

## On boundary conditions for multi-dimensional Brownian motions with symmetric resolvent densities

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### § 1. Introduction.

Let  $D$  be an arbitrary bounded domain of the  $N$ -dimensional Euclidean space  $R^N (N \geq 1)$ . A function  $G_\alpha(x, y)$  defined for  $\alpha > 0$ ,  $x, y \in D$ ,  $x \neq y$  will be called a *resolvent density* on  $D$ , if it satisfies that,  $G_\alpha(x, y) \geq 0$ ,  $\alpha \int_D G_\alpha(x, z) dz \leq 1$  and  $G_\alpha(x, y) - G_\beta(x, y) + (\alpha - \beta) \int_D G_\alpha(x, z) G_\beta(z, y) dz = 0$  for all  $\alpha > 0$ ,  $\beta > 0$  and  $x, y \in D$ ,  $x \neq y$ . Denote by  $G_\alpha^0(x, y)$  the resolvent density corresponding to the *absorbing barrier Brownian motion* on  $D$ <sup>1)</sup>.

Consider the family  $\mathcal{G}$  of all *conservative symmetric resolvent densities*<sup>2)</sup> on  $D$  possessing the following properties:

(G. a)  $G_\alpha(x, y)$  is written in the form

$$G_\alpha(x, y) = G_\alpha^0(x, y) + R_\alpha(x, y).$$

$R_\alpha(x, y)$  is a non-negative function of  $\alpha > 0$ ,  $x, y \in D$ , and  $\alpha$ -harmonic<sup>3)</sup> in  $x \in D$  for each  $\alpha > 0$  and  $y \in D$ .

(G. b) For any compact subset  $K$  of  $D$ ,  $\sup_{x \in K, y \in D} R_\alpha(x, y)$  is finite.

In [15], we constructed a particular element of  $\mathcal{G}$  and showed that it determines a continuous strong Markov process (called the *reflecting barrier Brownian motion*) on an extended state space  $D^*$ .

In the present paper, by studying the structure of Dirichlet spaces associated with elements of  $\mathcal{G}$ , we will answer the questions:

- (i) How many elements are there in  $\mathcal{G}$ ?
- (ii) In what sense is the resolvent density of [15] typical among  $\mathcal{G}$ ?

1) Cf. [5].

2) We will say that a resolvent density  $G_\alpha(x, y)$  is conservative (resp. symmetric) when  $\alpha \int_D G_\alpha(x, z) dz = 1$ ,  $\alpha > 0$ ,  $x \in D$  (resp.  $G_\alpha(x, y) = G_\alpha(y, x)$ ,  $\alpha > 0$ ,  $x, y \in D$ ).

3) We call a function on  $D$   $\alpha$ -harmonic when

$$\frac{1}{2} \sum_{i=1}^N \frac{\partial^2 u(x)}{\partial x_i^2} = \alpha u(x), \quad x \in D.$$

Our goal is to establish in section 5 and section 7 a one-to-one correspondence between  $\mathcal{G}$  and a class of Dirichlet spaces formed by functions on the Martin boundary of the domain  $D$ .

The present paper consists of nine sections.

Sections 2 and 3 will serve as preparations for later discussions. In section 2 we will introduce the notion of the Dirichlet space (relative to an  $L^2$ -space), in a slightly modified sense, due to Beurling and Deny [2]. In section 3, the Dirichlet space formed by every square integrable BLD function (denoted by  $\widehat{\text{BLD}}$ ) will be studied by making use of the Feller kernels on the Martin boundary.

With a given element  $G_\alpha(x, y) = G_\alpha^0(x, y) + R_\alpha(x, y)$  of the class  $\mathcal{G}$ , we associate a Dirichlet space  $(\mathcal{F}_D, \mathcal{E})$  relative to  $L^2(D)$  by

$$\mathcal{F}_D = \{u \in L^2(D); \mathcal{E}(u, u) = \lim_{\beta \rightarrow +\infty} \beta(u - \beta G_\beta u, u)_{L^2(D)} < +\infty\}.$$

In sections 4, 5 and 6, the space  $(\mathcal{F}_D, \mathcal{E})$  will be analyzed in details as outlined in the following.

Let  $\mathcal{F}_D^{(0)}$  (actually independent of  $\alpha > 0$ ) be the space spanned by  $\{G_\alpha^0 f, f \in \mathbf{B}(D)\}$  with respect to the norm  $\sqrt{\mathcal{E}^\alpha(u, u)} = \sqrt{\mathcal{E}(u, u) + \alpha(u, u)_{L^2(D)}}$  and  $\mathcal{H}_\alpha$  the space spanned by  $\{R_\alpha f, f \in \mathbf{B}(D)\}$ . For each  $\alpha > 0$ , spaces  $\mathcal{F}_D^{(0)}$  and  $\mathcal{H}_\alpha$  are orthogonal with respect to  $\mathcal{E}^\alpha$  and  $\mathcal{F}_D = \mathcal{F}_D^{(0)} \oplus \mathcal{H}_\alpha$ . Further the space  $(\mathcal{F}_D^{(0)}, \mathcal{E})$  is identical with the space  $\text{BLD}_0$  of BLD functions of potential type. The proof of these facts will be carried out in section 4 by making use of a Feller type expression of  $R_\alpha f: R_\alpha f(x) = H_\alpha^x \tilde{R}^\alpha \hat{H}_\alpha f$ .

Denote by  $M$  the Martin boundary of the domain  $D$ . Using the Feller kernels, we introduce by (3.14) and (3.15) respectively a bilinear form  $\mathbf{D}(\cdot, \cdot)$  for functions on  $M$  and a space  $\mathbf{H}_M$  of functions on  $M$ . Theorem 5.2 and 5.3 will characterize the above-mentioned Hilbert spaces  $\{(\mathcal{H}_\alpha, \mathcal{E}^\alpha), \alpha > 0\}$  by means of a Dirichlet space  $(\mathcal{F}_M, \mathcal{E}_M(\cdot, \cdot))$  satisfying the following conditions<sup>4)</sup>.

(B. 1)  $\mathcal{F}_M$  is a linear subspace of  $\mathbf{H}_M$ .  $\mathcal{F}_M$  contains every constant function on  $M$ .

(B. 2)  $\mathcal{E}_M$  is a bilinear form on  $\mathcal{F}_M$  which is written as  $\mathcal{E}_M(\varphi, \psi) = \mathbf{D}(\varphi \cdot \psi) + \mathbf{N}(\varphi, \psi)$ ,  $\varphi, \psi \in \mathcal{F}_M$ , where  $\mathbf{N}$  is a non-negative symmetric bilinear form on  $\mathcal{F}_M$  satisfying  $\mathbf{N}(1 \cdot 1) = 0$ . The space  $\mathcal{F}_M$  is complete with metric  $\mathcal{E}_M(\cdot, \cdot) + \lambda(\cdot, \cdot)_{L^2(M)}$ , for a  $\lambda > 0$ .

(B. 3) If  $\varphi \in \mathcal{F}_M$  and if  $\psi$  is a normal contraction of  $\varphi$  in the sense of [4], then  $\psi \in \mathcal{F}_M$  and  $\mathbf{N}(\psi, \psi) \leq \mathbf{N}(\varphi, \varphi)$ .

Conversely, for any pair  $(\mathcal{F}_M, \mathbf{N})$  satisfying the conditions (B. 1), (B. 2)

4) Conditions (B.1), (B.2) and (B.3) implies that  $(\mathcal{F}_M, \mathcal{E}_M)$  is a Dirichlet space relative to  $L^2(M)'$ , the space  $L^2(M)'$  being defined in section 3.

and (B. 3), we will construct in section 7 an element  $G_\alpha(x, y)$  of the class  $\mathbf{G}$  which corresponds to this pair  $(\mathcal{F}_M, N)$  in the manner of Theorem 5.2. In this way, we will establish a one-to-one correspondence between the class  $\mathbf{G}$  and the class of the pairs  $(\mathcal{F}_M, N)$ .

Section 6 will be concerned with the boundary condition. Consider again the Dirichlet space  $(\mathcal{F}_D, \mathcal{E})$  associated with a given element  $G_\alpha(x, y)$  of  $\mathbf{G}$ . Since  $2\mathcal{D}(\varphi, \varphi)$  for  $\varphi \in \mathbf{H}_M$  is nothing but an expression of the Dirichlet integral of the harmonic function with fine boundary function  $\varphi$  (see Doob [7] and Fukushima [13]), our results of sections 4 and 5 enable us in Theorem 6.1 to conclude that  $\text{BLD}_0 \subset \mathcal{F}^{(5)} \subset \widehat{\text{BLD}}$  and, for every  $u \in \mathcal{F}$ ,  $\mathcal{E}(u, u) \geq \frac{1}{2} \int_D (\text{grad } u, \text{grad } u)(x) dx$ . Furthermore, we can see that the space  $\mathcal{D} = G_\alpha(L^2(D))$  is a restriction of the domain  $\mathcal{D}(\mathcal{A})$  of the generalized Laplacian  $\mathcal{A}$  (denoted by the same symbol  $\mathcal{A}$  as the usual Laplacian), which is defined in terms of the space  $\widehat{\text{BLD}}$  (Definition 6.1). This restriction will be decided in terms of  $(\mathcal{F}_M, N)$  by the boundary condition (6.8). Formula (6.8) includes implicitly the notion of the (generalized) normal derivative in Doob's sense [7]. Moreover, (6.8) is analogous to a boundary condition by Feller [11; p. 560], where the Markov chains with a finite number of exit boundary points are treated.

The final two sections will be devoted to the study of several special cases. In section 8, we will be concerned with the subclass  $\mathbf{G}_1$  formed by those elements of  $\mathbf{G}$  for which the corresponding forms  $N(\cdot, \cdot)$  vanish identically on the corresponding spaces  $\mathcal{F}_M$ . We will see that a diffusion process on an extended state space corresponds to each element of  $\mathbf{G}_1$ . There are two extreme elements of  $\mathbf{G}_1$ : the cases when  $\mathcal{F}_M = \mathbf{H}_M$  and when  $\mathcal{F}_M$  contains only constant functions. We will see that the former case turns out to reconstruct the resolvent density of [15]. In section 9, we will examine the cases that the domain  $D$  is a disk and an interval<sup>6)</sup>.

Here are two remarks about our class  $\mathbf{G}$  of resolvent densities.

First, we note that there is a one-to-one correspondence between  $\mathbf{G}$  and a family of (equivalent classes of) Markov processes dominating the absorbing Brownian motion on  $D$ . Indeed, with each element  $G$ .  $(\cdot, \cdot)$  of  $\mathbf{G}$ , we can associate, exactly in the same manner as in [15; section 3], a right continuous strong Markov process  $X = (X_t, P_x, x \in D^*)$  whose state space  $D^*$  is the Martin-Kuramochi type completion of  $D$  with respect to the class of functions  $\{G_1(\cdot, y), y \in D\}$ .  $X$  has the following properties:

(X. 1)  $X$  is conservative on  $D$ :

5)  $\mathcal{F}$  is the refinement of the space  $\mathcal{F}_D$  (see (4.18)).

6) There, we can compare our boundary condition (6.8) with those of Wentzell [23] and Feller [12].

$$P_x(X_t \in D) = 1, \quad t > 0, \quad x \in D.$$

(X. 2) Let  $\tau$  be the first exit time from  $D$  of the path  $X_t$ , then  $(X_t, t < \tau, P_x, x \in D)$  is the absorbing Brownian motion on  $D$ .

(X. 3) For any Borel set  $E$  of  $D^*$ ,

$$\int_0^{+\infty} e^{-\alpha t} P_x(X_t \in E) dt = \int_{E \cap D} G_\alpha(x, y) dy, \quad \alpha > 0, \quad x \in D.$$

Conversely, suppose that a right continuous strong Markov process  $X$  on an enlarged state space  $D^*$  satisfies the conditions (X. 1) and (X. 2). Further we assume the existence of a symmetric, jointly continuous function  $G_\alpha(x, y)$ ,  $\alpha > 0$ ,  $x, y \in D$ ,  $x \neq y$  satisfying the condition (X. 3). Then, as one easily verifies, this function is an element of  $\mathcal{G}$ .

Second remark is about the relation between the class  $\mathcal{G}$  and the class of symmetric Brownian resolvents in the sense of T. Shiga and T. Watanabe [21]. By a Brownian resolvent, we mean a resolvent kernel  $\{G_\alpha(x, E), \alpha > 0, x \in D, E \subset D\}$  such that  $G_\alpha f(x) = \int_D G_\alpha(x, dy) f(y)$  satisfies the equation

$$\left( \alpha - \frac{1}{2} \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} \right) G_\alpha f(x) = f(x), \quad x \in D,$$

for any infinitely differentiable function  $f$  with compact support. A resolvent kernel  $\{G_\alpha(x, E)\}$  is said symmetric if, for any non-negative measurable functions  $f$  and  $g$ ,  $\int_D G_\alpha f(x) g(x) dx = \int_D f(x) G_\alpha g(x) dx \leq +\infty$ . Any symmetric resolvent kernel defines a symmetric resolvent (operator) on  $L^2(D)$  in the sense of section 2, so that we can associate with it a Dirichlet space relative to  $L^2(D)$ . It is obvious that each element of the class  $\mathcal{G}$  is a density function of a conservative symmetric Brownian resolvent (kernel). Conversely, we can prove that any conservative symmetric Brownian resolvent is of the class  $\mathcal{G}$ , as is outlined in the following. It is implied in the remark preceding Proposition A. 6 of [21] that the decomposition theorem (Theorem 4.3) of the present paper is still valid for the Dirichlet space associated with any symmetric Brownian resolvent. Hence, starting with a conservative symmetric Brownian resolvent (without assuming the existence of a density function), we can go along the same line as in section 5 and we can reconstruct in section 7 the resolvent considered, by showing that it has a density function of the class  $\mathcal{G}$ .

I wish to express my hearty thanks to T. Shiga and T. Watanabe for their valuable advices. They have shown me the manuscript of [21] before publication. T. Watanabe admitted me to mention one of his unpublished results that the space  $\mathcal{H}_\alpha$ , in our context, is contained in the space of  $\alpha$ -harmonic functions with finite Dirichlet integrals (Theorem 5.1). This made the arguments of section 5 simpler than those of the original version.

## § 2. Symmetric resolvents and Dirichlet spaces relative to $L^2$ -spaces.

Let  $(X, \mathcal{B}, m)$  be a measure space on a Hausdorff space  $X$  with the topological Borel field  $\mathcal{B}$ . We assume that  $m$  is finite:  $m(X) < +\infty$ . Denote by  $L^2(X)$  the space of all real-valued square integrable functions on  $X$  with the inner product  $(u, v)_X = \int_X u(x)v(x)m(dx)$ .

DEFINITION 2.1. A *symmetric resolvent* on  $L^2(X)$  is a family of symmetric linear operators  $\{G_\alpha, \alpha > 0\}$  on  $L^2(X)$  such that  $G_\alpha u$  is non-negative for any non-negative  $u \in L^2(X)$ ,  $\alpha G_\alpha 1 \leq 1$ ,  $G_\alpha - G_\beta + (\alpha - \beta)G_\alpha G_\beta = 0$  and  $G_\alpha u_n$  decreases to zero  $m$ -almost everywhere on  $X$  when  $u_n \in L^2(X)$  decreases to zero.

DEFINITION 2.2. Let  $u$  and  $v$  be measurable functions on  $X$ . We call  $u$  a *normal contraction* of  $v$  if the following inequalities are valid on  $X$ ;

$$|u(x)| \leq |v(x)|, \quad |u(x) - u(y)| \leq |v(x) - v(y)|.$$

DEFINITION 2.3. A function space  $(\mathcal{F}_X, \mathcal{E}_X(\cdot, \cdot))$  is called a *Dirichlet space relative to  $L^2(X)$* , if the following three conditions are satisfied.

(2.1)  $\mathcal{F}_X$  is a non-empty linear subset of  $L^2(X)$  and  $\mathcal{E}_X(\cdot, \cdot)$  is a non-negative symmetric bilinear form on  $\mathcal{F}_X$ .

(2.2) For some (or equivalently for every)  $\alpha > 0$ ,  $\mathcal{F}_X$  is a real Hilbert space with the inner product

$$\mathcal{E}_X^\alpha(u, v) = \mathcal{E}_X(u, v) + \alpha(u, v)_X,$$

two functions of  $\mathcal{F}_X$  being identified if they coincide  $m$ -almost everywhere on  $X$ .

(2.3) Every normal contraction operates on  $(\mathcal{F}_X, \mathcal{E}_X)$ ; if  $u$  is a normal contraction of  $v \in \mathcal{F}_X$ , then  $u \in \mathcal{F}_X$  and  $\mathcal{E}_X(u, u) \leq \mathcal{E}_X(v, v)$ .

Following Beurling and Deny [2] and Deny [4], let us state two theorems about a one-to-one correspondence between Dirichlet spaces and symmetric resolvents.

THEOREM 2.1. Let  $(\mathcal{F}_X, \mathcal{E}_X(\cdot, \cdot))$  be a Dirichlet space relative to  $L^2(X)$ .

(i) For each  $\alpha > 0$  and  $u \in L^2(X)$ , there is a unique element  $G_\alpha u$  of  $\mathcal{F}_X$  such that

$$(2.4) \quad \mathcal{E}_X^\alpha(G_\alpha u, v) = (u, v)_X \quad \text{for any } v \in \mathcal{F}_X.$$

(ii) The family of operators  $G_\alpha$ ,  $\alpha > 0$ , defined by (2.4) is a symmetric resolvent on  $L^2(X)$ .

(iii) For each  $\alpha > 0$ ,  $\{G_\alpha u; u \in L^2(X)\}$  is dense in  $\mathcal{F}_X$  with respect to the norm  $\mathcal{E}_X^\beta$  ( $\beta > 0$  being arbitrary).

We note that the non-negativity and the sub-Markov property of  $\alpha G_\alpha$ , where  $G_\alpha$  is defined by the equation (2.4), follow from the condition (2.3) of

the space  $(\mathcal{F}_X, \mathcal{E}_X)$ . Conversely, suppose that we are given a symmetric resolvent  $\{G_\alpha, \alpha > 0\}$  on  $L^2(X)$ . It is easy to see that  $G_\alpha$  on  $L^2(X)$  is a bounded operator with norm less than  $1/\alpha$  and consequently  $(G_\alpha u, u)_X$  is non-negative for any  $u \in L^2(X)$ <sup>7)</sup>. Put for  $\alpha \geq 0$  and  $u \in L^2(X)$ ,

$$(2.5) \quad \mathcal{E}_{X,\beta}^\alpha(u, u) = \beta(u - \beta G_{\beta+\alpha} u, u)_X$$

$$(2.6) \quad \mathcal{F}_{X,\beta}^\alpha(u, u) = (u - \beta G_{\beta+\alpha} u, u - \beta G_{\beta+\alpha} u)_X.$$

We then have,

$$(2.7) \quad -\frac{\partial}{\partial \beta} \mathcal{E}_{X,\beta}^\alpha(u, u) = \mathcal{F}_{X,\beta}^\alpha(u, u) \quad \text{and} \quad -\frac{\partial}{\partial \beta} \mathcal{F}_{X,\beta}^\alpha(u, u) \leq 0, \quad \beta > 0,$$

which leads us to the following theorem.

**THEOREM 2.2.** *Let  $\{G_\alpha, \alpha > 0\}$  be a symmetric resolvent on  $L^2(X)$ .*

(i)  $\mathcal{E}_{X,\beta}^\alpha(u, u)$  defined by (2.5) is non-negative and it is non-decreasing as  $\beta$  increases. If we set

$$(2.8) \quad \mathcal{E}_X(u, u) = \lim_{\beta \rightarrow +\infty} \mathcal{E}_{X,\beta}^0(u, u), \quad u \in L^2(X),$$

$$(2.9) \quad \mathcal{F}_X = \{u; u \in L^2(X), \quad \mathcal{E}_X(u, u) < +\infty\},$$

then  $(\mathcal{F}_X, \mathcal{E}_X(\cdot, \cdot))$  is a Dirichlet space relative to  $L^2(X)$ .

(ii) For  $u \in \mathcal{F}_X$  and  $\alpha > 0$ ,

$$\mathcal{E}_X^\alpha(u, u) (= \mathcal{E}_X(u, u) + \alpha(u, u)_X) = \lim_{\beta \rightarrow +\infty} \mathcal{E}_{X,\beta}^\alpha(u, u).$$

(iii)  $G_\alpha$  satisfies the equation (2.4) for the space  $(\mathcal{F}_X, \mathcal{E}_X(\cdot, \cdot))$  defined by (2.8) and (2.9).

Assertions (i) and (ii) of the theorem can be proved easily from (2.5) and (2.7). As for the statement (iii), note a consequence of (2.7):  $\beta G_\beta v$  converges to  $v$  strongly in  $L^2(X)$  if  $v$  is in  $\mathcal{F}_X$ . Hence we can conclude that the equation in statement (iii) is valid for every  $v \in \mathcal{F}_X$ .

The following lemma will be used in section 5.

**LEMMA 2.1.** *Suppose that  $(\mathcal{F}_X, \mathcal{E}_X)$  is a Dirichlet space and  $u \in \mathcal{F}_X$ . Denote by  $u_n$  the truncation of  $u$ :  $u_n(x) = u(x)$  for  $|u(x)| < n$ ,  $u_n(x) = n$  for  $u(x) \geq n$  and  $u_n(x) = -n$  for  $u(x) \leq -n$ . Then,*

(i)  $u_n \in \mathcal{F}_X$ , and  $\mathcal{E}_X(u_n, u_n)$  increases to  $\mathcal{E}_X(u, u)$  as  $n$  tends to infinity.

(ii)  $(u_n)^2 \in \mathcal{F}_X$  and  $\mathcal{E}_X((u_n)^2, (u_n)^2) \leq 4n^2 \mathcal{E}_X(u, u)$ .

**PROOF.** Since  $u_n$  is a normal contraction of  $u$ ,  $u_n$  is an element of  $\mathcal{F}_X$ . Obviously  $\mathcal{E}_X(u_n, u_n)$  is increasing and its limit is no greater than  $\mathcal{E}_X(u, u)$ . Define  $G_\beta$  and  $\mathcal{E}_{X,\beta}^0$  by (2.4) and (2.5) successively. Theorem 2.1 and 2.2 imply that, for any  $v \in \mathcal{F}_X$ ,  $\mathcal{E}_{X,\beta}^0(v, v)$  increases to  $\mathcal{E}_X(v, v)$  as  $\beta \rightarrow +\infty$ . Hence, we

7) By the resolvent equation,  $\frac{d}{d\alpha}(G_\alpha u, u)_X = -(G_\alpha u, G_\alpha u)_X \leq 0$ .

have  $\mathcal{E}_{X,\beta}^0(u_n, u_n) \leq \lim_{n \rightarrow +\infty} \mathcal{E}_X(u_n, u_n)$ . Letting  $n$  and  $\beta$  tend to infinity successively, we arrive at the statement (i). Assertion (ii) is an immediate consequence of the fact that  $(\frac{1}{2n}u_n)^2$  is a normal contraction of  $\frac{1}{2n}u_n$ .

From now on, we treat only the cases that the underlying space  $X$  is an Euclidean domain or its Martin boundary.

Suppose that  $G_\alpha(x, y)$ ,  $\alpha > 0$ ,  $x, y \in D$ ,  $x \neq y$  is a symmetric resolvent density on a bounded Euclidean domain  $D$ . Then, by

$$(2.10) \quad G_\alpha u(x) = \int_D G_\alpha(x, y)u(y)dy, \quad \alpha > 0, \quad u \in L^2(D),$$

we have a symmetric resolvent  $\{G_\alpha, \alpha > 0\}$  on  $L^2(D)$ .

DEFINITION 2.4. With the resolvent (2.10), we define a Dirichlet space  $(\mathcal{F}_D, \mathcal{E})$  relative to  $L^2(D)$  by formulae (2.8) and (2.9). We call  $(\mathcal{F}_D, \mathcal{E})$  the *Dirichlet space associated with the resolvent density  $G_\alpha(x, y)$  on  $D$* .

Denote by  $\mathbf{B}(D)(C_0^\infty(D))$  the space of all bounded measurable functions on  $D$  (resp. all infinitely differentiable functions with compact supports). By Theorem 2.2 (iii), we have

LEMMA 2.2. *Let  $G_\alpha(x, y)$  be a symmetric resolvent on  $D$ . Then,  $\{G_\alpha u, u \in C_0^\infty(D)\}$  and  $\{G_\alpha u, u \in \mathbf{B}(D)\}$  are the dense subsets of the associated Dirichlet space  $\mathcal{F}_D$  with metric  $\mathcal{E}^\beta(\cdot, \cdot)$  ( $\beta > 0$  being arbitrary).*

### § 3. Space of BLD functions which are square integrable. Integrations by the Feller kernel.

Properties of BLD functions were profoundly investigated by Deny and Lions [5] and Doob [7]. In this section, we will study BLD functions in terms of the associated Dirichlet spaces and the Feller kernels defined on the Martin boundary. Theorem 3.1 will state that the space of BLD functions of potential type is identical with the Dirichlet space associated with the resolvent density of the absorbing barrier Brownian motion. We will give two applications of this theorem to exhibit the properties of the Feller kernel. Finally, we will present some results concerning boundary properties of  $\alpha$ -harmonic functions with finite Dirichlet integrals, analogous to those by Doob [7]. Inequalities in Lemma 3.1 and equalities in the proof of the lemma will play basic roles in the following sections.

Throughout this section to section 8, we fix an arbitrary bounded domain  $D$  of  $R^N$ .

DEFINITION 3.1. Denote by  $\widehat{\text{BLD}}$  the space of all BLD functions which are square integrable on  $D$ . Precisely,  $u \in \widehat{\text{BLD}}$ , if and only if  $u \in L^2(D)$ , every first partial derivatives of  $u$  (in the sense of Schwartz's distribution) are in

$L^2(D)$  and  $u$  is fine continuous quasi-everywhere on  $D^8$ .

For  $u, v \in \widehat{\text{BLD}}$ , put

$$(u, v)_{D,1} = \frac{1}{2} \int_D (\text{grad } u, \text{grad } v)(x) dx.$$

The pair  $(\widehat{\text{BLD}}, (\cdot, \cdot)_{D,1})$  is a Dirichlet space relative to  $L^2(D)$  in the sense of Definition 2.3.

DEFINITION 3.2. Denote by  $\text{BLD}_0$  the closure of  $C_0^\infty(D)$  in the space  $(\widehat{\text{BLD}}, (\cdot, \cdot)_{D,1})$ .

Note that, for each  $\alpha > 0$ ,  $(u, u)_{D,1} + \alpha(u, u)_D$  gives a metric equivalent to  $(u, u)_{D,1}$  for the space  $\text{BLD}_0$  ([5]). In accordance with Doob [7], a function of  $\text{BLD}_0$  will be called a BLD function of potential type.

Let  $(\mathcal{F}_D^{(0)}, \mathcal{E}^{(0)})$  be the Dirichlet space associated with the resolvent density  $G_\alpha^0(x, y)$  of the absorbing barrier Brownian motion on  $D$  (see Definition 2.4). We put

$$(3.1) \quad \mathcal{F}^{(0)} = \{u \in \mathcal{F}_D^{(0)}, u \text{ is fine-continuous quasi-everywhere on } D\}.$$

We call  $\mathcal{F}^{(0)}$  the refinement of the space  $\mathcal{F}_D^{(0)}$ .

THEOREM 3.1.

(i) For each function  $u$  of  $\mathcal{F}_D^{(0)}$ , there exists a function of  $\mathcal{F}^{(0)}$ , which is equal to  $u$  almost everywhere.

(ii)  $\mathcal{F}^{(0)} = \text{BLD}_0$  and  $\mathcal{E}^{(0)}(u, u) = (u, u)_{D,1}$ ,  $u \in \mathcal{F}^{(0)}$ .

PROOF. On account of Lemma 2.2 and the remark in the preceding paragraph, it is sufficient to show that, for a fixed  $\alpha > 0$ ,

(a)  $\mathcal{R}^{(0)} = \{G_\alpha^0 u; u \in C_0^\infty(D)\}$  is contained in  $\text{BLD}_0$  and, for  $v \in \mathcal{R}^{(0)}$ ,  $\mathcal{E}^{(0), \alpha}(v, v) = (v, v)_{D,1} + \alpha(v, v)_D$ .

(b)  $\mathcal{R}^{(0)}$  is dense in the space  $\text{BLD}_0$  with respect to the norm  $(\cdot, \cdot)_{D,1} + \alpha(\cdot, \cdot)_D$ .

Consider a sequence of domains  $D_n$  which increases to  $D$ . Assume that boundaries  $\partial D_n$  are of class  $C^2$ . Approximate the function  $v = G_\alpha^0 u$ ,  $u \in C_0^\infty(D)$  by functions

$$v_n(x) = \begin{cases} G_\alpha^{(n)} u(x) & x \in D_n \\ 0 & x \in D - D_n, \end{cases} \quad n = 1, 2, \dots,$$

where  $G_\alpha^{(n)} u$  is defined by (2.10) for the resolvent density of absorbing Brownian motion on  $D_n$ . We can see that  $v_n \in \text{BLD}_0^{(9)}$ . By the equality

$$\alpha v_n(x) = -\frac{1}{2} \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} v_n(x) + u(x), \quad x \in D_n,$$

we have

$$(3.2) \quad (v_n, v_m)_{D,1} + \alpha(v_n, v_m)_D = (u, v_m)_D, \quad n \geq m.$$

8) By "quasi-everywhere" we means "except for a set of capacity zero".

9)  $G_\alpha^{(n)} u$  is in  $\text{BLD}_0$  for the domain  $D_n$  and hence,  $v_n \in \text{BLD}_0$  for  $D$  [5].



Since  $v_n$  converges to  $v$  uniformly on each compact set of  $D$ , the formula (3.2) implies that  $v_n$  is convergent in norm  $\sqrt{(\cdot, \cdot)_{D,1} + \alpha(\cdot, \cdot)_D}$  and the limiting function in  $\text{BLD}_0$  coincides with  $v$  almost everywhere. Hence,  $v \in \text{BLD}_0$  and  $(v, v)_{D,1} + \alpha(v, v)_D = (u, v)_D = \mathcal{E}^{(0),\alpha}(v, v)$ , completing the proof of assertion (a).

As for (b), assume that  $w \in \text{BLD}_0$  satisfies  $(w, v)_{D,1} + \alpha(w, v)_D = 0$  for all  $v = G_\alpha^0 u \in \mathcal{R}^{(0)}$ . Find  $w_n \in C_0^\infty(D)$  which converges to  $w$  in  $\text{BLD}_0$ , then we see that the left-hand side of the above equation is equal to  $\lim_{n \rightarrow +\infty} ((w_n, v)_{D,1} + \alpha(w_n, v)_D) = \lim_{n \rightarrow +\infty} (w_n, u)_D = (w, u)_D$ . Thus,  $w$  must vanish. The proof of the theorem is complete.

Now we are in a position to introduce several notions related to the Martin boundary  $M$  of the domain  $D$ . Let  $\mu(E)$  be the harmonic measure of the Borel set  $E$  of  $M$  relative to the fixed reference point  $x_0 \in D$ .

DEFINITION 3.3. If a function  $u$  on  $D$  has a fine limit  $\varphi(\xi)$  at  $\mu$ -almost every  $\xi \in M$ , we denote  $\varphi$  by  $\gamma u$  and call it a boundary function of  $u$ .

Doob [7] has proved that every BLD function has a boundary function in  $L^2(M)$  and that  $u$  is an element of  $\text{BLD}_0$  if and only if  $u$  is a BLD-function and  $(\gamma u)(\xi) = 0$  for almost all  $\xi \in M$ . Thus,

COROLLARY TO THEOREM 3.1.  $u$  belongs to  $\mathcal{F}^{(0)}$  if and only if  $u$  is a BLD function and  $u$  has a boundary function vanishing  $\mu$ -almost everywhere on  $M$ .

Let  $K(x, \xi) = K^\xi(x)$ ,  $x \in D$ , be the Martin kernel associated with  $\xi \in M$ .

Define, for  $\alpha > 0$ .

$$(3.3) \quad K_\alpha(x, \xi) = K_\xi^\alpha(x) = K^\xi(x) - \alpha \int_D G_\alpha^0(x, y) K^\xi(y) dy.$$

Put for  $\xi, \eta \in M$ ,  $\alpha > 0$ ,

$$(3.4) \quad U_\alpha(\xi, \eta) = \alpha (K^\xi, K_\alpha^\eta)_D \leq +\infty.$$

$U_\alpha(\xi, \eta)$  is non-decreasing in  $\alpha$  and we put

$$(3.5) \quad U(\xi, \eta) = \lim_{\alpha \rightarrow +\infty} U_\alpha(\xi, \eta) \leq +\infty.$$

We call  $U_\alpha$  and  $U$  the Feller kernels<sup>10)</sup>. For functions  $\varphi$  and  $\psi$  on  $M$ , we define

$$(3.6) \quad U_\alpha(\varphi, \psi) = \int_M \int_M U_\alpha(\xi, \eta) \varphi(\xi) \psi(\eta) \mu(d\xi) \mu(d\eta),$$

$$(3.7) \quad U(\varphi, \psi) = \int_M \int_M U(\xi, \eta) \varphi(\xi) \psi(\eta) \mu(d\xi) \mu(d\eta).$$

Finally, we set for  $\varphi \in L^1(M)$ ,

$$(3.8) \quad H\varphi(x) = \int_M K(x, \xi) \varphi(\xi) \mu(d\xi), \quad x \in D,$$

10) These kernels are symmetric  $\mu$ -almost everywhere (see [13] and footnote 15)).

$$(3.9) \quad H_\alpha \varphi(x) = \int_M K_\alpha(x, \xi) \varphi(\xi) \mu(d\xi), \quad x \in D.$$

If  $\varphi \in L^1(M)$ , then we have  $\gamma(H\varphi) = \varphi^{11}$ .

Here are two applications of Theorem 3.1.

**THEOREM 3.2.** *Let  $\varphi$  be a non-negative bounded measurable functions on  $M$ . Then, it holds that*

$$(3.10) \quad U(\varphi, \varphi) = \mathcal{E}^{(0)}(H\varphi, H\varphi).$$

Moreover, if  $U(\varphi, 1)$  is finite, then  $\varphi$  must vanish almost everywhere on  $M$ .

**PROOF.** It is evident that  $H\varphi \in L^2(D)$ . Identity (3.10) follows from  $U_\alpha(\varphi, \varphi) = \alpha(H_\alpha \varphi, H\varphi)_D = \alpha(H\varphi - \alpha G_\alpha^0 H\varphi, H\varphi)_D = \mathcal{E}_\alpha^{(0),0}(H\varphi, H\varphi)$ . Assume that  $U(\varphi, 1)$  is finite. Then  $U(\varphi, \varphi)$  is finite, and identity (3.10) implies that  $H\varphi$  must be an element of  $\mathcal{F}^{(0)}$ . Corollary to Theorem 3.1 now implies that  $\gamma(H\varphi) = \varphi = 0$ .

Theorem 3.2 will be used in the next section. In section 8, we will refer to the following theorem.

Let  $\tilde{D} = D \cup \{\infty\}$  be the one point compactification of  $D$ . For a Borel subset  $A$  of the Martin boundary  $M$ , we set  $\Pi_\beta^A(x) = H_\beta \chi_A(x)$ ,  $\chi_A(\xi)$  being the indicator function of the set  $A$ . Define a probability measure  $V_\beta^A$  on  $\tilde{D}$  by

$$(3.11) \quad \begin{cases} V_\beta^A(E) = \frac{\int_E \Pi_\beta^A(x) dx}{(\Pi_\beta^A, 1)_D}, & \text{if } E \text{ is a Borel set of } D \\ V_\beta^A(\{\infty\}) = 0. \end{cases}$$

**THEOREM 3.3.** *Suppose that  $\mu(A) > 0$ . As  $\beta$  tends to infinity, the sequence of measures  $V_\beta^A(dx)$  on  $\tilde{D} = D \cup \{\infty\}$  converges weakly to the  $\delta$ -measure concentrated at  $\{\infty\}$ .*

**PROOF.** By virtue of Theorem 3.2,  $\beta(\Pi_\beta^A, 1)_D = U_\beta(\chi_A, 1) \rightarrow +\infty$  as  $\beta$  tends to infinity. Hence, it suffices to prove that, for each open set  $E$  the closure of which is compact in  $D$ ,  $\beta \int_E \Pi_\beta^A(x) dx$  is bounded in  $\beta > 0$ . Choose a non-negative  $u \in C_0^\infty(D)$  with  $u = 1$  on the set  $E$ . Let  $v$  be an element of  $C_0^\infty(D)$  which is less than  $H\chi_A$  everywhere on  $D$  and equal to  $H\chi_A$  on the support of  $u$ . Then,

$$\begin{aligned} \beta \int_E \Pi_\beta^A(x) dx &\leq \beta(\Pi_\beta^A, u)_D = \beta(H\chi_A, u)_D - \beta^2(G_\beta^0 H\chi_A, u)_D \\ &\leq \beta(v, u)_D - \beta^2(G_\beta^0 v, u)_D. \end{aligned}$$

Owing to Theorem 3.1, the last term converges to  $(v, u)_{D,1}$  as  $\beta \rightarrow +\infty$ . The proof of Theorem 3.3 is complete.

Turning to the study of boundary properties of  $\alpha$ -harmonic functions, let

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11) Cf. Doob [6].

us introduce new spaces of functions on  $M$ . For a function  $\varphi$  on  $M$ , we put  $U_\alpha\varphi(\xi) = \int_M U_\alpha(\xi, \eta)\varphi(\eta)\mu(d\eta)$ . Define a new measure  $\mu'$  on  $M$  by

$$(3.12) \quad \mu'(A) = \int_A U_1 1(\xi)\mu(d\xi)$$

for the Borel set  $A$  of  $M$ . For functions  $\varphi$  and  $\psi$  on  $M$ , set

$$(3.13) \quad \begin{cases} (\varphi, \psi)_M = \int_M \varphi(\xi)\psi(\xi)\mu(d\xi) \\ (\varphi, \psi)'_M = \int_M \varphi(\xi)\psi(\xi)\mu'(d\xi) \end{cases}$$

$$(3.14) \quad \mathbf{D}(\varphi, \psi) = \frac{1}{2} \int_M \int_M (\varphi(\xi) - \varphi(\eta))(\psi(\xi) - \psi(\eta))U(\xi, \eta)\mu(d\xi)\mu(d\eta).$$

Denote by  $L^2(M)(L^2(M)')$  the space of measurable functions  $\varphi$  on  $M$  such as  $(\varphi, \varphi)_M < +\infty$  (resp.  $(\varphi, \varphi)'_M < +\infty$ ). We set

$$(3.15) \quad \mathbf{H}_M = \{\varphi; \varphi \in L^2(M)' \text{ and } \mathbf{D}(\varphi, \varphi) < +\infty\}.$$

$\mathbf{B}(D)$  ( $\mathbf{B}(M)$ ) will stand for the space of all bounded measurable functions on  $D$  (resp. on  $M$ ).

The next lemma collects the basic relations among these spaces and norms.

LEMMA 3.1.

(i)  $\mathbf{B}(M) \subset L^2(M)' \subset L^2(M)$  and there is a constant  $C > 0$  such that

$$(3.16) \quad (\varphi, \varphi)_M \leq C(\varphi, \varphi)'_M, \quad \text{for every } \varphi \in L^2(M)'.$$

(ii) For  $\varphi \in L^2(M)'$  and  $\alpha > 0$ ,

$$(3.17) \quad 0 \leq U_\alpha(\varphi, \varphi) \leq (\alpha \vee 1)(\varphi, \varphi)'_M.$$

(iii) For  $\varphi \in \mathbf{H}_M$  and  $\alpha > 0$ ,

$$(3.18) \quad (\varphi, \varphi)'_M \leq \left(1 \vee \frac{1}{\alpha}\right) \{\mathbf{D}(\varphi, \varphi) + U_\alpha(\varphi, \varphi)\}.$$

PROOF. The first inclusion in assertion (i) follows from  $(1_M, 1_M)'_M = (H1_M, H1_M)_D \leq$  the Lebesgue measure of  $D^{12}$ .  $U_1 1(\xi)$  is finite for  $\mu$ -almost all  $\xi \in M$ . It is lower semi-continuous and strictly positive everywhere on  $M$ . Hence, it suffices to set  $C = 1/\inf_{\xi \in M} U_1 1(\xi)$  to obtain estimate (3.16). Next, observe that

$U_\alpha 1(\xi)$  is increasing and  $\frac{1}{\alpha} U_\alpha 1(\xi)$  is decreasing as  $\alpha$  increases. The first and second inequalities in (3.17) and inequality (3.18) are the consequences of the following equalities (3.19), (3.20) and (3.21) respectively.

12)  $1_M$  denotes the function which is identically one on  $M$ .

$$(3.19) \quad U_\alpha(\varphi, \varphi) = \alpha(H_\alpha\varphi, H_\alpha\varphi)_D + \alpha^2(G_{0+}^0 H_\alpha\varphi, H_\alpha\varphi)_D,$$

$$(3.20) \quad U_\alpha(\varphi, \varphi) + \frac{1}{2} \int_M \int_M (\varphi(\xi) - \varphi(\eta))^2 U_\alpha(\xi, \eta) \mu(d\xi) \mu(d\eta) \\ = \int_M \varphi(\xi)^2 U_\alpha 1(\xi) \mu(d\xi), \quad \varphi \in L^2(M)',$$

$$(3.21) \quad D(\varphi, \varphi) + U_\alpha(\varphi, \varphi) = \frac{1}{2} \int_M \int_M (\varphi(\xi) - \varphi(\eta))^2 \\ \{U(\xi, \eta) - U_\alpha(\xi, \eta)\} \mu(d\xi) \mu(d\eta) + \int_M \varphi(\xi)^2 U_\alpha 1(\xi) \mu(d\xi), \quad \varphi \in \mathbf{H}_M.$$

Now, denote by  $\widehat{\text{BLD}}_{\alpha, h}$  the space of all  $\alpha$ -harmonic functions belonging to  $\widehat{\text{BLD}}$ . It is easy to see that  $\widehat{\text{BLD}}_{\alpha, h}$  is the orthogonal complement of  $\text{BLD}_0$  in the Hilbert space  $(\widehat{\text{BLD}}, (\cdot, \cdot)_{D, 1} + \alpha(\cdot, \cdot)_D)$ .

Our final assertions in this section are as follows.

THEOREM 3.4. Fix an  $\alpha > 0$ .

(i) Every bounded  $\alpha$ -harmonic function  $u$  on  $D$  has its boundary function  $\gamma u$  in  $\mathbf{B}(M)$  and  $u(x) = H_\alpha(\gamma u)(x)$ ,  $x \in D$ .

(ii) Every function  $u$  of  $\widehat{\text{BLD}}_{\alpha, h}$  has its boundary function  $\gamma u$  in  $\mathbf{H}_M$  and  $u(x) = H_\alpha(\gamma u)(x)$ ,  $x \in D$ . Further, it holds that

$$(3.22) \quad (u, u)_{D, 1} + \alpha(u, u)_D = D(\gamma u, \gamma u) + U_\alpha(\gamma u, \gamma u).$$

(iii) For  $\varphi \in L^2(M)'$ ,  $H_\alpha\varphi$  has  $\varphi$  as its boundary function. In particular, if  $\varphi \in \mathbf{H}_M$ , then  $H_\alpha\varphi \in \widehat{\text{BLD}}_{\alpha, h}$  and equality (3.22) holds for  $u = H_\alpha\varphi$  and  $\gamma u = \varphi$ .

(iv) For  $u \in \widehat{\text{BLD}}_{\alpha, h}$ , the following inequality holds.

$$(3.23) \quad (\gamma u, \gamma u)'_M \leq \left(1 \vee \frac{1}{\alpha}\right) \{(u, u)_{D, 1} + \alpha(u, u)_D\},$$

$\gamma u$  being the boundary function of  $u$ .

PROOF. (i) set

$$(3.24) \quad u_1 = u + \alpha G_{0+}^0 u.$$

$u_1$  is a bounded harmonic function. Hence,  $u_1$  has its boundary function, say  $\varphi$ , in  $\mathbf{B}(M)$  and  $u_1(x) = H\varphi(x)$ ,  $x \in D^{13}$ . Since  $\gamma(G_{0+}^0 u) = 0$ ,  $u$  has  $\varphi$  as its boundary function. By virtue of the equality  $u = u_1 - \alpha G_\alpha^0 u_1 = H\varphi - \alpha G_\alpha^0 H\varphi$  and identity (3.3), we obtain  $u = H_\alpha\varphi$ .

(ii) For  $u \in \widehat{\text{BLD}}_{\alpha, h}$ , define  $u_1$  by (3.24). Note that  $G_{0+}^0$  is a bounded operator on  $L^2(D)$ , so that  $u_1 \in L^2(D)$ . Therefore  $G_{0+}^0 u = G_\alpha^0 u_1 \in \text{BLD}_0$  (Theorem 3.1) and  $u_1 \in \widehat{\text{BLD}}$ . Hence we have

13) Cf. Doob [6].

$$\begin{aligned}
(u_1, u_1)_{D,1} + \alpha(u_1, u_1)_D &= (u, u)_{D,1} + \alpha(u, u)_D + (\alpha G_{0+}^0 u, \alpha G_{0+}^0 u)_{D,1} \\
&+ \alpha(\alpha G_{0+}^0 u, \alpha G_{0+}^0 u)_D = (u, u)_{D,1} + \alpha(u, u)_D + (\alpha G_{\alpha}^0 u_1, \alpha G_{0+}^0 u)_{D,1} \\
&+ \alpha(\alpha G_{\alpha}^0 u_1, \alpha G_{0+}^0 u)_D = (u, u)_{D,1} + \alpha(u, u)_D + (\alpha u_1, \alpha G_{0+}^0 u)_D,
\end{aligned}$$

so that

$$(3.25) \quad (u, u)_{D,1} + \alpha(u, u)_D = (u_1, u_1)_{D,1} + \alpha(u_1, u_1)_D.$$

Owing to Doob [7] and Fukushima [13],  $u_1$  has the boundary function (say  $\varphi$ ) in  $L^2(M)$  with  $(u_1, u_1)_{D,1} = \mathbf{D}(\varphi, \varphi)$ . Corollary Theorem 3.1 implies that  $\gamma(G_{0+}^0 u) = 0$ . Thus, in the same way as in the proof of statement (i), we have  $\gamma u = \varphi$  and  $u = H_{\alpha}\varphi$ . Identity (3.25) now implies (3.22). Further, in view of equality (3.22) and the preceding lemma,  $\varphi (= \gamma u)$  must be an element of  $\mathbf{H}_M$ .

(iii) By virtue of formulae (3.17) and (3.19), we see that  $H_{\alpha}\varphi \in L^2(D)$  for  $\varphi \in L^2(M)'$ . Hence,  $G_{0+}^0(H_{\alpha}\varphi) \in \text{BLD}_0$  and  $\gamma(H_{\alpha}\varphi) = \gamma(H\varphi) = \varphi$ . If, in addition,  $\mathbf{D}(\varphi, \varphi)$  is finite, then  $u_1 = H\varphi$  is BLD harmonic with  $(u_1, u_1)_{D,1} = \mathbf{D}(\varphi, \varphi)$  ([7]) and identity (3.25) is valid for  $u = H_{\alpha}\varphi$ .

(iv) is only the restatement of Lemma 3.1 (iii).

#### § 4. An expression of the symmetric resolvent density $G_{\alpha}(x, y)$ and a decomposition of the Dirichlet space associated with $G_{\alpha}(x, y)$ .

Throughout § 4, 5 and 6, we assume that we are given a resolvent  $G_{\alpha}(x, y)$  of  $\mathbf{G}$ :  $G_{\alpha}(x, y) = G_{\alpha}(x, y) + R_{\alpha}(x, y)$  is a conservative symmetric resolvent density and  $R_{\alpha}(x, y)$  satisfies the conditions (G, a) and (G, b) stated in the beginning of section 1.

Our first task in this section is to give an expression of  $R_{\alpha}f$ ,  $f \in \mathbf{B}(D)$ , which is analogous to that of Feller [11].

For a function  $\varphi$  on  $M$ ,  $H_{\alpha}\varphi$  defined by (3.9) can be rewritten in terms of the measure  $\mu'$  (see (3.12)) as

$$(4.1) \quad H_{\alpha}\varphi(x) = H_{\alpha}^x\varphi = \int_M K'_{\alpha}(x, \xi)\varphi(\xi)\mu'(d\xi), \quad x \in D,$$

with the function  $K'_{\alpha}(x, \xi)$ ,  $\alpha > 0$ ,  $x \in D$ ,  $\xi \in M$ , defined by

$$(4.2) \quad K'_{\alpha}(x, \xi) = \begin{cases} K_{\alpha}(x, \xi)/U_1 1(\xi) & \text{if } U_1 1(\xi) < +\infty, \\ 0 & \text{if } U_1 1(\xi) = +\infty. \end{cases}$$

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14) Theorem 1 of [13] states that  $U(\xi, \eta) = \frac{q}{2}\theta(\xi, \eta)$  when  $\xi$  and  $\eta$  are exit boundary points. Here,  $\theta$  is Naim's kernel [18] and  $q$  denotes either  $2\pi$  (if  $N=2$ ) or  $(N-2) \times \{\text{area of the unit sphere}\}$ . Since  $D$  is bounded,  $\mu$ -almost all points of  $M$  are exit (see footnote 15)) and Theorem 9.2 of [7] leads to this expression of the Dirichlet integral of the harmonic function. For one dimensional case, this expression is trivially true (see footnote 33)).

Indeed,  $U_1 1(\xi)$  is strictly positive everywhere and finite almost everywhere on  $M$ . For a signed measure  $\nu(dy)$  on  $D$ , let us put

$$(4.3) \quad \hat{H}_\alpha \nu(\xi) = \int_D K'_\alpha(x, \xi) \nu(dy), \quad \xi \in M.$$

$\hat{H}_\alpha$  brings signed measure on  $D$  into functions on  $M$ . For  $x \in D$ ,  $\hat{H}_\alpha^x$  will stand for  $K'_\alpha(x, \xi)$ . When  $\nu$  has a density function  $f \in \mathbf{B}(D)$ ,  $\hat{H}_\alpha \nu$  will be denoted by  $\hat{H}_\alpha f$ . Obviously,  $\hat{H}_\alpha f(\xi) = \int_D \hat{H}_\alpha^x(\xi) f(x) dx$ .

LEMMA 4.1. (i)  $\hat{H}_\alpha$  is a bounded linear operator from  $\mathbf{B}(D)$  into  $\mathbf{B}(M)$ , and

$$(4.4) \quad |\hat{H}_\alpha f(\xi)| \leq \left(1 \vee \frac{1}{\alpha}\right) \sup_{x \in D} |f(x)|$$

for  $f \in \mathbf{B}(D)$ ,  $\xi \in M$ .

(ii) The equation

$$(4.5) \quad \hat{H}_\alpha f - \hat{H}_\beta f + (\alpha - \beta) \hat{H}_\alpha G_\beta^0 f = 0$$

holds for every  $f \in \mathbf{B}(D)$ ,  $\alpha, \beta > 0$ .

(iii) The identity

$$(4.6) \quad (\alpha - \beta)(\varphi, \hat{H}_\alpha(H_\beta \psi))'_M = U_\alpha(\varphi, \psi) - U_\beta(\varphi, \psi)$$

holds for every  $\varphi, \psi \in \mathbf{B}(M)$ ,  $\alpha, \beta > 0$ .

PROOF. Note that  $\mu$ -almost all points  $\xi \in M$  are exit in the sense that  $K_\alpha(x, \xi) > 0$  for some  $\alpha > 0$  and some  $x \in D^{15}$ .  $U_\alpha(\xi, \eta)$  is symmetric if  $\xi$  and  $\eta$  are exit ([13; Lemma 2]). Therefore,  $\int_D K'_\alpha(y, \xi) dy = \hat{H}_\alpha(H1)(\xi)$  is either

$$\frac{1}{\alpha} \frac{U_\alpha 1(\xi)}{U_1 1(\xi)} \quad \text{or zero.}$$

Inequality (4.4) follows from this. The definition (3.3) of  $K_\alpha(x, \xi)$  and the resolvent equation for  $G_\alpha^0$  lead to equation (4.5) for bounded  $f$  with compact support. Identity (4.5) is valid for every  $f \in \mathbf{B}(D)$  by means of the bounded convergence theorem. Identity (4.6) follows from (3.3) and (4.5).

Now, let us state a representation theorem for

$$R_\alpha f(x) = \int_D R_\alpha(x, y) f(y) dy, \quad f \in \mathbf{B}(D).$$

Set  $A(M) = \hat{H}_\alpha(\mathbf{B}(D))$  for an  $\alpha > 0$ . In view of the preceding lemma,  $A(M)$  is independent of  $\alpha > 0$  and it is a linear subset of  $\mathbf{B}(M)$ .

To avoid confusion, we denote by  $1_D$  (resp.  $1_M$ ) the function on  $D$  (resp.  $M$ ) which is identically unity there. Note that  $1_M$  is, up to a set of  $\mu$ -measure

15) Let  $E$  be the set of all non-exit boundary points and put  $u = H\chi_E$ . Then  $\alpha G_\alpha^0 u = u$  for every  $\alpha > 0$ . Letting  $\alpha$  tend to zero, we obtain  $u = 0$  and  $\mu(E) = 0$ .

zero, an element of  $A(M)$  because of  $1_M = \hat{H}_1 1_D$   $\mu$ -almost everywhere.

THEOREM 4.1. For each  $\alpha > 0$ , the function  $R_\alpha f$ ,  $f \in \mathbf{B}(D)$ , is expressed as

$$(4.7) \quad R_\alpha f(x) = H_\alpha^x \tilde{R}^\alpha(\hat{H}_\alpha f), \quad x \in D,$$

with a non-negative linear operator  $\tilde{R}^\alpha$  from  $A(M)$  into  $L^\infty(M)$  satisfying the following conditions:

$$(4.8) \quad \mu\text{-ess sup}_{\xi \in M} |\tilde{R}^\alpha \varphi(\xi)| \leq \left(1 \vee \frac{1}{\alpha}\right) \sup_{\xi \in M} |\varphi(\xi)|, \quad \varphi \in A(M),$$

$$(4.9) \quad (\varphi, \tilde{R}^\alpha \psi)'_M = (\tilde{R}^\alpha \varphi, \psi)'_M, \quad \varphi, \psi \in A(M)$$

$$(4.10) \quad \lim_{\alpha \rightarrow +\infty} (1_M, \tilde{R}^\alpha 1_M)'_M = 0.$$

PROOF. On account of the conditions (G, a) and (G, b), for each  $x \in D$ ,  $R_\alpha(x, y)$  is bounded and  $\alpha$ -harmonic in  $y \in D$ . By Theorem 3.4 (i), there exists a boundary value

$$(4.11) \quad R_\alpha(x, \xi) = \text{fine-lim}_{y \rightarrow \xi} R_\alpha(x, y)$$

for  $\mu$ -almost all  $\xi \in M$  and

$$(4.12) \quad R_\alpha(x, y) = \int_M R_\alpha(x, \xi) \hat{H}_\alpha^y(\xi) \mu'(d\xi) \quad \text{for every } y \in D.$$

We set for  $\varphi \in A(M)$

$$(4.13) \quad R^\alpha \varphi(x) = \int_M R_\alpha(x, \xi) \varphi(\xi) \mu'(d\xi).$$

For any  $\varphi \in A(M)$ ,  $R^\alpha \varphi(x)$  is bounded and  $\alpha$ -harmonic on  $D$ . Indeed,  $\varphi$  can be written as  $\hat{H}_\alpha f$ ,  $f \in \mathbf{B}(D)$ , and, in view of identity (4.12),  $R^\alpha \varphi$  is expressed with this  $f$  as

$$(4.14) \quad R^\alpha \varphi(x) = R_\alpha f(x), \quad x \in D.$$

Thus, owing to Theorem 3.4 (i), there is a well defined function

$$(4.15) \quad \tilde{R}^\alpha \varphi = \gamma(R^\alpha \varphi), \quad \varphi \in A(M)$$

and this  $\tilde{R}^\alpha$  is a non-negative linear operator from  $A(M)$  into  $L^\infty(M)$ . Since  $1_M = \hat{H}_1 1_D \leq (\alpha \vee 1) \hat{H}_\alpha 1_D$   $\mu$ -almost everywhere, we see by (4.14) that  $R^\alpha 1_M(x) \leq (\alpha \vee 1) R_\alpha 1_D(x) \leq 1 \vee \frac{1}{\alpha}$ ,  $x \in D$ , and that  $\tilde{R}^\alpha 1_M(\xi) \leq 1 \vee \frac{1}{\alpha}$   $\mu$ -almost everywhere, which implies the estimate (4.8). Equality (4.9) is an immediate consequence of the expression (4.7) and the symmetry of  $R_\alpha(x, y)$ . Let us prove (4.10). We have for all  $\alpha > 0$

$$(4.16) \quad U_\alpha(\tilde{R}^\alpha 1_M, 1_M) = (1_M, 1_M)'_M < +\infty,$$

because the left-hand side of (4.16) is equal to

$$\alpha(\tilde{R}^{\alpha}1_M, \hat{H}_{\alpha}1_D)'_M = (1_M, \alpha\tilde{R}^{\alpha}(\hat{H}_{\alpha}1_D))'_M = (1_M, \gamma(\alpha R_{\alpha}1_D))'_M.$$

Let  $\{\alpha_n, n=1, 2, \dots\}$  be an arbitrary sequence of real numbers increasing to infinity. Then  $\tilde{R}^{\alpha_n}1_M(\xi)$  decreases to a non-negative function  $\varphi(\xi)$  for every  $\xi \in M$  except on a set of  $\mu$ -measure zero. We set  $\varphi(\xi)=0$  on the exceptional set. From (4.16), we have  $U_{\alpha}(\varphi, 1_M) \leq (1_M, 1_M)'_M$  for all  $\alpha > 0$ . Letting  $\alpha$  tend to infinity, we obtain  $U(\varphi, 1_M) < +\infty$ . Theorem 3.2 now implies that  $\varphi$  vanishes almost everywhere. Therefore,  $\lim_{n \rightarrow +\infty} (1_M, \tilde{R}^{\alpha_n}1_M)'_M = (1_M, \varphi)'_M = 0$ , completing the proof of (4.10).

Next, let  $(\mathcal{F}_D, \mathcal{E})$  be the Dirichlet space associated with our resolvent density  $G_{\alpha}(x, y) = G_{\alpha}^0(x, y) + R_{\alpha}(x, y)$ . Let us represent  $(\mathcal{F}_D, \mathcal{E})$  as a direct sum of a potential part and an  $\alpha$ -harmonic part. Our procedure is based on Theorem 4.1 and we will never use any classical tool such as Green's formula.

Put for  $\alpha > 0$ .

$$(4.17) \quad \mathcal{F}^* = \{G_{\alpha}^0 f, f \in \mathbf{B}(D)\}$$

$$\mathcal{H}_{\alpha}^* = \{R_{\alpha} f, f \in \mathbf{B}(D)\}.$$

Note that  $\mathcal{F}^*$  is independent of  $\alpha > 0$ . Let us show the following basic lemma.

LEMMA 4.2. (i)  $\mathcal{F}^* \subset \mathcal{F}_D$  and

$$\mathcal{E}(u, u) = \mathcal{E}^{(0)}(u, u) \quad \text{for } u \in \mathcal{F}^*.$$

Here,  $\mathcal{E}^{(0)}$  is the norm for the Dirichlet space  $\mathcal{F}_D^{(0)}$  associated with the resolvent density  $G_{\alpha}^0(x, y)$ .

(ii) For each  $\alpha > 0$ ,  $\mathcal{F}^*$  and  $\mathcal{H}_{\alpha}^*$  are orthogonal with respect to the inner product

$$\mathcal{E}^{\alpha}(u, v) = \mathcal{E}(u, v) + \alpha(u, v)_D.$$

PROOF. (i) Set  $u = G_{\alpha}^0 f, f \in \mathbf{B}(D)$ . Then  $u \in L^2(D)$  and

$$\begin{aligned} \mathcal{E}_{\beta}^0(u, u) &= \beta(u - \beta G_{\beta}^0 u, u)_D - \beta^2(R_{\beta} u, u)_D \\ &= \mathcal{E}_{\beta}^{(0),0}(u, u) - \beta^2(R_{\beta} u, u)_D. \end{aligned}$$

On the other hand, by virtue of Theorem 4.1 and Lemma 4.1,

$$\begin{aligned} \beta^2(R_{\beta} u, u)_D &= \beta^2(H_{\beta} \tilde{R}^{\beta}(\hat{H}_{\beta} G_{\alpha}^0 f), G_{\alpha}^0 f)_D = \beta^2(\tilde{R}^{\beta}(\hat{H}_{\beta} G_{\alpha}^0 f), \hat{H}_{\beta} G_{\alpha}^0 f)'_M \\ &= \frac{\beta^2}{(\beta-1)^2} (\tilde{R}^{\beta} \varphi_{\beta}, \varphi_{\beta})'_M, \quad \text{with } \varphi_{\beta} = (\hat{H}_1 - \hat{H}_{\beta})f. \end{aligned}$$

By Theorem 4.1, we have

$$|(\tilde{R}^{\beta} \varphi_{\beta}, \varphi_{\beta})'_M| \leq (\tilde{R}^{\beta} 1_M, 1_M)'_M \left( \sup_{x \in D} |f(x)| \right)^2 \xrightarrow{\beta \rightarrow +\infty} 0.$$

Thus,  $u \in \mathcal{F}_D$  and  $\mathcal{E}(u, u) = \mathcal{E}^{(0)}(u, u)$ .



(ii) Owing to the preceding assertion (i), we have for  $f, g \in \mathbf{B}(D)$  that

$$\begin{aligned} \mathcal{E}^\alpha(G_\alpha^0 f, R_\alpha g) &= \mathcal{E}^\alpha(G_\alpha^0 f, G_\alpha g) - \mathcal{E}^\alpha(G_\alpha^0 f, G_\alpha^0 g) \\ &= (G_\alpha^0 f, g)_D - \mathcal{E}^{(0), \alpha}(G_\alpha^0 f, G_\alpha^0 g) \\ &= (G_\alpha^0 f, g)_D - (G_\alpha^0 f, g)_D = 0. \end{aligned}$$

The proof of Lemma 4.2 is complete.

According to Lemma 2.2, functions

$$\{G_\alpha f = G_\alpha^0 f + R_\alpha f, f \in \mathbf{B}(D)\} \quad (\text{resp. } \{G_\alpha^0 f, f \in \mathbf{B}(D)\})$$

are dense in the Hilbert space  $\{\mathcal{F}_D, \mathcal{E}^\alpha\}$  (resp.  $\{\mathcal{F}_D^{(0)}, \mathcal{E}^{(0), \alpha}\}$ ). Hence, we immediately obtain the next theorem from Lemma 4.2.

THEOREM 4.2. (i)  $\mathcal{F}_D^{(0)} \subset \mathcal{F}_D$  and

$$\mathcal{E}(u, u) = \mathcal{E}^{(0)}(u, u) \quad \text{for } u \in \mathcal{F}_D^{(0)}.$$

(ii) For each  $\alpha > 0$ , the Hilbert space  $(\mathcal{F}_D, \mathcal{E}^\alpha)$  can be decomposed as a direct sum:

$$\mathcal{F}_D = \mathcal{F}_D^{(0)} \oplus \mathcal{H}_\alpha,$$

$\mathcal{H}_\alpha$  being the closure of  $\{R_\alpha f, f \in \mathbf{B}(D)\}$  in this space.

In order to refine Theorem 4.2, let us introduce the space

$$(4.18) \quad \mathcal{F} = \{u \in \mathcal{F}_D; u \text{ is fine continuous quasi-everywhere on } D\}.$$

We will call this the refinement of the Dirichlet space  $\mathcal{F}_D$ . We have then

THEOREM 4.3. (i) For each function  $u$  of  $\mathcal{F}_D$ , there exists a function of  $\mathcal{F}$  which coincides with  $u$  almost everywhere on  $D$ .

(ii)  $\mathcal{F}^{(0)} \subset \mathcal{F}$  and  $\mathcal{E}(u, u) = \mathcal{E}^{(0)}(u, u)$ ,  $u \in \mathcal{F}^{(0)}$ ,  $\mathcal{F}^{(0)}$  being the refinement of the space  $\mathcal{F}_D^{(0)}$  ((3.1)).

(iii) For each  $\alpha > 0$ , the space  $(\mathcal{F}, \mathcal{E}^\alpha)$  is represented as

$$\mathcal{F} = \mathcal{F}^{(0)} \oplus \mathcal{H}_\alpha,$$

with  $\mathcal{H}_\alpha = \{u \in \mathcal{F}; u \text{ is } \alpha\text{-harmonic on } D\}$ .  $R_\alpha(\mathbf{B}(D))$  is dense in  $(\mathcal{H}_\alpha, \mathcal{E}^\alpha)$ .

PROOF. Let  $\mathcal{H}_\alpha$  be the space of Theorem 4.2 (ii). Any function in  $\mathcal{H}_\alpha$  is  $\alpha$ -harmonic<sup>16)</sup>, and so, continuous on  $D$ . Hence, in view of Theorem 3.1 (i) and Theorem 4.2, we can see that statements (i) and (ii) of the present theorem hold and that  $\mathcal{F} = \mathcal{F}^{(0)} \oplus \mathcal{H}_\alpha$ . Take any  $\alpha$ -harmonic function  $u$  of  $\mathcal{F}$  and decompose  $u$  as  $u = u^{(1)} + u^{(2)}$ ,  $u^{(1)} \in \mathcal{F}^{(0)}$ ,  $u^{(2)} \in \mathcal{H}_\alpha$ . Then,  $u^{(1)}$  is  $\alpha$ -harmonic and belongs to the space  $\text{BLD}_0$  (Theorem 3.1 (ii)). Hence,  $u^{(1)} \in \text{BLD}_0 \cap \widehat{\text{BLD}}_{\alpha, h}$  and  $u^{(1)} = 0$ . This proves the last assertion of Theorem 4.3.

16) Any  $u \in \mathcal{H}_\alpha$  is a limit of  $\alpha$ -harmonic functions  $R_\alpha f_n$ ,  $f_n \in \mathbf{B}(D)$ , in  $L^2(D)$ . Hence,  $R_\alpha f_n(x)$  converges to  $u(x)$  uniformly on each compact subset of  $D$  and  $u$  is  $\alpha$ -harmonic (see [15; Lemma 2.2]).

**§ 5. The Dirichlet space  $(\mathcal{F}_M, \mathcal{E}_M)$  induced by  $(\mathcal{H}_\alpha, \mathcal{E}^\alpha)$ .**

In this section, the Hilbert space  $(\mathcal{H}_\alpha, \mathcal{E}^\alpha)$  appeared in Theorem 4.3 will be identified with a Dirichlet space formed by functions on the Martin boundary  $M$ .

For this purpose, we will employ the next theorem due to T. Watanabe, which permits us to conclude that each function  $u$  of  $\mathcal{H}_\alpha$  has its boundary function  $\gamma u$  in  $L^2(M)'$  and that  $u = H_\alpha(\gamma u)$ .

Take any symmetric Brownian resolvent  $\{\hat{G}_\alpha, \alpha > 0\}$  (see the final part of section 1 for the definition) and consider its associated Dirichlet space  $(\hat{\mathcal{F}}, \hat{\mathcal{E}})$  relative to  $L^2(D)$  in the sense of section 2. Set  $\hat{\mathcal{H}}_\alpha = \{u \in \hat{\mathcal{F}}; u \text{ is } \alpha\text{-harmonic}\}$ . Then,

THEOREM 5.1 (T. Watanabe).

$$\hat{\mathcal{H}}_\alpha \subset \widehat{\text{BLD}}_{\alpha, h} \text{ and } \hat{\mathcal{E}}(u, u) \geq (u, u)_{D, 1} \text{ for } u \in \hat{\mathcal{H}}_\alpha.$$

Combining this with Theorem 3.4 (ii), we are led to

COROLLARY. Every function  $u$  of  $\hat{\mathcal{H}}_\alpha$  has its boundary function  $\gamma u$  in  $\mathbf{H}_M$  (consequently in  $L^2(M)'$ ) and  $u(x) = H_\alpha(\gamma u)(x)$ ,  $x \in D$ . Further we have, for  $u \in \hat{\mathcal{H}}_\alpha$ ,  $\hat{\mathcal{E}}^\alpha(u, u) \geq \mathbf{D}(\gamma u, \gamma u) + \mathbf{U}_\alpha(\gamma u, \gamma u)$ .

Let us sketch the proof of Theorem 5.1. Take a function  $u \in \hat{\mathcal{H}}_\alpha$  and set  $\hat{\mathcal{E}}_\beta^\alpha(u, u) = \beta(u - \beta \hat{G}_\beta u, u)_D$ . Then, by definition,  $\hat{\mathcal{E}}(u, u) = \lim_{\beta \rightarrow +\infty} \hat{\mathcal{E}}_\beta^\alpha(u, u) < +\infty$ . It suffices for us to derive the inequality  $\lim_{\beta \rightarrow +\infty} \hat{\mathcal{E}}_\beta^\alpha(u, u) \geq \left(-\frac{1}{4} \Delta(u^2), 1\right)_D - \alpha(u, u)_D$ , since the right-hand side is nothing but  $(u, u)_{D, 1}$ .  $\hat{\mathcal{E}}_\beta^\alpha(u, u)$  can be expressed as  $\hat{\mathcal{E}}_\beta^\alpha(u, u) = \frac{1}{2}(f_\beta, 1)_D$ , with  $f_\beta = 2\beta u(u - \beta \hat{G}_\beta u) - \beta(u^2 - \beta \hat{G}_\beta u^2) + \beta u^2(1 - \beta \hat{G}_\beta 1)_D$ . It is easy to see that  $f_\beta$  is a non-negative function on  $D$  for each  $\beta > 0$ . On the other hand,  $\{\hat{G}_\beta, \beta > 0\}$ , being a Brownian resolvent, has the following property. If both  $|g|$  and  $\hat{G}_\beta g$  are locally integrable, then  $\beta(g - \beta \hat{G}_\beta g) \xrightarrow{\beta \rightarrow +\infty} -\frac{1}{2} \Delta g$  in the sense of Schwartz's distribution. Therefore,  $f_\beta$  converges (as distribution) to  $2u\left(-\frac{1}{2} \Delta u\right) + \frac{1}{2} \Delta(u^2)$  and we have, for any  $h \in C_0^\infty(D)$  such as  $0 \leq h \leq 1$ ,  $\lim_{\beta \rightarrow +\infty} \hat{\mathcal{E}}_\beta^\alpha(u, u) \geq \frac{1}{2}\left(-\frac{1}{2} \Delta(u^2) - u \cdot \Delta u, h\right)_D$ . The desired inequality follows from this.

Now, let us consider the space  $(\mathcal{H}_\alpha, \mathcal{E}^\alpha)$ ,  $\alpha > 0$ , of Theorem 4.3. Our main assertions are as follows.

THEOREM 5.2. (i) For each  $\alpha > 0$ , any function  $u$  of  $\mathcal{H}_\alpha$  has its boundary function  $\gamma u$  in  $L^2(M)'$  and  $u = H_\alpha(\gamma u)$ .

(ii) The function space

$$(5.1) \quad \mathcal{F}_M = \gamma \mathcal{H}_\alpha = \{\varphi; \varphi = \gamma u, u \in \mathcal{H}_\alpha\}$$

is independent of  $\alpha > 0$ . Set for  $\varphi, \psi \in \mathcal{F}_M$ ,

$$(5.2) \quad \mathcal{E}_M^{[\alpha]}(\varphi, \psi) = \mathcal{E}^\alpha(H_\alpha \varphi, H_\alpha \psi)$$

$$(5.3) \quad \mathcal{E}_M(\varphi, \psi) = \mathcal{E}_M^{[\alpha]}(\varphi, \psi) - U_\alpha(\varphi, \psi)$$

then  $\mathcal{E}_M(\varphi, \psi)$  is independent of  $\alpha > 0$ ,

(iii) For each  $\alpha > 0$ , the space  $(\mathcal{F}_M, \mathcal{E}_M^{[\alpha]})$  is a Dirichlet space relative to  $L^2(M)'$ .

(iv) The space  $(\mathcal{F}_M, \mathcal{E}_M)$  is a Dirichlet space relative to  $L^2(M)'$  and it satisfies the conditions (B. 1), (B. 2) and (B. 3) stated in section 1. Moreover, the bilinear form  $N(\cdot, \cdot)$  in (B. 2) and (B. 3) is given by the following formula.

$$(5.4) \quad N(\varphi, \varphi) = \lim_{n \rightarrow +\infty} \lim_{\alpha \rightarrow +\infty} \lim_{\mu \rightarrow +\infty} \frac{1}{2} \mu^2 \int_M \int_M \tilde{R}_\mu^\alpha(d\xi, d\eta) (\varphi_n(\xi) - \varphi_n(\eta))^2.$$

Here,  $\tilde{R}^\alpha(d\xi, d\eta)$  is a Radon measure on  $M \times M$  satisfying

$$(5.5) \quad \int_M \int_M \tilde{R}_\mu^\alpha(d\xi, d\eta) \varphi(\xi) \cdot \psi(\eta) = (\tilde{R}_\mu^\alpha \varphi, \psi)'_M,$$

$\varphi, \psi \in L^2(M)'$ , for the symmetric resolvent  $\{\tilde{R}_\mu^\alpha, \mu > 0\}$  on  $L^2(M)'$  associated with the Dirichlet space  $(\mathcal{F}_M, \mathcal{E}_M^{[\alpha]})$ .  $\varphi_n$  is a truncation of  $\varphi \in F_M: \varphi = (\varphi \wedge n) \vee (-n)$ .

PROOF OF THEOREM 5.2. The first assertion is involved in Corollary to Theorem 5.1, since  $\mathcal{H}_\alpha$  of Theorem 4.3 consists of all  $\alpha$ -harmonic functions in  $(\mathcal{F}_D, \mathcal{E})$ , which is a Dirichlet space associated with a symmetric Brownian resolvent having a density function in  $G$ .

The first part of (ii) is a consequence of Theorem 4.3 and Corollary to Theorem 3.1. Indeed, we have  $\mathcal{F}_M = \gamma(\mathcal{F}^{(0)} \oplus \mathcal{H}_\alpha) = \gamma \mathcal{F}$ , which is independent of  $\alpha > 0$ .

Now, let us prove the remaining assertions of Theorem 5.2 by a series of Lemmas.

The second part of statement (ii) is contained in Lemma 5.1. The third assertion will be proved in Lemma 5.3 by making use of Lemma 5.2. The last assertion is just Lemma 5.4.

LEMMA 5.1. (i) If  $\varphi_n \in \mathcal{F}_M$  converges to  $\varphi \in \mathcal{F}_M$  in norm  $\mathcal{E}_M^{[\alpha]}$ , then  $U_\alpha(\varphi_n, \varphi_n)$  converges to  $U_\alpha(\varphi, \varphi)$ .

(ii)  $\mathcal{E}_M(\varphi, \psi)$  defined by (5.3) for  $\varphi, \psi \in \mathcal{F}_M$  is independent of  $\alpha > 0$ .

(iii)  $1_M \in \mathcal{F}_M$  and  $\mathcal{E}_M(1_M, \varphi) = 0$  for any  $\varphi \in \mathcal{F}_M$ .

PROOF. (i) From the definition of  $\mathcal{E}_M^{[\alpha]}(\cdot, \cdot)$ , it follows that  $H_\alpha \varphi_n$  converges to  $H_\alpha \varphi$  in  $L^2(D)$ . Then, on account of identity (3.19) and the estimate  $(u, G_{0+}^0 u)_D \leq \sup_{x \in D} G_{0+}^0 1(x) \cdot (u, u)_D$  for  $u \in L^2(D)$ , we can see that statement (i) is valid.

17) Cf. footnote 16).

(ii) The desired identity is

$$\mathcal{E}_M^{[\alpha]}(\varphi, \varphi) - U_\alpha(\varphi, \varphi) = \mathcal{E}_M^{[\beta]}(\varphi, \varphi) - U_\beta(\varphi, \varphi)$$

for  $\alpha, \beta > 0$  and  $\varphi \in \mathcal{F}_M$ . Set  $\tilde{\mathcal{R}} = \gamma(R_\alpha(\mathbf{B}(D)))$ , then  $\tilde{\mathcal{R}} = \gamma(G_\alpha(\mathbf{B}(D)))$  and this is independent of  $\alpha > 0$ . Further,  $\tilde{\mathcal{R}}$  is dense in the space  $(\mathcal{F}_M, \mathcal{E}_M^{[\alpha]})$  for an arbitrary  $\alpha > 0$ , since  $R_\alpha(\mathbf{B}(D))$  is dense in  $(\mathcal{H}_\alpha, \mathcal{E}^\alpha(\cdot, \cdot))$ . Therefore, taking into account of the first assertion of this lemma, it suffices for us to show the above identity for  $\varphi \in \tilde{\mathcal{R}}$ . Let  $\varphi$  be  $\gamma(R_\alpha f)$  with an  $\alpha > 0$  and an  $f \in \mathbf{B}(D)$ . Then, it holds that

$$(5.6) \quad \begin{aligned} \mathcal{E}_M^{[\alpha]}(\varphi, \psi) &= \mathcal{E}^\alpha(R_\alpha f, H_\alpha \psi) = \mathcal{E}^\alpha(G_\alpha f, H_\alpha \psi) \\ &= (f, H_\alpha \psi)_D = (\hat{H}_\alpha f, \psi)'_M \quad \text{for any } \psi \in \mathcal{F}_M. \end{aligned}$$

On the other hand, the resolvent equation for  $G_\alpha$  implies that  $\varphi$  can be expressed as  $\gamma(R_\beta g)$  with  $\beta > 0$  and

$$(5.7) \quad g = f + (\beta - \alpha)G_\alpha^0 f + (\beta - \alpha)H_\alpha \varphi.$$

Hence, equations (4.5), (4.6) and (5.6) lead us to

$$\begin{aligned} \mathcal{E}_M^{[\beta]}(\varphi, \varphi) - U_\beta(\varphi, \varphi) &= (\hat{H}_\beta g, \varphi)'_M - U_\beta(\varphi, \varphi) \\ &= (\hat{H}_\alpha f, \varphi)'_M - U_\alpha(\varphi, \varphi) = \mathcal{E}_M^{[\alpha]}(\varphi, \varphi) - U_\alpha(\varphi, \varphi). \end{aligned}$$

(iii) From equation (5.6) and the identity  $\gamma(\alpha R_\alpha 1_D) = \gamma(\alpha G_\alpha 1_D) = 1_M$ , we have  $\mathcal{E}_M^{[\alpha]}(1_M, \psi) = \alpha(\hat{H}_\alpha 1_D, \psi)'_M = \alpha(H 1_M, H_\alpha \psi)_D = U_\alpha(1_M, \psi)$ ,  $\psi \in \mathcal{F}_M$ . The proof of Lemma 5.1 is complete.

Next, set for  $\alpha, \lambda > 0$  and  $u, v \in \mathcal{F}$ ,

$$(5.8) \quad \mathcal{E}^{\alpha, \lambda}(u, v) = \mathcal{E}^\alpha(u, v) + \lambda(\gamma u, \gamma v)'_M.$$

LEMMA 5.2. *Let us consider the space  $(\mathcal{F}, \mathcal{E}^{\alpha, \lambda}(\cdot, \cdot))$  with  $\alpha > 0$  and  $\lambda > 0$  fixed.*

- (i) *It is a real Hilbert space.*
- (ii) *It is decomposed as a direct sum:*

$$\mathcal{F} = \mathcal{F}^{(0)} \oplus \mathcal{H}_\alpha.$$

*Especially  $\mathcal{H}_\alpha$  is a closed subspace.*

(iii) *If a function  $v$  on  $D$  is a normal contraction of a function  $u \in \mathcal{F}$ , then  $v \in \mathcal{F}$  and  $\mathcal{E}^{\alpha, \lambda}(v, v) \leq \mathcal{E}^{\alpha, \lambda}(u, u)$ .*

PROOF. Theorem 4.3 and Corollary to Theorem 3.1 imply that each element  $u$  of  $\mathcal{F}$  is a sum of functions  $u^{(0)} \in \mathcal{F}^{(0)}$  and  $u^{(1)} \in \mathcal{H}_\alpha$  and that  $\mathcal{E}^{\alpha, \lambda}(u^{(0)}, u^{(1)}) = \mathcal{E}^\alpha(u^{(0)}, u^{(1)}) + \lambda(\gamma u^{(0)}, \gamma u^{(1)})'_M = 0$ .

Since  $\mathcal{E}^{\alpha, \lambda}(u, u) = \mathcal{E}^\alpha(u, u)$  for  $u \in \mathcal{F}^{(0)}$ , the space  $\mathcal{F}^{(0)}$  is closed in norm  $\mathcal{E}^{\alpha, \lambda}$ . Therefore, for the proof of assertions (i) and (ii), it suffices to show that  $\mathcal{H}_\alpha$

is complete with metric  $\mathcal{E}^{\alpha, \lambda}$ . Suppose that  $\{u_n\}$  forms a Cauchy sequence in  $\{\mathcal{H}_\alpha, \mathcal{E}^{\alpha, \lambda}\}$ . Then,  $u_n$  converges to a function  $u \in \mathcal{H}_\alpha$  with metric  $\mathcal{E}^\alpha$  and  $\gamma u_n$  converges in  $L^2(M)'$ -sence to a function  $\varphi$ . Since  $u_n$  converges in  $L^2(D)$ -sence, the convergence is the pointwise sence<sup>18)</sup>. On the other hand,  $u_n(x) = H_\alpha(\gamma u_n)(x) \xrightarrow{n \rightarrow +\infty} H_\alpha \varphi(x)$  for each  $x \in D$ . Hence  $u = H_\alpha \varphi$  and  $\gamma u = \varphi$ . The last statement of Lemma 5.2 follows from the facts that  $(\mathcal{F}, \mathcal{E})$  is a Dirichlet space and that  $|\gamma v(\xi)| \leq |\gamma u(\xi)|$  for  $\mu$ -almost all  $\xi \in M$ .

We will mention here the consequences of Lemma 5.2. Let  $\varphi$  be in  $L^2(M)'$ . Owing to Lemma 5.2 (i), there exists a unique element  $u_\varphi^{\alpha, \lambda}$  of  $\mathcal{F}$  such that the equation

$$(5.9) \quad \mathcal{E}^{\alpha, \lambda}(u_\varphi^{\alpha, \lambda}, v) = (\varphi, \gamma v)'_M \quad \text{holds for all } v \in \mathcal{F}.$$

By virtue of Lemma 5.2 (ii), we can conclude that

$$(5.10) \quad u_\varphi^{\alpha, \lambda} \in \mathcal{H}_\alpha,$$

since (5.9) implies  $\mathcal{E}^{\alpha, \lambda}(u_\varphi^{\alpha, \lambda}, v) = 0$  for all  $v \in \mathcal{F}^{(0)}$ . Furthermore,  $u_\varphi^{\alpha, \lambda}$  enjoys the property:

$$(5.11) \quad 0 \leq u_\varphi^{\alpha, \lambda} \leq 1 \quad \text{if } 0 \leq \varphi \leq 1.$$

We can see this from the final statement of Lemma 5.2 and the fact that  $u_\varphi^{\alpha, \lambda}$  is the unique element of  $\mathcal{F}$  minimizing the functional  $\Phi(v) = \mathcal{E}^\alpha(v, v) + \lambda(\gamma v - \frac{1}{\lambda} \varphi, \gamma v - \frac{1}{\lambda} \varphi)'_M$ .

Set, for  $\varphi \in L^2(M)'$ ,

$$(5.12) \quad \tilde{R}_\lambda^\alpha \varphi = \gamma u_\varphi^{\alpha, \lambda} (\in \mathcal{F}_M),$$

then we have

LEMMA 5.3. Fix an  $\alpha > 0$ .

(i) For each  $\lambda > 0$  and  $\varphi \in L^2(M)'$ ,  $\tilde{R}_\lambda^\alpha \varphi$  defined by (5.12) is the unique element of  $\mathcal{F}_M$  for which the equation

$$(5.13) \quad \mathcal{E}_M^{[\alpha]}(\tilde{R}_\lambda^\alpha \varphi, \psi) + \lambda(\tilde{R}_\lambda^\alpha \varphi, \psi)'_M = (\varphi, \psi)'_M$$

holds for every  $\psi \in \mathcal{F}_M$ .

(ii)  $\{\tilde{R}_\lambda^\alpha, \lambda > 0\}$  is a symmetric resolvent on  $L^2(M)'$  (see Definition 2.1.).

(iii)  $(\mathcal{F}_M, \mathcal{E}_M^{[\alpha]})$  is just the Dirichlet space relative to  $L^2(M)'$  associated with the above resolvent. In other words,  $\varphi \in L^2(M)'$  is an element of  $\mathcal{F}_M$  if and only if  $\lim_{\mu \rightarrow +\infty} \mathcal{E}_{M, \mu}^{[\alpha]}(\varphi, \varphi)$  is finite, and in this case the limit necessarily coincides with  $\mathcal{E}_M^{[\alpha]}(\varphi, \varphi)$ . Here,

$$(5.14) \quad \mathcal{E}_{M, \mu}^{[\alpha]}(\varphi, \psi) = \mu(\varphi - \mu \tilde{R}_\mu^\alpha \varphi, \psi)'_M.$$

18) Cf. footnote 16).

PROOF. (i) By (5.9), (5.10) and (5.12), the equation  $\mathcal{E}^{\alpha, \lambda}(H_\alpha \tilde{R}_\lambda^\alpha \varphi, H_\alpha \psi) = (\varphi, \psi)'_M$  holds for every  $\psi \in \mathcal{F}_M$ . Rewrite the left-hand side to obtain (5.13), which obviously characterize  $\tilde{R}_\lambda^\alpha \varphi$  in  $\mathcal{F}_M$ .

(ii) In view of (5.13),  $\tilde{R}_\lambda^\alpha$  is a bounded linear operator on  $L^2(M)'$ . Further, by (5.11), we have  $\lambda \tilde{R}_\lambda^\alpha 1 \leq 1$  and  $\tilde{R}_\lambda^\alpha \varphi \geq 0$  for  $\varphi \geq 0$ . Symmetry and the resolvent equation for  $\{\tilde{R}_\lambda^\alpha, \lambda > 0\}$  follow from assertion (i).

(iii) Note that, for each  $\lambda > 0$ , the space  $(\mathcal{F}_M, \mathcal{E}_M^{[\alpha]}(\cdot) + \lambda(\cdot, \cdot)'_M)$  is a real Hilbert space, since the space  $(\mathcal{A}_\alpha, \mathcal{E}^{\alpha, \lambda})$  is. Identity (5.13) for the resolvent  $\{\tilde{R}_\lambda^\alpha, \lambda > 0\}$  now implies assertion (iii).

LEMMA 5.4. (i) For  $\varphi \in \mathcal{F}_M$ ,  $\mathcal{E}_M(\varphi, \varphi)$  is expressed as  $\mathcal{E}_M(\varphi, \varphi) = \mathbf{D}(\varphi, \varphi) + \mathbf{N}(\varphi, \varphi)$  with  $\mathbf{D}(\varphi, \varphi)$  and  $\mathbf{N}(\varphi, \varphi)$  defined by (3.14) and (5.4) respectively. In particular,  $\mathcal{F}_M$  is a linear subspace of  $\mathbf{H}_M$ .

(ii)  $\mathcal{F}_M$  contains constant functions and  $\mathbf{N}(1, 1) = 0$ .

(iii) For each  $\lambda > 0$ ,  $(\mathcal{F}_M, \mathcal{E}_M(\cdot) + \lambda(\cdot, \cdot)'_M)$  is a real Hilbert space.

(iv) If  $\varphi$  is a normal contraction of  $\psi \in \mathcal{F}_M$ , then  $\varphi \in \mathcal{F}_M$  and  $\mathbf{N}(\varphi, \varphi) \leq \mathbf{N}(\psi, \psi)$ .

PROOF. (i) Take  $\varphi$  in  $\mathcal{F}_M$  and define  $\varphi_n$  by  $\varphi_n = (\varphi \wedge n) \vee (-n)$ ,  $n = 1, 2, \dots$ . Then,  $\varphi_n \in \mathcal{F}_M$  and

$$(5.15) \quad \lim_{n \rightarrow +\infty} \mathcal{E}_M(\varphi_n, \varphi_n) = \mathcal{E}_M(\varphi, \varphi).$$

Indeed, for any  $\alpha > 0$ ,  $(\mathcal{F}_M, \mathcal{E}_M^{[\alpha]})$  is a Dirichlet space (Lemma 5.3 (iii)) and therefore Lemma 2.1 implies that  $\varphi_n \in \mathcal{F}_M$  and  $\lim_{n \rightarrow +\infty} (\mathcal{E}_M(\varphi_n, \varphi_n) + \mathbf{U}_\alpha(\varphi_n, \varphi_n)) = \mathcal{E}_M(\varphi, \varphi) + \mathbf{U}_\alpha(\varphi, \varphi)$ . On the other hand,  $\mathbf{U}_\alpha(\varphi_n, \varphi_n)$  converges to  $\mathbf{U}_\alpha(\varphi, \varphi)$  because of identity (3.20).

We will compute  $\mathcal{E}_M(\varphi_n, \varphi_n)$ . Owing to Lemma 5.3 (iii), it holds that

$$(5.16) \quad \mathcal{E}_M(\varphi_n, \varphi_n) = \lim_{\mu \rightarrow +\infty} \mathcal{E}_{M, \mu}^{[\alpha]}(\varphi_n, \varphi_n) - \mathbf{U}_\alpha(\varphi_n, \varphi_n)$$

with  $\mathcal{E}_{M, \mu}^{[\alpha]}$  defined by (5.14). The right-hand side of (5.16) is independent of  $\alpha > 0$  (Lemma 5.1 (ii)). Rewrite  $\mathcal{E}_{M, \mu}^{[\alpha]}(\varphi_n, \varphi_n)$  as<sup>19)</sup>

$$(5.17) \quad \mathcal{E}_{M, \mu}^{[\alpha]}(\varphi_n, \varphi_n) = \frac{1}{2} \mu^2 \int_M \int_M \tilde{R}_\mu^\alpha(d\xi, d\eta) (\varphi_n(\xi) - \varphi_n(\eta))^2 + \mu(1 - \mu \tilde{R}_\mu^\alpha 1, \varphi_n^2)'_M.$$

On account of Lemma 2.1 and Lemma 5.1 (iii), we see that  $\varphi_n^2 \in \mathcal{F}_M$  and

$$(5.18) \quad \mathcal{E}_{M, \mu}^{[\alpha]}(1, \varphi_n^2) = (1 - \mu \tilde{R}_\mu^\alpha 1, \varphi_n^2)'_M \xrightarrow{\mu \rightarrow +\infty} \mathcal{E}_M(1, \varphi_n^2) + \mathbf{U}_\alpha(1, \varphi_n^2) = \mathbf{U}_\alpha(1, \varphi_n^2).$$

Combining (5.16) with (5.17) and (5.18) and employing equality (3.26), we arrive at

19) Cf. [2].

$$(5.19) \quad \mathcal{E}_M(\varphi_n, \varphi_n) = \frac{1}{2} \int_M \int_M (\varphi_n(\xi) - \varphi_n(\eta))^2 U_\alpha(\xi, \eta) \mu(d\xi) \mu(d\eta) \\ + \lim_{\mu \rightarrow +\infty} \mu^2 \int_M \int_M \tilde{R}_\mu^\alpha(d\xi, d\eta) (\varphi_n(\xi) - \varphi_n(\eta))^2$$

for each  $n$  and  $\alpha > 0$ .

Statement (i) of our lemma can be derived from (5.19) by letting  $\alpha$  and then  $n$  tend to infinity.

(ii) This assertion is immediate from Lemma 5.1 (iii) and formula (5.4).

(iii) By Lemma 5.3 (iii), the space  $\mathcal{F}_M$  is complete with metric  $\mathcal{E}_M(\cdot, \cdot) + U_\alpha(\varphi, \varphi) + \lambda(\cdot, \cdot)_M$  for  $\alpha > 0$ . In view of inequality (3.17), we arrive at conclusion (iii).

(iv) This is a consequence of Lemma 5.3 (iii) and formula (5.4).

The proof of Theorem 5.2 is now complete. We should point out here that the space  $(\mathcal{F}_M, \mathcal{E}_M)$  characterizes our resolvent density. Precisely,

**THEOREM 5.3.** *Consider two elements  $G_\alpha^{(i)}(x, y)$  of the class  $\mathbf{G}$ ,  $i=1, 2$ . We associate the space  $(\mathcal{F}_M^{(i)}, \mathcal{E}_M^{(i)})$  with  $G_\alpha^{(i)}(x, y)$  by means of Theorem 5.2,  $i=1, 2$ . Assume that  $(\mathcal{F}_M^{(1)}, \mathcal{E}_M^{(1)}) = (\mathcal{F}_M^{(2)}, \mathcal{E}_M^{(2)})$ , then  $G_\alpha^{(1)}(x, y) = G_\alpha^{(2)}(x, y)$ ,  $\alpha > 0$ ,  $x, y \in D$ .*

**PROOF.** Let  $(\mathcal{F}^{(i)}, \mathcal{E}^{(i), \alpha})$  and  $(\mathcal{H}^{(i)}, \mathcal{E}^{(i), \alpha})$  be the spaces of Theorem 4.3 associated with  $G_\alpha^{(i)}(x, y)$ ,  $i=1, 2$ . We have by assumption  $\mathcal{H}_\alpha^{(1)} = H_\alpha(\mathcal{F}_M^{(1)}) = H_\alpha(\mathcal{F}_M^{(2)}) = \mathcal{H}_\alpha^{(2)}$  and  $\mathcal{E}^{(1), \alpha}(u, u) = \mathcal{E}_M^{(1)}(\gamma u, \gamma u) + U_\alpha(\gamma u, \gamma u) = \mathcal{E}_M^{(2)}(\gamma u, \gamma u) + U_\alpha(\gamma u, \gamma u) = \mathcal{E}^{(2), \alpha}(u, u)$  for  $u \in \mathcal{H}_\alpha^{(1)}$ . By Theorem 4.3, we see that  $(\mathcal{F}^{(1)}, \mathcal{E}^{(1), \alpha}) = (\mathcal{F}^{(2)}, \mathcal{E}^{(2), \alpha})$  and that, for every  $u, v \in L^2(D)$ ,  $(G_\alpha^{(1)}u, v)_D = \mathcal{E}^{(2), \alpha}(G_\alpha^{(1)}u, G_\alpha^{(2)}v) = \mathcal{E}^{(1), \alpha}(G_\alpha^{(1)}u, G_\alpha^{(2)}v) = (u, G_\alpha^{(2)}v)_D$ , from which the conclusion of Theorem 5.3 follows.

## § 6. Boundary condition.

In the preceding two sections, we have investigated the structure of the Dirichlet space  $(\mathcal{F}_D, \mathcal{E})$  associated with a given element  $G_\alpha(x, y)$  in  $\mathbf{G}$ . Consider the space  $(\widehat{\text{BLD}}, (\cdot, \cdot)_{D,1})$  in section 3. On the ground of Theorem 3.1, 3.4, 4.3, 5.1 and 5.2, we can state the relation of the refinement  $\mathcal{F}^{(20)}$  of  $\mathcal{F}_D$  to the space  $\widehat{\text{BLD}}$  as follows.

**THEOREM 6.1.** (i)  $\text{BLD}_0 \subset \mathcal{F} \subset \widehat{\text{BLD}}$ ,

(ii) *Each function  $u$  of  $\mathcal{F}$  has its boundary function  $\gamma u$  in  $\mathbf{H}_M$  and it holds that*

$$(6.1) \quad \mathcal{E}(u, u) = (u, u)_{D,1} + N(\gamma u, \gamma u),$$

with a bilinear non-negative form  $N$  on  $\gamma\mathcal{F}$ . Moreover, if  $v$  is a normal contraction of  $u \in \mathcal{F}$ , then  $v \in \mathcal{F}$  and  $N(\gamma v, \gamma v) \leq N(\gamma u, \gamma u)$ .

**PROOF.** The first assertion is immediate from Theorem 3.1, 4.3 and 5.1.

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20) See (4.18).

Indeed, for a fixed  $\alpha > 0$ , the Hilbert space  $(\mathcal{F}, \mathcal{E}^\alpha)$  is a direct sum of the space  $\mathcal{F}^{(0)} = \text{BLD}_0$  and the space  $\mathcal{H}_\alpha$ , the latter being a subspace of  $\widehat{\text{BLD}}_{\alpha,h}$ . In order to prove equality (6.1), decompose  $u \in \mathcal{F}$  as  $u = u_1 + u_2$ ,  $u_1 \in \mathcal{F}^{(0)}$ ,  $u_2 \in \mathcal{H}_\alpha$ . Then,

$$(6.2) \quad (u_1, u_2)_{D,1} + \alpha(u_1, u_2)_D = 0.$$

By Theorem 3.1 and 4.3 (ii),

$$(6.3) \quad \mathcal{E}^\alpha(u_1, u_1) = (u_1, u_1)_{D,1} + \alpha(u_1, u_1)_D.$$

Combining Theorem 5.2 with Theorem 3.4 (iii), we see that  $u_2$  has its boundary function  $\gamma u_2 = \gamma u$  in  $\mathbf{H}_M$  and

$$(6.4) \quad \mathcal{E}^\alpha(u_2, u_2) = (u_2, u_2)_{D,1} + \alpha(u_2, u_2)_D + N(\gamma u, \gamma u).$$

Formula (6.2), (6.3) and (6.4) lead us to equality (6.1). The properties of  $N$  stated in this theorem are implied in Theorem 5.2.

Our next task is concerned with an expression of the boundary condition for the class  $\mathcal{G}$ .

DEFINITION 6.1. If, for a function  $u \in \widehat{\text{BLD}}$ , there exists an  $f \in L^2(D)$  such that the equation

$$(6.5) \quad (u, v)_{D,1} = (f, v)_D$$

holds for every  $v \in \text{BLD}_0$ , then we will write

$$(6.6) \quad -\frac{1}{2}\Delta u = -f.$$

The set of functions  $u$  satisfying the above property will be denoted by  $\mathcal{D}(\Delta)$ . We call such  $\Delta$  the *generalized Laplacian* with domain  $\mathcal{D}(\Delta)$ .

We notice that the equation (6.5) holds for all  $v \in \text{BLD}_0$  if and only if it does for all  $v \in C_0^\infty(D)$  (see the paragraph following Definition 3.2).

Thus,  $u$  is an element of  $\mathcal{D}(\Delta)$  if and only if  $u \in \widehat{\text{BLD}}$  and  $\sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} u$  in the sense of Schwartz's distribution is a function of  $L^2(D)$ . The notion  $\Delta$  in (6.6) is nothing but the Laplacian in the distribution sense.

For a given element  $G_\alpha(x, y)$  of  $\mathcal{G}$ , let us put

$$(6.7) \quad \mathcal{D} = G_\alpha(L^2(D)) = \left\{ u ; u = G_\alpha f = \int_D G_\alpha(\cdot, y) f(y) dy, \quad f \in L^2(D) \right\}.$$

The space  $\mathcal{D}$  does not depend on  $\alpha > 0$ . Let  $\mathcal{F}_M$  be the space of Theorem 5.2 and  $N$ , the form of (5.4). The next theorem will characterize the space  $\mathcal{D}$  (and consequently, the element of  $\mathcal{G}$ ).

THEOREM 6.2. A function  $u$  belongs to  $\mathcal{D}$  if and only if

- (1)  $u \in \mathcal{D}(\Delta)$ , and
- (2)  $u$  has its boundary function  $\gamma u$  in  $\mathcal{F}_M$  and it satisfies, for every  $\phi \in \mathcal{F}_M$ ,



$$(6.8) \quad \mathbf{D}(\gamma u, \phi) + \mathbf{N}(\gamma u, \phi) + \left(-\frac{1}{2}\Delta u, H\phi\right)_D = 0.$$

PROOF. Take a function  $u$  of  $\mathcal{D}$ . For  $\alpha > 0$ ,  $u$  is equal to  $G_\alpha f = G_\alpha^0 f + R_\alpha f$  with an  $f \in L^2(D)$ . The function  $u$  is an element of  $\mathcal{F}$  and we have  $\mathcal{E}^\alpha(u, v) = \mathcal{E}(u, v) + \alpha(u, v)_D = \mathcal{E}^{(0), \alpha}(G_\alpha^0 f, v) = (f, v)_D$  for every  $v \in \text{BLD}_0$ . On the other hand, according to the preceding theorem,  $u$  belongs to  $\widehat{\text{BLD}}$  and  $\mathcal{E}(u, v) = (u, v)_{D,1}$  for every  $v \in \text{BLD}_0$ . Therefore,  $u \in \mathcal{D}(\Delta)$  and

$$(6.9) \quad -\frac{1}{2}\Delta u = \alpha u - f.$$

Next, by making use of the identity

$$(6.10) \quad \mathcal{E}^\alpha(u, H_\alpha \phi) = \mathcal{E}^\alpha(R_\alpha f, H_\alpha \phi), \quad \phi \in \mathcal{F}_M,$$

we will derive formula (6.8). The left-hand side of (6.10) is equal to  $(f, H_\alpha \phi)_D$  and the right-hand side can be expressed in terms of  $\gamma u = \gamma(R_\alpha f)$  as  $\mathbf{D}(\gamma u, \phi) + U_\alpha(\gamma u, \phi) + \mathbf{N}(\gamma u, \phi)$  (Theorem 5.2). Hence it suffices to show

$$(6.11) \quad (f, H_\alpha \phi)_D - U_\alpha(\gamma u, \phi) = -\left(\frac{1}{2}\Delta u, H\phi\right)_D.$$

Note that  $(|g|, |H\phi|)_D$  is finite for  $g \in L^2(D)$  and  $\phi \in L^2(M)'$ . In fact, it is no greater than  $(|g|, |H_\alpha \phi|)_D + \alpha(|g|, G_{0+}^0 |H_\alpha \phi|)_D$ , which is finite because  $H_\alpha \phi \in L^2(D)$  (see (3.19)) and  $(|g|, G_{0+}^0 |H_\alpha \phi|)_D^2 \leq (\sup_{x \in \bar{D}} G_{0+}^0 1(x))^2 (g, g)_D (H_\alpha \phi, H_\alpha \phi)_D$ . Equation (6.11) now follows from (6.9) and a formal computation as follows:

$$(f, H_\alpha \phi)_D - U_\alpha(\gamma u, \phi) = (f, H\phi)_D - \alpha(G_\alpha^0 f, H\phi)_D - \alpha(R_\alpha f, H\phi)_D.$$

Conversely, suppose that a function  $u$  satisfies conditions (1) and (2) of our theorem. Set  $f = \alpha u - \frac{1}{2}\Delta u$ ,  $v = G_\alpha f$  and  $w = u - v$ . Then, we have  $\frac{1}{2}\Delta w = \alpha w$  or equivalently,

$$(6.12) \quad (w, v')_{D,1} + \alpha(w, v')_D = 0 \quad \text{for every } v' \in \text{BLD}_0.$$

Hence,  $w \in \widehat{\text{BLD}}_{\alpha, h}$  and  $w = H_\alpha(\gamma w)$  (Theorem 3.4). However,  $w$  satisfies the condition (6.8) for all  $\phi \in \mathcal{F}_M$ . Set  $\phi = \gamma w$ . Then we have  $U_\alpha(\gamma w, \gamma w) = 0$  which implies that  $w = H_\alpha(\gamma w) = 0$  in view of identity (3.19). Thus,  $u$  must be an element of  $\mathcal{D}$ .

## § 7. Construction of the symmetric resolvent density.

In the present section we are concerned with the converse problem to that of sections 4 and 5. For a given space  $(\mathcal{F}_M, \mathcal{E}_M)$  described just below, does there its associated resolvent density  $G_\alpha(x, y)$  of  $G$  exist? The answer is affirmative.

Define the bilinear form  $D$  and the function space  $H_M$  by (3.14) and (3.15) respectively. We start with a function space  $\mathcal{F}_M$  and a non-negative symmetric bilinear form  $N$  on  $\mathcal{F}_M$  satisfying the following conditions: (B. 1)  $\mathcal{F}_M$  is a linear subspace of  $H_M$  and it contains constant functions, (B. 2)  $\{\mathcal{F}_M, D(\cdot, \cdot) + N(\cdot, \cdot)\}$  is a Dirichlet space relative to  $L^2(M)'$  and  $N(1, 1) = 0$  and (B. 3) if  $\phi$  is a normal contraction of  $\varphi \in \mathcal{F}_M$ , then  $\phi \in \mathcal{F}_M$  and  $N(\phi, \phi) \leq N(\varphi, \varphi)$ .

For  $\varphi$  and  $\psi \in \mathcal{F}_M$ , set

$$(7.1) \quad \mathcal{E}_M(\varphi, \psi) = D(\varphi, \psi) + N(\varphi, \psi),$$

$$(7.2) \quad \mathcal{E}_M^{[\alpha]}(\varphi, \psi) = \mathcal{E}_M(\varphi, \psi) + U_\alpha(\varphi, \psi), \quad \alpha > 0.$$

By the assumption, the space  $\mathcal{F}_M$  is complete with the metric  $\mathcal{E}_M(\cdot, \cdot) + \lambda(\cdot, \cdot)'_M$  for each  $\lambda > 0$ . On the other hand, inequality (3.18) leads us to

$$(7.3) \quad (\varphi, \varphi)'_M \leq \left(1 \vee \frac{1}{\alpha}\right) \mathcal{E}_M^{[\alpha]}(\varphi, \varphi),$$

$$\varphi \in \mathcal{F}_M, \alpha > 0.$$

By virtue of inequality (7.3) and Lemma 3.1 (ii),  $\mathcal{E}_M^{[\alpha]}$  defines a metric on  $\mathcal{F}_M$  equivalent to  $\mathcal{E}_M(\cdot, \cdot) + \lambda(\cdot, \cdot)'_M$ ,  $\lambda > 0$ . Hence, the space  $(\mathcal{F}_M, \mathcal{E}_M^{[\alpha]})$  is a real Hilbert space for each  $\alpha > 0$ . Further, (7.3) implies

LEMMA 7.1. Fix an  $\alpha > 0$ . For each  $\varphi \in L^2(M)'$ , there is a unique element  $\tilde{R}^\alpha \varphi \in \mathcal{F}_M$  such that

$$(7.4) \quad \mathcal{E}_M^{[\alpha]}(\tilde{R}^\alpha \varphi, \psi) = (\varphi, \psi)'_M$$

for every  $\psi \in \mathcal{F}_M$ .

For  $\alpha > 0$  and  $y \in D$ , the function  $K'_\alpha(y, \xi)$  defined by (4.2) is in  $B(M)$  and so in  $L^2(M)'$  as a function of  $\xi \in M$  (Lemma 3.1 (i)).

DEFINITION 7.1. For  $x, y \in D$  and  $\alpha > 0$ , set

$$(7.5) \quad R_\alpha(x, y) = H_\alpha^x \tilde{R}^\alpha \hat{H}_\alpha^y$$

with  $\tilde{R}^\alpha$  of the preceding lemma (see section 4 for notations  $H_\alpha^x$  and  $\hat{H}_\alpha^y$ ). Further, we set

$$(7.6) \quad G_\alpha(x, y) = G_\alpha^0(x, y) + R_\alpha(x, y)$$

with above  $R_\alpha$  and the resolvent density  $G_\alpha^0$  of the absorbing barrier Brownian motion on  $D$ .

We will show the following theorem.

THEOREM 7.1. Suppose that a function space  $\mathcal{F}_M$  and a non-negative definite symmetric bilinear form  $N$  on  $\mathcal{F}_M$  satisfying conditions (B. 1), (B. 2) and (B. 3) are given. Then, the following statements hold.

(i)  $G_\alpha(x, y)$  defined by Definition 7.1 is an element of  $\mathbf{G}$ ; it is a conservative, symmetric resolvent density satisfying conditions (G. a) and (G. b).

(ii) Let  $(\mathcal{F}_D, \mathcal{E})$  be the Dirichlet space relative to  $L^2(D)$  associated with this  $G_\alpha(x, y)$ . For each  $\alpha > 0$ , decompose  $\{\mathcal{F}_D, \mathcal{E}^\alpha(\cdot) = \mathcal{E}(\cdot) + \alpha(\cdot)_D\}$  as  $\mathcal{F}_D = \mathcal{F}_D^{(0)} + \mathcal{H}_\alpha$  by means of Theorem 4.2.

Then, we have

$$\gamma \mathcal{H}_\alpha = \mathcal{F}_M \quad \text{and} \quad \mathcal{E}^\alpha(u, v) = \mathcal{E}_M^{\lceil \alpha \rceil}(\gamma u, \gamma v) \quad \text{for } u, v \in \mathcal{H}_\alpha.$$

Owing to Theorem 6.2, we obtain

COROLLARY TO THEOREM 7.1. Under the assumption of Theorem 7.1, there exists a unique element  $G_\alpha(x, y)$  of  $\mathbf{G}$  such that every function of  $G_\alpha(L^2(D))$  satisfies the boundary condition (6.8).

Before proceeding to the proof of Theorem 7.1, we prepare two lemmas. For  $\varphi \in \mathbf{B}(M)$  and  $\alpha > 0$ , we set

$$(7.7) \quad U'_\alpha \varphi(\xi) = \begin{cases} U_\alpha \varphi(\xi) / U_1 1(\xi) & \text{if } U_1 1(\xi) < +\infty, \\ 0 & \text{if } U_1 1(\xi) = +\infty. \end{cases}$$

Following the argument in the proof of Lemma 4.1,

$$(7.8) \quad U'_\alpha \varphi(\xi) = \alpha \hat{H}_\alpha(H\varphi)(\xi) \quad \text{for } \mu\text{-almost all } \xi \in M.$$

Further we have easily

$$(7.9) \quad 1_M \leq \left(1 \vee \frac{1}{\alpha}\right) U'_\alpha 1_M \quad \mu\text{-almost everywhere.}$$

LEMMA 7.2. Consider the operator  $\tilde{R}^\alpha$  of Lemma 7.1.

- (i) For each  $\alpha > 0$ ,  $\tilde{R}^\alpha$  is a positive linear operator.
- (ii)  $\tilde{R}^\alpha U'_\alpha 1_M = 1_M$ ,  $\alpha > 0$ .
- (iii)  $\tilde{R}^\alpha$  is a bounded operator on  $\mathbf{B}(M)$  with norm less than  $1 \vee \frac{1}{\alpha}$ .
- (iv)  $\tilde{R}^\alpha \varphi - \tilde{R}^\beta \varphi + \tilde{R}^\alpha (U'_\alpha - U'_\beta) \tilde{R}^\beta \varphi = 0$ ,  $\alpha, \beta > 0$ ,  
 $\varphi \in \mathbf{B}(M)^{21}$ .

PROOF. (i) We can see from condition (B. 3) and identity (3. 21) that every normal contraction operates on  $(\mathcal{F}_M, \mathcal{E}_M^{\lceil \alpha \rceil})$ ; if  $\varphi$  is a normal contraction of  $\phi \in \mathcal{F}_M$ , then  $\varphi \in \mathcal{F}_M$  and  $\mathcal{E}_M^{\lceil \alpha \rceil}(\varphi, \varphi) \leq \mathcal{E}_M^{\lceil \alpha \rceil}(\phi, \phi)$ . Thus,  $\tilde{R}^\alpha$  must be positive. (ii) and (iv). For  $\varphi, \psi \in \mathcal{F}_M$ , since  $\mathcal{E}_M(\varphi, 1_M) = 0$  and  $\mathcal{E}_M^{\lceil \alpha \rceil}(\varphi, \psi) = \mathcal{E}_M(\varphi, \psi) + (\varphi, U'_\alpha \psi)_M$ , equalities of (ii) and (iv) follow from equation (7.4) through simple computations. Assertion (iii) is a consequence of (i), (ii) and inequality (7.9).

LEMMA 7.3. Suppose that a function  $\varphi^x(\xi)$ ,  $x \in D$ ,  $\xi \in M$ , is jointly measurable in  $(x, \xi)$  and bounded in  $\xi$  for each  $x \in D$ . Let  $\nu$  be a signed measure on  $D$  such that  $\varphi^\nu(\xi) = \int_D \varphi^x(\xi) \nu(dx)$  is bounded in  $\xi \in M$ . Then, it holds that

21) See Neveu [20] for an analogous formula.

$$\int_D (\phi, \tilde{R}^\alpha \varphi^x)'_M \nu(dx) = (\phi, \tilde{R}^\alpha \varphi^\nu)'_M$$

for every  $\phi \in L^2(M)'$ .

PROOF. By equation (7.4),  $\tilde{R}^\alpha$  is symmetric on  $L^2(M)'$ . Integrating the identity

$$(\phi, \tilde{R}^\alpha \varphi^x)'_M = (\tilde{R}^\alpha \phi, \varphi^x)'_M$$

by  $\nu$ , we have

$$\int_D (\phi, \tilde{R}^\alpha \varphi^x)'_M \nu(dx) = (\tilde{R}^\alpha \phi, \varphi^\nu)'_M = (\phi, \tilde{R}^\alpha \varphi^\nu)'_M.$$

PROOF OF THEOREM 7.1 (i).

Condition (G. a).  $R_\alpha(x, y) = H_\alpha^x \tilde{R}^\alpha \hat{H}_\alpha^y$  is  $\alpha$ -harmonic in  $x \in D$  for each  $y \in D$ . Its non-negativity is due to Lemma 7.2 (i).

Condition (G. b). Take a compact set  $K$  of  $D$ . In view of Lemma 7.2 (iii),

$$\sup_{x \in D, y \in K} R_\alpha(x, y) \leq \sup_{\xi \in M, y \in K} \tilde{R}^\alpha \hat{H}_\alpha^y(\xi) \leq \left(1 \vee \frac{1}{\alpha}\right) \sup_{\xi \in M, y \in K} \hat{H}_\alpha^y(\xi) < +\infty.$$

Symmetry.  $R_\alpha(x, y) = (\hat{H}_\alpha^x, \tilde{R}^\alpha \hat{H}_\alpha^y)'_M$  is symmetric in  $x, y \in D$ , since  $\tilde{R}^\alpha$  is symmetric on  $L^2(M)'$ .

Conservativity. By Lemma 7.3, identity (7.8) and Lemma 7.2 (ii),

$$\begin{aligned} \alpha R_\alpha 1_D(x) &= \alpha \int_D (\hat{H}_\alpha^x, \tilde{R}^\alpha \hat{H}_\alpha^y)'_M dy \\ &= \alpha (\hat{H}_\alpha^x, \tilde{R}^\alpha (\hat{H}_\alpha 1_D))'_M = (\hat{H}_\alpha^x, \tilde{R}^\alpha U'_\alpha 1_M)'_M \\ &= H_\alpha^x 1_M(x) = 1 - \alpha G_\alpha^0 1_D(x) \end{aligned}$$

and, therefore,  $\alpha G_\alpha 1_D(x) = 1$ ,  $x \in D$ .

Resolvent equation. By making use of the resolvent equation for  $G_\alpha$  and equation (4.5), we can see that

$$G_\alpha(x, y) - G_\beta(x, y) + (\alpha - \beta) \int_D G_\alpha(x, z) G_\beta(z, y) dz$$

is equal to

$$\begin{aligned} (7.10) \quad & (\hat{H}_\alpha^x, \tilde{R}^\alpha \hat{H}_\alpha^y)'_M - (\hat{H}_\alpha^x, \tilde{R}^\beta \hat{H}_\beta^y)'_M \\ & + (\alpha - \beta) \int_D (\hat{H}_\alpha^x, \tilde{R}^\alpha \hat{H}_\alpha^z)'_M (\hat{H}_\beta^z, \tilde{R}^\beta \hat{H}_\beta^y)'_M dz \\ & + (\alpha - \beta) \int_D (\hat{H}_\alpha^x, \tilde{R}^\alpha \hat{H}_\alpha^z)'_M G_\beta^0(z, y) dz. \end{aligned}$$

By virtue of Lemma 7.3 and equations (4.5) and (4.6), (7.10) is seen to be identical with

$$(7.11) \quad (\hat{H}_\alpha^x, \tilde{R}^\alpha \hat{H}_\beta^y)'_M - (\hat{H}_\alpha^x, \tilde{R}^\beta \hat{H}_\beta^y)'_M + (\hat{H}_\alpha^x, \tilde{R}^\alpha (U'_\alpha - U'_\beta) \tilde{R}^\beta \hat{H}_\beta^y)'_M$$

which vanishes according to Lemma 7.2 (iv).

PROOF OF THEOREM 7.1 (ii). Set  $\mathcal{H}'_\alpha = H_\alpha(\mathcal{F}_M) = \{u; u = H_\alpha\varphi, \varphi \in \mathcal{F}_M\}$  and  $\mathcal{E}'^\alpha(u, v) = \mathcal{E}_M^{\alpha\alpha}(\gamma u, \gamma v)$  for  $u, v \in \mathcal{H}'_\alpha$ . It suffices to prove that  $(\mathcal{H}'_\alpha, \mathcal{E}'^\alpha)$  coincides with the space  $(\mathcal{H}_\alpha, \mathcal{E}^\alpha)$ . Space  $(\mathcal{H}'_\alpha, \mathcal{E}'^\alpha)$  is a real Hilbert space since  $(\mathcal{F}_M, \mathcal{E}_M^{\alpha\alpha})$  is. We can see that  $R_\alpha f, f \in \mathbf{B}(D)$ , belongs to  $\mathcal{H}'_\alpha$  and satisfies

$$(7.12) \quad \mathcal{E}'^\alpha(R_\alpha f, v) = (f, v)_D \quad \text{for every } v \in \mathcal{H}'_\alpha.$$

Indeed, according to Lemma 7.3,  $R_\alpha f(x) = H_\alpha^x(\tilde{R}^\alpha \hat{H}_\alpha f)$ . Hence  $R_\alpha f \in \mathcal{H}'_\alpha$  and

$$\begin{aligned} \mathcal{E}'^\alpha(R_\alpha f, H_\alpha \phi) &= \mathcal{E}_M^{\alpha\alpha}(\tilde{R}^\alpha \hat{H}_\alpha f, \phi) \\ &= (\hat{H}_\alpha f, \phi)'_M = (f, H_\alpha \phi)_D \quad \text{for } \phi \in \mathcal{F}_M. \end{aligned}$$

Evidently,  $R_\alpha f, f \in \mathbf{B}(D)$ , is an element of  $\mathcal{H}_\alpha$  and equation (7.12) is still valid if  $\mathcal{E}'^\alpha$  is replaced by  $\mathcal{E}^\alpha$  and  $\mathcal{H}'_\alpha$  by  $\mathcal{H}_\alpha$ . Thus,  $R_\alpha(\mathbf{B}(D))$  being dense in both spaces  $\mathcal{H}'_\alpha$  and  $\mathcal{H}_\alpha$ ,  $(\mathcal{H}'_\alpha, \mathcal{E}'^\alpha)$  must be identical with  $(\mathcal{H}_\alpha, \mathcal{E}^\alpha)$ .

### § 8. A class of diffusions including the reflecting Brownian motion.

In the preceding sections we have established a one-to-one correspondence between the class  $\mathbf{G}$  of symmetric resolvent densities and the class of pairs  $(\mathcal{F}_M, \mathbf{N})$  satisfying conditions (B. 1), (B. 2) and (B. 3).

Denote by  $\mathbf{G}_1$  the totality of  $G_\alpha(x, y)$  in  $\mathbf{G}$  such that the corresponding form  $\mathbf{N}(\cdot)$  vanishes identically on the corresponding space  $\mathcal{F}_M$ . According to those arguments in the preceding two sections, we can assert as follows.

THEOREM 8.1.

(i) *There is a one-to-one correspondence between the class  $\mathbf{G}_1$  and the class of function spaces  $\mathcal{F}_M$  satisfying*

(B<sub>1</sub>. 1)  *$\mathcal{F}_M$  contains every constant function on  $M$  and  $\mathcal{F}_M$  is a linear subspace of  $\mathbf{H}_M$ .*

(B<sub>1</sub>. 2)  *$\mathcal{F}_M$  is closed with the norm  $\mathbf{D}(\cdot) + \lambda(\cdot)'_M$  for a  $\lambda > 0$ .*

(B<sub>1</sub>. 3) *Every normal contraction of an element of  $\mathcal{F}_M$  is also an element of  $\mathcal{F}_M$ .*

(ii) *A linear space  $\mathcal{F}$  of functions on  $D$  with a bilinear form  $\mathcal{E}(\cdot)$  is the refinement<sup>22)</sup> of a Dirichlet space associated with an element of  $\mathbf{G}_1$  if and only if  $\mathcal{F}$  contains every constant function on  $D$ ,  $\text{BLD}_0 \subset \mathcal{F} \subset \widehat{\text{BLD}}$ ,  $\mathcal{E}(u, u) = (u, u)_{D,1}$  for every  $u \in \mathcal{F}$ ,  $\mathcal{F}$  is closed with norm  $\mathcal{E}^\alpha(u, u) = (u, u)_{D,1} + \alpha(u, u)_D$  for an  $\alpha > 0$ , and finally, every normal contraction of an element of  $\mathcal{F}$  is also an element of  $\mathcal{F}$ .*

(iii) *For any element  $G_\alpha(x, y)$  of  $\mathbf{G}_1$ , the function  $u = G_\alpha f$  ( $f \in \mathbf{B}(D)$ ,  $\alpha > 0$ ) belongs to the space  $\widehat{\text{BLD}}$  and*

22) See (4.18).

$$(8.1) \quad (u, u)_{D,1} + \alpha(u, u)_D = (u, f)_D.$$

PROOF. Conditions (B<sub>1</sub>.1), (B<sub>1</sub>.2) and (B<sub>1</sub>.3) of the first assertion are nothing but conditions (B.1), (B.2) and (B.3) with  $N(\cdot) = 0$ . Suppose that a space  $(\mathcal{F}, \mathcal{E})$  satisfies all conditions of statement (ii). Set  $\mathcal{F}_M = \gamma\mathcal{F}$ . Owing to Theorem 3.4,  $H_\alpha(\mathcal{F}_M)$  is the projection of  $(\mathcal{F}, \mathcal{E}^\alpha)$  to the space  $\widehat{\text{BLD}}_{\alpha,h}$  and  $\mathcal{F}_M$  is closed with norm  $D(\cdot) + U_\alpha(\cdot, \cdot)$ . Hence,  $\mathcal{F}_M$  satisfies conditions (B<sub>1</sub>.1) and (B<sub>1</sub>.2) (see the argument preceding Lemma 7.1). The same procedure as in Lemmas 5.2 and 5.3 can be applied to obtain the property (B<sub>1</sub>.3) for  $\mathcal{F}_M$ <sup>23)</sup>. Let  $G_\alpha(x, y)$  be the element of  $G_1$  which corresponds to this  $\mathcal{F}_M$  by means of assertion (i). Then, by virtue of Theorem 4.3 and 7.1, we can see that  $(\mathcal{F}, (\cdot, \cdot)_{D,1})$  is the refinement of the Dirichlet space associated with this  $G_\alpha(x, y)$ . Property (iii) follows from statement (ii).

Our main interest of this section lies on those Markov processes associated with elements of  $G_1$ . As was seen in the final argument of section 1, all the results of section 3 in the article [15] are valid for every resolvent of the class  $G$ .

Further, as far as the elements of the class  $G_1$  are concerned, all the statements of [15; Section 4] are valid, since we never used in [15] any special property of the resolvent density of the reflecting barrier Brownian motion except the above identity (8.1) (see [15; (4.11)]). Thus, we have the following generalization of [15; Theorem 2].

**THEOREM 8.2.** *For each element  $G_\alpha(x, y)$  of  $G_1$ , there exists a diffusion process (a strong Markov process with continuous paths)  $X = (X_t, P_x, x \in D^*)$  on an extended state space  $D^*$  and  $X$  has properties (X.1), (X.2) and (X.3) mentioned in the final part of section 1.*

Here are two extreme cases of elements in  $G_1$ .

(I). *Resolvent density of the reflecting Brownian motion.* This resolvent  $G_\alpha(x, y) = G_\alpha^0(x, y) + R_\alpha(x, y)$  was defined in [15] by means of the equation

$$(8.2) \quad (R_\alpha(x, \cdot), v)_{D,1} + \alpha(R_\alpha(x, \cdot), v)_D = v(x)$$

for every  $v \in \widehat{\text{BLD}}_{\alpha,h}$ . This is fitted for the case that  $\mathcal{F}_M = H_M$  and  $\mathcal{F} = \widehat{\text{BLD}}$ . Indeed, the same procedure as in the proof of [15; Lemma 2.10] is applicable to get from (8.2) the following equation for  $R_\alpha f, f \in B(D)$ ,

$$(8.3) \quad (R_\alpha f, v)_{D,1} + \alpha(R_\alpha f, v)_D = (f, v)_D$$

for all  $v \in \widehat{\text{BLD}}_{\alpha,h}$ . Equation (8.3) implies that the Dirichlet space  $\mathcal{F}$  associated with the resolvent satisfying (8.2) is just the space  $\widehat{\text{BLD}}$ . Note that the result

23) All assertions of Lemma 5.2 and 5.3 are still valid when we replace  $\mathcal{F}$  by the space in the latter statement of Theorem 8.1 (ii) and  $\mathcal{E}^\alpha(\cdot)$ , by  $(\cdot)_{D,1} + \alpha(\cdot)_D$ . Thus,  $\mathcal{F}_M (= \gamma\mathcal{F})$  is a Dirichlet space associated with a resolvent on  $L^2(M)$ '.

in the preceding section gives another method to construct the resolvent density satisfying (8.2)<sup>24)</sup>.

The boundary condition (6.8) for  $u \in G_\alpha(L^2(D))$  is now

$$(8.4) \quad D(\gamma u, \phi) + \left(\frac{1}{2}\Delta u, H\phi\right)_D = 0 \quad \text{for every } \phi \in \mathbf{H}_M.$$

Formula (8.4) means that  $u \in G_\alpha(L^2(D))$  has, as its generalized normal derivative of Doob [7] (in a slightly modified sense), a function identically vanishing on the boundary  $M$ .

(II). *The case when  $\mathcal{F}_M$  is trivial.* Let  $\mathcal{F}_M$  be the set of all constant functions on  $M$ .  $\mathcal{F}_M$  satisfies conditions (B<sub>1</sub>.1), (B<sub>1</sub>.2) and (B<sub>1</sub>.3) of Theorem 8.1 trivially. The corresponding resolvent in  $G_1$  is

$$(8.5) \quad G_\alpha(x, y) = G_\alpha^0(x, y) + \frac{\Pi_\alpha(x)\Pi_\alpha(y)}{\alpha(\Pi_\alpha, 1_D)_D},$$

with  $\Pi_\alpha(x) = H_\alpha 1_M(x)$ <sup>25)</sup>. In fact, by Definition 7.1,  $R_\alpha(x, y)$  is equal to  $H_\alpha^x \tilde{R}^\alpha \hat{H}_\alpha^y$  with  $\tilde{R}^\alpha \hat{H}_\alpha^y$  in  $\mathcal{F}_M$  satisfying equation (7.4) for  $\varphi = \hat{H}_\alpha^y$ . Hence  $\tilde{R}^\alpha \hat{H}_\alpha^y$  is a constant and

$$\tilde{R}^\alpha \hat{H}_\alpha^y = \frac{(\hat{H}_\alpha^y, 1_M)'_M}{U_\alpha(1_M, 1_M)} = \frac{\Pi_\alpha(y)}{\alpha(\Pi_\alpha, 1_D)}.$$

By virtue of Theorem 8.2, the corresponding process  $X$  to (8.5) is a diffusion. However, it may generally include branching points on  $D^* - D$  in Ray's sense<sup>26)</sup>. Suppose that, the relative boundary  $\partial D$  of  $D$  is so smooth that  $G_\alpha^0(x, y) \rightarrow 0$  and  $\Pi_\alpha(x) \rightarrow 1$  ( $\alpha > 0$ ) as  $x$  goes out of any compact subset of  $D$ . Then, the Martin-Kuramochi type completion  $D^*$  of  $D$  with respect to  $\{G_1(x, y)\}$  of (8.5) is just the one point compactification  $D \cup \{\infty\}$  of  $D$  and the extended resolvent density is given by

$$(8.6) \quad G_\alpha(\{\infty\}, y) = \lim_{x \rightarrow \{\infty\}} G_\alpha(x, y) = \frac{\Pi_\alpha(y)}{\alpha(\Pi_\alpha, 1_D)}, \quad \alpha > 0.$$

Hence, owing to Theorem 3.3, the measure  $\alpha G_\alpha(\{\infty\}, y)dy$  converges on  $D \cup \{\infty\}$  to the  $\delta$ -measure concentrated at  $\{\infty\}$  as  $\alpha$  tends to infinity. Thus, we can conclude, under the assumption on the smoothness of  $D$ , that to the resolvent (8.5) corresponds a continuous Hunt process on  $D \cup \{\infty\}$  (including no branching point).

Finally, we note that, besides above extreme cases (I) and (II), there may

24) Cf. [14].

25) The space  $\mathcal{D} = G_\alpha(L^2(D))$  for this resolvent is characterized as follows.  $u \in \mathcal{D}$  if and only if  $u \in \mathcal{D}(\mathcal{A})$ ,  $u$  has a constant boundary function and  $\int_D \Delta u(x) dx = 0$ .

26) Cf. [15].

be many elements of  $G_1^{27}$ . For instance,

(III) *Brownian motion on a torus.* Consider an open square  $D = \{(x_1, x_2); 0 < x_i < 1, i = 1, 2\} \subset R^2$ . The Martin boundary  $M$  of  $D$  consists of all its sides. Put

$$\mathcal{F}_M^T = \{\varphi \in \mathbf{H}_M; \varphi((x_1, 0)) = \varphi((x_1, 1)) \text{ and} \\ \varphi((0, x_2)) = \varphi((1, x_2)) \quad \mu\text{-almost everywhere}\}.$$

$\mathcal{F}_M^T$  satisfies conditions (B<sub>1</sub>. 1), (B<sub>1</sub>. 2), (B<sub>1</sub>. 3) of Theorem 8.1. Therefore, we can associate an element, say  $G_\alpha^*(x, y)$ , of the class  $G_1$  with the space  $\mathcal{F}_M^T$ . Let us show that the corresponding diffusion in Theorem 8.2 is the Brownian motion on the torus  $K = [0, 1) \times [0, 1)$ . Denote the resolvent density of the latter by  $G_\alpha^T(x, y)^{28}$ . Since the Martin-Kuramochi type completion of the domain  $D$  with respect to functions  $\{G_\alpha^T(\cdot, y), y \in D\}$  is just the torus  $K$ , it suffices for us to show that  $G_\alpha^*(x, y) = G_\alpha^T(x, y)$ ,  $x, y \in D$ .  $u(x) = \int_D G_\alpha^T(x, y)f(y)dy$ , with  $f((x_1, x_2)) = f_1(x_1) \cdot f_2(x_2)$ ,  $f_i \in C_0(0, 1)$ ,  $i = 1, 2$ , has the following properties.

(T.1)  $u$  and its first derivatives can be continuously extended to  $[0, 1] \times [0, 1]$ , periodically such as  $u((x_1, 0)) = u((x_1, 1))$ ,  $u((0, x_2)) = u((1, x_2))$ ,  $u_{x_2}((x_1, 0)) = u_{x_2}((x_1, 1))$ ,  $u_{x_1}((0, x_2)) = u_{x_1}((1, x_2))$ , for every  $x_1, x_2 \in [0, 1]^{29}$ .

$$(T.2) \quad \frac{1}{2} \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) u(x) = \alpha u(x) - f(x), \quad x \in D.$$

Hence,  $u \in \mathcal{D}(\Delta)$  and  $\gamma u \in \mathcal{F}_M^T$ . Further,

$$\mathbf{D}(\gamma u, \phi) + \left( -\frac{1}{2} \Delta u, H\phi \right)_D = (H(\gamma u), H\phi)_{D,1} + \frac{1}{2} (\Delta u, H\phi)_D \\ = \int_M \frac{\partial u}{\partial \mathbf{n}}(\xi) \phi(\xi) \sigma(d\xi) = 0 \quad \text{for any } \phi \in \mathcal{F}_M^T \text{ }^{30}.$$

Thus, by Theorem 6.2, we have  $u = G_\alpha^* f$  and consequently,  $G_\alpha^*(x, y) = G_\alpha^T(x, y)$ ,  $x, y \in D$ .

27) It is plausible that the class  $G_1$  is characterized by a family of partitions of the boundary  $M$ .

28)  $G_\alpha^T(x, y)$  is the Laplace transform of the transition density  $p(t, x, y) = \sum_{m, n = -\infty}^{\infty} g(t, x, (y_1 + m, y_2 + n))$ . Here,  $g(t, x, y)$  is the two dimensional Gauss kernel.

29)  $u_{x_i}$  denotes the derivative of  $u$  with respect to the variable  $x_i$  ( $i = 1, 2$ ).

30) The second equality for  $\phi \in \mathbf{H}_M$  is obtained in the similar manner as [15; footnote 5]).  $\mathbf{n}$  denotes the normal and  $\sigma$  denotes the linear Lebesgue measure on  $M$ .  $\sigma$  is absolutely continuous with respect to the measure  $\mu$ .



### §9. Cases of a circular disk and an interval.

(I) *The case of a circular disk.*

Let us examine the case when  $D$  is an open disk of radius 1. The Martin boundary  $M$  of  $D$  is, in this case, identified with its circle whose points can be characterized by the parameter  $\theta$ ;  $0 \leq \theta < 2\pi$ . The Feller kernel  $U(\cdot)$  is a constant multiple of  $\frac{1}{1 - \cos(\theta - \theta')}$  and the space  $H_M$  is given by

$$(9.1) \quad \begin{aligned} H_M &= \{ \varphi \in L^2(d\theta); D(\varphi, \varphi) \\ &= C \int_0^{2\pi} \int_0^{2\pi} (\varphi(\theta) - \varphi(\theta'))^2 \frac{d\theta d\theta'}{1 - \cos(\theta - \theta')} < +\infty \}, \end{aligned}$$

$C$  being a positive constant.

Suppose that the functions  $\sin \theta$  and  $\cos \theta$  belong to the space  $\mathcal{F}_M$  of Theorem 5.2. Then, by making use of formula (5.4), the bilinear form  $N(\cdot)$  of this theorem can be expressed explicitly as follows. For any continuously differentiable function  $\varphi \in \mathcal{F}_M$ ,

$$(9.2) \quad N(\varphi, \varphi) = \int_0^{2\pi} \varphi'(\theta)^2 \nu(d\theta) + \int_0^{2\pi} \int_0^{2\pi} (\varphi(\theta) - \varphi(\theta'))^2 \Phi(d\theta, d\theta'),$$

where,  $\nu$  is a finite measure on  $M$  and  $\Phi$  is a symmetric Radon measure on  $M \times M$  off the diagonal such that, for any  $\delta > 0$ ,

$$(9.3) \quad \iint_{|e^{i\theta} - e^{i\theta'}| > \delta} \Phi(d\theta, d\theta') < +\infty,$$

$$(9.4) \quad \iint_{|e^{i\theta} - e^{i\theta'}| \leq \delta} (1 - \cos(\theta - \theta')) \Phi(d\theta, d\theta') < +\infty.$$

We note that the convergence condition (9.4) for the Levy measure  $\Phi$  may not be satisfied in general<sup>31)</sup>. For instance, choose a measurable function  $a(\theta)$  bounded below and above by strictly positive constants and set

$$\Phi^*(\theta, \theta') = \frac{1}{(1 - \cos(\theta - \theta')) \left( 1 - \cos\left(\frac{\theta + \theta'}{2}\right) \right)},$$

$$N^*(\varphi, \varphi) = \int_0^{2\pi} \varphi'(\theta)^2 a(\theta) d\theta + \int_0^{2\pi} \int_0^{2\pi} (\varphi(\theta) - \varphi(\theta'))^2 \Phi^*(\theta, \theta') d\theta d\theta',$$

$$\mathcal{F}_M^\# = \{ \varphi \in H_M; \varphi \text{ is absolutely continuous and } N^*(\varphi, \varphi) \text{ is finite} \}.$$

The space  $\mathcal{F}_M^\#$  is non-trivial, since it contains the function  $\sin^2 \theta$ . The measure

31) In this sense, our boundary condition (6.8) for the disk is never included by the Wentzell boundary condition [23].

$\Phi^*(\theta, \theta')d\theta d\theta'$  satisfies condition (9.3), but does not satisfy (9.4). However, the pair  $(\mathcal{F}_M^*, N^*)$  clearly satisfies conditions (B.1), (B.2) and (B.3), and hence, on account of Theorem 7.1, we can construct a resolvent  $G_\alpha(x, y)$  of the class  $\mathcal{G}$  which corresponds to this pair (in the manner of Theorem 5.2). I don't know whether the closed disk  $\bar{D}$  is identified with the state space  $D^*$  (the Martin-Kuramochi type completion of  $D$  with respect to  $G_1(x, y)$ ) on which the associated strong Markov process moves.

(II) *One-dimensional case.*

In this case,  $D$  is a finite open interval  $(a, b)$  and the Martin boundary consists of two points  $a$  and  $b$ . We can express explicitly all the resolvents in the class  $\mathcal{G}$ .

(II<sub>1</sub>) *The case when  $\mathcal{F}_M$  is trivial; circular Brownian motion.*

This is one-dimensional case of section 8 (II). The corresponding resolvent is expressed as (8.5). The boundary condition is  $u(a) = u(b)$  and  $u'(a) = u'(b)$ <sup>32)</sup>. The corresponding process is a conservative diffusion on the one-point compactification of  $(a, b)$  and, as one easily sees, it is nothing but the Brownian motion on a circle.

(II<sub>2</sub>) *The case when  $\mathcal{F}_M$  is non-trivial.*

The space  $\mathcal{F}_M$  satisfying (B.1) and (B.2) necessarily consists of all functions on  $\{a, b\}$ .  $N(\varphi, \varphi)$  satisfying (B.2) and (B.3) is written as

$$(9.5) \quad N(\varphi, \varphi) = \kappa(\varphi(a) - \varphi(b))^2$$

with a non-negative constant  $\kappa$ . Thus, this case is completely determined by each  $\kappa \geq 0$ . Take a  $\kappa \geq 0$ . By means of one-dimensional Brownian measure and Brownian hitting time to  $a$  and  $b$ , we set  $H_\alpha^x(a) = E_x(e^{-\alpha\sigma_a}; \sigma_a < \sigma_b)$  and  $H_\alpha^x(b) = E_x(e^{-\alpha\sigma_b}; \sigma_b < \sigma_a)$ ,  $a < x < b$ . Rewriting formulae (7.4) and (7.5), we can derive the following expression of the corresponding resolvent density.

$$(9.6) \quad G_\alpha(x, y) = G_\alpha^0(x, y) + (H_\alpha^x(a), H_\alpha^x(b))A^\alpha \begin{pmatrix} H_\alpha^y(a) \\ H_\alpha^y(b) \end{pmatrix},$$

where  $A^\alpha$  is the inverse of the regular matrix

$$(9.7) \quad \mathcal{M}^\alpha = \begin{pmatrix} U^{ab} + \kappa + U_\alpha^{aa}, & -U^{ab} - \kappa + U_\alpha^{ab} \\ -U^{ab} - \kappa + U_\alpha^{ab}, & U^{ab} + \kappa + U_\alpha^{bb} \end{pmatrix}.$$

Here,  $U^{ab} = U(a, b)\mu(\{a\})\mu(\{b\})$ ,  $U_\alpha^{aa} = U_\alpha(a, a)\mu(\{a\})\mu(\{a\})$  and so on<sup>33)</sup>. Theorem

32) See footnote 25).

33)  $U_\alpha^{ab} = U_\alpha^{ba} = \frac{b-a}{2}(1 - \sqrt{2\alpha} \operatorname{cosech} \sqrt{2\alpha}(b-a))$ ,

$U_\alpha^{aa} = U_\alpha^{bb} = \frac{b-a}{2}(\sqrt{2\alpha} \operatorname{coth} \sqrt{2\alpha}(b-a) - 1)$  and

$U^{ab} = U^{ba} = \frac{b-a}{2}$ .

6.2 states that  $u \in G_\alpha(L^2(a, b))$  if and only if (1)  $u \in \mathcal{D}(\Delta)$ , (2)  $u(x)$  has limits at  $a$  and  $b$  and

$$2(u(a)-u(b))(U^{ab}+\kappa)+\int_a^b \Delta u(x)H_0^x(a)dx=0$$

$$-2(u(a)-u(b))(U^{ab}+\kappa)+\int_a^b \Delta u(x)H_0^x(b)dx=0.$$

It is easy to see that these conditions (1) and (2) are equivalent to the following simple conditions: (1)'  $u$ ,  $u'$  and  $u''$  are square integrable. Here  $u'$  and  $u''$  are the Radon-Nikodym derivatives. (2)'  $u$  and  $u'$  have limits at  $a$  and  $b$  and

$$u'(a)+(u(b)-u(a))\kappa=0$$

$$u'(b)+(u(a)-u(b))\kappa=0.$$

Thus, as for the one-dimensional case, the boundary condition (6.8) is reduced to Feller's one [12] applied to the class  $G$ .

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