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Pluricanonical systems on algebraic surfaces of general type

Dedicated to Professor S. Iyanaga on his 60th birthday

By Kunihiko KODAIRA

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By a minimal non-singular algebraic surface of general type we shall mean a non-singular algebraic surface free from exceptional curves (of the first kind) of which the bigenus P_2 and the Chern number c_1^2 are both positive, where c_1 denote the first Chern class of the surface (see § 3). Let S denote a minimal non-singular algebraic surface of general type defined over the field of complex numbers and let K be a canonical divisor on S. The number of non-singular rational curves E on S satisfying the equation: $KE=0$ is smaller than the second Betti number of S, where KE denotes the intersection multiplicity of K and E. We define $\mathcal{E}$ to be the union of all the non-singular rational curves E with $KE=0$ on S and represent it as a sum: $\mathcal{E}=\sum_{\nu}\mathcal{E}_{\nu}$ of its *connected components* $\mathcal{E}_{\nu}$. Obviously $\mathcal{E}$ may be an empty set. Consider a holomorphic map $\Phi: z\rightarrow\Phi(z)$ of S into a projective n-space $\boldsymbol{P}^n$. We shall say that Φ is *biholomorphic modulo* $\mathcal{E}$ if and only if Φ is biholomorphic on $S-\mathcal{E}$ and $\Phi^{-1}\Phi(z)=\mathcal{E}_{\nu}$ for $z\in\mathcal{E}_{\nu}$. For any positive integer m, we let Φ_{mK} denote the *rational map* of S into $\boldsymbol{P}^n$ defined by the pluri-canonical system $|mK|$, where $n=\dim|mK|$. Note that, if $|mK|$ has no base point, then Φ_{mK} is a holomorphic map. D. Mumford proved that, for every sufficiently large integer m, the pluri-canonical system $|mK|$ has no base point and Φ_{mK} is biholomorphic modulo $\mathcal{E}$ (see Mumford [6]; compare also Zariski [9], Matsusaka and Mumford [5]). His proof is based on results of Zariski [9] and covers the abstract case. On the other hand, it has been shown by Šafarevič [8] that Φ_{9K} is a birational map. The main purpose of this paper is to prove the following theorem:

THEOREM. *For every integer $m\geqq 4$, the pluri-canonical system $|mK|$ has no base point and Φ_{mK} is a holomorphic map. For every integer $m\geqq 6$, the map Φ_{mK} is biholomorphic modulo $\mathcal{E}$.*

§ 1. Notation.

Let S be a non-singular algebraic surface defined over the field $\boldsymbol{C}$ of complex numbers. We shall denote by x, y, z points on S, by $C, C_1, \cdots, \Theta, \cdots$ irreducible curves on S, by $X, Y, D, D_1, \cdots$ divisors on S and by m, n, h, i, j, k rational integers. We say that a divisor $D=\sum_i n_i C_i$ is *positive* and write $D>0$ if the coefficients n_i are positive. For any divisors D and X on S we denote by DX the intersection multiplicity of D and X. We write D^2 for DD. We indicate by the symbol $\approx$ linear equivalence. We let $[D]$ denote the complex line bundle over S determined by the divisor D.

Let F be a complex line bundle over S. By a local holomorphic section of F we shall mean a holomorphic section of F defined over an open subset of S. Let $\varphi: z \rightarrow \varphi(z)$ be a local holomorphic section of F. We choose a sufficiently fine finite covering $\{U_j\}$ of S and denote by $\varphi_j(z)$ the fibre coordinate of $\varphi(z)$ over U_j, provided that $z \in U_j$. Let x be a point on S and let (z_1, z_2) denote a local coordinate of the center x on S. We call x a *zero* of φ of order h if

$$\varphi_j(x)=0, \qquad (\partial^{m+n}\varphi_j/\partial z_1{}^m \partial z_2{}^n)(x)=0 \qquad \text{for } m+n \leqq h-1$$

and if at least one partial derivative $(\partial^h \varphi_j/\partial z_1{}^n \partial z_2{}^{h-n})(x)$ of order h does not vanish, provided that $x \in U_j$. We denote by $\mathcal{O}$ the sheaf over S of germs of holomorphic functions and by $\mathcal{O}(F)$ the sheaf over S of germs of holomorphic sections of F. Moreover we denote by the symbol

$$\mathcal{O}(F-hx-ky-\cdots)$$

the subsheaf of $\mathcal{O}(F)$ consisting of germs of those holomorphic sections of F of which the points $x, y, \cdots$ are zeros of respective orders $\geqq h, \geqq k, \cdots$. We remark that $\mathcal{O}(-x)$ is the sheaf of the ideals of the point x and that

$$\mathcal{O}(F-hx-ky-\cdots)=\mathcal{O}(F) \underset{\mathcal{O}}{\otimes} \mathcal{O}(-x)^h \mathcal{O}(-y)^k \cdots.$$

Let $\boldsymbol{C}^n$ denote the vector space of n complex variables. The stalks of the quotient sheaf $\mathcal{O}/\mathcal{O}(-x)^h$ are

$$(\mathcal{O}/\mathcal{O}(-x)^h)_z=\begin{cases} \boldsymbol{C}^{h(h+1)/2}, & \text{if } z=x, \\ 0 & \text{otherwise.} \end{cases}$$

To indicate this we write

$$\boldsymbol{C}_x^{h(h+1)/2}=\mathcal{O}/\mathcal{O}(-x)^h.$$

Then, for instance, we have

$$\mathcal{O}(F)/\mathcal{O}(F-hx-ky) \cong \boldsymbol{C}_x^{h(h+1)/2} \oplus \boldsymbol{C}_y^{k(k+1)/2}. \tag{1}$$

For any holomorphic section ψ of a complex line bundle over S, we denote by (ψ) the divisor of ψ. Let D be a *positive divisor* on S. Obviously D is the divisor (ψ) of a holomorphic section ψ of the complex line bundle $[D]$. We say that x is a point of D and write $x \in D$ if and only if x is a zero of ψ. We define the multiplicity of a point x of D to be m if x is a zero of ψ of order m. Moreover we call x a simple point or a *multiple point* of D according as $m=1$ or $m \geqq 2$. We shall say that a local holomorphic section φ of F defined on an open subset $W \subset S$ is *divisible* by D if φ_j/ψ_j is holomorphic on $U_j \cap W$ for every neighborhood U_j. We denote by $\mathcal{O}(F-D)$ the sheaf over S of germs of those holomorphic sections of F which are divisible by D. We have the isomorphism:

$$\mathcal{O}(F-D) \cong \mathcal{O}(F-[D]).$$

We define

$$\mathcal{O}(F-D-hx-ky-\cdots) = \mathcal{O}(F-D) \cap \mathcal{O}(F-hx-ky-\cdots).$$

Note that, if x is a point of D of multiplicity $m \geqq h$, then

$$\mathcal{O}(F-D-hx-ky-\cdots) = \mathcal{O}(F-D-ky-\cdots). \tag{2}$$

We denote by $|F|$ the complete linear system consisting of the divisors (φ) of holomorphic sections $\varphi \in H^0(S, \mathcal{O}(F))$, $\varphi \neq 0$, and define

$$\dim |F| = \dim H^0(S, \mathcal{O}(F)) - 1.$$

Note that $|[D]| = |D|$. Letting $\{\varphi_0, \varphi_1, \cdots, \varphi_n\}$ be a base of the linear space $H^0(S, \mathcal{O}(F))$, we define a *rational map*

$$\Phi_F : z \to \Phi_F(z) = (\varphi_0(z), \varphi_1(z), \cdots, \varphi_n(z))$$

of S into $\boldsymbol{P}^n$. We call z a *base point* of the complete linear system $|F|$ if $z \in D$ for all divisors $D \in |F|$. It is obvious that, if $|F|$ has no base point, then Φ_F is a holomorphic map. We let K denote *either* the canonical bundle of S *or* a canonical divisor on S. We denote by p_g, P_m and q, respectively, the geometric genus, the m-genus and the irregularity of S. Note that

$$P_m = \dim |mK| + 1, \qquad m = 1, 2, 3, \cdots.$$

For any divisor X on S we let $\pi(X)$ denote the virtual genus of X defined by the formula:

$$2\pi(X) - 2 = X^2 + KX.$$

Every complex line boundle F over S is determined by a divisor D on $S: F = [D]$. We let $F^2 = D^2$. Moreover, for any divisor X on S, we define

$$FX = DX, \qquad F = [D].$$

§ 2. Vanishing theorems.

Let F be a complex line bundle over S and let C denote an irreducible curve on S. We define the restriction to C of the sheaf $\mathcal{O}(F)$ to be the quotient sheaf:

$$\mathcal{O}(F)_C = \mathcal{O}(F)/\mathcal{O}(F-C).$$

For any element φ of $\mathcal{O}(F)$ we denote by φ_C the element of $\mathcal{O}(F)_C$ corresponding to φ.

Let $\tilde{C}$ denote the non-singular model of C and let μ be the holomorphic birational map of $\tilde{C}$ onto C. Moreover let μ^*F denote the complex line bundle over $\tilde{C}$ induced from F. For any complex line bundle $\mathfrak{f}$ over $\tilde{C}$ we denote by $c(\mathfrak{f})$ the Chern class of $\mathfrak{f}$ which can be regarded as an integer. We have

$$c(\mu^*F) = FC.$$

Letting $\mathfrak{d}$ be an effective divisor on $\tilde{C}$, we denote by $\mathcal{O}(\mathfrak{f}-\mathfrak{d})$ the sheaf over $\tilde{C}$ of germs of holomorphic sections of $\mathfrak{f}$ which are divisible by $\mathfrak{d}$. Let $\mathfrak{c}$ denote the *conductor* of C on $\tilde{C}$. We have the exact sequence

$$0 \longrightarrow \mathcal{O}(\mu^*F-\mathfrak{c}) \xrightarrow{\ \mu\ } \mathcal{O}(F)_C \longrightarrow M \longrightarrow 0, \tag{3}$$

where M is a sheaf over C such that the stalk M_z is zero for every simple point z of C. In forming the exact sequence (3) we regard $\mathcal{O}(\mu^*F-\mathfrak{c})$ as a sheaf over C by means of the map $\mu: \tilde{C} \to C$ (see [2], § 1).

In what follows we denote by $\boldsymbol{C}\{t\}$ the ring of convergent power series in a variable t with coefficients in $\boldsymbol{C}$. Let x be a point of C of multiplicity m. The inverse image $\mu^{-1}(x)$ consists of a finite number of points $p_1, \cdots, p_\lambda, \cdots, p_r$ on $\tilde{C}$. We introduce a local coordinate (w, z) of the center x on S which is "general" with respect to C (we write w, z in place of z_1, z_2). Then, for each point p_λ, we find a local uniformization variable t_λ of the center p_λ on $\tilde{C}$ such that, in a neighborhood of p_λ, the map μ takes the following form

$$\mu: t_\lambda \to (w, z) = (P_\lambda(t_\lambda), t_\lambda^{m_\lambda}), \qquad P_\lambda(t_\lambda) \in t_\lambda^{m_\lambda}\boldsymbol{C}\{t_\lambda\},$$

where m_λ is a positive integer and $t_\lambda^{m_\lambda}\boldsymbol{C}\{t_\lambda\}$ denotes the ideal of $\boldsymbol{C}\{t_\lambda\}$ generated by $t_\lambda^{m_\lambda}$. It is clear that

$$R(w, z) = \prod_{\lambda=1}^{r} \prod_{k=0}^{m_\lambda-1} (w - P_\lambda(\varepsilon_\lambda^k z^{1/m_\lambda})), \qquad \varepsilon_\lambda = e^{2\pi i/m_\lambda},$$

is a polynomial of the form

$$w^m + A_1(z)w^{m-1} + \cdots + A_m(z), \qquad A_k(z) \in z^k\boldsymbol{C}\{z\},$$

and the equation:

$$R(w, z)=w^{m}+A_1(z)w^{m-1}+\cdots+A_m(z)=0$$

is a minimal equation of C on a neighborhood of x. We let

$$B_h(w, z)=w^{h}+A_1(z)w^{h-1}+\cdots+A_h(z).$$

We define

$$\sigma_\lambda dt_\lambda=d(t_\lambda^{m_\lambda})/\partial_w R(P_\lambda(t_\lambda), t_\lambda^{m_\lambda}),$$

where $\partial_w R(w, z)=\partial R(w, z)/\partial w$. The exponent c_λ in the expansion

$$\sigma_\lambda=t_\lambda^{-c_\lambda}(a_{\lambda 0}+a_{\lambda 1}t_\lambda+a_{\lambda 2}t_\lambda^2+\cdots), \qquad a_{\lambda 0}\neq 0,$$

is a non-negative integer and, by definition,

$$\mathfrak{c}=c_1 p_1+\cdots+c_\lambda p_\lambda+\cdots+c_r p_r+\cdots.$$

Since the complex line bundle F is locally trivial, the restriction to the point x of the exact sequence (3) is reduced to

$$0\longrightarrow\bigoplus_{\lambda=1}^{r}\mathcal{O}(-\mathfrak{c})_{p_\lambda}\xrightarrow{\ \mu\ }(\mathcal{O}_C)_x\longrightarrow M_x\longrightarrow 0.$$

For any convergent power series $f=f(w, z)$ in w and z, we denote by f_C the restriction of f to C. Obviously the stalk $(\mathcal{O}_C)_x$ consists of the restrictions f_C of elements f of $\mathcal{O}_x$. It is clear that $\mathcal{O}(-\mathfrak{c})_{p_\lambda}=t_\lambda^{c_\lambda}\boldsymbol{C}\{t_\lambda\}$. Hence an arbitrary element of the ring $\bigoplus_{\lambda=1}^{r}\mathcal{O}(-\mathfrak{c})_{p_\lambda}$ can be written in the form

$$\xi=\sum_{\lambda=1}^{r}\xi_\lambda(t_\lambda), \qquad \xi_\lambda(t_\lambda)\in t_\lambda^{c_\lambda}\boldsymbol{C}\{t_\lambda\}.$$

LEMMA 1. *For any element $\xi=\sum_{\lambda=1}^{r}\xi_\lambda(t_\lambda)$ of the ring $\bigoplus_{\lambda=1}^{r}\mathcal{O}(-\mathfrak{c})_{p_\lambda}$, there exists one and only one element f of $\mathcal{O}_x$ of the form*

$$f=\sum_{h=0}^{m-1}f_h(z)w^{m-1-h}, \qquad f_h(z)=\sum_{n=0}^{\infty}f_{hn}z^n,$$

which satisfies the equation:

$$f_C=\mu\xi.$$

Moreover the coefficients f_{hn} of f are given by the formula

$$f_{hn}=\frac{1}{2\pi i}\sum_{\lambda=1}^{r}\oint\xi_\lambda(t_\lambda)B_h(P_\lambda(t_\lambda), t_\lambda^{m_\lambda})t_\lambda^{-(n+1)m_\lambda}\sigma_\lambda dt_\lambda. \tag{4}$$

For a proof of this lemma, see [2], Appendix I.

For any integer h, we denote by h^+ the *positive part* of h, i.e., $h^+=\max\{h, 0\}$.

LEMMA 2. *Let k be a non-negative integer and let*

$$\mathfrak{d}_x=\sum_{\lambda=1}^{r}(k-m+1)^+ m_\lambda p_\lambda.$$

Then we have

$$\mu \bigoplus_{\lambda=1}^{r} \mathcal{O}(-\mathfrak{c}-\mathfrak{d}_x)_{p_\lambda} \subset (\mathcal{O}(-kx)_C)_x . \tag{5}$$

PROOF. We take an arbitrary element ξ of $\bigoplus_\lambda \mathcal{O}(-\mathfrak{c}-\mathfrak{d}_x)_{p_\lambda}$ and, with the aid of the above lemma, determine an element f of $\mathcal{O}_x$ satisfying the equation: $f_C=\mu\xi$. Let $d_\lambda=(k-m+1)^+m_\lambda$. We then have

$$\xi=\sum_{\lambda=1}^{r}\xi_\lambda(t_\lambda), \qquad \xi_\lambda(t_\lambda)\in t_\lambda^{c_\lambda+d_\lambda}\boldsymbol{C}\{t_\lambda\} .$$

Since

$$\xi_\lambda(t_\lambda)B_h(P_\lambda(t_\lambda),\ t_\lambda^{m_\lambda})t_\lambda^{-(n+1)m_\lambda}\sigma_\lambda \in t_\lambda^{(h-n-1)m_\lambda+d_\lambda}\boldsymbol{C}\{t_\lambda\}$$

and

$$(h-n-1)m_\lambda+d_\lambda \geqq 0 \qquad \text{for } m-1-h+n \leqq k-1 ,$$

we infer from (4) that

$$f_{hn}=0 , \qquad \text{for } m-1-h+n\leqq k-1 .$$

It follows that $f\in\mathcal{O}(-kx)_x$, q. e. d.

We remark that, in the case in which x is a simple point of C, the formula (5) is reduced to the equality

$$\mu\mathcal{O}(-\mathfrak{d}_x)_p=(\mathcal{O}(-kx)_C)_x , \qquad p=\mu^{-1}(x) .$$

THEOREM 1. *Let C be an irreducible curve on S and let F denote a complex line bundle over S. Moreover let x and y be distinct points of C with respective multiplicities m and n and let h and k denote non-negative integers. If*

$$FC-C^2-KC>(h-m+1)^+m+(k-n+1)^+n ,$$

then the cohomology group $H^1(C, \mathcal{O}(F-hx-ky)_C)$ vanishes.

PROOF. In view of Lemma 2 and the above remark, we have the exact sequence

$$0\to\mathcal{O}(\mu^*F-\mathfrak{c}-\mathfrak{d}_x-\mathfrak{d}_y)\to\mathcal{O}(F-hx-ky)_C\to M''\to 0 ,$$

where $\mathfrak{d}_x$ and $\mathfrak{d}_y$ are effective divisors on $\tilde{C}$ of respective degrees $(h-m+1)^+m$ and $(k-n+1)^+n$ and M'' is a sheaf over C such that the stalk M''_z vanishes for every simple point z of C. Hence we obtain the exact sequence

$$\cdots\to H^1(\tilde{C}, \mathcal{O}(\mu^*F-\mathfrak{c}-\mathfrak{d}_x-\mathfrak{d}_y))\to H^1(C, \mathcal{O}(F-hx-ky)_C)\to 0 .$$

Let $\mathfrak{k}$ denote the canonical bundle of $\tilde{C}$. Since

$$\mathfrak{k}=\mu^*([C]+K)-[\mathfrak{c}]$$

(see [2], § 2), we have

$$c(\mu^*F-[\mathfrak{c}+\mathfrak{d}_x+\mathfrak{d}_y]-\mathfrak{k})=FC-C^2-KC-(h-m+1)^+m-(k-n+1)^+n>0 .$$

Hence, using the duality theorem, we infer that

$$H^1(\tilde{C}, \mathcal{O}(\mu^*F-\mathfrak{c}-\mathfrak{d}_x-\mathfrak{d}_y))=0\,.$$

Combining this with the above exact sequence, we conclude that

$$H^1(C, \mathcal{O}(F-hx-ky)_C)=0\,,$$

q. e. d.

THEOREM 2. *Let F be a complex line bundle over S with $F^2>0$. If there exists a positive integer m such that the complete linear system $|mF|$ has no base point, then the cohomology group $H^1(S, \mathcal{O}(F+K))$ vanishes.*

PROOF. Let $\{\varphi_0, \varphi_1, \cdots, \varphi_n\}$ be a base of the linear space $H^0(S, \mathcal{O}(mF))$. Since, by hypothesis, $|mF|$ has no base point,

$$\Phi : z \to \Phi(z)=(\varphi_0(z), \varphi_1(z), \cdots, \varphi_n(z))$$

is a holomorphic map of S into a projective n-space $\boldsymbol{P}^n$. Suppose that the image $\Phi(S)$ is a curve in $\boldsymbol{P}^n$. Then, for any pair of general hyperplanes L_1 and L_2 in $\boldsymbol{P}^n$, the intersection $\Phi(S)\cap L_1\cap L_2$ is empty. The inverse images $D_1=\Phi^{-1}(L_1)$ and $D_2=\Phi^{-1}(L_2)$ are divisors belonging to $|mF|$. It follows that $m^2F^2=D_1D_2=0$. This contradicts that $F^2>0$. Thus we see that *the image $\Phi(S)$ is a surface in $\boldsymbol{P}^n$.*

Let $\{U_j\}$ be a finite covering of S by small open subsets U_j. The complex line bundle F is determined by a 1-cocycle $\{f_{jk}\}$ composed of non-vanishing holomorphic functions $f_{jk}=f_{jk}(z)$ with respective domains $U_j\cap U_k$. Let $\varphi_{\lambda j}(z)$ denote the fibre coordinate of $\varphi_\lambda(z)$ over U_j and let

$$a_j(z)=\Big(\sum_{\lambda=0}^{n}|\varphi_{\lambda j}(z)|^2\Big)^{1/m}, \qquad \text{for } z\in U_j\,.$$

Since $|mF|$ has no base point, $a_j(z)$ is positive. Moreover, since

$$\varphi_{\lambda j}(z)=f_{jk}(z)^m\varphi_{\lambda k}(z)\,, \qquad \text{on } U_j\cap U_k\,,$$

we have

$$a_j(z)=|f_{jk}(z)|^2 a_k(z)\,, \qquad \text{on } U_j\cap U_k\,.$$

We let

$$\gamma=\frac{i}{2\pi}\sum_{\alpha,\beta=1}^{2}\gamma_{\alpha\beta}(z)dz^\alpha\wedge d\bar{z}^\beta=\frac{i}{2\pi}\partial\bar{\partial}\log a_j(z)\,, \qquad i=\sqrt{-1}\,,$$

on each open set $U_j\subset S$. The real d-closed (1, 1)-form γ thus defined belongs to the Chern class $c(F)$ of F (see [**3**], Lemma). The (1, 1)-form $m\gamma$ is induced from a standard Kähler form on $\boldsymbol{P}^n$ by the holomorphic map $\Phi: S\to\boldsymbol{P}^n$, while the image $\Phi(S)$ is a surface. Consequently, there exists a proper analytic subset N of S such that the Hermitian matrix $(\gamma_{\alpha\beta}(z))$ *is positive definite for every point $z\in S-N$.* Hence, applying a differential geometric method of [**3**], we infer that $H^1(S, \mathcal{O}(F+K))$ vanishes (see Mumford [**7**]).

§ 3. Composition series of pluri-canonical divisors.

Let S be a non-singular algebraic surface and let K denote a canonical divisor on S.

DEFINITION. We call S a minimal non-singular algebraic surface of general type if and only if S is free from exceptional curves (of the first kind) and

$$P_2 = \dim |2K| + 1 \geqq 1 , \qquad K^2 \geqq 1 . \tag{6}$$

We remark that, if S is free from exceptional curves of the first kind and if either $P_2 = 0$ or $K^2 \leqq 0$, then S is one of the following five types of surfaces: projective plane, ruled surface, $K3$ surface, abelian variety, elliptic surface (see [4], Enriques [1], Šafarevič [8]).

In what follows in this paper we let S denote a minimal non-singular algebraic surface of general type.

By a *divisorial cycle* on S we shall mean a linear combination $\sum r_i C_i$ of a finite number of irreducible curves C_i on S with *rational* coefficients r_i. We say that a divisorial cycle $\sum r_i C_i$ is positive if the coefficients r_i are positive. We indicate by the symbol $\sim$ homology with respect to rational coefficients. For any divisorial cycles ξ and η on S we denote by $\xi\eta$ the intersection multiplicity of ξ and η. We write ξ^2 for $\xi\xi$. Since, by hypothesis, $K^2 \geqq 1$, the following lemma is an immediate consequence of Hodge's index theorem (see Zariski [9], § 6):

LEMMA 3. *Let ζ be a divisorial cycle on S. If $K\zeta = 0$ and if $\zeta \nsim 0$, then ζ^2 is negative.*

In connection with this lemma, we note that *every positive divisorial cycle on S is not homologous to zero.*

We have *the inequality: $KC \geqq 0$ for every irreducible curve C on S.* Moreover *the equality: $KC = 0$ holds if and only if C is a non-singular rational curve with $C^2 = -2$* (see Mumford [6]). In fact, since, by hypothesis, $P_2 \geqq 1$, the bicanonical system $|2K|$ contains a positive divisor D. If $KC < 0$, then $DC < 0$ and therefore C^2 is a negative integer, while $C^2 + KC = 2\pi(C) - 2$. Hence $\pi(C) = 0$, $C^2 = -1$ and thus C is an exceptional curve of the first kind. If $KC = 0$, then, by Lemma 3, we have

$$2\pi(C) - 2 = C^2 + KC = C^2 < 0 .$$

This proves that $\pi(C) = 0$ and $C^2 = -2$.

THEOREM 3. *The number of those irreducible curves E on S which satisfy the equation: $KE = 0$ is smaller than the second Betti number b_2 of S.*

PROOF. Let $E_1, \cdots, E_i, \cdots, E_n$ be irreducible curves on S such that $KE_i = 0$. For our purpose it suffices to show that the curves E_i are homologically independent. Assume a homology

$$\sum_{i=1}^{k} r_i E_i \sim \sum_{i=k+1}^{n} r_i E_i, \qquad r_i \geqq 0.$$

Then we have

$$\Big(\sum_{1}^{k} r_i E_i\Big)^2 = \Big(\sum_{k+1}^{n} r_i E_i\Big)^2 = \sum_{i=1}^{k} \sum_{j=k+1}^{n} r_i r_j E_i E_j \geqq 0.$$

Hence we infer from Lemma 3 that the coefficients r_i vanish, q. e. d.

We denote by $\mathcal{E}$ the sum of all the irreducible curves E_i on S satisfying $KE_i = 0$:

$$\mathcal{E} = E_1 + \cdots + E_i + \cdots + E_b, \qquad b < b_2.$$

Obviously the vanishing of KE_i implies that *the canonical bundle K is trivial on the non-singular rational curve E_i.*

Let e be a positive integer such that $\dim |eK| \geqq 0$ and let D denote a pluri-canonical divisor belonging to the system $|eK|$.

LEMMA 4. *If D is a sum: $D = X + Y$ of two positive divisors X and Y, then we have the inequality:*

$$XY \geqq 1.$$

PROOF. We let

$$X = rK + \xi, \qquad r = KX/K^2,\ K\xi = 0,$$

$$Y = sK + \eta, \qquad s = KY/K^2,\ K\eta = 0,$$

where ξ and η are divisorial cycles. Since $X + Y = D \approx eK$, we have a homology: $\xi + \eta \sim 0$. Hence we obtain

$$XY = rsK^2 - \xi^2.$$

On the other hand, r and s are non-negative and, since the positive divisors X and Y are not homologous to zero, if $\xi \sim 0$ then rs is positive. If $\xi \nsim 0$, then, by Lemma 3, ξ^2 is negative. Consequently, XY is a positive integer, q. e. d.

We represent the pluri-canonical divisor D as a sum:

$$D = \sum_{i=1}^{n} C_i = C_1 + \cdots + C_i + \cdots + C_n$$

of irreducible curves C_i and let

$$D_i = C_1 + C_2 + \cdots + C_i.$$

We call the representation: $\sum_{i=1}^{n} C_i$ a *composition series*. Since $KD = eK^2 \geqq K^2 \geqq 1$, at least one irreducible component Θ of D satisfies the inequality: $K\Theta \geqq 1$.

LEMMA 5. *Let Θ be an irreducible component of D with $K\Theta \geqq 1$. There exists a composition series $D = \sum_{i=1}^{n} C_i$ with $C_1 = \Theta$ satisfying the condition*

(α) $\qquad KC_1 \geqq 1\,, \quad D_{i-1}C_i \geqq 1 \qquad$ for $i=2, 3, \cdots, n$.

PROOF. We choose the components $C_2, C_3, \cdots$ of D successively by induction. Suppose that we have chosen $C_1=\Theta, C_2, \cdots, C_{i-1}$ such that

$$D_{j-1}C_j \geqq 1\,, \qquad \text{for } j=2, 3, \cdots, i-1\,,$$

and let

$$D = D_{i-1}+Z_i\,,$$

where $D_{j-1}=C_1+\cdots+C_{j-1}$. If $Z_i>0$, then, by Lemma 4, $D_{i-1}Z_i \geqq 1$ and therefore at least one irreducible curve $C \leqq Z_i$ has $D_{i-1}C \geqq 1$. Hence, letting $C_i=C$, we obtain

$$D_{i-1}C_i \geqq 1\,,$$

q. e. d.

LEMMA 6. *Let E_1 and E_2 be irreducible curves on S satisfying the condition that $KE_1=KE_2=E_1E_2=0$. If D is a sum:*

$$D=X+Y+E_1+E_2$$

of E_1, E_2 and two positive divisors X, Y and if $KX>0$, $KY>0$, then XY is non-negative.

PROOF. We write

$$X=rK+r_1E_1+r_2E_2+\xi\,, \qquad K\xi=E_1\xi=E_2\xi=0\,,$$

$$Y=sK+s_1E_1+s_2E_2+\eta\,, \qquad K\eta=E_1\eta=E_2\eta=0\,,$$

where ξ and η are divisorial cycles. Since $E_1^2=E_2^2=-2$, the coefficients r, s, r_ν, s_ν, $\nu=1, 2$, are given by the formulae:

$$K^2r=KX, \quad K^2s=KY, \quad -2r_\nu=E_\nu X, \quad -2s_\nu=E_\nu Y\,.$$

The linear equivalence $X+Y+E_1+E_2 \approx eK$ implies that

$$1+r_1+s_1=0\,, \qquad 1+r_2+s_2=0\,, \qquad \xi+\eta\sim 0\,.$$

Hence we obtain

$$XY=rsK^2-2r_1s_1-2r_2s_2+\xi\eta=rsK^2+\sum_{n=1}^{2}2r_\nu(r_\nu+1)-\xi^2 \geqq rsK^2-1-\xi^2\,.$$

Since, by hypothesis, r and s are positive and, by Lemma 3, $\xi^2 \leqq 0$, this proves that $XY>-1$, while XY is an integer. Consequently XY is non-negative, q. e. d.

We write the curve $\mathcal{E}=E_1+E_2+\cdots+E_b$ as a sum:

$$\mathcal{E}=\mathcal{E}_1+\cdots+\mathcal{E}_\nu+\cdots+\mathcal{E}_\kappa$$

of *connected components* $\mathcal{E}_\nu$. We shall say that a positive divisor X meets D if there exists a point z such that $z\in X$, $z\in D$. Since $DE_i=eKE_i=0$, if E_i meets D, then E_i is a component of D. Hence, if $\mathcal{E}_\nu$ meets D, then $\mathcal{E}_\nu<D$.

LEMMA 7. *If* $\mathcal{E}_\lambda+\mathcal{E}_\nu<D$, $\lambda\neq\nu$, *then there exists a composition series* $D=\sum_{i=1}^{n}C_i$ *with* $C_{n-1}<\mathcal{E}_\lambda$, $C_n<\mathcal{E}_\nu$, *which satisfies the condition*

$$(\beta)\qquad D_{i-1}C_i\geqq 0,\qquad KC_i+D_{i-1}C_i\geqq 1,\qquad \text{for } i=1,2,\cdots,n.$$

PROOF. We may assume that $E_1<\mathcal{E}_\lambda$, $E_2<\mathcal{E}_\nu$. Suppose that we have chosen $C_1,\cdots,C_j,\cdots,C_{i-1}$ satisfying

$$(\beta_j)\qquad D_{j-1}C_j\geqq 0,\quad KC_j+D_{j-1}C_j\geqq 1,\qquad \text{for } j=1,2,\cdots,i-1,$$

in such a manner that

$$D=D_{i-1}+X_i+E_1+E_2,\qquad X_i\geqq 0,$$

where $D_{j-1}=C_1+\cdots+C_{j-1}$. Then we have two alternatives: *either* $KX_i=0$ *or* there is an irreducible curve $C\leqq X_i$ satisfying the condition:

$$(7)\qquad D_{i-1}C\geqq 0,\qquad KC+D_{i-1}C\geqq 1.$$

In fact, since $KD_{i-1}\geqq KC_1\geqq 1$, if $KX_i>0$, then, by Lemma 6, $D_{i-1}X_i$ is non-negative. It follows that either there is an irreducible curve $C\leqq X_i$ with $D_{i-1}C\geqq 1$ or every irreducible curve $C\leqq X_i$ satisfies the equation: $D_{i-1}C=0$. If $D_{i-1}C\geqq 1$ for an irreducible curve $C\leqq X_i$, then the curve C satisfies (7). The inequality: $KX_i>0$ implies that an irreducible curve $C\leqq X_i$ satisfies $KC\geqq 1$. If $D_{i-1}C=0$, then this curve C satisfies (7).

If there exists an irreducible curve $C\leqq X_i$ satisfying (7), then, letting $C_i=C$ and $D_i=D_{i-1}+C_i$, we get

$$(\beta_i)\qquad D_{i-1}C_i\geqq 0,\qquad KC_i+D_{i-1}C_i\geqq 1,$$

and

$$D=D_i+X_{i+1}+E_1+E_2,\qquad X_{i+1}\geqq 0.$$

Thus we choose $C_1,\cdots,C_i,\cdots,C_h$ satisfying

$$D_{i-1}C_i\geqq 0,\qquad KC_i+D_{i-1}C_i\geqq 1,\qquad \text{for } i=1,2,\cdots,h,$$

where $D_{i-1}=C_1+\cdots+C_{i-1}$, such that

$$D=C_1+\cdots+C_h+X+E_1+E_2,\qquad X\geqq 0,\ KX=0.$$

Now, with the aid of Lemma 4, we extend the series $C_1+\cdots+C_h$ to a composition series

$$D=C_1+\cdots+C_h+C_{h+1}+\cdots+C_n$$

such that

$$(8)\qquad (C_1+\cdots+C_h+\cdots+C_{i-1})C_i\geqq 1,\qquad \text{for } i=h+1,\cdots,n.$$

Note that $C_i<\mathcal{E}$ for $i=h+1,\cdots,n$. If $C_jC_{j+1}=0$ for an integer j, $h<j<n$, then the inequalities (8) are not affected by the permutation: $C_j\to C_{j+1}$, $C_{j+1}\to C_j$.

Moreover $E_1 < \mathcal{E}_\lambda$ and $E_2 < \mathcal{E}_\nu$ appear among the irreducible components C_i, $i = h+1, \cdots, n$. Hence, by means of an appropriate permutation of the components $C_{h+1}, C_{h+2}, \cdots, C_n$, we obtain a composition series $D = \sum_{i=1}^{n} C_i$ satisfying the condition (β) such that $C_{n-1} < \mathcal{E}_\lambda$, $C_n < \mathcal{E}_\nu$, q. e. d.

In a similar manner we obtain the following

LEMMA 8. *If $\mathcal{E}_\lambda < D$, then there exists a composition series: $D = \sum_{i=1}^{n} C_i$ with $C_n < \mathcal{E}_\lambda$ which satisfies the above condition (β).*

§ 4. Pluri-canonical systems.

In this section we denote by K the *canonical bundle* of S. Let e be a positive integer such that $\dim |eK| \geqq 0$ and let D be a member of $|eK|$. Moreover let m denote an integer $> e$. For any composition series:

$$D = C_1 + C_2 + \cdots + C_i + \cdots + C_n ,$$

we let

$$Z_i = C_i + C_{i+1} + \cdots + C_n , \qquad Z_{n+1} = 0 ,$$

and define

$$F_i = mK - [Z_i] .$$

Then, by a simple calculation, we obtain

(9) $$F_{i+1}C_i - C_i^2 - KC_i = (m-e-1)KC_i + D_{i-1}C_i .$$

Let x and y be distinct points of D and let

$$\Xi_i = \mathcal{O}(mK - Z_i - hx - ky) = \mathcal{O}(mK - Z_i) \cap \mathcal{O}(mK - hx - ky) ,$$

where h and k are non-negative integers. We consider the *ascending chain:*

$$\Xi_1 \subset \Xi_2 \subset \cdots \subset \Xi_i \subset \cdots \subset \Xi_{n+1} = \mathcal{O}(mK - hx - ky) .$$

We assume that the multiplicities of the points x and y of D are not smaller than h and k, respectively, and that

$$\Xi_{i+1}/\Xi_i \cong \mathcal{O}(F_{i+1} - h_i x - k_i y)_{C_i} ,$$

where h_i and k_i are non-negative integers.

LEMMA 9. *If*

$$(m-e-1)KC_i + D_{i-1}C_i > \frac{1}{4}(h_i+1)^2 + \frac{1}{4}(k_i+1)^2 , \qquad \textit{for } i = 1, 2, \cdots, n ,$$

then we have the inequalities

(10) $$\dim H^1(S, \mathcal{O}((m-e)K)) \geqq \dim H^1(S, \Xi_i) , \qquad i = 2, 3, \cdots, n+1 .$$

PROOF. According as $x \in C_i$ or $x \notin C_i$, we define m_i to be the multiplicity of the point x of C_i or zero. Similarly, according as $y \in C_i$ or $y \notin C_i$, we define

n_i to be the multiplicity of the point y of C_i or zero. Since

$$\frac{1}{4}(h_i+1)^2+\frac{1}{4}(k_i+1)^2 \geqq (h_i-m_i+1)^+m_i+(k_i-n_i+1)^+n_i,$$

we infer from Theorem 1 and the formula (9) that

$$H^1(S, \Xi_{i+1}/\Xi_i) \cong H^1(C_i, \mathcal{O}(F_{i+1}-h_ix-k_iy)_{C_i})=0.$$

It follows that the sequences

$$H^1(S, \Xi_i) \to H^1(S, \Xi_{i+1}) \to 0$$

are exact, while

$$\Xi_1=\mathcal{O}(mK-D) \cong \mathcal{O}((m-e)K).$$

Hence we obtain the inequalities (10), q. e. d.

LEMMA 10. *There exists an integer m_0 such that*

(11) $$\dim H^1(S, \mathcal{O}((m-e)K)) = \dim H^1(S, \mathcal{O}(mK)), \qquad \textit{for } m \geqq m_0.$$

(see Zariski [**9**]).

PROOF. With the aid of Lemma 5, we choose a composition series: $D=\sum_{i=1}^{n} C_i$ satisfying the condition (α) and let

$$\Xi_i=\mathcal{O}(mK-Z_i).$$

We have

$$\Xi_{i+1}/\Xi_i \cong \mathcal{O}(F_{i+1})/\mathcal{O}(F_{i+1}-C_i)=\mathcal{O}(F_{i+1})_{C_i}.$$

Assume that $m \geqq e+2$. Then it follows from the condition (α) that

$$(m-e-1)KC_i+D_{i-1}C_i \geqq 1.$$

Hence, by Lemma 9, we have the inequality

$$\dim H^1(S, \mathcal{O}((m-e)K)) \geqq \dim H^1(S, \mathcal{O}(mK)).$$

Hence we infer readily the existence of an integer m_0 such that the equality (11) holds for $m \geqq m_0$, q. e. d.

For any point $x \in S$, we have the exact sequence

$$0 \to \mathcal{O}(mK-x) \to \mathcal{O}(mK) \to \boldsymbol{C}_x \to 0$$

(see (1)) and the corresponding exact cohomology sequence

(12) $$0 \to H^0(S, \mathcal{O}(mK-x)) \to H^0(S, \mathcal{O}(mK)) \to \boldsymbol{C}$$
$$\to H^1(S, \mathcal{O}(mK-x)) \to H^1(S, \mathcal{O}(mK)) \to 0 \to \cdots.$$

THEOREM 4. *Let e be a positive integer such that $P_e \geqq 2$, $eK^2 \geqq 2$. If $m \geqq e+2$ and if $m \geqq m_0$, then, for every point $x \in S$, the sequence*

(13) $$0 \to H^0(S, \mathcal{O}(mK-x)) \to H^0(S, \mathcal{O}(mK)) \to \boldsymbol{C} \to 0$$

is exact.

PROOF. Since $\dim |eK| = P_e - 1 \geqq 1$, we find a divisor $D \in |eK|$ such that $x \in D$.

I) The case in which $x \notin \mathcal{E}$. We choose a composition series: $D = \sum_{i=1}^{n} C_i$ satisfying the condition (α) and let

$$\Xi_i = \mathcal{O}(mK - Z_i - x).$$

We find an integer h such that $x \in C_h$, $x \notin Z_{h+1}$. Since $C_h \not< \mathcal{E}$, we have $KC_h \geqq 1$. Moreover, *we may assume that* $KC_h \geqq 2$ *if* $h = 1$. In fact, since, by hypothesis,

$$KD = eK^2 \geqq 2,$$

if $KC_h = 1$, then there exists an irreducible curve $\Theta \leqq D - C_h$ with $K\Theta \geqq 1$. In view of Lemma 5, we may assume that $C_1 = \Theta$. It follows that $h \geqq 2$.

Since $x \in Z_i$ for $i \leqq h$, we have

$$\Xi_i = \mathcal{O}(mK - Z_i), \qquad \text{for } i \leqq h.$$

We have the commutative diagram:

$$\begin{array}{ccc} \mathcal{O}(mK - Z_h) & \subsetneq & \mathcal{O}(mK - Z_{h+1} - x) \\ \wr\| & & \wr\| \\ \mathcal{O}(F_{h+1} - C_h) & \subsetneq & \mathcal{O}(F_{h+1} - x). \end{array}$$

Hence we obtain the isomorphism:

$$\Xi_{h+1}/\Xi_h \cong \mathcal{O}(F_{h+1} - x)_{C_h}.$$

Thus we infer that

$$\Xi_{i+1}/\Xi_i \cong \mathcal{O}(F_{i+1} - \delta_{ih} x)_{C_i},$$

where δ_{ih} denotes Kronecker's delta. Since $m - e \geqq 2$ and $KC_h \geqq 1 + \delta_{h1}$, it follows from the condition (α) that

$$(m - e - 1)KC_i + D_{i-1}C_i \geqq 1 + \delta_{ih}.$$

Hence, by Lemma 9, we have the inequality

$$\dim H^1(S, \mathcal{O}((m-e)K)) \geqq \dim H^1(S, \mathcal{O}(mK - x)).$$

Combining this with (11) and (12), we infer the exactness of (13).

II) The case in which $x \in \mathcal{E}_\lambda$. With the aid of Lemma 8, we choose a composition series: $D = \sum_{i=1}^{n} C_i$ with $C_n < \mathcal{E}_\lambda$ which satisfies the condition (β) and let

$$\Xi_i = \mathcal{O}(mK - Z_i).$$

Since $\Xi_{i+1}/\Xi_i \cong \mathcal{O}(F_{i+1})_{C_i}$ and

$$(m - e - 1)KC_i + D_{i-1}C_i \geqq KC_i + D_{i-1}C_i \geqq 1,$$

we have, by Lemma 9,

$$\dim H^1(S, \mathcal{O}((m-e)K)) \geqq \dim H^1(S, \Xi_n).$$

Combined with (11), this proves that

$$\dim H^1(S, \mathcal{O}(mK)) \geqq \dim H^1(S, \mathcal{O}(mK-C_n)). \tag{14}$$

Since K is trivial on C_n, we have the exact sequence

$$0 \to \mathcal{O}(mK-C_n) \to \mathcal{O}(mK) \to \mathcal{O}_{C_n} \to 0.$$

Moreover C_n is a non-singular rational curve. Hence we obtain the exact sequence

$$0 \to H^0(S, \mathcal{O}(mK-C_n)) \to H^0(S, \mathcal{O}(mK)) \to \boldsymbol{C}$$
$$\to H^1(S, \mathcal{O}(mK-C_n)) \to H^1(S, \mathcal{O}(mK)) \to 0.$$

Combining this with (14), we infer that the sequence

$$0 \to H^0(S, \mathcal{O}(mK-C_n)) \to H^0(S, \mathcal{O}(mK)) \to \boldsymbol{C} \to 0$$

is exact, while every holomorphic section $\varphi \in H^0(S, \mathcal{O}(mK))$ is reduced to a constant on $\mathcal{E}_\lambda$. Hence the exactness of (13) follows.

THEOREM 5. *The cohomology group $H^1(S, \mathcal{O}(mK))$ vanishes for every integer $m \geqq 2$.*

PROOF. Let e be a positive integer such that $P_e \geqq 2$, $eK^2 \geqq 2$. The existence of such an integer e is obvious by the Riemann-Rock theorem. Let $k = m-1$ and choose a positive integer n such that $nk \geqq e+2+m_0$. By Theorem 4, the sequence

$$0 \to H^0(S, \mathcal{O}(nkK-x)) \to H^0(S, \mathcal{O}(nkK)) \to \boldsymbol{C} \to 0$$

is exact for every point $x \in S$. It follows that the complete linear system $|nkK|$ has no base point, while $(kK)^2 = k^2K^2 > 0$. Hence, by Theorem 2,

$$H^1(S, \mathcal{O}(mK)) = H^1(S, \mathcal{O}(kK+K)) = 0,$$

q. e. d.

COROLLARY. *The pluri-genera P_m, $m \geqq 2$, are given by the formula:*

$$P_m = \frac{1}{2} m(m-1)K^2 + p_g - q + 1. \tag{15}$$

THEOREM 6. *Let e be a positive integer such that $P_e \geqq 2$, $eK^2 \geqq 2$. If $m \geqq e+2$, then the pluri-canonical system $|mK|$ has no base point and the map Φ_{mK} is holomorphic.*

PROOF. It follows from Theorem 5 that $m_0 = e+2$, where m_0 is the integer appeared in (11). Hence we infer from Theorem 4 that, if $m \geqq e+2$, then $|mK|$ has no base point and, consequently, Φ_{mK} is a holomorphic map, q. e. d.

For any pair of distinct points x and y on S, we have the exact sequence

$$0 \to \mathcal{O}(mK-x-y) \to \mathcal{O}(mK) \to \boldsymbol{C}_x \oplus \boldsymbol{C}_y \to 0$$

(see (1)) and the corresponding exact cohomology sequence

$$\cdots \to H^0(S, \mathcal{O}(mK)) \to \boldsymbol{C}^2 \to H^1(S, \mathcal{O}(mK-x-y)) \to \cdots .$$

We shall say that x and y are *distinct modulo* $\mathcal{E}$ if x and y are distinct and *not* contained in one and the same connected component of $\mathcal{E}$.

THEOREM 7. *Let e be a positive integer such that $P_e \geqq 3$, $eK^2 \geqq 2$. If $m \geqq e+3$, then, for any pair of points x and y on S which are distinct modulo $\mathcal{E}$, the sequence*

$$0 \to H^0(S, \mathcal{O}(mK-x-y)) \to H^0(S, \mathcal{O}(mK)) \to \boldsymbol{C}^2 \to 0 \tag{16}$$

is exact.

PROOF. Since $\dim |eK| = P_e - 1 \geqq 2$, we find a divisor $D \in |eK|$ such that $x \in D$, $y \in D$.

I) The case in which $x, y \notin \mathcal{E}$. With the aid of Lemma 5, we choose a composition series: $D = \sum_{i=1}^{n} C_i$ satisfying the condition (α) and let

$$\Xi_i = \mathcal{O}(mK - Z_i - x - y) .$$

We find h and j such that $x \in C_h$, $x \notin Z_{h+1}$, $y \in C_j$, $y \notin Z_{j+1}$. Then we have

$$\Xi_{i+1}/\Xi_i \cong \mathcal{O}(F_{i+1} - \delta_{ih}x - \delta_{ij}y)_{C_i} .$$

Since $C_h \not< \mathcal{E}$, $C_j \not< \mathcal{E}$, we have $KC_h \geqq 1$, $KC_j \geqq 1$ and, as was mentioned in the proof of Theorem 4, we may assume that $KC_1 \geqq 2$ if h is equal to 1. The condition (α) implies therefore that

$$(m-e-1)KC_i + D_{i-1}C_i \geqq 1 + \delta_{ih} + \delta_{ij} .$$

Hence, by Lemma 9 and Theorem 5, $H^1(S, \Xi_{n+1})$ vanishes. It follows that the sequence (16) is exact.

II) The case in which $x \in \mathcal{E}_\lambda$, $y \in \mathcal{E}_\nu$, $\lambda \neq \nu$. With the aid of Lemma 7, we choose a composition series: $D = \sum_{i=1}^{n} C_i$ with $C_{n-1} < \mathcal{E}_\lambda$, $C_n < \mathcal{E}_\nu$ which satisfies the condition (β) and let

$$\Xi_i = \mathcal{O}(mK - Z_i) .$$

Since $\Xi_{i+1}/\Xi_i \cong \mathcal{O}(F_{i+1})_{C_i}$ and

$$(m-e-1)KC_i + D_{i-1}C_i \geqq KC_i + D_{i-1}C_i \geqq 1 ,$$

we infer from Lemma 9 and Theorem 5 that

$$H^1(S, \mathcal{O}(mK - C_{n-1} - C_n)) = 0 . \tag{17}$$

Since K is trivial on C_{n-1} and on C_n, we have the exact sequence

$$0 \to \mathcal{O}(mK - C_{n-1} - C_n) \to \mathcal{O}(mK) \to \mathcal{O}_{C_{n-1}} \oplus \mathcal{O}_{C_n} \to 0 .$$

Combining this with (17), we infer that the sequence

$$0 \to H^0(S, \mathcal{O}(mK-C_{n-1}-C_n)) \to H^0(S, \mathcal{O}(mK)) \to \boldsymbol{C}^2 \to 0$$

is exact, while every holomorphic section $\varphi \in H^0(S, \mathcal{O}(mK))$ is reduced to a constant on each connected component of $\mathcal{E}$. Hence the exactness of (16) follows.

III) The case in which $x \notin \mathcal{E}$, $y \in \mathcal{E}_\lambda$. We choose a composition series: $D=\sum_{i=1}^{n} C_i$ with $C_n < \mathcal{E}_\lambda$ satisfying the condition (β) and let

$$\Xi_i = \mathcal{O}(mK-Z_i-x).$$

We find h such that $x \in C_h$, $x \notin Z_{h+1}$. Then we have

$$\Xi_{i+1}/\Xi_i \cong \mathcal{O}(F_{i+1}-\delta_{ih}x)_{C_i}.$$

Moreover, since $KC_h \geqq 1$, we have

$$(m-e-1)KC_i+D_{i-1}C_i \geqq 1+\delta_{ih}.$$

Hence, by Lemma 9 and Theorem 5, we get

$$H^1(S, \mathcal{O}(mK-C_n-x))=0.$$

Combining this with the exact sequence

$$0 \to \mathcal{O}(mK-C_n-x) \to \mathcal{O}(mK) \to \mathcal{O}_{C_n} \oplus \boldsymbol{C}_x \to 0,$$

we infer that the sequence

$$0 \to H^0(S, \mathcal{O}(mK-C_n-x)) \to H^0(S, \mathcal{O}(mK)) \to \boldsymbol{C}^2 \to 0$$

is exact. Hence the exactness of (16) follows, q. e. d.

Now we consider the exact sequence

$$0 \to \mathcal{O}(mK-2x) \to \mathcal{O}(mK) \to \boldsymbol{C}_x^3 \to 0.$$

THEOREM 8. *Let e be a positive integer such that $P_e \geqq 4$, $eK^2 \geqq 2$. If $m \geqq e+3$ and if $x \notin \mathcal{E}$, then the sequence*

$$(18) \qquad 0 \to H^0(S, \mathcal{O}(mK-2x)) \to H^0(S, \mathcal{O}(mK)) \to \boldsymbol{C}^3 \to 0$$

is exact.

PROOF. Since, by hypothesis, $\dim|eK| = P_e-1 \geqq 3$, we find a divisor $D \in |eK|$ such that *x is a multiple point of D*. We choose a composition series: $D=\sum_{i=1}^{n} C_i$ satisfying the condition (α) and let

$$\Xi_i = \mathcal{O}(mK-Z_i-2x).$$

We find h such that $x \in C_h$, $x \notin Z_{h+1}$. As was mentioned in the proof of Theorem 4, we may assume that $KC_h \geqq 2$ if $h=1$. To prove the exactness of (18) it suffices to show the vanishing of $H^1(S, \Xi_{n+1})$.

i) If x is a multiple point of C_h, then

$$\Xi_i = \mathcal{O}(mK - Z_i)\,, \qquad \text{for } i \leqq h\,,$$

and therefore

$$\Xi_{i+1}/\Xi_i \cong \mathcal{O}(F_{i+1} - 2\delta_{ih}x)_{C_i}\,.$$

Since $m-e \geqq 3$ and $KC_1 \geqq 1+\delta_{h1}$, it follows from the condition (α) that

$$(m-e-1)KC_i + D_{i-1}C_i \geqq 1+2\delta_{ih}\,.$$

Hence, by Lemma 9 and Theorem 5, $H^1(S, \Xi_{n+1})$ vanishes.

ii) If x is a simple point of C_h, then we find an integer $j<h$ such that

$$x \in C_j\,, \qquad x \notin C_{j+1} + C_{j+2} + \cdots + C_{h-1}\,.$$

Since x is a simple point of Z_{j+1}, we have the isomorphism:

$$\mathcal{O}(mK - Z_{j+1} - 2x) \cong \mathcal{O}(F_{j+1} - x)\,.$$

Moreover we have the commutative diagram:

$$\begin{array}{ccc} \mathcal{O}(mK-Z_j) & \subsetneq & \mathcal{O}(mK-Z_{j+1}-2x) \\ \wr\| & & \wr\| \\ \mathcal{O}(F_{j+1}-C_j) & \subsetneq & \mathcal{O}(F_{j+1}-x)\,. \end{array}$$

Hence Ξ_{j+1}/Ξ_j is isomorphic to $\mathcal{O}(F_{j+1}-x)_{C_j}$. Thus we see that

$$\Xi_{i+1}/\Xi_i \cong \mathcal{O}(F_{i+1} - (\delta_{ij} + 2\delta_{ih})x)_{C_i}\,.$$

Moreover, since $KC_j \geqq 1$, $KC_h \geqq 1$, the condition (α) implies that

$$(m-e-1)KC_i + D_{i-1}C_i \geqq 1 + \delta_{ij} + 2\delta_{ih}\,.$$

Hence, by Lemma 9 and Theorem 5, $H^1(S, \Xi_{n+1})$ vanishes, q. e. d.

Let Φ be a holomorphic map of S into a complex manifold. We shall say that Φ is *one-to-one modulo* $\mathcal{E}$ if any only if

$$\Phi^{-1}\Phi(z) = \begin{cases} z\,, & \text{for } z \in S - \mathcal{E}\,, \\ \mathcal{E}_\lambda & \text{for } z \in \mathcal{E}_\lambda\,. \end{cases}$$

Moreover we say that Φ is *biholomorphic modulo* $\mathcal{E}$ if Φ is one-to-one modulo $\mathcal{E}$ and biholomorphic on $S-\mathcal{E}$.

THEOREM 9. *Let e be a positive integer such that $P_e \geqq 3$, $eK^2 \geqq 2$. For every integer $m \geqq e+3$, the map Φ_{mK} is holomorphic and one-to-one modulo $\mathcal{E}$.*

PROOF. We infer from Theorem 7 that Φ_{mK} is holomorphic and $\Phi_{mK}(x) \neq \Phi_{mK}(y)$ for any pair of points x, y on S which are distinct modulo $\mathcal{E}$. Moreover the image $\Phi_{mK}(\mathcal{E}_\lambda)$ of each component $\mathcal{E}_\lambda$ is a point, since K is trivial on $\mathcal{E}_\lambda$, q. e. d.

The exactness of the sequence (18) implies that Φ_{mK} is biholomorphic in a neighborhood of x on S. Hence we infer from Theorems 8 and 9 the following

THEOREM 10. *Let e be a positive integer such that $P_e \geqq 4$, $eK^2 \geqq 2$. For* *every integer* $m \geqq e+3$, *the map* Φ_{mK} *is holomorphic and biholomorphic modulo* $\mathcal{E}$.

LEMMA 11. *If* $p_g=0$, *then* $q=0$.

PROOF. We have the Noether formula:

$$8q+K^2+b_2=12p_g+10,$$

where b_2 denotes the second Betti number of S. Since $K^2 \geqq 1$, this formula proves that, if $p_g=0$, then $q \leqq 1$. Suppose that $q=1$. Then there exists a holomorphic map Ψ of S onto an elliptic curve Δ such that the inverse image $C=\Psi^{-1}(u)$ of any general point $u \in \Delta$ is an irreducible non-singular curve. Since $C^2=0$, C and K are homologically independent. It follows that $b_2 \geqq 2$. This contradicts the Noether formula. Thus we infer that $q=0$, q. e. d.

LEMMA 12. *If* $K^2=1$, *then* $p_g \leqq 2$ *and* $q \leqq 1$.

PROOF. i) Assume that $p_g \geqq 2$. Any general member of $|K|$ is an irreducible non-singular curve of genus 2. To prove this we let D denote a general member of $|K|$. The general member D has an irreducible component C with $C^2 \geqq 0$. Since $KD=K^2=1$, we have

$$D=C+X, \qquad X \geqq 0, \qquad KC=1, \qquad KX=0,$$

while

$$C^2=2\pi(C)-2-KC, \qquad C^2+CX=KC.$$

Hence we infer that $CX=0$ and therefore, by Lemma 4, $X=0$. Thus we see that $D=C$. If follows that $\pi(C)=2$. By a theorem of Bertini, C has no singular point outside the base points of $|K|$, while, since $C^2=1$, any base point of $|K|$ is a simple point of C. Hence C is a non-singular curve. It is clear that

$$\dim H^0(C, \mathcal{O}([C])_C) \leqq 1.$$

Combining this with the exact sequence

$$0 \to H^0(S, \mathcal{O}) \to H^0(S, \mathcal{O}(C)) \to H^0(C, \mathcal{O}([C])_C) \to \cdots$$

we obtain the inequality

$$p_g=\dim H^0(S, \mathcal{O}(C)) \leqq 2.$$

ii) Since $P_2 \geqq p_g$, we infer from (15) that

$$q=K^2+1+p_g-P_2 \leqq 2.$$

iii) Now we assume that $q=2$ and derive a contradiction. There exist on S two linearly independent holomorphic 1-forms φ_1 and φ_2.

If $\varphi_1 \wedge \varphi_2=0$, then there exists a holomorphic map Ψ of S onto a non-singular algebraic curve Δ of genus 2 such that the inverse image $\Theta_u=\Psi^{-1}(u)$ of any general point $u \in \Delta$ is an irreducible non-singular curve. Since, by Lemma 11, $p_g \geqq 1$, the canonical system $|K|$ contains a positive divisor D.

Since $KD=K^2=1$, we have a composition series:

$$D=C+\sum_{i=2}^{n} E_i\,, \qquad KC=1,\ KE_i=0\,.$$

Since $\Theta_u^2=0$, $K\Theta_u$ is positive, while $K\Theta_u=2\pi(\Theta_u)-2$ is even. Moreover the projection $\Psi(E_i)$ of each rational curve E_i is a point on Δ. Hence $C\Theta_u=K\Theta_u\geqq 2$ and therefore C is a covering of Δ with at least two sheets. It follows that

$$2\pi(C)-2\geqq 4\pi(\Delta)-4\geqq 4\,.$$

This contradicts that

$$2\pi(C)-2=C^2+KC=2KC-\sum_i CE_i\leqq 2\,.$$

If $\varphi_1\wedge\varphi_2\neq 0$, then φ_1 and φ_2 define a holomorphic map Φ of S *onto* the Albanese variety A attached to S. The canonical divisor $(\varphi_1\wedge\varphi_2)$ is an irreducible non-singular curve of genus 2. To prove this we let

$$(\varphi_1\wedge\varphi_2)=C+X\,, \qquad X\geqq 0,\ KC=1,\ KX=0\,.$$

Suppose that the restrictions φ_{1C} and φ_{2C} of φ_1 and φ_2 to C are linearly dependent. Then $\Phi(C)$ is either a point or an elliptic curve on A. If $X>0$, then X is composed of non-singular rational curves $E_i<\mathcal{E}$. Hence $\Phi(X)$ consists of a finite number of points on A. Consequently, there exists an irreducible non-singular curve Γ on A which meets neither $\Phi(C)$ nor $\Phi(X)$. It follows that $K\Phi^{-1}(\Gamma)=0$ and therefore $\Phi^{-1}(\Gamma)$ is composed of rational curves. This contradicts that $\pi(\Gamma)\geqq 1$.

Thus φ_{1C} and φ_{2C} are linearly independent and therefore the genus of the non-singular model of C is not smaller than 2, while

$$2\pi(C)-2=C^2+KC=2-CX$$

and, by Lemma 4, CX is positive if $X>0$. Hence we infer that C is a non-singular curve of genus 2 and $X=0$. It follows that $(\varphi_1\wedge\varphi_2)=C$.

The Euler number of S is equal to the sum of the indices of the singular points of the *covariant* vector field φ_1. Since $(\varphi_1\wedge\varphi_2)=C$, the vector field φ_1 has no singular point outside C. We may assume that φ_{1C} has two simple zeros x and y on C. Since φ_{2C} does not vanish at x, we can choose a local coordinate (w, z) of the center x on S such that

$$\varphi_2=dz\,, \qquad \varphi_1\wedge\varphi_2=w\,dw\wedge dz\,.$$

It follows that

$$\varphi_1=w\,dw+\rho z\,dz\,, \qquad \rho\neq 0\,,$$

where ρ is a holomorphic function of z. This shows that x is a singular point of φ_1 of index 1. Thus the vector field φ_1 has exactly two singular points of index 1 and therefore the Euler number $\chi(S)$ of S is equal to 2. This contradicts the Noether formula:

$$\chi(S)+K^2=12(p_g-q+1),$$

q. e. d.

Theorem 11. *The bigenus of S is not smaller than two: $P_2 \geqq 2$.*

Proof. i) If $p_g \geqq 2$, then it is obvious that $P_2 \geqq p_g \geqq 2$.

ii) If $p_g=1$, then, by the Noether formula, $q \leqq 2$. If, moreover, $K^2=1$, then, by Lemma 12, $q \leqq 1$. Hence, using (15), we obtain

$$P_2=K^2+p_g-q+1 \geqq 2.$$

iii) If $p_g=0$, then, by Lemma 11, $q=0$ and therefore

$$P_2=K^2+1 \geqq 2.$$

Thus we see that $P_2 \geqq 2$. Moreover, using (15), we get

$$P_3=2K^2+P_2 \geqq 4.$$

Hence we infer from Theorems 6 and 10 the following

Theorem 12. *For every integer $m \geqq 4$, the pluricanonical system $|mK|$ has no base point and the map Φ_{mK} is holomorphic. For every integer $m \geqq 6$, the map Φ_{mK} is holomorphic and biholomorphic modulo $\mathcal{E}$.*

If $p_g \geqq 4$, then, by Lemma 12, $K^2 \geqq 2$. Hence we infer from Theorem 10 the following

Theorem 13. *If $p_g \geqq 4$, then, for every integer $m \geqq 4$, the map Φ_{mK} is holomorphic and biholomorphic modulo $\mathcal{E}$.*

§ 5. Birational embeddings.

It has been shown by Šafarevič [8] that, if $p_g \geqq 4$, then Φ_{3K} is a birational map. In this section we prove in the context of this paper that, if $p_g \geqq 4$, then $|3K|$ has no base point and Φ_{3K} is a holomorphic birational map.

Let Λ denote the set of those irreducible curves C on S which satisfy the inequality: $KC \leqq 1$.

Lemma 13. *If $K^2 \geqq 2$, then Λ is a finite set.*

Proof. In view of Theorem 3 it suffices to consider the subset Λ_1 of Λ consisting of irreducible curves C with $KC=1$. We choose a base $\{K, B_1, \cdots, B_i, \cdots, B_h\}$ of divisorial cycles on S such that $B_1, \cdots, B_i, \cdots$ are divisors satisfying the conditions

$$B_i^2<0, \qquad KB_i=0, \qquad B_iB_k=0 \qquad \text{for } i \neq k.$$

For each curve $C \in \Lambda_1$, we have a homology

$$C \sim r_0K+\sum_{i=1}^{h} r_iB_i, \qquad r_0=1/K^2, \qquad r_i=B_iC/B_i^2.$$

We have

$$C^2 = 1/K^2 + \sum_{i=1}^{h} r_i^2 B_i^2 \leqq 1/K^2 \leqq 1/2$$

and

$$C^2 = 2\pi(C) - 2 - KC = 2\pi(C) - 3 .$$

Hence we infer that $C^2 = -1$ or -3 and that

$$(19) \qquad - \sum_{i=1}^{n} r_i^2 B_i^2 < 4 .$$

The homology class of C contains no irreducible curve other than C. In fact, if Θ is an irreducible curve on S and if $\Theta \sim C$, then $\Theta C = C^2 < 0$ and therefore Θ coincides with C. Moreover $r_i B_i^2 = B_i C$, $i = 1, 2, \cdots, h$, are rational integers. Hence we infer from (19) the finiteness of the set Λ_1, q. e. d.

THEOREM 14. *If $p_g \geqq 4$, then the tri-canonical system $|3K|$ has no base point and Φ_{3K} is a holomorphic birational map.*

PROOF. Since, by hypothesis, $p_g \geqq 4$, we have, by Lemma 12, $K^2 \geqq 2$. Hence, by Theorem 6, the tri-canonical system $|3K|$ has no base point and Φ_{3K} is a holomorphic map. Moreover, by Lemma 13, Λ is a finite set. Let $\mathcal{C}$ denote the union of the curves $C \in \Lambda$. To prove that Φ_{3K} is a birational map, it suffices to show that, for any pair of distinct points $x, y \in S - \mathcal{C}$, the sequence

$$(20) \qquad 0 \to H^0(S, \mathcal{O}(3K - x - y)) \to H^0(S, \mathcal{O}(3K)) \to \boldsymbol{C}^2 \to 0$$

is exact.

We denote by $|K - x - y|$ the linear subsystem of $|K|$ consisting of those divisors $D \in |K|$ which pass through x and y in the sense that $x \in D$, $y \in D$. It is obvious that

$$\dim |K - x - y| \geqq p_g - 3 \geqq 1 .$$

Let D be a general member of $|K - x - y|$. We choose a composition series: $D = \sum_{i=1}^{n} C_i$ satisfying the condition (α) and let

$$\Xi_i = \mathcal{O}(3K - Z_i - x - y) .$$

We find h and j such that $x \in C_h$, $x \notin Z_{h+1}$, $y \in C_j$, $y \notin Z_{j+1}$. Since, by hypothesis, $x \notin \mathcal{C}$, $y \notin \mathcal{C}$, we have $KC_h \geqq 2$, $KC_j \geqq 2$. Moreover, *we may assume that $KC_1 \geqq 3$ if $h = j = 1$.* To show this we suppose that $h = j = 1$ for *every* composition series: $D = \sum_{i=1}^{n} C_i$ satisfying the condition (α). Then, in view of Lemma 5, KC_i vanishes for $i \geqq 2$. Thus the composition series has the form

$$D = C + E_2 + \cdots + E_i + \cdots + E_n , \qquad E_i < \mathcal{E} ,$$

where $C = C_1$. Since D is a general member of $|K - x - y|$ and since $E_i^2 = -2$, the sum: $\sum_{i=2}^{n} E_i$ is the fixed component of $|K-x-y|$. Let $C'+\sum_{i=2}^{n} E_i$ be another general member of $|K-x-y|$. Then C and C' intersect at x and y and therefore

$$KC = C^2 + \sum_{i=2}^{n} E_i C \geqq C^2 = CC' \geqq 2 .$$

Suppose that $KC=2$. Then $C^2=CC'=2$, and, by Lemma 4, $D=C$. It follows that $C \cap C' = x \cup y$. By a theorem of Bertini, the general member C has no singular point outside the base points x and y, while, since $CC'=2$, x and y are simple points of C. Thus C is a non-singular curve. It is clear that $\pi(C)=3$. Thus C is non-rational and therefore

$$\dim H^0(C, \mathcal{O}(K)_C) \leqq KC = 2 .$$

Since, by hypothesis, $p_g \geqq 4$, this contradicts the exact sequence

$$0 \to \boldsymbol{C} \to H^0(S, \mathcal{O}(K)) \to H^0(C, \mathcal{O}(K)_C) \to \cdots .$$

Thus we see that $KC_1 \geqq 3$.

We have

$$\Xi_{i+1}/\Xi_i \cong \mathcal{O}(3K - \delta_{ih}x - \delta_{ij}y)_{C_i} .$$

Since $KC_h \geqq 2$, $KC_j \geqq 2$ and $KC_1 \geqq 3$ if $h=j=1$, the condition (α) implies that

$$KC_i + D_{i-1}C_i \geqq 1 + \delta_{ih} + \delta_{ij} .$$

Hence, by Lemma 9 and Theorem 5, $H^1(S, \Xi_{n+1})$ vanishes and, consequently, the sequence (20) is exact, q. e. d.

University of Tokyo

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