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Notes on differenntial theoretic characterization of regular local rings

Dedicated to Professor Shôkichi Iyanaga on his 60th birthday

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Let R be a locality over a field k, i.e., R is a quotient ring of an affine domain A over a field k with respect to a prime ideal. We have then two types of differential theoretic characterization for R to be a regular local ring. One is given by the author in [**3**] and another one is given by E. Kunz [**2**]. It says as follows: R is a regular local ring if and only if $D_k(R)$ is a free R-module, modulo some assumptions. Kunz's criterion asserts that R is regular if and only if $D(R)$ (absolute differential module) is free. The first criterion is essentially equivalent to the Jacobian criterion of simple loci and the second one to the mixed Jacobian criterion, initiated by Zariski [**4**]. This fact about the first one was discussed in [**3**], but not in a satisfactory way and the fact about the second one has not been noticed, to the best of the authors' knowledge, in any literature yet. In this paper we shall make clear precise relations which exist among these criterions. This gives also an improvement of the theorems in [**3**].

1. Let S and R be commutative rings with 1, and assume that S is an R-algebra. We denote by $D_R(S)$ the module of R-differentials of S. Any ring S is an algebra over a prime ring Π. So we denote $D_\Pi(S)$ simply by $D(S)$. This is canonically isomorphic to $D_{S^p}(S)$ if the characteristic of S is a positive prime number p. For these notations and fundamental properties of $D_R(S)$ we refer to [**3**].

For the sake of convenience we list fundamental results which are necessary in the sequel.

PROPOSITION 1 (Theorem 1 of [**3**]). *Let S be an R-algebra and let T be an S-algebra. Then any R-derivation of S into a T-module V can be extended to an R-derivation of T into V if and only if we have the following split exact sequence*

$$0 \to T \otimes D_R(S) \xrightarrow{\varphi_{R;S,T}} D_R(T) \to D_S(T) \to 0$$

PROPOSITION 2 (Proposition 3 of Exposé 13 [**1**])[1]. *Let L be a field finitely generated over a field K and R a subring of K. Let N be the kernel of the homomorphism $\varphi_{R;K,L}$. Then if R is contained in K^p (p is the characteristic of K. If $p=0$ we understand R is a prime ring Z) we have the equality*

$$\dim_L D_K(L) - \dim_L N = \mathrm{tr.d.}_K L\,.$$

PROPOSITION 3 (Theorem 5 of Exposé 17 [**1**]). *Let R be a local ring containing a field K and let M be the maximal ideal of R. Assume that the residue field R/M is a separable extension of K. Then we have the exact sequence*

$$0 \to M/M^2 \to (R/M)\otimes D_K(R) \to D_K(R/M) \to 0\,.$$

PROPOSITION 4. *Let S be a local integral domain with the quotient field K, and R a subring of S. Assume that $D_R(S)$ is a finite S-module. Then $D_R(S)$ is a free module if and only if the rank of $D_R(S)$ is equal to the rank of $D_R(K)$.*

Proof is easy and is omitted.

2. THEOREM 1. *Let $\mathfrak{P}$ be a prime ideal of the polynomial ring $k[X_1, \cdots, X_n]$ and let $A = k[X_1, \cdots, X_n]/\mathfrak{P}$. We assume that transcendence degree of the quotient field K of A over k is r (hence* rank *$\mathfrak{P} = n-r$). Let $\mathfrak{Q}$ be any prime ideal of $k[X_1, \cdots, X_n]$ containing $\mathfrak{P}$ and set $\mathfrak{q} = \mathfrak{Q}/\mathfrak{P}$. We set $R = A_{\mathfrak{q}}$ and let M be the maximal ideal of R. Then the following conditions* (1) *and* (2) *are equivalent.*

(1) *$D_k(R)$ is a free R-module and the quotient field K of R is a separable extension of k.*

(2) *There exist $n-r$ polynomials $f_1(X), \cdots, f_{n-r}(X)$ such that the rank of the matrix $(\partial f_i(X)/\partial X_j)$ $(i=1, \cdots, n-r\,;\ j=1, \cdots, n)$ is $n-r$ modulo $\mathfrak{Q}$.*

If R satisfies one of the above conditions, then

(3) *R is a regular local ring.*

Conversely if R is a regular local ring and

(4) *the residue field R/M is a separable extension of k,*

then R satisfies the above conditions (1) *and* (2). *Moreover any regular system of parameters of R form a part of a separating transcendence basis of K over k.*

First we give some remarks.

REMARKS 1. $D_k(R)$ can be a free module without R being a regular local ring if K is not separable over k. (See Example 1 of [3].)

2. R can be a regular local ring without $D_k(R)$ being a free module if R/M is not separable over k. (See Example 2 of [3].)

3. (2) implies (3) but not necessarily (4) as we can see easily.

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1) Strictly speaking this is a slight generalization of the quoted Proposition. But the same proof is applicable.

4. If a regular local ring R has the separable residue class field over k, then the quotient field K is a separable extension of k. Hence if K is not separable over k and R is regular, the residue field cannot be a separable extension of k. This fact is sometimes useful to know that a ring is not a regular local ring.

PROOF OF THEOREM 4. It is proved in [4] that (2) implies (3) and (3), (4) implies (2)[2]. We shall give here the equivalence of (1) and (2).

Assume the condition (2). Without loss of generalities we can suppose that $\det|\partial f_i/\partial X_j| \not\equiv 0$ modulo $\mathfrak{Q}$ $(j=r+1, \cdots, n)$. We shall denote by x_i the class of X_i modulo $\mathfrak{P}$. If we denote by d the differential operator d_k^A in $D_k(A)$, then

$$0=df_i(x)=\sum_{j=1}^{n}(\partial f_i/\partial x_j)dx_j\,.$$

By our assumption $\det|\partial f_i/\partial x_j|$ $(j=r+1, \cdots, n)$ is not contained in $\mathfrak{q}$. Hence the above linear equations are solvable in $dx_{r+1}, \cdots, dx_n$ in $D_k(R)=R\underset{A}{\otimes}D_k(A)$ and we see that

$$D_k(R)=Rdx_1+\cdots+Rdx_r\,.$$

It remains to prove that $dx_1, \cdots, dx_r$ are linearly independent over R and K is separably algebraic over $k(x_1, \cdots, x_r)$. Since the quotient field K is a finitely generated extension of transcendence degree r and $D_k(K)=K\underset{R}{\otimes}D_k(R)$ is generated by $1\otimes dx_1, \cdots, 1\otimes dx_r$, they are a free base of $D_k(K)$ over K. Hence K is separably algebraic over $k(x_1, \cdots, x_r)$ by Th. 2 of Exposé 13 [1], and $dx_1, \cdots, dx_r$ are linearly independent over R.

Now assume the condition (1). Then $D_k(R)$ is a free module of rank, say, s, with the basis $dx_1, \cdots, dx_s$. $D_k(K)=K\otimes D_k(R)$ is a vector space of dimension r on account of the assumption on K. So s must be equal to r. Let us set $S=k[X]_{\mathfrak{Q}}$, and $\mathfrak{P}^*=\mathfrak{P}S$. Then we have the exact sequence

$$\mathfrak{P}^*/\mathfrak{P}^{*2}\xrightarrow{\rho}(S/\mathfrak{P}^*)\otimes D_k(S)\longrightarrow D_k(S/\mathfrak{P}^*)\longrightarrow 0$$

(cf. Proposition 9 of [3]). By definition $S/\mathfrak{P}^*=R$ and the third term is a free module of rank r over R. Hence there exists an R-homomorphism $\varphi: D_k(R)\to R\otimes D_k(S)$ such that

$$R\otimes D_k(S)=\mathrm{Im}\,(\rho)\oplus\mathrm{Im}\,(\varphi)\,.$$

From this we have

$$(R/M)\otimes D_k(S)=(R/M)\otimes\mathrm{Im}\,(\rho)\oplus(R/M)\otimes\mathrm{Im}\,(\varphi)\,.$$

$D_k(S)$ is a free module of rank n over S and hence the left hand side is a

2) These results can also be proved without the help of [4].

vector space of dimension n and the second term of the right hand side has dimension r since $D_k(R)\cong \mathrm{Im}(\varphi)$. Hence $(R/M)\otimes \mathrm{Im}(\rho)$ has dimension $n-r$. This implies the existence of $n-r$ polynomials $f_1^*, \cdots, f_{n-r}^*$ in $\mathfrak{P}^*$ such that $1\otimes \bar{d}f_1^*, \cdots, 1\otimes \bar{d}f_{n-r}^*$ are linearly independent, where $\bar{d}$ stands for d_k^S. From this we readily deduce the conclusion (2).

Now assume that R is regular and has a separable residue field. Then we have by proposition 3 the exact sequence

$$0 \longrightarrow M/M^2 \longrightarrow (R/M)\otimes D_k(R) \longrightarrow D_k(R/M) \longrightarrow 0\,.$$

Hence for any regular system $(u_1, \cdots, u_s)$ of parameters of R, $du_1, \cdots, du_s$ form a part of the free basis of $D_k(R)$. Hence $u_1, \cdots, u_s$ form a part of separating transcendence basis of K/k.

COROLLARY. *If the ground field k is perfect the following three conditions are equivalent*

(1) *R is a regular local ring.*

(2) *$D_k(R)$ is a free module.*

(3) *Jacobian criterion* ((2) *of Theorem* 1) *holds.*

3. We shall retain the notations in the preceding paragraph and assume k is a non-perfect field of characteristic p. Let k' be a field of definition for A such that (1) $k'\supset k^p$ and (2) $[k':k^p]<\infty$, where we mean by "a field of definition for A" a field k' such that $\mathfrak{P}$ is generated by elements in $k'[X_1, \cdots, X_n]$. Let $(\alpha_i, i\in I)$ be a p-basis of k over k' and set $k_0=k^p(\alpha_i, i\in I)$. Let $\beta_1, \cdots, \beta_s$ be a p-basis of k' over k^p. Then $(\alpha_i, i\in I: \beta_1, \cdots, \beta_s)$ form a p-basis of k and k' and k_0 are linearly disjoint over k^p. If we denote by $D(A)$ the differential module $D_{k^p}(A)$, we have the following split exact sequence by Proposition 1.

$$0 \longrightarrow A\otimes D(k_0) \longrightarrow D(A) \longrightarrow D_{k_0}(A) \longrightarrow 0\,.$$

Since $D(k_0)$ is a free module we see easily that $D(A_\mathfrak{q})$ is a free module if and only if $D_{k_0}(A_\mathfrak{q})$ is a free module for any prime ideal $\mathfrak{q}$ of A.

E. Kunz proved in [2] that $D(A_\mathfrak{q})$ (hence $D_{k_0}(A_\mathfrak{q})$) is a free module if and only if $A_\mathfrak{q}$ is a regular local ring. We shall show that $D_{k_0}(A_\mathfrak{q})$ is a free module if and only if the mixed Jacobian criterion of Zariski [4] holds for $\mathfrak{Q}$.

The key point of the proof is the following

LEMMA 1. *Notations and assumptions being as before, the rank of $D_{k_0}(K)$ is equal to $r+s$.*

PROOF. We have the following commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & 0 & \longrightarrow & K\otimes D(k_0) & \longrightarrow & D(K) & \longrightarrow & D_{k_0}(K) \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow id & & \\
0 & \longrightarrow & N & \longrightarrow & K\otimes D(k) & \longrightarrow & D(K) & \longrightarrow & D_k(K) \longrightarrow 0 \\
 & & & & \downarrow & & & & \\
 & & & & K\otimes D(k') & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

From this diagram we can easily deduce the following exact sequence

$$0 \longrightarrow N \longrightarrow K\otimes D(k') \longrightarrow D_{k_0}(K) \longrightarrow D_k(K) \longrightarrow 0 .$$

All the factors in this sequence are of finite dimensional over K, hence we have

$$\begin{aligned}\dim D_{k_0}(K) &= \dim (K\otimes D(k'))+\dim D_k(K)-\dim N \\ &= s+r .\end{aligned}$$

(Cf. Proposition 2).

We shall consider the differential module $D_{k_0}(k[X])$. It is easily seen that this is a free module with the basis $dX_1, \cdots, dX_n$; $d\beta_1, \cdots, d\beta_s$. Let ∂_i and δ_i be derivation of $k[X]$ over k_0 such that

$$\begin{aligned} &\partial_i(X_j)=\delta_{ij}, \qquad \partial_i(\beta_t)=0 \\ &\delta_i(X_t)=0, \qquad \delta_i(\beta_j)=\delta_{ij}. \end{aligned}$$

For a set of polynomials $f_1, \cdots, f_t$ in $k[X]$, we denote by $J(f_1, \cdots, f_t;\ X_1, \cdots, X_n;\ \beta_1, \cdots, \beta_s)$ the matrix

$$\begin{pmatrix} \partial_1 f_1, \cdots, \partial_n f_1, & \delta_1 f_1, \cdots, \delta_s f_1 \\ \vdots \qquad\quad \vdots & \vdots \qquad\quad \vdots \\ \partial_1 f_t, \cdots, \partial_n f_t, & \delta_1 f_t, \cdots, \delta_s f_t \end{pmatrix}$$

This matrix will be called the mixed Jacobian matrix.

THEOREM 2. *Retaining the notations as above $D(R)$ is a free module if and only if there exist $n-r$ polynomials $f_1, \cdots, f_{n-r}$ in $\mathfrak{P}$ such that the rank of the matrix $J(f_1, \cdots, f_{n-r};\ X_1, \cdots, X_n;\ \beta_1, \cdots, \beta_s)$ is $n-r$ module $\mathfrak{Q}$.*

PROOF. We shall express the ring $A=k[X_1, \cdots, X_n]/\mathfrak{P}$ as an affine ring over k_0. Let $(F_1, \cdots, F_m)$ be a basis of $\mathfrak{P}$ in $k'[X]$. If we denote by M_λ the monomials in the indeterminates X's we can represent F's in the form

$$F_i(X)=\sum_\lambda a_{i\lambda}M_\lambda,\ a_{i\lambda}\in k'$$

and

$$a_{i\lambda}=\sum\alpha^{i\lambda}_{t_1}\cdots{}_{t_s}\beta_1^{t_1}\cdots\beta_s^{t_s}$$

where $\alpha^{i\lambda}_{t_1\cdots t_s}\in k^p$ and $0\leqq t_i\leqq p-1$. We set

$$\bar{F}_i=\sum\alpha^{i\lambda}_{t_1\cdots t_s}Y_1^{t_1}\cdots Y_s^{t_s}M_\lambda\,.$$

Then we have

$$A=k_0[X_1,\cdots,X_m;\ Y_1,\cdots,Y_s]/(\bar{F}_1,\cdots,\bar{F}_m;\ Y_i^p-\beta_i^p\ (i=1,\cdots,s))\,.$$

We denote by $\tilde{\mathfrak{P}}$ the ideal in $k_0[X;Y]$ generated by $\bar{F}'s$ and $Y_i^p-\beta_i^p$ $(i=1,\cdots,s)$. If we denote by ψ the natural homomorphism $k_0[X,Y]\to k[X]$ such that $\psi(X_i)=X_i$, $\psi(Y_i)=\beta_i$ and $\psi(a)=a$ for $a\in k_0$, then we have $\tilde{\mathfrak{P}}=\psi^{-1}(\mathfrak{P})$. Let us set $\tilde{\mathfrak{Q}}=\psi^{-1}(\mathfrak{Q})$. Then we have clearly $R=A_q=\tilde{S}/\tilde{\mathfrak{P}}\tilde{S}$ where $\tilde{S}=k_0[X;Y]_{\tilde{\mathfrak{Q}}}$. If we set $\tilde{\mathfrak{P}}^*=\tilde{\mathfrak{P}}\tilde{S}$, we have the exact sequence

$$\tilde{\mathfrak{P}}^*/\tilde{\mathfrak{P}}^{*2}\xrightarrow{\tilde{\rho}}R\otimes D_{k_0}(\tilde{S})\longrightarrow D_{k_0}(R)\longrightarrow 0\,.$$

Now assume that $D_{k_0}(R)$ is a free module. Then the rank of $D_{k_0}(R)$ is equal to the rank of $D_{k_0}(K)$, and this is equal to $r+s$ by Proposition 1. Then we have also

$$R\otimes D_{k_0}(\tilde{S})=\mathrm{Im}\,(\tilde{\rho})\oplus D_{k_0}(R)\,.$$

$R\otimes D_{k_0}(\tilde{S})$ is a free module of rank $n+s$ and hence $\mathrm{Im}\,(\tilde{\rho})$ is also a projective, hence a free R-module and whose rank is exactly $n-r$. By applying $\otimes(R/M)$ we get $(R/M)\otimes D_{k_0}(S)=(R/M)\otimes\mathrm{Im}\,(\tilde{\rho})\oplus D_{k_0}(R)$. This implies the existence of polynomials $\bar{f}_1,\cdots,\bar{f}_{n-r}$ in $\tilde{\mathfrak{P}}$ such that $1\otimes d^*\bar{f}_1,\cdots,1\otimes d^*\bar{f}_{n-r}$ are linearly independent in $R\otimes D_{k_0}(S)$, where d^* stands for $d^S_{k_0}$. The equivalency of the linear independency of $1\otimes d^*\bar{f}_1,\cdots,1\otimes d^*\bar{f}_{n-r}$ to the statement that the mixed Jacobian matrix $J(f_1,\cdots,f_{n-r};\ X_1,\cdots,X_n;\ \beta_1,\cdots,\beta_s)$ has the rank $n-r$ where $f_i=\psi(\bar{f}_i)$ follows easily from the

LEMMA 2. (1) $$\psi\cdot\frac{\partial}{\partial X_i}=\partial_i\cdot\psi$$

(2) $$\psi\cdot\frac{\partial}{\partial Y_j}=\delta_j\cdot\psi$$

where $\dfrac{\partial}{\partial X_i}$ *and* $\dfrac{\partial}{\partial Y_j}$ *are derivations of* $k_0[X;Y]$ *over* k_0 *in the ordinary esnse.*

Now assume that the existence of $(n-r)$-polynomials $f_1,\cdots,f_{n-r}$ in $\mathfrak{P}$ such that $J(f_1,\cdots,f_{n-r};\ X_1,\cdots,X_n;\ \beta_1,\cdots,\beta_s)$ has the rank $n-r$. Then the above argument implies that we have the exact sequence

$$(R/M)\otimes\mathrm{Im}\,(\tilde{\rho})\xrightarrow{j}(R/M)\otimes D_{k_0}(S)\longrightarrow(R/M)\otimes D_{k_0}(R)\longrightarrow 0\,.$$

$\mathrm{Im}\,(j)$ has at least of dimension $n-r$, hence $(R/M)\otimes D_{k_0}(R)$ has at most of imension $r+s$ and $D_{k_0}(R)$ is at most of rank $r+s$. Since $D_{k_0}(K)$ has the rank

$r+s$ this implies that $D_{k_0}(R)$ is also of rank $r+s$ and $D_{k_0}(R)$ is a free module by Proposition 1.

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