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Algebraic varieties without deformation and the Chow variety

Dedicated to Professor Shôkichi Iyanaga on his 60th birthday

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The purpose of this note is to indicate two simple facts, of which the first one is almost obvious, once it is formulated:

THEOREM 1. *If a complex projective non-singular variety V has no deformation, then V is biregularly equivalent to a projective variety defined over an algebraic number field.*

Here we say that V has *no deformation* if $H^1(V, \Theta)=0$, where Θ denotes the sheaf of germs of holomorphic sections of the tangent bundle of V. Calabi and Vesentini [**1**] have proved that $H^1(V, \Theta)=0$ if V is the quotient S/Γ of an irreducible bounded symmetric domain S of dimension >1 by a discontinuous group Γ operating freely on S such that S/Γ is compact. (See also [**4**], [**5**], [**11**].) Therefore Th. 1 shows that such a quotient has a model defined over an algebraic number field.

To state the second fact, let $C_p(N, m, d)$ denote the set of all the Chow points of positive cycles of dimension m and degree d in the projective space of dimension N, defined with respect to a universal domain of characteristic $p \geqq 0$. It is well known that $C_p(N, m, d)$ is a Zariski closed set in a certain projective space, which is defined by equations with coefficients in the prime field. Then one can ask the following question:

(Q) *Is every absolutely irreducible component of $C_p(N, m, d)$ defined over the prime field?*[1)]

The answer is negative if the characteristic is 0:

THEOREM 2. *There exist positive integers N, m, d such that $C_0(N, m, d)$ has a component which is not defined over the rational number field.*

Such a component will be obtained so as to contain the Chow point of a certain variety without deformation. This is why we present these two theorems together. We shall also show in the last section that the answer to the question (Q) is still negative even if the characteristic is positive.

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1) I thank S. Lichtenbaum for reminding me of this question.

1. We shall denote by $c(X)$ the Chow point of a positive cycle X in a projective space, and by P^N the projective space of dimension N. A variety or a curve will always mean an absolutely irreducible one. In this section, the universal domain is the complex number field $\boldsymbol{C}$.

PROPOSITION 1. *Let V be a non-singular variety of dimension m and degree d in P^N, and B a component of $C_0(N, m, d)$ containing $c(V)$. Suppose that V has no deformation. Then there exists a Zariski open subset B' of B such that if $c(X) \in B'$, X is a non-singular variety biregularly isomorphic to V.*

PROOF. Let k_0 be a finitely generated extension of the rational number field $\boldsymbol{Q}$ over which $c(V)$ and B are rational. Let x be a generic point of B over k_0, and W the cycle such that $c(W) = x$. We shall now prove

(1.1) *W is a non-singular variety biregularly isomorphic to V.*

The non-singularity is obvious, since V is a specialization of W over k_0. Put $u = c(V)$. We can find a generic specialization y of x over k_0 and a finitely generated extension k of k_0, such that u is a specialization of y over k and $k(y)$ is a regular extension of k of dimension one. Let E be a complete non-singular curve with a generic point z over k such that $k(z) = k(y)$, and f a morphism of E into B, rational over k, such that $f(z) = y$. Take a point v on E such that $f(v) = u$. For each $t \in E$, let W_t denote the cycle such that $c(W_t) = f(t)$. Then W_t is the unique specialization of W_z compatible with the specialization $z \rightarrow t$ over k. Let E' be the set of all t on E such that W_t is a non-singular variety. Then E' is a Zariski open subset of E rational over k, containing both z and v. Let w be a generic point of W_z over $k(z)$. (Note that W_z is defined over $k(z)$, since $c(W_z) = f(z)$.) Let H be the locus of $z \times w$ on $E' \times P^N$ over k. Then we see easily that the intersection product of H with $t \times P^N$ on $E' \times P^N$ is $t \times W_t$ for every $t \in E'$. By [**10**, Ch. VI, § 2, Th. 6], every point of $t \times W_t$ is simple on H. Thus we have a non-singular variety H which may be regarded as a fibre bundle with E' as base and the W_t as fibres. By the theorem of Frölicher and Nijenhuis [**2**] (cf. also [**3**]), our assumption $H^1(V, \Theta) = 0$ implies that there exists a neighborhood M of v on E' such that V $(= W_v)$ is biregularly isomorphic to W_t for every $t \in M$. We can find a generic point s of E over k lying in M. Let q be a biregular morphism of W_s to V. Since $f(s)$ is a generic specialization of x over k_0, there exists an automorphism σ of $\boldsymbol{C}$ over k_0 such that $f(s)^\sigma = x$. Then q^σ is a biregular morphism of W $(= (W_s)^\sigma)$ to V. This proves (1.1).

Now for every $c(X) \in B$, $(c(X), c(V))$ is a specialization of $(c(W), c(V))$ over k_0. Over this specialization, the graph of q^σ can be specialized to the graph of a biregular morphism of X onto V, at least for all $c(X)$ in some Zariski open subset of B (see for example [**9**, § 12.3, Prop. 24]). This proves our proposition.

PROOF OF THEOREM 1. Let V, B and B' be as in Prop. 1, and k the smallest field of definition for B. Then k is an algebraic number field of finite degree. We can find a point v of B' algebraic over k. Let X be such that $c(X)=v$. Then X is biregularly isomorphic to V and defined over $k(v)$. This proves Th. 1.

One could actually prove Th. 1 more directly. For example, take the locus U of $c(V)$ over the algebraic closure of $\boldsymbol{Q}$, and construct a fibre variety of which the base is a Zariski open subset of U and the generic fibre at $c(V)$ is V. Specializing $c(V)$ to an algebraic point on U close to $c(V)$, one obtains the desired conclusion.

2. The universal domain being still $\boldsymbol{C}$, let V be a projective variety. A subfield h of $\boldsymbol{C}$ is called *the bottom field* (resp. *the strong bottom field*) for V if the following condition is satisfied.

(2.1) *An automorphism σ of $\boldsymbol{C}$ is the identity mapping on h if and only if V^σ is birationally* (*resp. biregularly*) *equivalent to V.*

Such a field h is unique for V if it exists. The following three assertions are immediate consequences of this definition.

(2.2) *If W is birationally* (*resp. biregularly*) *equivalent to V, and h is the bottom field* (*resp. the strong bottom field*) *for V, then h is the bottom field* (*resp. the strong bottom field*) *for W.*

(2.3) *Every field of definition for V contains the bottom field* (*resp. the strong bottom field*) *for V, if the latter exists.*

(2.4) *If V is defined over an algebraic number field, then the bottom field and the strong bottom field for V exist and are algebraic number fields of finite degree.*

We say that V is a *minimal model* if every rational mapping of a variety X into V is defined at all the simple points of X. Then we can easily verify

(2.5) *The bottom field and the strong bottom field for V exist and coincide, if either one of them exists and V is a non-singular minimal model.*

PROPOSITION 2. *Let V and B be as in Prop.* 1. *Suppose that V has no deformation. Then the bottom field and the strong bottom field for V exist and are contained in the smallest field of definition for B.*

PROOF. By (2.2), (2.4) and Th. 1, we know that the strong bottom field for V exists and is an algebraic number field of finite degree. Denote it by h. Let k_0 be a finitely generated extension of h over which B and $c(V)$ are rational. Let x be a generic point of B over k_0, and W the variety such that $c(W)=x$. By (1.1) and (2.2), h is the strong bottom field for W. Let k be the smallest field of definition for B. Since W is defined over $k(x)$, we have $h\subset k(x)$ by (2.3). Since both h and k are algebraic over $\boldsymbol{Q}$ and k is algebraically closed in $k(x)$, we have $h \subset k$, which completes the proof for the strong bottom field. The same reasoning applies also to the bottom field.

3. In order to prove Th. 2, it is sufficient, in view of Prop. 2, to show the existence of a non-singular projective variety without deformation whose strong bottom field is not $\boldsymbol{Q}$. We shall obtain such a variety as a quotient $\mathfrak{H}^r/\Gamma$ of the product of r copies of the upper half plane $\mathfrak{H}=\{z \in \boldsymbol{C} \mid \mathrm{Im}(z)>0\}$ by a discontinuous group Γ. To define Γ, we consider a totally real algebraic number field F of degree g, and a division quaternion algebra D over F. Let $\mathfrak{p}_{\infty 1}, \cdots, \mathfrak{p}_{\infty g}$ be the archimedean primes of F. Suppose that D is unramified at $\mathfrak{p}_{\infty 1}, \cdots, \mathfrak{p}_{\infty r}$ and ramified at $\mathfrak{p}_{\infty r+1}, \cdots, \mathfrak{p}_{\infty g}$. Put $D_{\boldsymbol{R}}=D \otimes_{\boldsymbol{Q}} \boldsymbol{R}$. Then $D_{\boldsymbol{R}}$ can be identified with $M_2(\boldsymbol{R})^r \times \boldsymbol{K}^{g-r}$, the product of r copies of the total matrix algebra of degree 2 over the real number field $\boldsymbol{R}$ and $g-r$ copies of the division ring $\boldsymbol{K}$ of real quaternions. Let $\mathfrak{o}$ be a maximal order in D, and $\Gamma(\mathfrak{o})$ the group of all units in $\mathfrak{o}$ whose reduced norm to F is 1. For every $\alpha \in \Gamma(\mathfrak{o})$, let $\begin{bmatrix} a_\nu & b_\nu \\ c_\nu & d_\nu \end{bmatrix}$ $(\nu=1, \cdots, r)$ be the projections of α to the first r factors $M_2(\boldsymbol{R})$ of $D_{\boldsymbol{R}}$. Define the action of α on $\mathfrak{H}^r$ by

$$\alpha(z_1, \cdots, z_r)=(w_1, \cdots, w_r),\ w_\nu=(a_\nu z_\nu+b_\nu)(c_\nu z_\nu+d_\nu)^{-1} \qquad (z_\nu \in \mathfrak{H}).$$

Then $\Gamma(\mathfrak{o})$ gives a discontinuous group operating on $\mathfrak{H}^r$. Now we assume

(3.1) *The multiplicative group of invertible elements of D has no element of finite order other than ± 1.*

Then $\mathfrak{H}^r/\Gamma(\mathfrak{o})$ is biregularly isomorphic to a non-singular projective variety V. By [5, Th. 7.4], V has no deformation if $r>1$. Assume further

(3.2) $r=g-1>1$, *and $\mathfrak{p}_{\infty g}$ corresponds to the identity mapping of F.*

The results of [7, 4.9 and 4.10] tell us that, under the assumptions (3.1) and (3.2), the composite of F and the bottom field for V is an unramified class field over F corresponding to an ideal group $I(D/F)$ in F generated by the following three types of ideals:

(I) the principal ideals (a) such that a is totally positive;

(II) the squares of all ideals in F;

(III) the prime ideals in F which are ramified in D.

In view of these facts, we construct our counterexample in the following way. Take any totally real algebraic number field F of degree $g>2$ whose class number is even. (For example, $F=\boldsymbol{Q}(\sqrt{10}, \sqrt{13})$.) We can find a prime ideal $\mathfrak{p}$ in F which is generated by a totally positive element and decomposes in every quadratic extension of F containing non-trivial roots of unity. There exists a quaternion algebra D over F which is ramified exactly at $\mathfrak{p}$ and $\mathfrak{p}_{\infty g}$, where $\mathfrak{p}_{\infty g}$ is as in (3.2). By our choice of $\mathfrak{p}$, we see that (3.1) is satisfied, and $I(D/F)$ is different from the whole ideal group of F. Hence a model for $\mathfrak{H}^r/\Gamma(\mathfrak{o})$, with any maximal order $\mathfrak{o}$ in D, has the bottom field different from $\boldsymbol{Q}$. This completes the proof of Th. 2.

It is possible to obtain many more examples of varieties without deformation and with bottom fields $\neq \boldsymbol{Q}$, by considering the quotient of a bounded symmetric domain, other than $\mathfrak{H}^r$, by a discontinuous group. This can be done, for example, by making the results of [**7**, § 2] more precise, or extending them to the case of congruence subgroup of the discontinuous group in question (cf. also [**8**]).

4.[2] Another type of counterexample can be obtained by considering the topological structure of varieties. We first prove an elementary

PROPOSITION 3. *Let V and V' be non-singular projective varieties whose Chow points belong to the same component of $C_0(N, m, d)$. Then there exists a diffeomorphism of V onto V'.*

PROOF. Let B, k_0, z, E', f, w and H be as in the proof of Prop. 1. We see that V, as a fibre of H, is diffeomorphic to a generic fibre W_z with a generic point z of E' over k. (Note that we do not need the condition $H^1(V, \Theta) = 0$.) Let B_0 be the set of all simple points r of B such that $r = c(X)$ with a non-singular variety X. Then B_0 is a Zariski open subset of B rational over k_0. Let K be the locus of $f(z) \times w$ over k_0 on $B_0 \times P^N$. By the same type of argument as in the proof of Prop. 1, we see that K is non-singular, and $K \rightarrow B_0$ defines a fibre bundle whose fibres are the varieties X such that $c(X) \in B_0$. If $c(X)$ and $c(Y)$ belong to B_0, then X and Y are diffeomorphic. Since $c(W_z)$ $(= f(z))$ belongs to B_0, we see that V is diffeomorphic to the fibres of K. Applying the same reasoning to V', we obtain our assertion.

Now Serre [**6**] gave an example of a non-singular projective variety V defined over an algebraic number field k such that V is not homeomorphic to V^σ for some isomorphism σ of k into $\boldsymbol{C}$. Then the component of $C_0(N, m, d)$ containing $c(V)$ can not be defined over $\boldsymbol{Q}$, since it can not contain $c(V^\sigma) = c(V)^\sigma$ in view of Prop. 3. It may be interesting to investigate the deformation of this V.

5. Let us now consider $C_p(N, m, d)$ for positive p. Since $C_0(N, m, d)$ is defined by equations with rational coefficients, we can naturally speak of its reduction modulo p for a prime number p.

PROPOSITION 4. *There exists a finite set $S(N, m, d)$ of rational primes such that $C_p(N, m, d)$ is exactly the reduction of $C_0(N, m, d)$ modulo p if p is not contained in $S(N, m, d)$.*

This can easily be verified by applying the following two simple principles

2) The possibility of using Serre's example was kindly suggested by G. Washnitzer.

to the defining equations of $C_0(N, m, d)$.

(5.1) *Reduction modulo p of the intersection of algebraic sets is the intersection of the reduced sets for all except a finite number of p.*

(5.2) *Reduction modulo p of the projection of an algebraic set is the projection of the reduced set for all except a finite number of p.*

For more precise statements of these facts, see, for example, [**9**, Ch. III, Prop. 19, Prop. 20].

Now let B be a component of $C_0(N, m, d)$ which is not defined over $\boldsymbol{Q}$. Let $B_1, \cdots, B_r$ be all the components of $C_0(N, m, d)$ with $B_1 = B$. Let $y_i = c(B_i)$ for $i = 1, \cdots, r$. Then both $\boldsymbol{Q}(y_1)$ and $\boldsymbol{Q}(y_1, \cdots, y_r)$ are algebraic number fields of degree > 1. We can find a prime ideal $\mathfrak{p}$ in $\boldsymbol{Q}(y_1, \cdots, y_r)$ such that: (i) y_1 modulo $\mathfrak{p}$ is not rational over the prime field; (ii) $\mathfrak{p}(B_1)$[3] is absolutely irreducible and not contained in $\mathfrak{p}(B_i)$ for $i > 1$; and (iii) the rational prime p divisible by $\mathfrak{p}$ does not belong to $S(N, m, d)$. Then $\mathfrak{p}(B_1)$ is a component of $C_p(N, m, d)$ which is not defined over the prime field. Thus the answer to the question (Q) is negative at least for infinitely many prime characteristics.

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3) $\mathfrak{p}(X)$ means the cycle obtained from X by reduction modulo $\mathfrak{p}$.