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A characterization of the simple groups PSL $(2, q)$

Dedicated to Professor Shôkichi Iyanaga

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1. The purpose of this paper is to prove the following theorem.

THEOREM A. *Let G be a finite group which satisfies the following two conditions:*

(i) *G is semi-simple, and*

(ii) *if X is a cyclic subgroup of even order and if Y is a non-identity subgroup of G, then*

$$X \supseteqq Y \quad \textit{implies} \quad N_G(X) \supseteqq N_G(Y).$$

Then G is isomorphic to either PGL $(2, q)$ *or* PSL $(2, q)$ *for some prime power* $q > 3$.

Notation and terminology are standard. PGL $(2, q)$ is the factor group of the group of all 2×2 *non-singular* matrices over the finite field of q elements by its center. The group PSL $(2, q)$ is similarly defined by replacing non-singular by unimodular. If $q > 3$, PSL $(2, q)$ is a simple group. See [**3**]. If q is even, PGL $(2, q)$ coincides with PSL $(2, q)$. Otherwise PSL $(2, q)$ is the only proper normal subgroup of PGL $(2, q)$. Thus, both groups are *semi-simple*, i.e. they contain no proper solvable normal subgroups. $N_G(X)$ is the normalizer and $C_G(X)$ is the centralizer of a subset X in G.

It is not hard to show that the groups PGL $(2, q)$ and PSL $(2, q)$ satisfy the condition (ii). Thus the converse of Theorem A holds.

In [**1**], Brauer, Wall and the author studied a finite group of even order which satisfies the property that two distinct maximal cyclic subgroups of even order have a trivial intersection. Such a group satisfies the condition (ii). So Theorem A may be considered as a generalization of the result of [**1**].

In the proof of Theorem A we use the structure theorems of finite groups in which the centralizers of elements of order 2 are always nilpotent. Re-

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quired results are contained in Suzuki [12]. These results are deep and difficult to prove, so the proof of Theorem A is not elementary. In the next section we prove a technical result (Theorem B), which is another characterization of the projective transformation groups of the projective line. Actually Theorem B is more than necessary in order to prove Theorem A. Instead of using Theorem B, we may use the result of [1] to finish the proof in case of Lemma 6 below.

It is possible to generalize Theorem A eliminating the assumption (i). But the result becomes much more complicated to state. We do not go into this generalization, although it is not difficult to supply missing arguments for discussing groups which satisfy (ii) but not (i). We state a remark on the assumption (ii). Since every subgroup of a cyclic group is characteristic,

$$X \supseteqq Y \neq 1 \quad \text{implies} \quad N_G(X) \subseteqq N_G(Y),$$

for any cyclic subgroup X of G. Thus the condition (ii) is a containment opposite to the natural one, and yields that

$$X \supseteqq Y \neq 1 \quad \text{implies} \quad N_G(X) = N_G(Y)$$

for a cyclic subgroup X of even order.

2. The following proposition is a consequence of a result of Gorenstein and Walter [6]. Let G be a simple group and S a Sylow 2-group of G. If the center of S contains an involution j such that $C_G(j)$ is a dihedral group, then G is isomorphic to PSL $(2, q)$ for an odd prime power $q > 3$. We record here a generalization of this result.

We define a *D-group* as a group G which contains a proper subgroup A such that

$$x^2 = 1 \quad \text{for all} \quad x \in G - A.$$

By definition an elementary abelian 2-group is a D-group. If a group is a D-group but is not an elementary abelian 2-group, then it is defined as a D^*-group. The following lemma is easy to prove.

LEMMA 1. *A D-group G contains a normal abelian subgroup A of index* 2 *and any element of G—A is an involution and inverts every element of A. If G is a D*-group, the subgroup A which satisfies the above properties is unique.*

It follows from Lemma 1 that the center of a D^*-group is the set of elements of order $\leqq 2$ of A.

THEOREM B. *Let G be a group. Suppose that G contains a subgroup H which satisfies the following two conditions:*

(1) *H is a D-group of order divisible by* 4, *and*

(2) $H = C_G(j)$ *for any involution j of the center of H.*

Then, if G is not solvable, G contains a normal subgroup N such that the order of N is either odd or twice an odd number, and that

$$G/N \cong \mathrm{PSL}(2, q) \quad \text{or} \quad \mathrm{PGL}(2, q)$$

for some prime power $q > 3$.

REMARKS. Since a dihedral group is a D-group, Theorem B is a generalization of a result of [**6**] mentioned earlier. The normal subgroup N is solvable by Feit-Thompson [**4**]. This solvability of groups of odd order will be used implicitly in the proof. It is however not essential. The prime power q may be odd or even. It is not difficult to find the structure of a solvable group G which satisfies the assumptions of Theorem B. Most of them are either 2-closed or $2'$-closed. But there is a class of groups with 2-length 2.

PROOF. If H is an elementary abelian 2-group then G is isomorphic to $\mathrm{PSL}(2, q)$ for $q = |H|$. This is a result of Fowler [**5**], and is a particular case of a theorem of [**11**].

We may, therefore, assume that H is a D^*-group. Let A be the unique normal abelian subgroup of index 2 of H, and S a Sylow 2-group of H. We distinguish two cases according as S is a Sylow 2-group of G or not.

Suppose that S is a Sylow 2-group of G. Define $B = A \cap S$. Any involution j of B is contained in the center of H and by assumption (2) we have $C_G(j) = H$. If B is cyclic, S is a dihedral group. Hence G satisfies the assumptions of Theorem 1 of [**6**]. Since the alternating group A_7 does not satisfy the assumptions (1) and (2), the assertion follows from [**6**]. Suppose that B is not cyclic. Let u be an element of $S-B$ and define

$$U = \langle u, Z(H) \rangle ,$$

where $Z(H)$ is the center of H. By assumption (1), U is elementary abelian and $U = C_H(u)$. By assumption we have $|U| \geqq 8$. If S_1 is a Sylow 2-group of G which contains U, then

$$|U : U \cap Z(S_1)| \leqq 2 .$$

Hence there is an involution j of $Z(H)$ such that $C_G(j) \supseteqq S_1$. This forces S_1 to coincide with S. Therefore S is the unique Sylow 2-group of G, which contains U. This yields that $u \in S-B$ is not conjugate to any involution of B. Hence the focal group of S in G is contained in B (cf. Higman [8]). This implies that G contains a normal subgroup G_1 of index 2 such that $G_1 \cap H = A$. The group B is a Sylow 2-group of G_1. Hence the centralizer of any involution in G_1 is abelian. By the main theorem of [**10**], G_1 is a direct product $L \times C$ where $L \cong \mathrm{PSL}(2, q)$, $q = |B|$, and C is an abelian group of odd order.

An element u of $S-B$ inverts every element of C. Since B is elementary abelian, u centralizes B. The element u normalizes L. It is easy to see that the group $\langle u, L\rangle$ is a direct product of L and a cyclic group of order 2. Hence G contains a normal subgroup N such that $G/N \cong L$ and N contains C as a normal subgroup of index 2.

Suppose finally that S is not a Sylow 2-group of G. Let T be a Sylow 2-group of G, which contains S. Consider $T_1 = N_T(S)$. Since $T \neq S$, T_1 is larger than S. Let t be an element of $T_1 - S$ such that $t^2 \in S$. If B denotes $A \cap S$ as before, t normalizes $B \cap B^t$. If $|B| > 2$, then $B \cap B^t$ contains an involution j which commutes with t. This is not the case, since $H = C_G(j)$. Hence we have $|S| = 4$ and $|B| = 2$. Let j be the involution of B. Then $S = C_T(j)$ is of order 4. By Lemma 4 of [**9**], T is either a dihedral or a semi-dihedral group. Suppose that T is a dihedral group. If u denotes the involution of the center of T, the group $C_G(u)$ contains a normal 2-complement U. Since j commutes with u, j normalizes U. Since no element of odd order centralizes the group $\langle u, j\rangle$, j inverts every element of U. This forces U to be abelian and Theorem 1 of [**6**] yields the assertion.

If T is not a dihedral group, G contains a normal subgroup G_1 of index 2 such that a Sylow 2-group of G_1 is a generalized or an ordinary quaternion group. By Brauer-Suzuki [**2**], G_1 contains a normal subgroup N such that the order of N is twice an odd integer. A Sylow 2-group of G_1/N is dihedral and since G is not solvable involutions of G_1/N form a single conjugate class. Let V be a normal cyclic subgroup of order 4 of T. $N_G(V)$ possesses a normal 2-complement W. The involution j normalizes W. Since u centralizes W, j inverts every element of W. Hence W is abelian. The centralizer of an involution of G_1/N is a homomorphic image of $N_G(V)$. Therefore G_1/N satisfies the assumptions of Theorem 1 of [**6**]. We have

$$G/N \cong \mathrm{PGL}\,(2, q)\,,$$

since Sylow 2-groups of G/N are dihedral groups.

3. We begin the proof of Theorem A. Suppose that the centralizer of an involution of G is always nilpotent. Then the structure of G is known. Theorems 1 and 2 of [**12**], together with (i) yield that a minimal normal subgroup of G is a simple group in the list given in Theorem 1 of [**12**]. It is not hard to check the validity of Theorem A in this case.

We assume therefore that the centralizer of some involution j of G is not nilpotent. Denote

$$H = C_G(j) \quad \text{and} \quad J = \langle j\rangle\,.$$

We prove several propositions about H.

LEMMA 2. *H contains an abelian normal 2-complement A. A is not trivial. If B is a non-identity subgroup of A, then we have $N_G(B) = H$.*

PROOF. By assumption, H is not nilpotent. Hence H contains elements of odd order >1. Let x be any one of those elements and set

$$X=\langle x\rangle \quad \text{and} \quad Y=\langle x, j\rangle .$$

Since x commutes with j, the condition (ii) yields that

$$N_G(X)=N_G(Y)=N_G(J)=C_G(j)=H .$$

Therefore any subgroup of odd order of H is normal. The assertions of Lemma 2 are obvious.

LEMMA 3. *Let U be a cyclic 2-group $\neq 1$ of H. If $C_A(U)\neq 1$, then $N_G(U)=H$ and U centralizes A.*

PROOF. Let X be a non-identity cyclic subgroup of $C_A(U)$. Define $Y=\langle X, U\rangle$. By assumption Y is cyclic and

$$N_G(U)=N_G(Y)=N_G(X) .$$

By Lemma 2, $N_G(X)=H$. This yields Lemma 3.

We use the following further notation:

S: a Sylow 2-group of H,

$B=C_S(A)$.

LEMMA 4. *B is a normal subgroup of H and S/B is cyclic.*

PROOF. By Lemma 2, we have $C_G(A)=C_H(A)$ and H normalizes $C_H(A)$. By definition $B=S\cap C_G(A)$ and so

$$C_G(A)=B\times A .$$

This proves that B is a normal subgroup of H. If x is an element $\neq 1$ of A, then $C_G(x)$ is a part of $C_G(A)$ by Lemma 3. Hence $C_G(x)=C_G(A)$. If $X=\langle x\rangle$, then

$$S/B\cong H/C_G(A)=N_G(X)/C_G(X) .$$

If the order of x is a prime number $N_G(X)/C_G(X)$ is cyclic. This proves Lemma 4.

LEMMA 5. *There is a cyclic subgroup U of S such that $S=UB$ and $U\cap B=1$. Furthermore if v is the involution of U, then v inverts every element of $C_G(A)$.*

PROOF. By Lemma 4, there is a cyclic subgroup U such that $S=UB$. Let v be the involution of U. Set $V=\langle v\rangle$. If $v\in B$, then by Lemma 3, V is a normal subgroup of H. By the assumption (ii), U is a normal subgroup of H also. This implies that U centralizes A. This is not the case since H is not nilpotent. Hence $v\notin B$ and $B\cap U=1$. By Lemma 3, v centralizes no non-identity element of A. This implies that v inverts every element of A.

Let x be an element of B. By Lemma 3 again, v normalizes $\langle x\rangle$. Hence we have

$$v^{-1}xv = x^s, \quad \text{or} \quad (vx)^2 = x^{1+s}.$$

If $x^{1+s} \neq 1$, then $\langle vx \rangle$ is a normal subgroup of H by (ii). This yields that $vx \in C_G(A)$. But this is not the case, because $v \notin B$. Hence v inverts every element of $C_G(A)$.

LEMMA 6. *If* $|S:B| = 2$, *then* H *satisfies the assumptions of Theorem B.*

PROOF. By Lemma 5, H is a D^*-group if $|S:B| = 2$. The condition (2) in Theorem B is satisfied by Lemma 3.

By Lemma 6, Theorem A follows from Theorem B if $|S:B| = 2$. So in the rest of this section we assume that

$$|S:B| > 2$$

and try to derive a contradiction.

LEMMA 7. *B is elementary abelian.*

PROOF. Since $|S:B| > 2$, the involution v of U is a square in S; that is, $v = w^2$ for some w of S. If B contains an element x of order 4, we have $w^{-1}xw = x^s$ and

$$v^{-1}xv = w^{-2}xw^2 = x^r \quad \text{where} \quad r = s^2.$$

This contradicts Lemma 5 as $r \not\equiv -1 \pmod 4$.

LEMMA 8. *If an element t normalizes $V = \langle v \rangle$, then*

$$t^{-1}xt = x \quad \text{or} \quad xv \quad \text{for all} \quad x \in B.$$

PROOF. Let u be a generator of U. By assumption (ii), the element t normalizes U and $\langle uv \rangle$. Note that, by Lemmas 3 and 7, u commutes with x and $(ux)^2 = u^2$. Hence t normalizes the subgroup $\langle u, ux \rangle$. This group contains exactly 3 elements of order 2, namely v, x and vx. Lemma 8 follows immediately.

LEMMA 9. *Define $T = N_G(V)$ where $V = \langle v \rangle$. Then the following assertions hold:*

(a) $$|T:S| = 2,$$

(b) *T is a Sylow 2-group of G, and*

(c) *V is weakly closed in T.*

PROOF. By Lemma 5, we have $N_H(V) = S$. Let x be an involution of B. Then by Lemma 3, $C_G(x)$ coincides with H. This implies that

$$C_T(x) = C_G(x) \cap T = H \cap N_G(V) = S.$$

By Lemma 8 we have $|T:S| \leqq 2$. By Lemmas 3 and 7, S is abelian. Hence V is a characteristic subgroup of S. If $T = S$, then $N_G(S) = S$. By the transfer theorem of Burnside, G contains a normal 2-complement. This is not the case. The first assertion (a) holds.

Again let x be an involution of B. Then $C_T(x)=S$ as shown above. This, together with (a) and Lemma 8, implies that x is conjugate to xv in T. Furthermore the center of T is contained in S. Hence v is the only involution of the center of T. Thus V is a characteristic subgroup of T. Hence we have $N_G(T)=T$. This implies (b).

Suppose that (c) is false. Let $W=\langle w\rangle$ be a conjugate subgroup of V which is contained in T. Then W is not contained in S. Hence, if x is an involution of B, we have

$$w^{-1}xw=xv\,.$$

So $v=z^2$ where $z=xw$. Set $Z=\langle z\rangle$. Then by (ii) Z is a normal subgroup of T. We may suppose that U centralizes Z. If a generator u of U does not commute with z, we replace U by $\langle ux\rangle$. Then U is the center of T. This implies that $C_T(W)=W\times U$. Since U is a cyclic 2-group of order $\geqq 4$, V is a characteristic subgroup of $C_T(W)$. On the other hand $C_G(W)$ is a Sylow 2-group of G. Since $C_T(W)$ is a proper subgroup of T, it is a proper subgroup of $R=C_G(W)$. Hence $N_R(V)$ contains $C_T(W)$ as a proper subgroup. This is a contradiction.

We can finish the proof of Theorem A as follows. Since V is a weakly closed subgroup of the center of T, we can apply the transfer theorem of P. Hall-Wielandt (cf. [7], p. 212). Since $T=N_G(V)$, G contains a normal 2-complement. This contradicts the assumption (i).

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