+(r, \varepsilon)=\oint_{\substack{|z|=\sqrt{r^2+\varepsilon^2}\\ -\mathrm{Im}[z]\leqq-\varepsilon}}\langle R_0(z)VR_1(z)f, g\rangle dz,$$

$$\theta_-(r, \varepsilon)=\oint_{\substack{|z|=\sqrt{r^2+\varepsilon^2}\\ \mathrm{Im}[z]\leqq-\varepsilon}}\langle R_0(z)VR_1(z)f, g\rangle dz.$$

Since $R_0(z)VR_1(z)=R_0(z)-R_1(z)$, and $R_0(z)$ is holomorphic at $z=\lambda_\nu$, we see

$$\Sigma_+\oint_{\Gamma(\lambda_\nu)}\langle R_0(z)VR_1(z)f, g\rangle dz=2\pi i\Sigma_+\langle P_\nu f, g\rangle,$$

$$\Sigma_-\oint_{\Gamma(\lambda_\nu)}\langle R_0(z)VR_1(z)f, g\rangle dz=2\pi i\Sigma_-\langle P_\nu f, g\rangle.$$

On the other hand, by the assumption on B, it follows that

$$\|BR_0(z)^*g\|\leqq \text{const}\,(1+|z|)^{-\delta}\|g\|_{\delta_3,r_3},\ \frac{1}{2}<\delta<\delta_3.$$

If we notice that $\{I+Q(z)\}^{-1}$ is uniformly bounded, we see from (4.7) that $AR_1(z)f=A_1(z)R_0(z)f$ is also estimated by similar inequalities. Hence we have

$$|\theta_+(r, \varepsilon)+\theta_-(r, \varepsilon)|=\oint_{\substack{|z|=\sqrt{r^2+\varepsilon^2}\\ |\mathrm{Im}[z]|\geqq\varepsilon}}\langle A_1(z)R_0(z)f, BR_0(z)^*g\rangle dz|$$

$$\leqq \text{const}\,(1+r)^{-2\delta+1}\|f\|_{\delta_3,r_3}\|g\|_{\delta_3,r_3}.$$

By the passage to the limit, $\theta(r)=\lim_{\varepsilon\to+0}\{\theta_+(r, \varepsilon)+\theta_-(r, \varepsilon)\}$ exists, and is estimated for sufficiently large r by $|\theta(r)|\leqq\text{const}\,(1+r)^{-2\delta+1}$. Thus

$$\lim_{\varepsilon\to+0}\int_{-r}^{r}\Big\{\langle R_0(\lambda+i\varepsilon)VR_1(\lambda+i\varepsilon)f, g\rangle-\langle R_0(\lambda-i\varepsilon)VR_1(\lambda-i\varepsilon)f, g\rangle\Big\}d\lambda$$

$$=2\pi i\Big\{\Sigma_+\langle P_\nu f, g\rangle+\Sigma_-\langle P_\nu f, g\rangle\Big\}+0(r^{-2\delta+1}).$$

Summarizing these results, we can finally get

$$\langle E_1((a, r))f, g\rangle = \langle E_0((a, r))f, g\rangle - \sum_{\nu=1}^{N} \langle P_\nu f, g\rangle + 0(r^{-2\delta+1}).$$

Let r tend to $+\infty$. Then, since $\langle E_0((a, r))f, g\rangle \to \langle f, g\rangle$ and $\langle E_1((a, r))f, g\rangle \to \langle E_1((a, b))f, g\rangle$, it follows from $\delta > \frac{1}{2}$ that[14)]

$$\langle E_1((a, b))f, g\rangle = \langle f, g\rangle - \sum_{\nu=1}^{N} \langle P_\nu f, g\rangle.$$

Since $\mathfrak{B}_{\delta_3,\gamma_3}$ is dense in $\mathfrak{H}$, this yields (4.10), and the theorem is proved.

5. Final remark

In this section we consider the case when V is dissipative. Namely, let

$$V = V_1 + iV_2. \tag{5.1}$$

where the operator V_1 is symmetric and the operator V_2 is positive definite, i.e., $V_2 = A^*A$. We assume that $V_1, A \in \mathfrak{M}_{\delta_2,\gamma_2}$, where $\delta_2 > \frac{3}{4}$ and $0 < \gamma_2 < 1$. For the sake of simplicity, we assume moreover that V_1 is a *small perturbation.* Then we know by [4] that there exists a unitary operator U such that

$$U^*(L_0 + V_1)U = L_0. \tag{5.2}$$

We see more: U is a bounded operator acting on $\mathfrak{B}_{\delta'_2,\gamma'_2}$, where $\delta'_2(<\delta_2)$ and $\gamma'_2(<\gamma_2)$ can be chosen as close to δ_2 and γ_2 as we wish. So the operator AU belongs again to the class $\mathfrak{M}_{\delta'_2,\gamma'_2}$. Thus from the beginning, we can set L_1 as follows:

$$L_1 = L_0 + V = L_0 + iA^*A, \qquad A \in \mathfrak{M}_{\delta_2,\gamma_2}. \tag{5.3}$$

In this case we have the following lemma.

LEMMA 5.1. *There exists no singular point of $Q(z)$ on the lower edge of (a, b). The points a and b are also non-singular points of $Q(z)$.*

PROOF. Let $\tilde{\varphi} + Q(\lambda - i0)\tilde{\varphi} = 0$, where $Q(z) = iA[R_0(z)A^*]$. Then it follows that

$$\begin{aligned} 0 &= \|\tilde{\varphi}\|^2 + \lim_{\varepsilon\to+0} \langle iAR_0(\lambda - i\varepsilon)A^*\tilde{\varphi}, \tilde{\varphi}\rangle \\ &= \|\tilde{\varphi}\|^2 + \lim_{\varepsilon\to+0} i\int \frac{|[A^*\tilde{\varphi}](x)|^2}{x-(\lambda - i\varepsilon)}dx \\ &= \|\tilde{\varphi}\|^2 + \pi|[A^*\tilde{\varphi}](\lambda)|^2 + iP\int \frac{|[A^*\tilde{\varphi}](x)|^2}{x-\lambda}dx. \end{aligned}$$

14) To factor V as B^*A is only needed to get the above estimate of $\theta_\pm$. Otherwise, we could only get the factor $(1+r)^{-\delta+1}$ which would not work for the present purpose.

Since $P\int\frac{|[A^*\tilde{\varphi}](x)|^2}{x-\lambda}dx$ is real, it follows that

$$\|\tilde{\varphi}\|^2+\pi|[A^*\tilde{\varphi}](\lambda)|^2=0.$$

This implies $\tilde{\varphi}(x)\equiv 0$, which is to be proved. The lemma is proved.

Thus, by Lemma 1.4, both L_1 and L_1^* have no eigenvalue on the real segment $[a, b]$. Moreover we see that $Y^{(-)}(\Delta)$, $X^{(+)}(\Delta)$, $Y^{*(+)}(\Delta)$ and $X^{*(-)}(\Delta)$ exist even for $\Delta=(a, b)$. Put

$$W^{(+)}=I-X^{(+)}((a, b)), \qquad Z^{(-)}=I-Y^{(-)}((a, b)). \tag{5.4}$$

$$W^{*(-)}=I-X^{*(-)}((a, b)), \qquad Z^{*(+)}=I-Y^{*(+)}((a, b)). \tag{5.4}^*$$

Since $W^{(+)*}=Z^{*(+)}$, the range of $W^{(+)}$ is orthogonal to the null space of $Z^{*(+)}$, which includes all the root subspaces of L_1^* corresponding to the discrete eigenvalues. It might be expected that the null space of $Z^{*(+)}$ coincides with the direct sum of the root subspaces of L_1^*, and the range of $W^{(+)}$ is the orthogonal complement of the null space of $Z^{*(+)}$. However we do not have now any affirmative answers.

In conclusion, let us consider again equation (3.1). As we see easily, $-iL_1=-iL_0+A^*A$ is the infinitesimal generator of the semi-group $\exp\{-iL_1t\}$ $(t\leqq 0)$. Consequently, for any initial data $u_0\in\mathfrak{D}$, the solution $u(t)$ is obtained by

$$u(t)=\exp\{-iL_1t\}u_0 \qquad (t\leqq 0). \tag{5.5}$$

Here, if we assume u_0 belongs to the range of $W^{(+)}$, i.e., $u_0=W^{(+)}f_0$ $(f_0\in\mathfrak{D})$, then $u(t)$ is represented by

$$u(t)=W^{(+)}\exp\{-iL_0t\}f_0 \qquad (t\leqq 0). \tag{5.6}$$

Let us consider the function $\exp\{iL_0t\}W^{(+)}\exp\{-iL_0t\}f_0$. We have from (5.4)

$$\begin{aligned}&[\exp\{iL_0t\}W^{(+)}\exp\{-iL_0t\}f_0](x)-f_0(x)\\ &=-\lim_{\varepsilon\to+0}\int\frac{v_1(x, y; y-i0)}{x-(y-i\varepsilon)}\exp\{i(x-y)t\}f_0(y)dy\\ &=-P\int\frac{v_1(x, y; y-i0)}{x-y}\exp\{i(x-y)t\}f_0(y)dy+\pi i v_1(x, x; x-i0)f_0(x).\end{aligned}$$

On the other hand it is easily verified that

$$\lim_{-t\to\infty}P\int\frac{\exp\{i(x-y)t\}}{x-y}dy=-\pi i.$$

Hence we get

$$\begin{aligned}&\operatorname*{s-lim}_{t\to-\infty}[\exp\{iL_0t\}W^{(+)}\exp\{-iL_0t\}f_0](x)-\{1+2\pi i v_1(x, x; x-i0)\}f_0(x)\\ &=\operatorname*{s-lim}_{t\to-\infty}P\int\frac{\{v_1(x, y; y-i0)-v_1(x, x; x-i0)\}f_0(y)}{y-x}\exp\{i(x-y)t\}dy.\end{aligned}$$

Applying Lemma 3.2 (2), we see that the right side is equal to zero. This shows that

$$\underset{t\to-\infty}{\text{s-lim}}\,[u(x, t)-\exp\{-ixt\}\{1+2\pi i v_1(x, x; x-i0)\}f_0(x)]=0.$$

Analogously to (3.13) it is not difficult to see

$$\{1+2\pi i v_1(x, x; x-i0)\}f_0(x)=[Z^{(-)}W^{(+)}f_0](x). \tag{5.7}$$

Hence we conclude that the solution $u(t)$ to equation (3.1) with the initial condition $u(0)=W^{(+)}f_0$ $(f_0\in\mathfrak{D})$ has the following asymptotic behavior

$$\lim_{t\to-\infty}\|u(t)-\exp\{-iL_0t\}Z^{(-)}W^{(+)}f_0\|=0. \tag{5.8}$$

Let us remark that, since we cannot say anything about the uniqueness of the solution for $t\geqq 0$ (see foot-note 12)), we have no information, in this case, about the asymptotic behavior of the solution $u(t)$ for $t\to+\infty$.

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