J. Math. Soc. Japan
Vol. 20, Nos. 1-2, 1968

On the classification of sufficiently connected manifolds

Dedicated to Professor S. Iyanaga on his 60th birthday

By Itiro TAMURA

(Received Aug. 29, 1967)

Smale theory enables us to give a handlebody presentation for a simply-connected closed oriented differentiable m-manifold ($m \geqq 6$) and to establish a diffeomorphism between two such manifolds by proving that they are h-cobordant. In consequence we can achieve the determination of differentiable manifolds with a certain homotopy type, by firstly fixing up explicit handle-body presentations and secondly considering h-cobordism classes among them.

In this note we perform this for simply-connected closed oriented differentiable m-manifolds such that homology groups are trivial except in dimensions k, $m-k$ and for which certain cohomology operations vanish, where $m=2n$, $k=n-1$, $n \geqq 6$, or $m=2n+1$, $k=n-1$, $n \geqq 7$.

Chief results will be stated as Theorems 1, 2 and 3. However, in order to have a proper understanding of the form of our theorems, we pick up here some results in them:

Let M^{2n} be a simply connected closed oriented differentiable $2n$-manifold ($n \geqq 6$, $n \equiv 4, 6, 7 \bmod 8$) such that homology groups are trivial except in dimensions $n-1$, $n+1$ and that $Sq^2(H^{n-1}(M^{2n}; Z_2))=0$. Then M^{2n} is diffeomorphic to a connected sum of an S^{n-1}-bundle over S^{n+1}, copies of $S^{n-1}\times S^{n+1}$ and a homotopy sphere. In case $n \equiv 7 \bmod 8$, this presentation is unique up to diffeomorphism.

Let M^{2n+1} be a simply connected closed oriented differentiable $(2n+1)$-manifold ($n \geqq 7$, $n \equiv 6, 7 \bmod 8$) such that homology groups are trivial except in dimensions $n-1$, $n+2$ and that $\Phi(H^{n-1}(M^{2n+1}; Z_2))=0$. Then M^{2n+1} is diffeomorphic to a connected sum of an S^{n-1}-bundle over S^{n+2}, copies of $S^{n-1}\times S^{n+2}$ and a homotopy sphere. In case $n \equiv 6 \bmod 8$, this presentation is unique up to diffeomorphism.

1. Presentations.

Let M^m be a simply-connected closed oriented differentiable m-manifold ($m \geqq 7$) such that

This research was partially supported by the D. S. I. R.

$$H_q(M^m; z) = \begin{cases} Z & q=0,\ m\,, \\ Z+Z+\cdots+Z & q=k,\ m-k \quad (k<m-k)\,, \\ 0 & \text{otherwise}, \end{cases}$$

where $3 \leq k < m/2$.

Let us consider a presentation of M^m by "elementary" manifolds.

According to Smale [9], there exists a nice function f on M^m whose critical points are $a_0, a_1, a_2, \cdots, a_r, b_1, b_2, \cdots, b_r, b_0$ such that

$$f(a_0) = \text{index of } a_0 = 0\,,$$
$$f(a_i) = \text{index of } a_i = k \qquad i=1, 2, \cdots, r\,,$$
$$f(b_i) = \text{index of } b_i = m-k \qquad i=1, 2, \cdots, r\,,$$
$$f(b_0) = \text{index of } b_0 = m\,.$$

It follows that M^m has a decomposition by handles corresponding to critical points:

$$M^m = \underset{(a_0)}{D^m} \cup \underset{(a_1)}{(D_1^k \times D_1^{m-k})} \cup \cdots \cup \underset{(a_r)}{(D_r^k \times D_r^{m-k})} \cup \underset{(b_1)}{(D_1^{m-k} \times D_1^k)} \cup \cdots \cup \underset{(b_r)}{(D_r^{m-k} \times D_r^k)} \cup \underset{(b_0)}{D^m}$$

where r denotes the Betti number of dimensions k, $m-k$.

In this case the handle corresponding to a_i (resp. b_i) represents a homology class in $H_k(M^m; Z)$ (resp. $H_{m-k}(M^m; Z)$), which will be denoted simply by a_i (resp. b_i). Then $a_1, a_2, \cdots, a_r$ (resp. $b_1, b_2, \cdots, b_r$) represent a basis of $H_k(M^m; Z)$ (resp. $H_{m-k}(M^m; Z)$).

Let $a_1', a_2', \cdots, a_r' \in H_k(M^m; Z)$ be another basis characterized by the property

$$(*) \qquad a_i' \circ b_j = \delta_{ij}\,,$$

where $a_i' \circ b_j$ denotes the intersection number of a_i' and b_j. Define

$$W = f^{-1}[0, m/2] = D^m \cup (D_1^k \times D_1^{m-k}) \cup \cdots \cup (D_r^k \times D_r^{m-k})\,,$$

then $a_1, a_2, \cdots, a_r$ and $a_1', a_2', \cdots, a_r'$ are bases of $H_k(W; Z) = H_k(M^m; Z)$. Let

$$f_i : S^k \to \text{Int } W \qquad i=1, 2, \cdots, r\,,$$

be mappings such that

$$[f_i(S^k)] = a_i' \qquad i=1, 2, \cdots, r\,,$$

where $[f_i(S^k)]$ denotes the homology class represented by $f_i(S^k)$. By Whitney's imbedding theorem [15], we may suppose that f_i $(i=1, 2, \cdots, r)$ are imbeddings and

$$f_i(S^k) \cap f_j(S^k) = \emptyset \qquad i \neq j\,.$$

Let $N(f_i)$ be a tubular neighbourhood of $f_i(S^k)$ in Int W $(i=1, 2, \cdots, r)$ such that

$$N(f_i)\cap N(f_j)=\emptyset \qquad i\neq j\,.$$

$N(f_i)$ is a D^{m-k}-bundle over S^k: $(N(f_i), S^k, D^{m-k}, \bar{p}_j)$. Let

$$\hat{W}=N(f_1)\natural N(f_2)\natural\cdots\natural N(f_r)\subset \mathrm{Int}\, W$$

be a boundary connected sum of $N(f_1)$, $N(f_2)$, $\cdots$, $N(f_r)$ which is realized in Int W connecting $N(f_i)$ and $N(f_{i+1})$ by imbedded $D^1\times D^{m-1}$ $(i=1, 2, \cdots, r-1)$. Then $\hat{W}$ is an m-dimensional submanifold of W which is a handlebody whose handles represent homology classes $a_1', a_2', \cdots, a_r'$ in W. The inclusion maps

$$\partial W\to W-\mathrm{Int}\,\hat{W}, \qquad \partial\hat{W}\to W-\mathrm{Int}\,\hat{W}$$

are homotopy equivalences, because the inclusion map $\hat{W}\to W$ is a homotopy equivalence [**9**; Lemma 4.2]. Hence $W-\mathrm{Int}\,\hat{W}$ defines an h-cobordism between ∂W and $\partial\hat{W}$. Since ∂W and $\partial\hat{W}$ are simply-connected $(m-1)$-manifolds $(m-1\geqq 5)$, by Smale's theorem, we have

$$W-\mathrm{Int}\,\hat{W}=\partial\hat{W}\times I=\partial W\times I\,,$$

which implies that $\hat{W}$ is diffeomorphic to W [**9**; Theorem 4.1]. Therefore, replacing W by $\hat{W}$, we have a decomposition as follows:

$$M^m-\mathrm{Int}\,D^m=(N(f_1)\natural N(f_2)\natural\cdots\natural N(f_r))\cup(D_1^{m-k}\times D_1^k)\cup\cdots\cup(D_r^{m-k}\times D_r^k)\,.$$

Attaching maps of handles $D_i^{m-k}\times D_i^k$ $(i=1, 2, \cdots, r)$ will be denoted by

$$\begin{aligned} g_i:\partial D_i^{m-k}\times D_i^k &\to\partial(N(f_1)\natural\cdots\natural N(f_r)) \\ &=\partial N(f_1)\#\cdots\#\partial N(f_r)\,. \end{aligned}\qquad i=1, 2, \cdots, r\,,$$

(For this relation between the connected sum and the boundary connected sum, see [**6**].)

The homotopy type of $M^m-\mathrm{Int}\,D^m$ is obviously given by

$$M^m-\mathrm{Int}\,D^m\simeq(S_1^k\vee S_2^k\vee\cdots\vee S_r^k)\cup e_1^{m-k}\cup e_2^{m-k}\cup\cdots\cup e_r^{m-k}\,.$$

Now to simplify the situation let us put the hypothesis:

(H1) $\qquad M^m-\mathrm{Int}\,D^m\simeq S_1^k\vee S_2^k\vee\cdots\vee S_r^k\vee S_1^{m-k}\vee S_2^{m-k}\vee\cdots\vee S_r^{m-k}\,.$

Let $\bar{g}_i$ denote the map $\bar{g}_i=g_i|\partial D_i^{m-k}\times 0$:

$$\bar{g}_i:\partial D_i^{m-k}\times 0\to\partial N(f_1)\#\cdots\#\partial N(f_r) \qquad i=1, 2, \cdots, r\,.$$

Since $k<m-k$, the sphere bundle over sphere $(\partial N(f_i), S^k, S^{m-k-1}, p_i)$ admits a cross section, which implies

$$\partial N(f_i)\simeq(S^k\vee S^{m-k-1})\cup e^{m-1}\,.$$

Therefore by [**4**] we have

$$\{\bar{g}_i\} \in \pi_{m-k-1}(\partial N(f_1) \# \cdots \# \partial N(f_r))$$
$$= \pi_{m-k-1}(S_1^k \vee S_2^k \vee \cdots \vee S_r^k \vee S_1^{m-k-1} \vee S_2^{m-k-1} \vee \cdots \vee S_r^{m-k-1}) .$$
$$= \sum_i \pi_{m-k-1}(S_i^k) + Z + Z + \cdots + Z .$$

Let $\varepsilon_i : S^{m-k-1} \to p_i^{-1}(x_i)$ $(x_i \in S^k)$ be a natural imbedding of S^{m-k-1} in $N(f_i)$ $(i=1, 2, \cdots, r)$, then it follows from (H1) and $(*)$ that $\bar{g}_i$ is homotopic to ε_i. In order that these homotopies can be realized by isotopies, let us put the following hypothesis:

(H2) $$3k \geqq m+3 .$$

Then, according to Haefliger's theorem [3], there exists an imbedding

$$\bar{g}_1' : D^{m-k} \to N(f_1) \natural \cdots \natural N(f_r) ,$$

which bounds $\bar{g}_1(\partial D_1^{m-k} \times 0)$:

$$\bar{g}_1'(D^{m-k}) \cap (\partial N(f_1) \# \cdots \# \partial N(f_r)) = \bar{g}_1'(\partial D^{m-k}) = \bar{g}_1(\partial D_1^{m-k} \times 0) .$$

We may take that $(D_1^{m-k} \times 0) \cup \bar{g}_1'(D^{m-k}) \cup \bar{g}'(D^{m-k})$ is the natural sphere imbedded in $M^m - D^m$. Making use of Smale's theorem, it is easily verified that the closure of the complement of a tubular neighbourhood of $\bar{g}_1'(D^{m-k})$ in $N(f_1) \natural \cdots \natural N(f_r)$ is diffeomorphic to $N(f_2) \natural N(f_3) \natural \cdots \natural N(f_r)$. Hence there exists an imbedding

$$\bar{g}_2' : D^{m-k} \to N(f_1) \natural \cdots \natural N(f_r) ,$$

which bounds $\bar{g}_2(\partial D_2^{m-k} \times 0)$, such that

$$\bar{g}_1'(D^{m-k}) \cap \bar{g}_2'(D^{m-k}) = \emptyset .$$

By repeating this method, we obtain imbeddings

$$\bar{g}_i' : D^{m-k} \to N(f_1) \natural \cdots \natural N(f_r) \qquad i = 1, 2, \cdots, r ,$$

such that

$$\bar{g}_i'(D^{m-k}) \cap (\partial N(f_1) \# \cdots \# \partial N(f_r)) = \bar{g}_i'(\partial D^{m-k}) = \bar{g}_i(\partial D_i^{m-k} \times 0) ,$$
$$\bar{g}_i'(D^{m-k}) \cap \bar{g}_j'(D^{m-k}) = \emptyset \qquad i \neq j$$

and that $(D_i^{m-k} \times 0) \cup \bar{g}_i'(D^{m-k})$ is the natural sphere imbedded in $M^m - \mathrm{Int}\, D^m$. By Whitney's theorem, we may assume that

$$f_i(S^k) \cap \bar{g}_j'(D^{m-k}) = \emptyset \qquad i \neq j$$

and that $f_i(S^k)$ and $\bar{g}_i'(D^{m-k})$ intersect regularly at one point.

Let $(\bar{B}, S^k, D^{m-k}, p)$, $(\bar{B}', S^{m-k}, D^k, p')$ be disk bundles over spheres. Total spaces $\bar{B}$, $\bar{B}'$ are oriented differentiable m-manifolds with differentiable structure defined by the natural differentiable structures of S^k, D^{m-k}, S^{m-k} and D^k.

Let $\bar{B} \vee \bar{B}'$ denote the oriented differentiable m-manifold obtained as a quotient space of the disjoint sum $\bar{B} \cup \bar{B}'$ by identifying $p^{-1}(D^k) = D^k \times D^{m-k}$ $(D^k \subset S^k)$ and $p^{-1}(D^{m-k}) = D^{m-k} \times D^k$ $(D^{m-k} \subset S^{m-k})$ in such a way that $(x, y) = (y, x)$ $(x \in D^k,\ y \in D^{m-k})$. $\bar{B} \vee \bar{B}'$ is called the plumbing manifold of B and B'.

Let $N(g_i)$ be a sufficiently thin tubular neighbourhood of $(D_i^{m-k} \times 0) \cup \bar{g}_i'(D^{m-k})$ in $M^m - D^m$ $(i = 1, 2, \cdots, r)$. Then there exists a natural imbedding

$$(N(f_1) \vee N(g_1)) \natural (N(f_2) \vee N(g_2)) \natural \cdots \natural (N(f_r) \vee N(g_r)) \rightarrow M^m - \operatorname{Int} D^m$$

which is a homotopy equivalence. This implies that $(M^m - \operatorname{Int} D^m) - \operatorname{Int}(N(f_1) \vee N(g_1)) \natural \cdots \natural (N(f_r) \vee N(g_r))$ defines an h-cobordism between $\partial(M^m - \operatorname{Int} D^m)$ and $\partial(N(f_1)) \vee (N(f_1) \vee N(g_1)) \sharp \cdots \sharp \partial(N(f_r) \vee N(g_r))$. Therefore, making use of Smale's theorem, the following proposition holds:

PROPOSITION 1. *Under the hypotheses* (H1) *and* (H2), *we have*

$$M^m - \operatorname{Int} D^m = (N(f_1) \vee N(g_1)) \natural (N(f_2) \vee N(g_2)) \natural \cdots \natural (N(f_r) \vee N(g_r)) .$$

As shown in [8], $\partial(\bar{B} \vee \bar{B}')$ is homeomorphic to S^{m-1}:

$$\partial(\bar{B} \vee \bar{B}') \in \theta^{m-1} ,$$

where θ^q $(q \geqq 5)$ denotes the group of differentiable structures on S^q [6]. Now let us put the hypothesis:

(H3) For any $\bar{B}$, $\bar{B}'$, $\partial(\bar{B} \vee \bar{B}') = S^{m-1}$,

where $\bar{B}$, $\bar{B}'$ represent $N(f_i)$, $N(g_i)$ respectively. This hypothesis is equivalent to the existence of a closed oriented differentiable m-manifold M' such that

$$M' - \operatorname{Int} D^m = \bar{B} \vee \bar{B}' .$$

Then we obtain the following proposition.

PROPOSITION 2. *Under the hypotheses* (H1), (H2) *and* (H3), *we have a presentation of* M^m:

$$M^m = M_1 \sharp M_2 \sharp \cdots \sharp M_r \sharp \tilde{S}^m \qquad \tilde{S}^m \in \theta^m ,$$

where M_i *is a closed oriented differentiable m-manifold obtained from the plumbing manifold of* D^{m-k}*-bundle over* S^k *and* D^k*-bundle over* S^{m-k} *by attaching* D^m *to the boundary* $(i = 1, 2, \cdots, r)$.

2. Lemmas.

Dimensions which satisfy $m-k > k$ and (H2) are as follows:

(I) m odd $\geqq 9$, $k = [m/2]$ (i. e. $m-k = k+1$),

(II) m even $\geqq 12$, $k = (m/2)-1$ (i. e. $m-k = k+2$),

(III) m odd $\geqq 15$, $k = [m/2]-1$ (i. e. $m-k = k+3$),

etc.
The classification of differentiable manifolds of case (I) is considered in [11], [13]. In the following let us consider the cases (II), (III).

The following lemma is easily verified.

LEMMA 1. (i) *If m is odd, $k=[m/2]$, then the hypothesis* (H1) *is always satisfied.*

(ii) *If m is even $\geqq 8$, $k=(m/2)-1$, then the hypothesis* (H1) *is equivalent to*

$$Sq^2=0\,,$$

where $Sq^2: H^k(M^m; Z_2)\to H^{k+2}(M^m; Z_2)$.

(iii) *If m is odd $\geqq 11$, $k=[m/2]-1$, then the hypothesis* (H1) *is equivalent to*

$$\Phi=0\,,$$

where $\Phi: H^k(M^m; Z_2)\to H^{k+3}(M^m; Z_2)$ is the cohomology operation of the second kind [1].

Let $(\bar{B}, S^{m-k}, D^k, \bar{p})$ be the D^k-bundle over S^{m-k} whose characteristic map is $\mu\in\pi_{m-k-1}(SO(k))$. Let B' denote the closed oriented differentiable m-manifold obtained from two copies of $\bar{B}$ glueing their boundaries by the identity map. Then B' is the total space of the S^k-bundle over S^{m-k} whose characteristic map is $\iota_*(\mu): (B', S^{m-k}, S^k, p')$, where $\iota_*: \pi_{m-k-1}(SO(k))\to\pi_{m-k-1}(SO(k+1))$ is the natural homomorphism. It is obvious that

$$B'-\operatorname{Int} D^m=p'^{-1}(D^{m-k})\cup\bar{B}\,.$$

Hence we have the following lemma.

LEMMA 2. $B'-\operatorname{Int} D^m=(S^k\times D^{m-k})\mathbin{\underline{\vee}}\bar{B}$.

Similarly for $(\bar{B}'', S^k, D^{m-k}, \bar{p}'')$ and $(B''', S^k, S^{m-k}, p''')$ with the same characteristic map in $\pi_{k-1}(SO)$, we have the following lemma.

LEMMA 3. $B'''-\operatorname{Int} D^m=\bar{B}''\mathbin{\underline{\vee}}(S^{m-k}\times D^k)$.

According to Bott [2], we have

$$\pi_{k-1}(SO)=\begin{cases} Z_2 & k\equiv 1, 2 \quad \bmod 8\,,\\ 0 & k\equiv 3, 5, 6, 7 \bmod 8\,.\end{cases}$$

Thus, by Lemma 2, the following lemma holds:

LEMMA 4. *The hypothesis* (H3) *is satisfied for* $k\equiv 3, 5, 6, 7 \bmod 8$.

In order to consider the cases $k=1, 2 \bmod 8$, we need the following lemma, due to Seiya Sasao.

LEMMA S. *Let $K=(S^k\vee S^{m-k})\cup_g e^m$ be the CW-complex $\bar{B}\mathbin{\underline{\vee}}\bar{B}'/\partial(\bar{B}\mathbin{\underline{\vee}}\bar{B}')$ formed from the plumbing manifold $\bar{B}\mathbin{\underline{\vee}}\bar{B}'$ of total spaces of D^{m-k}-bundle over S^k and D^k-bundle over S^{m-k} $(k<m-k)$ by shrinking its boundary $\partial(\bar{B}\mathbin{\underline{\vee}}\bar{B}')$ to a point and let $\{g\}=g'+g''+e\in\pi_{m-1}(S^k\vee S^{m-k})\cong\pi_{m-1}(S^k)+\pi_{m-1}(S^{m-k})+Z$ $(g'\in\pi_{m-1}(S^k)$, $g''\in\pi_{m-1}(S^{m-k})$, $e\in Z)$. Then we have $g''=J(\alpha)$, where α is the*

characteristic map of B *and* $J:\pi_{k-1}(SO(m-k))\rightarrow\pi_{m-1}(S^{m-k})$ *is the Whitehead J-homomorphism.*

PROOF. Let $T(B)$ be the Thom complex of B and let $j:\bar{B}\vee\bar{B}'/\partial(\bar{B}\vee\bar{B}')\rightarrow T(B)$ be the map defined by $j|\mathrm{Int}\,B=$ identity and $j(\bar{B}\vee\bar{B}-\mathrm{Int}\,B)=$ (one point). Since $k<m-k$, $T(B)\simeq S^{m-k}\cup_{J(\alpha)} e^m$, which implies $j(S^k)$ is homotopic to zero in $T(B)$. Hence it is easy to see that there exists a homotopy equivalence $K/S^k\rightarrow T(B)$. This completes the proof.

Now let us introduce a following condition on $N(f_i)$ given in section 1:

[T] $\quad N(f_i)$ $(i=1, 2, \cdots, r)$ are trivial in cases $k=1, 2 \bmod 8$.

Since, as proved by J. F. Adams, $J:\pi_{k-1}(SO(m-k))\rightarrow\pi_{m-1}(S^{m-k})$ is monomorphism in these cases, Lemma S shows that the condition [T] is homotopy type invariant.

Then we have the following lemma which is a direct consequence of Lemma 2.

LEMMA 4′. *The hypothesis* (H3) *is satisfied for* $k=1, 2 \bmod 8$, *if the condition* [T] *holds.*

Let (B_i, S^{m-k}, S^k, p_i) be the S^k-bundle over S^{m-k} with characteristic map $\mu_i\in\pi_{m-k-1}(SO(k+1))$, which admits a cross section $(i=1, 2, \cdots, 2q)$. Then the following lemma holds.

LEMMA 5. *If there exists a* $(q\times q)$*-matrix with integer coefficients* (ξ_{ij}) *such that*

$$\det(\xi_{ij})=\pm 1,$$

$$\begin{pmatrix}\mu_{q+1}\\ \mu_{q+2}\\ \vdots\\ \mu_{2q}\end{pmatrix}=\begin{pmatrix}\xi_{11} & \xi_{12} & \cdots & \xi_{1q}\\ \xi_{21} & \xi_{22} & \cdots & \xi_{2q}\\ \vdots & \vdots & & \vdots\\ \xi_{q1} & \xi_{q2} & \cdots & \xi_{qq}\end{pmatrix}\begin{pmatrix}\mu_1\\ \mu_2\\ \vdots\\ \mu_q\end{pmatrix},$$

then we have

$$B_1\# B_2\#\cdots\# B_q=B_{q+1}\# B_{q+2}\#\cdots\# B_{2q}.$$

PROOF. Let $(\bar{B}_i, S^{m-k}, D^{k+1}, \bar{p}_i)$ denote the D^{k+1}-bundle over S^{m-k} associated with $B_i (i=1, 2, \cdots, 2q)$ and let $\overline{W}$ be a boundary connected sum of $\bar{B}_1, \bar{B}_2, \cdots, \bar{B}_q$:

$$\overline{W}=\bar{B}_1\natural\bar{B}_2\natural\cdots\natural\bar{B}_q.$$

$\overline{W}$ has the homotopy type of $S_1^{m-k}\vee S_2^{m-k}\vee\cdots\vee S_q^{m-k}$. Let a_i denote the homology class of $H_{m-k}(\overline{W}; Z)$ represented by zero section of $\bar{B}_i$ $(i=1, 2, \cdots, q)$ and let

$$h_i: S^{m-k}\rightarrow\mathrm{Int}\,\overline{W}\qquad i=1, 2, \cdots, q$$

be mappings such that

$$[h_i(S^{m-k})]=\sum_{j=1}^{q}\xi_{ij}a_j\qquad i=1, 2, \cdots, q.$$

Since B_i admits a cross section ($i=1, 2, \cdots, q$), we may suppose that h_i ($i=1, 2, \cdots, q$) are imbeddings such that

$$h_i(S^{m-k}) \cap h_j(S^{m-k}) = \emptyset \qquad i \neq j .$$

Let $N(h_i)$ be a tubular neighbourhood of $h_i(S^{m-k})$ in Int $\overline{W}$ ($i=1, 2, \cdots, q$) such that

$$N(h_i) \cap N(h_j) = \emptyset \qquad i \neq j .$$

Then it is easy to see that

$$N(h_i) = \bar{B}_{q+i} \qquad i=1, 2, \cdots, q .$$

Denote by $\hat{W}$ the boundary connected sum of $N(h_1), N(h_2), \cdots, N(h_q)$ in Int $\overline{W}$:

$$\hat{W} = N(h_1) \natural N(h_2) \natural \cdots \natural N(h_q) \subset \overline{W} .$$

Since the natural inclusion map $\hat{W} \to \overline{W}$ is a homotopy equivalence, $\overline{W}$—Int $\hat{W}$ defines an h-cobordism between $\partial \overline{W}$ and $\partial \hat{W}$. The lemma then follows by Smale's theorem.

In case $m-k \equiv 0 \bmod 4$, m even, $k=(m/2)-1$ or $m-k \equiv 0 \bmod 4$, m odd, $[m/2]-1$, according to Kervaire [**5**], we have

$$\pi_{m-k-1}(SO(k+1)) \cong Z .$$

Let us denote $\mu_i = n_i \rho$ ($i=1, 2, \cdots, 2q$), where ρ is a generator of $\pi_{m-k-1}(SO(k+1))$. Then the following lemma holds:

LEMMA 6. *In case* $m-k \equiv 0 \bmod 4$, *m even*, $k=(m/2)-1$ *or* $m-k \equiv 0 \bmod 4$, *m odd*, $[m/2]-1$, *we have*

$$B_1 \# B_2 \# \cdots \# B_q = B_{q+1} \# B_{q+2} \# \cdots \# B_{2q} ,$$

if and only if

$$G.C.D.(n_1, n_2, \cdots, n_q) = G.C.D.(n_{q+1}, n_{q+2}, \cdots, n_{2q}) .$$

PROOF. Let $\bar{\alpha}_i$ be a generator of $H^{m-k}(\bar{B}_i; Z)$ dual to a_i ($i=1, 2, \cdots, 2q$). Then the Pontrjagin class of $\bar{B}_i$ is given by

$$p_{(m-k)/4}(\bar{B}_i) = c n_i \bar{\alpha}_i ,$$

where

$$c = \begin{cases} 12 & m-k=8 , \\ 2((2j-1)\,!) & m-k=4j,\ j \text{ odd} \geqq 3 , \\ (2j-1)\,! & m-k=4j,\ j \text{ even} \geqq 4 , \end{cases}$$

(cf. [**7**], [**10**]). Thus it follows that

$$p_{(m-k)/4}(B_i) = c n_i \alpha_i ,$$

where α_i is a generator of $H^{m-k}(B_i; Z)$ represented by a cross section.

Now suppose that there exists a diffeomorphism

$$h: B_{q+1} \# B_{q+2} \# \cdots \# B_{2q} \to B_1 \# B_2 \# \cdots \# B_q\,,$$

then we have

$$h^*(\alpha_j) = \sum_{i=1}^{q} \xi_{ij}\alpha_{q+i} \qquad j=1, 2, \cdots, q\,.$$

where α_i (resp. α_{q+i}) is an element of $H^{m-k}(B_1 \# B_2 \# \cdots \# B_q; Z)$ (resp. $H^{m-k}(B_{q+1} \# B_{q+2} \# \cdots \# B_{2q}; Z)$) represented by a cross section of B_i (resp. B_{q+i}). Obviously we have

(**) $$\det(\xi_{ij}) = \pm 1\,.$$

Since

$$h^*(p_{(m-k)/4}(B_1 \# \cdots \# B_q)) = p_{(m-k)/4}(B_{q+1} \# \cdots \# B_{2q})\,,$$

it follows that

(***) $$\begin{pmatrix} n_{q+1} \\ n_{q+2} \\ \vdots \\ n_{2q} \end{pmatrix} = \begin{pmatrix} \xi_{11} & \xi_{12} & \cdots & \xi_{1q} \\ \xi_{21} & \xi_{22} & \cdots & \xi_{2q} \\ \vdots & \vdots & & \vdots \\ \xi_{q1} & \xi_{q2} & \cdots & \xi_{qq} \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ \vdots \\ n_q \end{pmatrix}.$$

Conversely if there exists a matrix (ξ_{ij}) which satisfies (**), (***), then $B_1 \# \cdots \# B_q$ and $B_{q+1} \# \cdots \# B_{2q}$ are diffeomorphic by Lemma 5. On the other hand, the existence of such matrix is equivalent to

$$G.C.D.(n_1, n_2, \cdots, n_q) = G.C.D.(n_{q+1}, n_{q+2}, \cdots, n_{2q})\,.$$

3. Classification of $2n$-dimensional case.

Let M^{2n} be a simply-connected closed oriented differentiable $2n$-manifolds ($n \geqq 6$, $n \not\equiv 1, 5 \bmod 8$) such that

$$H_q(M^{2n}; Z) = \begin{cases} Z & q = 0,\ 2n \\ Z+Z+\cdots+Z & q = n-1,\ n+1\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

$$Sq^2(H^{n-1}(M^{2n}; Z_2)) = 0\,,$$

and that the condition [T] holds in cases $n \equiv 2, 3 \bmod 8$. Then, by Proposition 2, Lemmas 1, 4 and 4′, M^{2n} has a presentation by a connected sum of elementary manifolds.

According to Kervaire [5], we have isomorphisms and exact sequences as follows:

$$\pi_6(SO(5)) = \pi_6(SO(6)) = 0\,.$$

In case $n \equiv 0 \bmod 8$, $n \geqq 8$,

$$\pi_n(SO(n-1)) = Z_2 + Z_2, \qquad \pi_n(SO(n)) = Z_2 + Z_2 + Z_2\,,$$

$$0 \to \pi_n(SO(n-1)) \xrightarrow{\iota_*} \pi_n(SO(n)) \to Z_2 \to 0 .$$

In cases $n \equiv 2,\ 6 \bmod 8,\ n \geqq 10,$

$$\pi_n(SO(n-1)) \cong Z_8, \qquad \pi_n(SO(n)) \cong Z_4 ,$$

$$\pi_n(SO(n-1)) \xrightarrow{\iota_*} \pi_n(SO(n)) \to 0 .$$

In cases $n \equiv 3,\ 7, \bmod 8,\ n \geqq 7,$

$$\pi_n(SO(n-1)) \cong Z , \qquad \pi_n(SO(n)) \cong Z ,$$

$$0 \to \pi_n(SO(n-1)) \xrightarrow{\iota_*} \pi_n(SO(n)) \to 0 .$$

In case $n \equiv 4 \bmod 8,\ n \geqq 12,$

$$\pi_n(SO(n-1)) \cong Z_2 , \qquad \pi_n(SO(n)) \cong Z_2+Z_2 ,$$

$$0 \to \pi_n(SO(n-1)) \xrightarrow{\iota_*} \pi_n(SO(n)) \to Z_2 \to 0 .$$

Denote by B_m (resp. $B_{m,m'}$) the total space of the S^{n-1}-bundle over S^{n+1} with characteristic map $m\rho$ (resp. $m\rho+m'\rho'$) $\in \iota_*\pi_n(SO(n-1))$, where ρ (resp. ρ, ρ') is a system of generators of $\iota_*\pi_n(SO(n-1))$. Then we have closed oriented differentiable $2n$-manifolds as follows:
In case $n=6$,

$$S^7 \times S^5 .$$

In case $n \equiv 0 \bmod 8,\ n \geqq 8,$

$$B_{m,m'} \qquad m,\ m'=0 \text{ or } 1 .$$

In cases $n \equiv 2,\ 6 \bmod 8,\ n \geqq 10,$

$$B_m \qquad -1 \leqq m \leqq 2 .$$

In cases $n \equiv 3,\ 7 \bmod 8,\ n \geqq 7$

$$B_m \qquad m \in Z .$$

In case $n \equiv 4 \bmod 8,\ n \geqq 12$

$$B_m \qquad m=0,\ 1 .$$

These manifolds are our elementary manifolds. By Lemmas 5, 6, the following relations hold:
In case $n \equiv 0 \bmod 8,\ n \geqq 8,$

$$B_{m,m'} \# B_{m'',m'''} = B_{m,m'} \# B_{m+m'',m'+m'''} .$$

In cases $n \equiv 2,\ 6 \bmod 8,\ n \geqq 10$; $n \equiv 3,\ 7 \bmod 8,\ n \geqq 7$; $n \equiv 4 \bmod 8,\ n \geqq 12,$

$$B_{m_1} \# B_{m_2} \# \cdots \# B_{m_q} = B_m \# B_0 \# \cdots \# B_0 \qquad m=G.C.D.(m_1,\ m_2,\ \cdots,\ m_q) .$$

The groups θ^q have been determined by Kervaire-Milnor [**6**]:

$$\theta^{12} \cong 0,\ \theta^{14} = Z_2,\ \theta^{15} \cong Z_{8128} + Z_2,\ \theta^{16} \cong Z_2,\ \theta^{17} = Z_{16},\ \theta^{18} = Z_{16}$$

etc.

Therefore we obtain the following theorem.

THEOREM 1. *Let M^{2n} be a simply-connected closed oriented differentiable $2n$-manifold ($n \geqq 6$, $n \not\equiv 1, 5 \bmod 8$) such that*

$$H_q(M^{2n}; Z) = \begin{cases} Z & q = 0,\ 2n, \\ Z + Z + \cdots + Z & q = n-1,\ n+1, \\ 0 & \text{otherwise}, \end{cases}$$

$$Sq^2(H^{n-1}(M^{2n}; Z_2)) = 0$$

and that the condition [T] *holds in cases $n \equiv 2, 3 \bmod 8$. Then M^{2n} has a presentation as follows:*

In case $n = 6$,

$$M^{12} = (S^7 \times S^5) \# (S^7 \times S^5) \# \cdots \# (S^7 \times S^5)\,.$$

This presentation is unique.

In case $n \equiv 0 \bmod 8$, $n \geqq 8$,

$$M^{2n} = B_{m,m'} \# (S^{n+1} \times S^{n-1}) \# \cdots \# (S^{n+1} \times S^{n-1}) \# \tilde{S}^{2n} \qquad m, m' = 0 \text{ or } 1\,,$$

$$\text{or } B_{0,1} \# B_{1,0} \# (S^{n+1} \times S^{n-1}) \# \cdots \# (S^{n+1} \times S^{n-1}) \# \tilde{S}^{2n}\,.$$

In cases $n \equiv 2, 6 \bmod 8$, $n \geqq 10$,

$$M^{2n} = B_m \# (S^{n+1} \times S^{n-1}) \# \cdots \# (S^{n+1} \times S^{n-1}) \# \tilde{S}^{2n} \qquad 0 \leqq m \leqq 2\,.$$

In cases $n \equiv 3, 7 \bmod 8$, $n \geqq 7$,

$$M^{2n} = B_m \# (S^{n+1} \times S^{n-1}) \# \cdots \# (S^{n+1} \times S^{n-1}) \# \tilde{S}^{2n} \qquad m \geqq 0\,.$$

This presentation is unique modulo θ^{2n}.

In case $n \equiv 4 \bmod 8$, $n \geqq 12$,

$$M^{2n} = B_m \# (S^{n+1} \times S^{n-1}) \# \cdots \# (S^{n+1} \times S^{n-1}) \# \tilde{S}^{2n} \qquad m = 0,\ 1\,.$$

In above all cases, $\tilde{S}^{2n} \in \theta^{2n}$.

COROLLARY 1. *The topological manifold $S^7 \times S^5$ admits a unique differentiable structure.*

COROLLARY 2. *In case $n = 6$, homotopy, homeomorphy and diffeomorphy classification are the same.*

Since rational Pontrjagin classes are invariants of combinatorial structures compatible with differentiable structures [**12**], we have

COROLLARY 3. *In cases $n \equiv 3, 7 \bmod 8$, $n \geqq 7$, the compatible combinatorial structures are characterized by Betti numbers and Pontrjagin classes.*

4. Classification of $(2n+1)$-dimensional case.

Let M^{2n+1} be a simply-connected closed oriented differentiable $(2n+1)$-manifold ($n \geqq 7$, $n \not\equiv 1, 5 \bmod 8$) such that

$$H_q(M^{2n+1}; Z) = \begin{cases} Z & q=0,\ 2n+1, \\ Z+Z+\cdots+Z & q=n-1,\ n+2, \\ 0 & \text{otherwise,} \end{cases}$$

$$\Phi(H^{n-1}(M^{2n+1}; Z_2)) = 0$$

and that the condition [T] holds in cases $n \equiv 2, 3 \bmod 8$. Then, by Proposition 2, Lemmas 1, 4 and 4′, M^{2n+1} has a presentation by a connected sum of elementary manifolds.

According to Kervaire [**5**], we have isomorphisms and exact sequences as follows:

$$\pi_8(SO(6)) \cong Z_{24}, \qquad \pi_8(SO(7)) \cong Z_2+Z_2,$$

$$\pi_8(SO(6)) \xrightarrow{\iota_*} \pi_8(SO(7)) \to Z_2 \to 0.$$

In case $n \equiv 0 \bmod 8$, $n \geqq 8$,

$$\pi_{n+1}(SO(n-1)) \cong Z_2+Z_2, \qquad \pi_{n+1}(SO(n)) \cong Z_2+Z_2+Z_2,$$

$$0 \to \pi_{n+1}(SO(n-1)) \xrightarrow{\iota_*} \pi_{n+1}(SO(n)) \to Z_2 \to 0.$$

In cases $n \equiv 2, 6, \bmod 8$, $n \geqq 10$,

$$\pi_{n+1}(SO(n-1)) \cong Z+Z_2, \qquad \pi_{n+1}(SO(n)) \cong Z,$$

$$\pi_{n+1}(SO(n-1)) \xrightarrow{\iota_*} \pi_{n+1}(SO(n)) \to 0.$$

In case $n \equiv 3 \bmod 8$, $n \geqq 11$,

$$\pi_{n+1}(SO(n-1)) \cong Z_{12}, \qquad \pi_{n+1}(SO(n)) \cong Z_2,$$

$$\pi_{n+1}(SO(n-1)) \xrightarrow{\iota_*} \pi_{n+1}(SO(n)) \to Z_2 \to 0.$$

In case $n \equiv 4 \bmod 8$, $n \geqq 12$,

$$\pi_{n+1}(SO(n-1)) \cong Z_2+Z_2, \qquad \pi_{n+1}(SO(n)) \cong Z_2+Z_2,$$

$$0 \to \pi_{n+1}(SO(n-1)) \xrightarrow{\iota_*} \pi_{n+1}(SO(n)) \to 0.$$

In case $n \equiv 7 \bmod 8$, $n \geqq 15$,

$$\pi_{n+1}(SO(n-1)) \cong Z_{12}+Z_2, \qquad \pi_{n+1}(SO(n)) \cong Z_2+Z_2,$$

$$\pi_{n+1}(SO(n-1)) \xrightarrow{\iota_*} \pi_{n+1}(SO(n)) \to Z_2 \to 0 .$$

Denote by B_m (resp. $B_{m,m'}$) the total space of the S^{n-1}-bundle over S^{n+2} with characteristic map $m\rho$ (resp. $m\rho+m'\rho'$) $\in \iota_*\pi_{n+1}(SO(n-1))$, where ρ (resp. ρ, ρ') is a system of generators of $\iota_*\pi_{n+1}(SO(n-1))$. Then we have closed oriented differentiable $(2n+1)$-manifolds follows:
In cases $n\equiv 0$, 4 mod 8, $n \geqq 8$,

$$B_{m,m'} \qquad m, m'=0 \text{ or } 1 .$$

In cases $n\equiv 2$, 6 mod 8, $n\geqq 10$,

$$B_m \qquad m\in Z .$$

In case $n\equiv 3$ mod 8, $n\geqq 11$,

$$S^{n+2}\times S^{n-1} .$$

In case $n\equiv 7$ mod 8, $n\geqq 7$,

$$B_m \qquad m=0, 1 .$$

These manifolds are our elementary manifolds. By Lemmas 5, 6, the following relations hold:
In cases $n\equiv 0$, 4 mod 8, $n\geqq 8$,

$$B_{m,m'} \# B_{m'',m'''} = B_{m,m'} \# B_{m+m'',m'+m'''} .$$

In cases $n\equiv 2$, 6 mod 8, $n\geqq 10$; $n\equiv 7$ mod 8, $n\geqq 7$,

$$B_{m_1}\# B_{m_2}\# \cdots \# B_{m_q} = B_m \# B_0 \# \cdots \# B_0 \qquad m=G.C.D.\ (m_1, m_2, \cdots, m_q) .$$

Therefore we obtain the following theorem.

THEOREM 2. *Let M^{2n+1} be a simply-connected closed oriented differentiable $(2n+1)$-manifold ($n\geqq 7$, $n\not\equiv 1$, 5 mod 8) such that*

$$H_q(M^{2n+1}; Z)=\begin{cases} Z & q=0,\ 2n+1 , \\ Z+Z+\cdots+Z & q=n-1,\ n+2 \\ 0 & \text{otherwise,} \end{cases}$$

$$\Phi(H^{n-1}(M^{2n+1}; Z_2))=0$$

and that the condition [T] *holds in cases* $n\equiv 2$, 3 mod 8. *Then* M^{2n+1} *has a presentation as follows:*
In cases $n\equiv 0$, 4 mod 8, $n\geqq 8$,

$$M^{2n+1}=B_{m,m'}\#(S^{n+2}\times S^{n-1})\# \cdots \#(S^{n+2}\times S^{n-1})\#\tilde{S}^{2n+1} \qquad m, m'=0 \text{ or } 1 ,$$

$$\text{or} \quad B_{0,1}\# B_{1,0}\#(S^{n+2}\times S^{n-1})\# \cdots \#(S^{n+2}\times S^{n-1})\#\tilde{S}^{2n+1} .$$

In cases $n\equiv 2$, 6 mod 8, $n\geqq 10$,

$$M^{2n+1}=B_m\#(S^{n+2}\times S^{n-1})\# \cdots \#(S^{n+2}\times S^{n-1})\#\tilde{S}^{2n+1} \qquad m\geqq 0 .$$

This presentation is unique modulo θ^{2n+1}.
In case $n\equiv 3 \bmod 8$, $n\geqq 11$,

$$M^{2n+1}=(S^{n+2}\times S^{n-1})\#(S^{n+2}\times S^{n-1})\#\cdots\#(S^{n+2}\times S^{n-1})\#\tilde{S}^{2n+1}.$$

This presentation is unique modulo θ^{2n+1}.
In case $n\equiv 7 \bmod 8$, $n\geqq 7$,

$$M^{2n+1}=B_m\#(S^{n+2}\times S^{n-1})\#\cdots\#(S^{n+2}\times S^{n-1})\#\tilde{S}^{2n+1} \qquad m=0,\ 1.$$

In above all cases $\tilde{S}^{2n+1}\in\theta^{2n+1}$.

COROLLARY 4. *In cases* $n\equiv 2,\ 6 \bmod 8$, $n\geqq 10$, *the compatible combinatorial structures are characterized by Betti numbers and Pontrjagin classes.*

COROLLARY 5. *In case* $n\equiv 3 \bmod 8$, $n\geqq 11$, *homotopy and homeomorphy classifications are the same, which are characterized by Betti numbers.*

5. Uniqueness.

Let B be the total space of a S^{n-1}-bundle over S^{n+1} and $\bar{B}$ the total space of the D^n-bundle over S^{n+1} associated with B ($n\equiv 3, 5, 6, 7 \bmod 8$). Suppose that

$$B\#(S^{n+1}\times S^{n-1})\#\cdots\#(S^{n+1}\times S^{n-1})\#\tilde{S}^{2n}$$
$$=B\#(S^{n+1}\times S^{n-1})\#\cdots\#(S^{n+1}\times S^{n-1}),$$

where $\tilde{S}^{2n}\in\theta^{2n}$. Then there exists a diffeomorphism

$$\phi: B\#(S^{n+1}\times S^{n-1})\#\cdots\#(S^{n+1}\times S^{n-1})-\mathrm{Int}\,D^{2n}$$
$$\rightarrow B\#(S^{n+1}\times S^{n-1})\#\cdots\#(S^{n+1}\times S^{n-1})-\mathrm{Int}\,D^{2n}$$

such that

$$\tilde{S}^{2n}=D^{2n}\bigcup_{\phi'}D^{2n},$$

where

$$\phi'=\phi\,|\,\partial(B\#(S^{n+1}\times S^{n-1})\#\cdots\#(S^{n+1}\times S^{n-1})-\mathrm{Int}\,D^{2n}).$$

Let W be the closed oriented differentiable $(2n+1)$-manifold with boundary obtained from two copies of $\bar{B}\natural(S^{n+1}\times D^n)\natural\cdots\natural(S^{n+1}\times D^n)$ glueing their boundaries by ϕ. Then obuiously we have

$$\partial W=\tilde{S}^{2n}.$$

It is easy to see that

$$H_q(W;Z)=\begin{cases} Z & q=0 \\ Z+Z+\cdots+Z & q=n,\ n+1, \\ 0 & \text{otherwise.}\end{cases}$$

Each element of $H_n(W;Z)$ is represented by an imbedded n-sphere whose normal bundle is trivial, since $\pi_{n-1}(SO(n+1))=0$. Therefore, by surgery [6], $\tilde{S}^{2n}$

bounds a contractible manifold. Hence we obtain the following lemma.

LEMMA 7. *In cases* $n \equiv 3, 5, 6, 7 \bmod 8$,

$$B \# (S^{n+1} \times S^{n-1}) \# \cdots \# (S^{n+1} \times S^{n-1}) \# \tilde{S}^{2n}$$
$$= B \# (S^{n+1} \times S^{n-1}) \# \cdots \# (S^{n+1} \times S^{n-1}), \qquad \tilde{S}^{2n} \in \theta^{2n}$$

implies

$$\tilde{S}^{2n} = S^{2n}.$$

Now let B' be the total space of a S^{n-1}-bundle over S^{n+2}, then we obtain the following lemma similarly.

LEMMA 8. *In cases* $n \equiv 3, 5, 6, 7 \bmod 8$,

$$B' \# (S^{n+2} \times S^{n-1}) \# \cdots \# (S^{n+2} \times S^{n-1}) \# \tilde{S}^{2n+1}$$
$$= B' \# (S^{n+2} \times S^{n-1}) \# \cdots \# (S^{n+2} \times S^{n-1}), \qquad \tilde{S}^{2n+1} \in \theta^{2n+1},$$

implies

$$\tilde{S}^{2n+1} = S^{2n+1}.$$

By Lemmas 7 and 8, we have the following theorem.

THEOREM 3. (i) *The presentation in cases* $n \equiv 3, 7 \bmod 8$, $n \geqq 7$ *of Theorem 1 is unique.*

(ii) *The presentation in case* $n \equiv 6 \bmod 8$, $n \geqq 10$ *of Theorem 2 is unique.*

(iii) *The presentation in case* $n \equiv 3 \bmod 8$, $n \geqq 11$ *of Theorem 2 is unique.*

COROLLARY 6. *In case* $n \equiv 3 \bmod 8$, $n \geqq 11$, *the topological manifold* $S^{n+2} \times S^{n-1}$ *admits exactly the same number of differentiable structure as* S^{2n+1}.

COROLLARY 7. (i) *In cases* $n \equiv 3, 7 \bmod 8$, $n \geqq 7$ *the topological manifold* $S^{n+1} \times S^{n-1}$ *admits at least differentiable structures of order of* θ^{2n}.

(ii) *In case* $n \equiv 6 \bmod 8$, $n \geqq 10$, *the topological manifold* $S^{n+2} \times S^{n-1}$ *admits at least differentiable structures of order of* θ^{2n+1}.

Let us consider a diagram

$$\begin{array}{ccccc}
 & & \pi_q(SO(n-1)) & \xrightarrow{\iota_*} & \pi_q(SO(n)) \\
 & \Delta \nearrow & \downarrow J & & \downarrow J \\
\pi_{q+1}(S^{n-1}) & \xrightarrow{P} & \pi_{n+q-1}(S^{n-1}) & \xrightarrow{E} & \pi_{n+q}(S^n),
\end{array}$$

where

$$P\alpha = [\alpha, \iota_{n-1}],$$
$$P = -J\Delta, \qquad EJ = -J\iota_*.$$

Let B_λ be the total space of the S^{n-1}-bundle over S^{q+1} with characteristic map $\iota_*(\lambda) \in \pi_q(SO(n))$. Then, according to Whitehead-James [14], B_λ and $B_{\lambda'}$, have the same homotopy type, if and only if

$$lJ\lambda = \pm lJ\lambda',$$

where $l: J\pi_q(SO(n-1)) \to J\pi_q(SO(n-1))/P\pi_{q+1}(S^{n-1})$. Since $J: \pi_8(SO(7)) \to \pi_{15}(S^7) \cong Z_2+Z_2+Z_2$ is injective, B_1 in case $n=7$ of Theorem 2 has a different homotopy type as $S^9 \times S^6$. Hence by Theorem 2 and Lemma 8 we have

COROLLARY 8. *The topological manifold $S^9 \times S^6$ admits exactly* 16256 *differentiable structures.*

Similarly, $B_{0,1}$, $B_{1,0}$ and $B_{1,1}$ in case $n=8$ of Theorem 1 have different homotopy types as $S^9 \times S^7$. On the other hand, by the method in the proof of Lemma 7, it is proved that $(S^9 \times S^7) \# \tilde{S}^{16} = S^9 \times S^7$, $\tilde{S}^{16} \in \theta^{16}$ implies $\tilde{S}^{16} = S^{16}$. Hence by Theorem 1, we have

COROLLARY 9. *The topological manifold $S^9 \times S^7$ admits exactly* 2 *differentiable structures.*

6. Some more complicated cases.

Let M^{2n} be a torsion free, $(n-2)$-connected closed oriented differentiable $2n$-manifold ($n \geqq 6$, $n \not\equiv 1, 5 \bmod 8$) such that

$$Sq^2(H^{n-1}(M^{2n}; Z_2)) = 0$$

and that the tubular neighbourhood of any imbedding of S^{n+1} in M^{2n} is trivial in cases $n \equiv 2, 3 \bmod 8$. (It is easy to see that this triviality condition is homotopy type invariant by the following argument and Lemma S of section 2.) Then, by Smale's theorem, M^{2n} has a decomposition by handles:

$$M^{2n} = D^{2n} \cup (D_1^{n-1} \times D_1^{n+1}) \cup \cdots \cup (D_r^{n-1} \times D_r^{n+1}) \cup (D_1^n \times D_1^n) \cup \cdots$$
$$\cup (D_{r'}^n \times D_{r'}^n) \cup (D_1^{n+1} \times D_1^{n-1}) \cup \cdots \cup (D_r^{n+1} \times D_r^{n-1}) \cup D^{2n},$$

where r (resp. r') is the Betti number of dimensions $n-1$, $n+1$ (resp. n). Let $a_1, a_2, \cdots, a_r \in H_{n-1}(M^{2n}; Z)$, $b_1, b_2, \cdots, b_r \in H_{n+1}(M^{2n}; Z)$ be bases whose intersection numbers are

$$a_i \circ b_j = \delta_{ij}.$$

Let W denote the handlebody as follows:

$$W = D^{2n} \cup (D_1^{n-1} \times D_1^{n+1}) \cup \cdots \cup (D_r^{n-1} \times D_r^{n+1})$$
$$\cup (D_1^n \times D_1^n) \cup \cdots \cup (D_{r'}^n \times D_{r'}^n) \subset M,$$

then we have

$$W \cong S_1^{n-1} \vee S_2^{n-1} \vee \cdots \vee S_r^{n-1} \vee S_1^n \vee S_2^n \vee \cdots \vee S_{r'}^n.$$

Let

$$f_i: S^{n-1} \to \operatorname{Int} W \qquad i = 1, 2, \cdots, r,$$
$$h_i: S^n \to \operatorname{Int} W \qquad i = 1, 2, \cdots, r'$$

be mappings such that

$$[f_i(S^{n-1})] = a_i \qquad i = 1, 2, \cdots, r$$

and that $[h_i(S^n)]$ $(i = 1, 2, \cdots, r')$ is a basis of $H_n(W; Z)$. By Whitney's imbedding theorem, we may assume that f_i $(i = 1, 2, \cdots, r)$ and h_i $(i = 1, 2, \cdots, r')$ are imbeddings and that

$$f_i(S^{n-1}) \cap f_j(S^{n-1}) = \emptyset \qquad i \neq j,$$

$$f_i(S^{n-1}) \cap h_j(S^n) = \emptyset$$

$$h_i(S^n) \cap h_0(D^{2n}) = h_i(D^n_-) \qquad i = 1, 2, \cdots, r',$$

$$h_i(D^n_+) \cap h_j(D^n_+) = \emptyset \qquad i \neq j,$$

where $h_0 : D^{2n} \to \mathrm{Int}\, W - \sum_{i=1}^{r} f_i(S^{n-1})$ is an imbedding and D^n_+ (resp. D^n_-) denotes the upper (resp. lower) hemi-sphere of S^n. Let $N(f_i)$ $(i = 1, 2, \cdots, r)$ and $N(h_i)$ $(i = 1, 2, \cdots, r')$ be tubular neighbourhood of f_i and h_i respectively such that

$$N(f_i) \cap N(f_j) = \emptyset \qquad i \neq j,$$

$$N(f_i) \cap (N(h_j) \cup h_0(D^{2n})) = \emptyset,$$

$$N(h_j) \cap N(h_j) \subset h_0(D^{2n}) \qquad i \neq j.$$

Then $W' = h_0(D^{2n}) \cup N(h_1) \cup \cdots \cup N(h_{r'})$ is a handlebody. Since the natural imbedding

$$N(f_1) \natural N(f_2) \natural \cdots \natural N(f_r) \natural W' \to \mathrm{Int}\, W$$

is a homotopy equivalence, $W - \mathrm{Int}\,(N(f_1) \natural \cdots \natural N(f_r) \natural W')$ defines the h-cobordism between ∂W and $\partial N(f_1) \# \cdots \# \partial N(f_r) \# \partial W'$, which implies by Smale's theorem that

$$W = N(f_1) \natural N(f_2) \natural \cdots \natural N(f_r) \natural W'.$$

Hence the following decomposition holds:

$$M^{2n} - \mathrm{Int}\, D^{2n} = (N(f_1) \natural \cdots \natural N(f_r) \natural W') \cup (D^{n+1}_1 \times D^{n-1}_1) \cup \cdots \cup (D^{n+1}_r \times D^{n-1}_r).$$

We may suppose that the handle $D^{n+1}_i \times D^{n-1}_i$ represents b_i $(i = 1, 2, \cdots, r)$. Let

$$g_i : \partial D^{n+1}_i \times D^{n-1}_i \to \partial N(f_1) \# \cdots \# \partial N(f_r) \# \partial W' \qquad i = 1, 2, \cdots, r$$

be attaching maps of handles $(D^{n+1}_i \times D^{n-1}_i)$, then, since $\partial W' \simeq S^{2n-1}$, $g_i | \partial D^{n+1}_i \times 0$ is homotopic to a natural imbedding of the fibre in the total space. Making use of the method as in section 1, it follows that

$$M^{2n} - \mathrm{Int}\, D^{2n} = (N(f_1) \veebar B_1) \natural \cdots \natural (N(f_r) \veebar B_r) \natural W',$$

where B_i is the total space of a D^{n-1}-bundle over S^{n+1} $(i = 1, 2, \cdots, r)$. By Lemmas 4 and 4′, we have $\partial(N(f_i) \veebar B_i) = S^{2n-1}$ $(i = 1, 2, \cdots, r)$, which implies

$\partial W' = S^{2n-1}$.

Therefore we obtain the following theorem.

THEOREM 4. *Let M^{2n} be a differentiable manifold as above, then we have a connected sum decomposition*

$$M^{2n} = M' \# M'' ,$$

where M' is a differentiable manifold as in Theorem 1 and M'' is an $(n-1)$-connected closed oriented differentiable $2n$-manifold.

Let M^{2n+1} be a torsion free, $(n-2)$-connected closed oriented differentiable $(2n+1)$-manifold ($n \geqq 7$, $n \not\equiv 1, 5 \bmod 8$) such that

$$Sq^2(H^{n-1}(M^{2n+1}; Z_2)) = 0 , \qquad Sq^2(H^n(M^{2n+1}; Z_2)) = 0 ,$$

$$\Phi(H^{n-1}(M^{2n+1}; Z_2)) = 0$$

and that the tubular neighbourhood of any imbedding of S^{n-1} (resp. S^n) in M^{2n} is trivial in cases $n \equiv 2, 3 \bmod 8$ (resp. in case $n \equiv 1 \bmod 8$). Then the following theorem holds.

THEOREM 5. $$M^{2n+1} = M' \# M'' ,$$

where M' is a differentiable manifold as in Theorem 2 and M' is a torsion free, $(n-1)$-connected closed oriented differentiable $(2n+1)$-manifold.

The proof is similar as that of Theorem 4. So we omit it.

Particularly, by Theorem 5 and [**11**; Theorem 7], we have the following theorem.

THEOREM 6. *Let M^{15} be a torsion free, 5-connected closed oriented differentiable 15-manifold whose Steenrod operations and the secondary operation Φ vanish. Then we have*

$$M^{15} = B_m \# B'_{m'} \# (S^9 \times S^6) \# \cdots \# (S^9 \times S^6) \# (S^8 \times S^7) \# \cdots \# (S^8 \times S^7) \# \tilde{S}^{15} ,$$

$$\{\tilde{S}^{15}\} \in \theta^{15}/m'(\theta^{15}(\partial\pi)) \qquad m' \text{ odd} ,$$

$$\in \theta^{15}/(m'/2)(\theta^{15}(\partial\pi)) \qquad m' \text{ even} ,$$

where B_m is a differentiable manifold in case $n=7$ of Theorem 2, $B'_{m'}$ is the total space of a S^7-bundle over S^8 whose characteristic map is $m'\iota_*(\rho')$ (ρ' : a generator of $\pi_7(SO(7))$ and $\theta^{15}(\partial\pi)$ is the subgroup of θ^{15} consisting of elements which bound π-manifolds. This presentation is unique.

Uniqueness follows from [**11**; Theorem 7], making use of the surgery as in section 5.

University of Tokyo

References

[1] J. Adem, The iteration of the Steenrod square in algebraic topology, Proc. Nat. Acad. Sci. U. S. A., **38** (1952), 720–726.

[2] R. Bott, The stable homotopy of the classical groups, Ann. of Math., **70** (1959), 313–337.

[3] A. Haefliger, Plongements différentiables de variétés dans variétés, Comment. Math. Helv., **36** (1961), 47–82.

[4] P. J. Hilton, On the homotopy groups of the union of spheres, J. London Math. Soc., **30** (1955), 154–172.

[5] M. Kervaire, Some non-stable homotopy groups of Lie groups, Illinois J. Math., **4** (1960), 161–169.

[6] M. Kervaire and J. Milnor, Groups of homotopy spheres, I, Ann. of Math., **77** (1963), 504–537.

[7] J. Milnor, Some consequences of a theorem of Bott, Ann. of Math., **68** (1958), 444–449.

[8] J. Milnor, Differentiable structures on spheres, Amer. J. Math., **81** (1959), 962–972.

[9] S. Smale, On the structure of manifolds, Amer. J. Math., **84** (1962), 387–399.

[10] I. Tamura, On Pontrjagin classes and homotopy types of manifolds, J. Math. Soc. Japan, **9** (1957), 250–262.

[11] I. Tamura, Classification des variétés différentiables, $(n-1)$-connexes, sans torsion, de dimension $2n+1$, Séminaire Cartan 1962/63.

[12] R. Thom, Les classes caractéristiques de Pontrjagin des variétés triangulées, Symposium international de topologia algebraica, Mexico, 1958, 54–67.

[13] C. T. C. Wall, Classification problems in differential topology, I, II, Q, III, IV, VI, Topology, **2** (1963), 253–262, 263–272, 281–298, **3** (1965) 291–304, **5** (1966), 73–94, **6** (1967) 273–296.

[14] J. H. C. Whitehead and I. M. James, The homotopy theory of sphere bundles over spheres, Proc. London Math. Soc., **4** (1954), 196–218.

[15] H. Whitney, The self-intersections of a smooth n-manifold in $2n$-space, Ann. of Math., **45** (1944), 220–246.