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On the pre-closedness of the potential operator

Dedicated to Professor Iyanaga on his 60th birthday

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§1. **Introduction.** Let X be a separable, locally compact, non-compact Hausdorff space, and B be the completion with respect to the maximum norm of the space $C_0(X)$ of real-valued continuous functions with compact supports defined in X. G. A. Hunt [1] introduced the notion of the potential operator V as a positive linear operator on $D(V) \subseteqq B$ with $D(V) \supseteqq C_0(X)$ into B satisfying the "principle of positive maximum"[1]:

(1) For any $f \in C_0(X)$, we have $\sup_{f(x)>0} (Vf)(x) = \sup_{x \in X} (Vf)(x)$ if the latter supremum is positive.

The fundamental result of Hunt reads as follows:

THEOREM. *Let V satisfy* (1) *and the condition that*

(2) $V \cdot C_0(X)$ *is dense in* B.

Then, there exists a uniquely determined semi-group $\{T_t; t \geqq 0\}$ *of class* (C_0) *of positive contraction linear operators* T_t *on* B *into* B *such that*

(3) $AVf = -f$, $f \in C_0(X)$, *for the infinitesimal generator* A *of* T_t.

An operator-theoretical proof of this theorem was given in K. Yosida [2], showing that the resolvent $J_\lambda = (\lambda I - A)^{-1}$, $\lambda > 0$, of A is the continuous extension to the whole space B of the operator $\hat{J}_\lambda$ defined by

(4) $$\lambda Vf + f \to Vf, \qquad f \in C_0(X),$$

with an additional remark that

(5) V^{-1} exists and $V^{-1} = -A$ if and only if V is closed.

The purpose of the present note is to show that *the restriction $V|C_0(X)$ of V to $C_0(X)$ is pre-closed so that its smallest closed extention, which shall be*

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1) This principle, sometimes called as the "weak principle of positive maximum", is proved on page 220 of [2] in the course of the proof of:

(1)′ For any $f \in C_0(X)$, the condition $(Vf)(x_0) = \sup_{x \in X} (Vf)(x)$ implies $f(x_0) \geqq 0$.

It is also proved on the same page that (1)′ is a consequence of (1) and (2).

denoted by the same letter V, satisfies the true Poisson equation for the potential of functions:

(5)′ $$V=-A^{-1}.$$

The closure property of V is important, since, as in [2], we can prove:

(6) for any $\lambda>0$, the inverse $(\lambda V+I)^{-1}$ exists as a continuous linear operator on B into B so that $J_\lambda=(\lambda I-A)^{-1}=V(\lambda V+I)^{-1}$. In particular, for any $g\in B$, there exists a uniquely determined f in the domain $D(V)$ of V with $\lambda Vf+f=g$.

Thus, applying the closed range theorem and its corollary in K. Yosida [3] to the closed linear operator $(\lambda V+I)$, we obtain:

(7) for any $\lambda>0$, the inverse $(\lambda V^*+I^*)^{-1}$, of the dual operator (λV^*+I^*) of $(\lambda V+I)$ exists as a continuous linear operator on the dual space B^* of B into B^*. In particular, for any measure $\gamma\in B^*$, there exists a uniquely determined measure $\varphi\in D(V^*)$ such that $\lambda V^*\varphi+\varphi=\gamma$.

Moreover, since the domain $D(A)$ of the infinitesimal generator A is dense in B, we have $(A^*)^{-1}=(A^{-1})^*$ by the denseness in B of the range $R(A)=D(V)$. For the proof, see p. 224 in K. Yosida [3]. Therefore, by (5)′, we have *the true Poisson equation for the potential of measures:*

(5)″ $$V^*=-(A^*)^{-1}.$$

§2. **Proofs of the pre-closedness of the operator** $V|C_0(X)$. We have to prove $g=0$ from $f_n\in C_0(X)$, $s-\lim_{n\to\infty}f_n=0$ and $s-\lim_{n\to\infty}Vf_n=g$.

THE FIRST PROOF (by Tanaka). From (33) in [2], we have $AVf=-f$. Thus, by the closure property of the operator A, we have $Ag=0$. By $J_\lambda=(\lambda I-A)^{-1}$, we have $(I-\lambda J_\lambda)g=-J_\lambda Ag=0$ so that $g=s-\lim_{\lambda\downarrow 0}\lambda J_\lambda g$. That the latter $s-\lim$ is $=0$ is proved in the paragraph following (35) of [2].

THE SECOND PROOF (by Yoshida). By (32) in [2], we have $Vf=\lambda J_\lambda Vf+J_\lambda f$ so that $g=s-\lim_{n\to\infty}Vf_n=\lambda J_\lambda g$. Hence $g=0$ as in the first proof.

THE THIRD PROOF (by Watanabe). It is straightforward in the sense that it only makes use of the principle of positive maximum. It reads as follows.

Suppose $g(x_0)>0$ for some $x_0\in X$. Let K be any compact set of X such that $K\ni x_0$. Let $h\in C_0(X)$ be such that $h(x)=1$ on K. Then $\|V(f_nh)\|\leqq\|f_n\|\cdot\|Vh\|$. Hence $s-\lim_{n\to\infty}V(1-h)f_n=g$. Choose n so large that

(8) $$|(V(1-h)f_n)(x)-g(x)|<\frac{1}{4}g(x_0)\qquad\text{for all } x.$$

Then

$$(V(1-h)f_n)(x_0) \geqq \frac{3}{4} g(x_0) > 0, \text{ and support } ((1-h)f_n) \subseteqq X-K.$$

By the principle of positive maximum, there exists thus a point $x_1 \in X-K$ such that $(V(1-h)f_n)(x_1) \geqq (V(1-h)f_n)(x_0)$ and so, by (8), $|g(x_1)| \geqq \frac{1}{2} g(x_0)$. Since K was arbitrary, this contradicts to the fact that $g(x) \in B$ tends to zero at infinity. Therefore g must be $\leqq 0$ everywhere. In the same way, we can prove $g \geqq 0$ everywhere. This prove $g=0$.

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